

PROOF OF SUN'S CONJECTURAL SUPERCONGRUENCE INVOLVING CATALAN NUMBERS

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ABSTRACT. We confirm a conjectural supercongruence involving Catalan numbers, which is one of the 100 selected open conjectures on congruences of Sun. The proof makes use of hypergeometric series identities and symbolic summation method.

1. **Introduction.** In 2003, Rodriguez-Villegas [14] conjectured the following four supercongruences associated to certain elliptic curves:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{-3}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

where $p \geq 5$ is a prime and $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. These four supercongruences were first proved by Mortenson [12, 13] by using the Gross-Koblitz formula. Guo, Pan and Zhang [3] established some interesting q -analogues of the above four supercongruences. For more q -analogues of congruences, one can refer to [1, 2, 4, 5, 10].

Recall that the Euler numbers are defined as

$$\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!},$$

and the n th Catalan number is given by

$$C_n = \frac{1}{n+1} \binom{2n}{n},$$

which plays an important role in various counting problems. We refer to [17] for many different combinatorial interpretations of the Catalan numbers.

In 2016, Z.-H. Sun [18] proved that for any prime $p \geq 5$,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} C_{2k}}{64^k} \equiv (-1)^{\frac{p-1}{2}} - 3p^2 E_{p-3} \pmod{p^3},$$

which was originally conjectured by Z.-W. Sun [19].

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Mao and Z.-W. Sun [11] showed that for any prime $p \geq 5$,

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_{2k}}{64^k} \equiv (-1)^{\frac{p-1}{2}} 2^{p-1} \pmod{p^2}. \tag{1}$$

Z.-W. Sun [22, Conjecture 11] also conjectured an extension of (1) as follows.

Conjecture 1.1 (Sun, 2019). *For any prime $p \geq 5$, we have*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_{2k}}{64^k} \equiv (-1)^{\frac{p-1}{2}} (2^{p-1} - (2^{p-1} - 1)^2) \pmod{p^3}. \tag{2}$$

The main purpose of the paper is to prove (2). Our proof is based on hypergeometric series identities and symbolic summation method.

Theorem 1.2. *The supercongruence (2) is true.*

We establish two preliminary results in the next section. The proof of Theorem 1.2 will be given in Section 3.

2. Preliminary results. In order to prove Theorem 1.2, we need the following two key results.

Proposition 2.1. *For any prime $p \geq 5$, we have*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(2k-1)2^k 4^k} \equiv (-1)^{\frac{p-1}{2}} (2^{p-1} - q_p(2)) \pmod{p^2}, \tag{3}$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(2k-1)^3 16^k} \equiv 2 - 2q_p(2) - p(q_p(2)^2 - 4q_p(2) + 3) \pmod{p^2}, \tag{4}$$

where $q_p(2)$ is the Fermat quotient $(2^{p-1} - 1)/p$.

Remark. Z.-W. Sun [20, (1.7)] and [21, (1.7), (3.3), (3.4)] has proved the following closely related results:

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2 4^k} &\equiv (-1)^{\frac{p+1}{2}} \frac{q_p(2)^2}{2} \pmod{p}, \\ \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)16^k} &\equiv -2q_p(2) - pq_p(2)^2 + \frac{5p^2}{12} B_{p-3} \pmod{p^3}, \\ \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)^2 16^k} &\equiv -2q_p(2)^2 + \frac{2p}{3} q_p(2)^3 - \frac{p}{6} B_{p-3} \pmod{p^2}, \\ \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)^3 16^k} &\equiv -\frac{4}{3} q_p(2)^3 - \frac{1}{6} B_{p-3} \pmod{p}, \end{aligned}$$

where the Bernoulli numbers are given by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Before proving Proposition 2.1, we establish the following lemma.

Lemma 2.2. *For any integer $n \geq 2$, we have*

$$\sum_{k=0}^n \frac{(-n)_k (n-1)_k}{(1)_k \left(\frac{1}{2}\right)_k} = \frac{(-1)^{n-1}}{2n-1}, \tag{5}$$

$$\sum_{k=0}^n \frac{(-n)_k (n-1)_k \left(-\frac{1}{2}\right)_k}{(1)_k^2 \left(\frac{1}{2}\right)_k} = \frac{4n(n-1)}{2n-1}, \tag{6}$$

where $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$.

Proof. Recall Gauss' theorem [16, (1.7.6), page 28]:

$${}_2F_1 \left[\begin{matrix} a & b \\ & c \end{matrix} ; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{7}$$

provided that $\Re(c-a-b) > 0$. Letting $a = -n, b = n-1$ and $c = \frac{1}{2}$ in (7) gives

$${}_2F_1 \left[\begin{matrix} -n & n-1 \\ & \frac{1}{2} \end{matrix} ; 1 \right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{3}{2}-n\right)} = \frac{(-1)^{n-1}}{2n-1},$$

which is (5).

Also, we have the following transformation formula of hypergeometric series [16, (2.5.11), page 76]:

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} a & b & -n \\ & e & f \end{matrix} ; 1 \right] = \frac{(e-a)_n (f-a)_n}{(e)_n (f)_n} \\ & \times {}_3F_2 \left[\begin{matrix} 1-s & a & -n \\ & 1+a-e-n & 1+a-f-n \end{matrix} ; 1 \right], \end{aligned} \tag{8}$$

where $s = e + f - a - b + n$. Letting $a = n-1, b = -\frac{1}{2}, e = x$ and $f = \frac{3}{2} - x$ in (8) yields

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} n-1 & -\frac{1}{2} & -n \\ & x & \frac{3}{2}-x \end{matrix} ; 1 \right] = \frac{(x+1-n)_n \left(\frac{5}{2}-x-n\right)_n}{(x)_n \left(\frac{3}{2}-x\right)_n} \\ & \times {}_3F_2 \left[\begin{matrix} -2 & n-1 & -n \\ & -x & x-\frac{3}{2} \end{matrix} ; 1 \right]. \end{aligned} \tag{9}$$

Furthermore, we can evaluate the terminating hypergeometric series on the right-hand side of (9):

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} -2 & n-1 & -n \\ & -x & x-\frac{3}{2} \end{matrix} ; 1 \right] \\ & = \frac{4x^4 - 12x^3 + (-8n^2 + 8n + 11)x^2 + (12n^2 - 12n - 3)x + 4n(n-1)(n^2 - n - 1)}{x(x-1)(2x-1)(2x-3)}. \end{aligned}$$

It follows that

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} n-1 & -\frac{1}{2} & -n \\ & x & \frac{3}{2}-x \end{matrix} ; 1 \right] = \frac{(x+1-n)_n \left(\frac{5}{2}-x-n\right)_n}{(x)_n \left(\frac{3}{2}-x\right)_n} \\ & \times \frac{4x^4 - 12x^3 + (-8n^2 + 8n + 11)x^2 + (12n^2 - 12n - 3)x + 4n(n-1)(n^2 - n - 1)}{x(x-1)(2x-1)(2x-3)}. \end{aligned} \tag{10}$$

Letting $x \rightarrow 1$ on both sides of (10) and noting that

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{4x^4 - 12x^3 + (-8n^2 + 8n + 11)x^2 + (12n^2 - 12n - 3)x + 4n(n-1)(n^2 - n - 1)}{x(2x-1)(2x-3)} \\ = -4n^2(n-1)^2, \end{aligned}$$

and

$$\lim_{x \rightarrow 1} \frac{(x+1-n)_n \left(\frac{5}{2} - x - n\right)_n}{(x-1)(x)_n \left(\frac{3}{2} - x\right)_n} = -\frac{1}{n(n-1)(2n-1)},$$

we arrive at

$${}_3F_2 \left[\begin{matrix} n-1 & -\frac{1}{2} & -n \\ & 1 & \frac{1}{2} \end{matrix}; 1 \right] = \frac{4n(n-1)}{2n-1},$$

which proves (6). □

Proof of (3). We can rewrite (5) as

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{(-n)_k (n-1)_k}{(1)_k \left(\frac{1}{2}\right)_k} &= \frac{(-1)^{n-1}}{2n-1} - \frac{(-n)_n (n-1)_n}{(1)_n \left(\frac{1}{2}\right)_n} \\ &= \frac{(-1)^{n-1}}{2n-1} (1 + 4^{n-1}(2n-2)) \\ &= (-1)^{n-1} \left(2^{2n-2} - \frac{2^{2n-2} - 1}{2n-1} \right). \end{aligned} \tag{11}$$

Letting $n = \frac{p+1}{2}$ in (11) gives

$$\sum_{k=0}^{(p-1)/2} \frac{\left(\frac{-1-p}{2}\right)_k \left(\frac{-1+p}{2}\right)_k}{(1)_k \left(\frac{1}{2}\right)_k} = (-1)^{\frac{p-1}{2}} (2^{p-1} - q_p(2)).$$

Since for $0 \leq k \leq \frac{p-1}{2}$,

$$\left(\frac{-1-p}{2}\right)_k \left(\frac{-1+p}{2}\right)_k \equiv \left(-\frac{1}{2}\right)_k^2 \pmod{p^2}, \tag{12}$$

we have

$$\sum_{k=0}^{(p-1)/2} \frac{\left(-\frac{1}{2}\right)_k^2}{(1)_k \left(\frac{1}{2}\right)_k} \equiv (-1)^{\frac{p-1}{2}} (2^{p-1} - q_p(2)) \pmod{p^2}. \tag{13}$$

Note that

$$\frac{\left(\frac{1}{2}\right)_k}{(1)_k} = \frac{\binom{2k}{k}}{4^k}, \tag{14}$$

$$\frac{\left(-\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_k} = \frac{1}{1-2k}. \tag{15}$$

Then the proof of (3) follows from (13)–(15). □

Proof of (4). We can rewrite (6) as

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{(-n)_k (n-1)_k \left(-\frac{1}{2}\right)_k}{(1)_k^2 \left(\frac{1}{2}\right)_k} &= \frac{4n(n-1)}{2n-1} - \frac{(-n)_n (n-1)_n \left(-\frac{1}{2}\right)_n}{(1)_n^2 \left(\frac{1}{2}\right)_n} \\ &= \frac{1}{2n-1} \left(4n(n-1) + (-1)^n \binom{2n-2}{n} \right). \end{aligned} \tag{16}$$

Letting $n = \frac{p+1}{2}$ in (16) and using (12), we obtain

$$\sum_{k=0}^{(p-1)/2} \frac{\left(-\frac{1}{2}\right)_k^3}{(1)_k^2 \left(\frac{1}{2}\right)_k} \equiv \frac{1}{p} \left(p^2 - 1 + (-1)^{\frac{p+1}{2}} \binom{p-1}{\frac{p+1}{2}} \right) \pmod{p^2}.$$

For $0 \leq k \leq p-1$, we have

$$\begin{aligned} \binom{p-1}{k} &\equiv (-1)^k \left(1 - p \sum_{i=1}^k \frac{1}{i} + p^2 \sum_{1 \leq i < j \leq k} \frac{1}{ij} \right) \\ &= (-1)^k \left(1 - pH_k + \frac{p^2}{2} (H_k^2 - H_k^{(2)}) \right) \pmod{p^3}, \end{aligned} \tag{17}$$

where

$$H_k^{(r)} = \sum_{j=1}^k \frac{1}{j^r},$$

with the convention that $H_k = H_k^{(1)}$. It follows that

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\left(-\frac{1}{2}\right)_k^3}{(1)_k^2 \left(\frac{1}{2}\right)_k} &\equiv \frac{p}{2} \left(H_{\frac{p+1}{2}}^2 - H_{\frac{p+1}{2}}^{(2)} + 2 \right) - H_{\frac{p+1}{2}} \\ &= \frac{p}{2} \left(H_{\frac{p-1}{2}}^2 + 4H_{\frac{p-1}{2}} - H_{\frac{p-1}{2}}^{(2)} + 6 \right) - H_{\frac{p-1}{2}} - 2 \pmod{p^2}. \end{aligned} \tag{18}$$

By [7, (41)] and [19, Lemma 2.4], we have

$$H_{\frac{p-1}{2}} \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2}, \tag{19}$$

and

$$H_{\frac{p-1}{2}}^{(2)} \equiv 0 \pmod{p}. \tag{20}$$

Substituting (19) and (20) into (18) gives

$$\sum_{k=0}^{(p-1)/2} \frac{\left(-\frac{1}{2}\right)_k^3}{(1)_k^2 \left(\frac{1}{2}\right)_k} \equiv 2q_p(2) - 2 + p(q_p(2)^2 - 4q_p(2) + 3) \pmod{p^2}. \tag{21}$$

Finally, applying (14) and (15) to the left-hand side of (21), we reach

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(2k-1)^3 16^k} \equiv 2 - 2q_p(2) - p(q_p(2)^2 - 4q_p(2) + 3) \pmod{p^2},$$

as desired. □

3. Proof of Theorem 1.2.

Lemma 3.1. *For any non-negative integer n , we have*

$$\sum_{k=0}^n \frac{(-n)_k (n+1)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(1)_k^2 \left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_k} = \frac{\binom{2n}{n}}{4^n}, \tag{22}$$

and

$$\begin{aligned} & \sum_{k=0}^n \frac{(-n)_k (n+1)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(1)_k^2 \left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_k} \sum_{j=1}^k \frac{1}{(2j-1)^2} \\ &= -\frac{\binom{2n}{n}}{4^n} \left(3 + \sum_{k=1}^n \frac{1}{(2k-1)^2} \right) + \frac{2}{2n+1} \sum_{k=0}^n \frac{\binom{2k}{k}}{(2k-1)^2 4^k} \\ & \quad - \frac{4^n}{(2n+1)\binom{2n}{n}} \sum_{k=0}^n \frac{\binom{2k}{k}^2}{(2k-1)^3 16^k}. \end{aligned} \tag{23}$$

Proof. Recall that (see [16, (2.4.2.2), page 65])

$${}_4F_3 \left[\begin{matrix} d & 1+f-g & \frac{f}{2} & \frac{f+1}{2} \\ & 1+f & \frac{1+f+d-g}{2} & 1 + \frac{f+d-g}{2} \end{matrix}; 1 \right] = \frac{\Gamma(g-f)\Gamma(g-d)}{\Gamma(g)\Gamma(g-f-d)}. \tag{24}$$

Letting $d = -n, f = \frac{1}{2}$ and $g = -n + \frac{1}{2}$ in (24), we obtain

$${}_4F_3 \left[\begin{matrix} -n & n+1 & \frac{1}{4} & \frac{3}{4} \\ & 1 & \frac{1}{2} & \frac{3}{2} \end{matrix}; 1 \right] = \frac{\Gamma(-n)\Gamma(\frac{1}{2})}{\Gamma(-n+\frac{1}{2})\Gamma(0)} = \frac{\binom{2n}{n}}{4^n},$$

which is (22).

On the other hand, (23) can be discovered and proved by symbolic summation package **Sigma** due to Schneider [15]. One can refer to [9] for the same approach to finding and proving identities of this type. \square

Proof of (2). Recall that (see [8, (4.4)])

$$\left(\frac{1+p}{2}\right)_k \left(\frac{1-p}{2}\right)_k \equiv \left(\frac{1}{2}\right)_k^2 \left(1 - p^2 \sum_{j=1}^k \frac{1}{(2j-1)^2}\right) \pmod{p^4}. \tag{25}$$

Letting $n = \frac{p-1}{2}$ in (22) and using (25), we obtain

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(1)_k^2 \left(\frac{3}{2}\right)_k} &\equiv \frac{1}{2^{p-1}} \binom{p-1}{\frac{p-1}{2}} \\ & \quad + p^2 \sum_{k=0}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(1)_k^2 \left(\frac{3}{2}\right)_k} \sum_{j=1}^k \frac{1}{(2j-1)^2} \pmod{p^4}, \end{aligned} \tag{26}$$

where we have utilized the fact $\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k / \left(\frac{3}{2}\right)_k \in \mathbb{Z}_p$ for $0 \leq k \leq \frac{p-1}{2}$.

From (25), we deduce that

$$\left(\frac{1+p}{2}\right)_k \left(\frac{1-p}{2}\right)_k \equiv \left(\frac{1}{2}\right)_k^2 \pmod{p^2}. \tag{27}$$

Letting $n = \frac{p-1}{2}$ in (23) and using (27) gives

$$\begin{aligned} & \sum_{k=0}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{\left(1\right)_k^2 \left(\frac{3}{2}\right)_k} \sum_{j=1}^k \frac{1}{(2j-1)^2} \\ & \equiv -\frac{1}{2^{p-1}} \binom{p-1}{\frac{p-1}{2}} \left(3 + \sum_{k=1}^{(p-1)/2} \frac{1}{(2k-1)^2} \right) + \frac{2}{p} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(2k-1)^2 4^k} \\ & - \frac{2^{p-1}}{p \binom{p-1}{\frac{p-1}{2}}} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(2k-1)^3 16^k} \pmod{p^2}. \end{aligned} \tag{28}$$

Substituting (28) into (26) yields

$$\begin{aligned} & \sum_{k=0}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{\left(1\right)_k^2 \left(\frac{3}{2}\right)_k} \equiv \frac{1}{2^{p-1}} \binom{p-1}{\frac{p-1}{2}} - \frac{p^2}{2^{p-1}} \binom{p-1}{\frac{p-1}{2}} \left(3 + \sum_{k=1}^{(p-1)/2} \frac{1}{(2k-1)^2} \right) \\ & + 2p \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(2k-1)^2 4^k} - \frac{2^{p-1} p}{\binom{p-1}{\frac{p-1}{2}}} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(2k-1)^3 16^k} \pmod{p^4}. \end{aligned} \tag{29}$$

Furthermore, by (17), (19) and (20) we have

$$\begin{aligned} \binom{p-1}{\frac{p-1}{2}} & \equiv (-1)^{\frac{p-1}{2}} \left(1 - pH_{\frac{p-1}{2}} + \frac{p^2}{2} \left(H_{\frac{p-1}{2}}^2 - H_{\frac{p-1}{2}}^{(2)} \right) \right) \\ & \equiv (-1)^{\frac{p-1}{2}} \left(1 + 2pq_p(2) + p^2 q_p(2)^2 \right) \pmod{p^3}. \end{aligned} \tag{30}$$

By (20) and the Wolstenholme's theorem [6, page 114], we have

$$\sum_{k=1}^{(p-1)/2} \frac{1}{(2k-1)^2} = H_{p-1}^{(2)} - \frac{1}{4} H_{\frac{p-1}{2}}^{(2)} \equiv 0 \pmod{p}. \tag{31}$$

Setting $2^{p-1} = a$ and $q_p(2) = (a-1)/p$, and then substituting (3), (4), (30) and (31) into (29), we arrive at

$$\begin{aligned} & \sum_{k=0}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{\left(1\right)_k^2 \left(\frac{3}{2}\right)_k} \equiv (-1)^{\frac{p-1}{2}} \left(\frac{a^3 - 2a^2 + 4a - 2}{2a-1} + \frac{3(a-1)^2 p^2}{a(2a-1)} \right) \\ & = (-1)^{\frac{p-1}{2}} \left(a - (a-1)^2 + \frac{3(a-1)^3}{2a-1} + \frac{3(a-1)^2 p^2}{a(2a-1)} \right) \pmod{p^3}. \end{aligned}$$

By the Fermat's little theorem, we have $a-1 \equiv 0 \pmod{p}$, and so

$$\sum_{k=0}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{\left(1\right)_k^2 \left(\frac{3}{2}\right)_k} \equiv (-1)^{\frac{p-1}{2}} (a - (a-1)^2) \pmod{p^3}. \tag{32}$$

Note that

$$\frac{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{\left(1\right)_k^2} = \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k}, \tag{33}$$

$$\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k} = \frac{1}{2k+1}. \tag{34}$$

Then the proof of (2) follows from (32)–(34). □

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REFERENCES

- [1] V. J. W. Guo, [Proof of a generalization of the \(B.2\) supercongruence of Van Hamme through a \$q\$ -microscope](#), *Adv. in Appl. Math.*, **116** (2020).
- [2] V. J. W. Guo and J.-C. Liu, [\$q\$ -Analogues of two Ramanujan-type formulas for \$1/\pi\$](#) , *J. Difference Equ. Appl.*, **24** (2018), 1368–1373.
- [3] V. J. W. Guo, H. Pan and Y. Zhang, [The Rodriguez-Villegas type congruences for truncated \$q\$ -hypergeometric functions](#), *J. Number Theory*, **174** (2017), 358–368.
- [4] V. J. W. Guo and M. J. Schlosser, A family of q -hypergeometric congruences modulo the fourth power of a cyclotomic polynomial, preprint, [arXiv:1909.10294](#).
- [5] V. J. W. Guo and M. J. Schlosser, [Some new \$q\$ -congruences for truncated basic hypergeometric series: Even powers](#), *Results Math.*, **75** (2020), 15pp.
- [6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, Oxford, 2008.
- [7] E. Lehmer, [On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson](#), *Ann. of Math. (2)*, **39** (1938), 350–360.
- [8] J.-C. Liu, [On Van Hamme's \(A.2\) and \(H.2\) supercongruences](#), *J. Math. Anal. Appl.*, **471** (2019), 613–622.
- [9] J.-C. Liu, [Semi-automated proof of supercongruences on partial sums of hypergeometric series](#), *J. Symbolic Comput.*, **93** (2019), 221–229.
- [10] J.-C. Liu and F. Petrov, [Congruences on sums of \$q\$ -binomial coefficients](#), *Adv. in Appl. Math.*, **116** (2020), 11pp.
- [11] G.-S. Mao and Z.-W. Sun, [New congruences involving products of two binomial coefficients](#), *Ramanujan J.*, **49** (2019), 237–256.
- [12] E. Mortenson, [A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function](#), *J. Number Theory*, **99** (2003), 139–147.
- [13] E. Mortenson, [Supercongruences between truncated \${}_2F_1\$ hypergeometric functions and their Gaussian analogs](#), *Trans. Amer. Math. Soc.*, **355** (2003), 987–1007.
- [14] F. Rodriguez-Villegas, Hypergeometric families of Calabi-Yau manifolds, in *Calabi-Yau Varieties and Mirror Symmetry*, Fields Inst. Commun., 38, Amer. Math. Soc., Providence, RI, 2003, 223–231.
- [15] C. Schneider, Symbolic summation assists combinatorics, *Sém. Lothar. Combin.*, **56** (2006/07), 36pp.
- [16] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.
- [17] R. P. Stanley, *Enumerative Combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, 62, Cambridge University Press, 1999.
- [18] Z.-H. Sun, [Super congruences involving Bernoulli polynomials](#), *Int. J. Number Theory*, **12** (2016), 1259–1271.
- [19] Z.-W. Sun, [Super congruences and Euler numbers](#), *Sci. China Math.*, **54** (2011), 2509–2535.
- [20] Z.-W. Sun, [On congruences related to central binomial coefficients](#), *J. Number Theory*, **131** (2011), 2219–2238.
- [21] Z.-W. Sun, [\$p\$ -adic congruences motivated by series](#), *J. Number Theory*, **134** (2014), 181–196.
- [22] Z.-W. Sun, [Open conjectures on congruences](#), *Nanjing Univ. J. Math. Biquarterly*, **36** (2019), 1–99.

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