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THE MAHLER MEASURE OF $(x+1/x)(y+1/y)(z+1/z)+\sqrt{k}$

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Abstract. In this paper we study the Mahler measures of reciprocal polynomials $(x+1/x)(y+1/y)(z+1/z)+\sqrt{k}$ for $k=16$, $k=-104\pm 60\sqrt{3}$, 4096 and $k=-2024\pm 765\sqrt{7}$. We prove six conjectural identities proposed by Samart in [16].

1. Introduction

Let f be a nonzero Laurent polynomial in $\mathbb{C}[T_1,\dots,\ T_n,T_1^{-1},\dots,T_n^{-1}]$. The logarithmic Mahler measure of f is defined as

$$m(f)=\int_0^1\cdots\int_0^1 \log|f(\mathrm{e}^{2\pi i\theta_1},\ ...,\ \mathrm{e}^{2\pi i\theta_n})|d\theta_1\cdots d\theta_n.$$

There have been a lot of numerical evidence and theoretical results implying that there exist non-trivial relations between Mahler measure of certain kind of Laurent polynomials and special values of L-functions. The first example is provided by Smyth[17]:

$$m(x+y+1)=\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)=L'(\chi_{-3},1),$$

where χ_D denotes the Jacobi symbol $\left(\frac{D}{\cdot}\right)$.

Suppose P is a two-variable Laurent polynomial. If $P(x,y)=0$ happens to define an elliptic curve E over $\mathbb{Q}$, then $m(P)$ can sometimes be related to $L(E,2)$ by Beilinson's regulator maps. It is Deninger who first noticed this and conjectured in [5] that

$$m(1+X+1/X+Y+1/Y)=L'(E_{15},0),$$

where E_{15} is an elliptic curve with conductor 15. This conjecture is proved by Rogers and Zudilin in [11] and [12]. After Deninger's discovery, Boyd proposed hundreds of conjectures in this direction [4].

In 2008, Bertin [1, 2] investigated the Mahler measure of three-variable Laurent polynomials:

$$m(x+x^{-1}+y+y^{-1}+z+z^{-1}+k).$$

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Later, Rogers [10] considered the following functions:

$$\begin{aligned} f_2(k) =& 2m\left(\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)+\sqrt{k}\right), \\ f_4(k) =& 4m(x^4+y^4+z^4+1+k^{1/4}xyz), \end{aligned}$$

where $k \in \mathbb{C}$ is a constant. Samart's work shows that there are some appealing connections between the special values of the above functions and the special value of L-functions of certain modular forms [15]. Actually Samart made many more conjectures of this type in [15] and [16]. The following five identities conjectured in [15] have been proved by Guo, Peng and Qin in [7]:

$$\begin{aligned} f_2(-64) &= 2L'(g_{32},0)+2L'(\chi_{-4},-1), \\ f_2(-512) &= L'(g_{64},0)+L'(\chi_{-8},-1), \\ f_4(-1024) &= \frac{8}{5}(5L'(g_{20},0)+2L'(\chi_{-4},-1)), \\ f_4(-12288) &= \frac{40}{9}(L'(g_{36},0)+2L'(\chi_{-3},-1)), \\ f_4(-82944) &= \frac{40}{13}(L'(g_{52},0)+2L'(\chi_{-4},-1)), \end{aligned}$$

where g_N denotes some newform with rational coefficients in the space of cusp forms of weight 3 and level N.

In this article, we prove six identities conjectured in 2015 in [16]:

$$\begin{aligned} f_2(16) =& 8M_{12}, \\ f_2(-104-60\sqrt{3}) =& \frac{1}{6}(4M_{12\otimes(-4)}+36M_{12}+15d_3+8d_4), \\ f_2(-104+60\sqrt{3}) =& \frac{1}{2}(4M_{12\otimes(-4)}-36M_{12}+15d_3-8d_4), \\ f_2(4096) =& \frac{4}{7}(M_{7\otimes(-4)}+8d_4), \\ f_2(-2024-765\sqrt{7}) =& \frac{1}{14}(4M_{7\otimes(-4)}+384M_7+32d_4+11d_7), \\ f_2(-2024+765\sqrt{7}) =& \frac{1}{2}(4M_{7\otimes(-4)}-384M_7-32d_4+11d_7). \end{aligned}$$

Here, $d_k = L'(\chi_{-k},-1)$, $M_N = L'(g_N,0)$, $M_{N\otimes D} = L'(g_N\otimes\chi_D,0)$ and $g_N\otimes\chi_D$ is the quadratic twist of g_N by χ_D.

In addition to the complicated computation of the lattice sums, which is mainly due to the distinct forms of the integral basis of the number fields $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-7})$, some techniques that work well in [7] fail in our cases. For example, the method of finding the identities of the corresponding modular forms fails, as the L-functions are no longer equal to each other in one of our cases. In order to prove the identities of the special values of L-functions, we have to develop another method, see Theorem 4.2 in this paper. On the other hand, it seems that the identities like $f_2(1) = 8M_7$ in [16] cannot be proved by any of our previous techniques, although they appear very similar to the ones we have verified. In fact we are still not able to prove them.

2. Modular functions

As usual we denote the *Dedekind eta function* by $\eta(\tau)$. Then we have $\Delta(\tau) = \eta(\tau)^{24}$, where Δ is the *modular discriminant*. Define

$$s_2(q(\tau)) = -\frac{\Delta(\tau+\frac{1}{2})}{\Delta(2\tau+1)}.$$

Write $\mathfrak{f}(\tau)$ for the classical Weber modular function $e^{-\frac{\pi i}{24}}\frac{\eta((\tau+1)/2)}{\eta(\tau)}$. The following relation holds (cf. [15], Lemma 2.2)

$$s_2(q(\tau)) = \mathfrak{f}(2\tau)^{24}.$$

Let $\mathfrak{f}_1(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)}$, $\mathfrak{f}_2(\tau) = \sqrt{2}\frac{\eta(2\tau)}{\eta(\tau)}$. We have $\mathfrak{f}(\tau)\mathfrak{f}_1(\tau)\mathfrak{f}_2(\tau) = \sqrt{2}$.

Moreover, $\mathfrak{f}(\tau)^{24}$, $-\mathfrak{f}_1(\tau)^{24}$, $-\mathfrak{f}_2(\tau)^{24}$ are exactly the three roots of the cubic equation

$$(x-16)^3 - j(\tau)x = 0,$$

where $j(\tau)$ is the classical j-invariant. One can see more details on page 288 of [3].

Lemma 2.1. *We have the following special values of* $\mathfrak{f}(\tau)^{24}$*:*

(1) $\mathfrak{f}((1+\sqrt{-3})/2)^{24} = s_2(q((1+\sqrt{-3})/4)) = 16.$
(2) $\mathfrak{f}(1+\sqrt{-3})^{24} = s_2(q((1+\sqrt{-3})/2)) = -104-60\sqrt{3}.$
(3) $\mathfrak{f}((3+\sqrt{-3})/3)^{24} = s_2(q((3+\sqrt{-3})/6)) = -104+60\sqrt{3}.$
(4) $\mathfrak{f}(\sqrt{-7})^{24} = s_2(q(\sqrt{-7}/2)) = 4096.$
(5) $\mathfrak{f}(1+\sqrt{-7})^{24} = s_2(q((1+\sqrt{-7})/2)) = -2024-765\sqrt{7}.$
(6) $\mathfrak{f}((7+\sqrt{-7})/7)^{24} = s_2(q((7+\sqrt{-7})/14)) = -2024+765\sqrt{7}.$

Proof. (1) We have

$$\begin{pmatrix} \mathfrak{f}(\tau+1) \\ \mathfrak{f}_1(\tau+1) \\ \mathfrak{f}_2(\tau+1) \end{pmatrix} = \begin{pmatrix} 0 & \zeta^{-1} & 0 \\ \zeta^{-1} & 0 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix} \begin{pmatrix} \mathfrak{f}(\tau) \\ \mathfrak{f}_1(\tau) \\ \mathfrak{f}_2(\tau) \end{pmatrix} \tag{2.1}$$

where $\zeta = e^{\frac{\pi i}{24}}$ (cf. page 1647 of [19]). Setting $x=16$ in equation $(x-16)^3 - j(\tau)x = 0$, we get $j(\tau) = 0$, which implies τ can be taken as $\frac{-1+\sqrt{-3}}{2}$.

It follows from $(x-16)^3 = 0$ that

$$\mathfrak{f}(\tau)^{24} = 16, -\mathfrak{f}_1(\tau)^{24} = 16, -\mathfrak{f}_2(\tau)^{24} = 16,$$

and we are done.

(2) Setting $x = -104 \pm 60\sqrt{3}$ in equation $(x-16)^3 - j(\tau)x = 0$, one gets $j(\tau) = 54000$. The three roots of the equation $(x-16)^3 - 54000x = 0$ are

$$\{x_1 = 256, x_2 = -104-60\sqrt{3}, x_3 = -104+60\sqrt{3}\}.$$

By Weber's table VI [18], $\mathfrak{f}(\sqrt{-3}) = \sqrt[3]{2}$, i.e., $\mathfrak{f}(\sqrt{-3})^{24} = 2^8 = 256$, hence

$$\left\{-\mathfrak{f}_1(\sqrt{-3})^{24}, -\mathfrak{f}_2(\sqrt{-3})^{24}\right\} = \left\{-104+60\sqrt{3}, -104-60\sqrt{3}\right\}.$$

By numerical computations, one can deduce that $\mathfrak{f}_1(\sqrt{-3})^{24} = 104+60\sqrt{3}$ and $\mathfrak{f}_2(\sqrt{-3})^{24} = 104-60\sqrt{3}$.

Using (2.1) we obtain

$$\mathfrak{f}(1+\sqrt{-3})^{24} = -\mathfrak{f}_1(\sqrt{-3})^{24} = -104-60\sqrt{3}.$$

(3) Note that the transformation $\tau \mapsto -\frac{1}{\tau}$ fixes $\mathfrak{f}$, and exchanges $\mathfrak{f}_1$, $\mathfrak{f}_2$. Hence, $\mathfrak{f}_1(-1/\sqrt{-3})^{24} = \mathfrak{f}_2(\sqrt{-3})^{24} = 104 - 60\sqrt{3}$, which implies

$$\mathfrak{f}\left(1 + \frac{-1}{\sqrt{-3}}\right)^{24} = -\mathfrak{f}_1\left(\frac{-1}{\sqrt{-3}}\right)^{24} = -104 + 60\sqrt{3}.$$

(4) By Weber's table VI [18], we have $\mathfrak{f}(\sqrt{-7}) = \sqrt{2}$, which implies

$$\mathfrak{f}(\sqrt{-7})^{24} = 4096.$$

(5) & (6): Setting $x = -2024 \pm 765\sqrt{7}$ in equation $(x-16)^3 - j(\tau)x = 0$, we get $j(\tau) = 16581375$. Solving the equation $(x-16)^3 - 16581375x = 0$ one obtains three roots:

$$\left\{x_1 = 4096,\ x_2 = -2024 + 765\sqrt{7},\ x_3 = -2024 - 765\sqrt{7}\right\}.$$

From above, we have known that $\mathfrak{f}(\sqrt{-7})^{24} = 4096$. Thus

$$\left\{-\mathfrak{f}_1(\sqrt{-7})^{24}, -\mathfrak{f}_2(\sqrt{-7})^{24}\right\} = \{-2024 + 765\sqrt{7}, -2024 - 765\sqrt{7}\}.$$

By numerical computation, one can decide that

$$-\mathfrak{f}_1(\sqrt{-7})^{24} = -2024 - 765\sqrt{7}$$

and

$$-\mathfrak{f}_2(\sqrt{-7})^{24} = -2024 + 765\sqrt{7}.$$

Therefore

$$\mathfrak{f}\left(1 + \sqrt{-7}\right)^{24} = -\mathfrak{f}_1(\sqrt{-7})^{24} = -2024 - 765\sqrt{7}.$$

Moreover, we have

$$-\mathfrak{f}_1(\sqrt{-7}/7)^{24} = -2024 + 765\sqrt{7},$$

which implies

$$\mathfrak{f}\left(1 + \frac{\sqrt{-7}}{7}\right)^{24} = -\mathfrak{f}_1\left(\sqrt{-7}/7\right)^{24} = -2024 + 765\sqrt{7}.$$

□

Lemma 2.2. ([15], Proposition 2.1(i)). *Assume that* $q \in (0,1)$. *If* $\mathrm{Im}(\tau) \geq \frac{1}{2}$, *then*

$$f_2(s_2(q(\tau))) = \frac{2\mathrm{Im}(\tau)}{\pi^3}(-A + 16B)$$

where

$$A = \sum_{m,n\in\mathbb{Z}}{}' \frac{4(m\mathrm{Re}(\tau)+n)^2}{|m\tau+n|^6} - \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{|m\tau+n|^4},$$

$$B = \sum_{m,n\in\mathbb{Z}}{}' \frac{4(4m\mathrm{Re}(\tau)+n)^2}{|4m\tau+n|^6} - \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{|4m\tau+n|^4}.$$

The restriction on q can be removed and the lower bound of $\mathrm{Im}(\tau)$ is unnecessary. Applying a similar argument as in [8], one can show that the identity in this lemma holds everywhere. Indeed, both $f_2(k)$ and $\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A+16B)$ are the real parts of holomorphic functions at least when $k \in \mathbb{C} \setminus [-64, 64]$ and they coincide on a non-discrete set of points, see [10] and [15].

The L-function with respect to a quadratic Dirichlet character χ is denoted by $L(s,\chi)$. Let χ_{-4} be the character $\left(\frac{-1}{\cdot}\right)$, $\chi_{-8}(\cdot)=\left(\frac{-2}{\cdot}\right)$, $\chi_8(\cdot)=\left(\frac{2}{\cdot}\right)$ and $\chi_{-3}(\cdot)=\left(\frac{-3}{\cdot}\right)$, etc.

We will use the notations as in [6]. For any integer n, write $L_n(s)=L(\chi_n,s)$, in particular, $L_1(s)=\zeta(s)$ the Riemann zeta function. The following results prove to be crucial to our computation.

Lemma 2.3. (Glasser & Zucker [6], Table VI). *For any complex s with $Re(s)>1$,*

(1) $\sum'_{x,y\in\mathbb{Z}}\frac{1}{(x^2+3y^2)^s}=2(1+2^{1-2s})L_1(s)L_{-3}(s)$

(2) $\sum'_{x,y\in\mathbb{Z}}\frac{1}{(x^2+7y^2)^s}=2(1-2^{1-s}+2^{1-2s})L_1(s)L_{-7}(s)$

(3) $\sum'_{x,y\in\mathbb{Z}}\frac{1}{(x^2+12y^2)^s}=(1+2^{-2s}+2^{2-4s})L_1(s)L_{-3}(s)+L_{-4}(s)L_{12}(s)$

(4) $\sum'_{x,y\in\mathbb{Z}}\frac{1}{(x^2+28y^2)^2}=$
$(1-2^{1-s}+3\cdot2^{-2s}-2^{2-3s}+2^{2-4s})L_1(s)L_{-7}(s)+L_{-4}(s)L_{28}(s)$

Remark 1. The following results are useful for our computation: $L_1(2)=\frac{\pi^2}{6}$, $L_{12}(2)=\frac{\sqrt{3}}{18}\pi^2$, $L_{28}(2)=\frac{2\pi^2}{7\sqrt{7}}$.

Remark 2. For the sake of brevity, let us do the following convention of notations in this paper. Let $R(l,m)$ be any function of l and m. We denote

$$\sum_{(a,b)}R(l,m)=\sum_{\substack{(l,m)\equiv(a,b)\\ (\text{mod } 2)}}R(l,m),$$

where $a,b\in\mathbb{Z}$. As usual, the symbol $\sum'_{(a,b)}R(l,m)$ means the sum does not include $(a,b)=(0,0)$. Similarly, the notation $\Sigma'_{x\equiv y(d)}$ means $\Sigma'_{x\equiv y\ (\text{mod } d)}$, where the primed summation sign means to sum over all integer pairs except $(x,y)=(0,0)$.

3. Computation of $s_2(k)$ for $k=16,-104\pm60\sqrt{3}$

Following Samart's method, we will calculate A and B in the statement of Lemma 2.2 for our case, where $\tau=\frac{1+\sqrt{-3}}{4}$ by Lemma 2.1(1).

$$\begin{aligned}
A&=\sum_{m,n\in\mathbb{Z}}{}'\left(\frac{16((m+n)+3n)^2}{((m+n)^2+3n^2)^3}-\frac{16}{((m+n)^2+3n^2)^2}\right)\\
&=\sum_{l,n\in\mathbb{Z}}{}'\left(\frac{16(l+3n)^2}{(l^2+3n^2)^3}-\frac{16}{(l^2+3n^2)^2}\right)\quad(\text{let } l=m+n)\\
&=\sum_{l,n\in\mathbb{Z}}{}'\left(\frac{16(l^2+9n^2)}{(l^2+3n^2)^3}-\frac{16}{(l^2+3n^2)^2}\right)\quad\left(\sum{}'_{l,n\in\mathbb{Z}}ln/(l^2+3n^2)^3=0\right)\\
&=\sum_{l,n\in\mathbb{Z}}{}'\left(\frac{16(l^2+9n^2)}{(l^2+3n^2)^3}-\frac{32\left(l^2+3n^2\right)}{(l^2+3n^2)^3}+\frac{16}{(l^2+3n^2)^2}\right)\\
&=\sum_{l,n\in\mathbb{Z}}{}'\frac{-16(l^2-3n^2)}{(l^2+3n^2)^3}+\sum_{l,n\in\mathbb{Z}}{}'\frac{16}{(l^2+3n^2)^2}=A_M+A_D,
\end{aligned}$$

and

$$\begin{aligned}B &= \sideset{}{'}\sum_{m,n\in\mathbb{Z}}\left(\frac{4(m+n)^2}{(n^2+2mn+4m^2)^3}-\frac{1}{(n^2+2mn+4m^2)^2}\right)\\ &= \sideset{}{'}\sum_{l,m\in\mathbb{Z}}\left(\frac{4l^2}{(l^2+3m^2)^3}-\frac{1}{(l^2+3m^2)^2}\right)\\ &= \sideset{}{'}\sum_{l,m\in\mathbb{Z}}\left(\frac{2(l^2-3m^2)}{(l^2+3m^2)^3}+\frac{1}{(l^2+3m^2)^2}\right)=B_M+B_D.\end{aligned}$$

Thus,

$$-A_D+16B_D=-\sideset{}{'}\sum_{l,n\in\mathbb{Z}}\frac{16}{\left(l^2+3n^2\right)^2}+\sideset{}{'}\sum_{l,m\in\mathbb{Z}}\frac{16}{(l^2+3m^2)^2}=0.$$

Let

$$f(\tau)=\sum_{x,y\in\mathbb{Z}}\left(\frac{x^2-3y^2}{2}\right)q^{x^2+3y^2}=\eta(2\tau)^3\eta(6\tau)^3.$$

By lemma 2.7 in [15], $f(\tau)\in S_3(\Gamma_0(12),\chi_{-3})$. Thus,

$$\begin{aligned}-A_M+16B_M &= \sideset{}{'}\sum_{l,m\in\mathbb{Z}}\left(\frac{16(l^2-3m^2)}{\left(l^2+3m^2\right)^3}+\frac{32(l^2-3m^2)}{(l^2+3m^2)^3}\right)\\ &= \sideset{}{'}\sum_{l,m\in\mathbb{Z}}\frac{48(l^2-3m^2)}{(l^2+3m^2)^3}=96L(f,3)\\ &= \frac{16\pi^3\sqrt{3}}{3}L'(f,0).\end{aligned}$$

Therefore, we obtain:

$$\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A+16B)=\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A_M+16B_M)=8L'(f,0),$$

as desired.

3.1. $k=-104-60\sqrt{3}$**.** In this subsection, $K=\mathbb{Q}(\sqrt{-3})$. By Lemma 2.1, we know that

$$s_2\left(q\left(\frac{1+\sqrt{-3}}{2}\right)\right)=\mathfrak{f}(1+\sqrt{-3})^{24}=-104-60\sqrt{3}.$$

So in this case

$$\tau=\frac{1+\sqrt{-3}}{2},\ \mathrm{Im}(\tau)=\frac{\sqrt{3}}{2},\ \mathrm{Re}(\tau)=\frac{1}{2}.$$

To ease the notation, we set

$$\begin{aligned}A_M &= \sideset{}{'}\sum_{m\equiv l(2)}\tfrac{32(l^2-3m^2)}{(l^2+3m^2)^3}, & B_M &= \sideset{}{'}\sum_{l,m\in\mathbb{Z}}\tfrac{2(l^2-12m^2)}{(l^2+12m^2)^3},\\ A_D &= \sideset{}{'}\sum_{m\equiv l(2)}\tfrac{16}{(l^2+3m^2)^2}, & B_D &= \sideset{}{'}\sum_{l,m\in\mathbb{Z}}\tfrac{1}{(l^2+12m^2)^2},\\ C &= \sideset{}{'}\sum_{l,m\in\mathbb{Z}}\tfrac{1}{(l^2+3m^2)^2}, & X &= \sum_{(1,1)}\tfrac{1}{(l^2+3m^2)^2}.\end{aligned}$$

By Samart's formulas in Lemma 2.2, we have

$$\begin{aligned} A &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{64(2n+m)^2}{((2n+m)^2+3m^2)^3} - \frac{16}{((2n+m)^2+3m^2)^2}\right) \\ &= \sum_{m\equiv l(2)}{}' \left(\frac{64l^2}{(l^2+3m^2)^3} - \frac{16}{(l^2+3m^2)^2}\right) \quad (l=2n+m) \\ &= \sum_{m\equiv l(2)}{}' \frac{32(l^2-3m^2)}{(l^2+3m^2)^3} + \sum_{m\equiv l(2)}{}' \frac{16}{(l^2+3m^2)^2} = A_M + A_D, \end{aligned}$$

and

$$\begin{aligned} B &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4(n+2m)^2}{(16m^2+4mn+n^2)^3} - \frac{1}{(16m^2+4mn+n^2)^2}\right) \\ &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4(n+2m)^2}{((n+2m)^2+12m^2)^3} - \frac{1}{((n+2m)^2+12m^2)^2}\right) \\ &= \sum_{l,m\in\mathbb{Z}}{}' \left(\frac{4l^2}{(l^2+12m^2)^3} - \frac{1}{(l^2+12m^2)^2}\right) \quad (l=n+2m) \\ &= \sum_{l,m\in\mathbb{Z}}{}' \frac{2(l^2-12m^2)}{(l^2+12m^2)^3} + \sum_{l,m\in\mathbb{Z}}{}' \frac{1}{(l^2+12m^2)^2} = B_M + B_D. \end{aligned}$$

3.1.1. *The part of the L-Series of a modular form.* We continue to set

$$f(\tau) = \sum_{x,y\in\mathbb{Z}} \left(\frac{x^2-3y^2}{2}\right) q^{x^2+3y^2} = \eta(2\tau)^3\eta(6\tau)^3 \in S_3(\Gamma_0(12),\chi_{-3}).$$

From the proof of Lemma 2.4 in [15], we know that $h = \sum_{x\equiv y \pmod 2} \left(\frac{x^2-3y^2}{2}\right) q^{x^2+3y^2} = 0$. As a multiple of the L-function associate to h, $A_M = 0$. Thus

$$f(\tau) = \sum_{x\not\equiv y(2)} \left(\frac{x^2-3y^2}{2}\right) q^{x^2+3y^2}.$$

On the other hand, we let

$$f\otimes\chi_{-4}(\tau) = \sum_{x\not\equiv y(2)} \left(\frac{x^2-3y^2}{2}\right) \chi_{-4}(x^2+3y^2) q^{x^2+3y^2}.$$

It follows from [15] (cf. page 248, last line) that $f\otimes\chi_{-4}(\tau) \in S_3(\Gamma_0(48),\chi_{-3})$. We see that

$$f\otimes\chi_{-4}(\tau) = \sum_{\substack{(x,y)\equiv(1,0)\\ \pmod 2}} \left(\frac{x^2-3y^2}{2}\right) q^{x^2+3y^2} - \sum_{\substack{(x,y)\equiv(0,1)\\ \pmod 2}} \left(\frac{x^2-3y^2}{2}\right) q^{x^2+3y^2},$$

and

$$\sum_{\substack{(x,y)\equiv(1,0)\\ \pmod 2}} \left(\frac{x^2-3y^2}{2}\right) q^{x^2+3y^2} = \frac{1}{2}\Big(f(\tau) + f\otimes\chi_{-4}(\tau)\Big).$$

Therefore,

$$\sum_{l,m\in\mathbb{Z}}{}' \frac{2(l^2-12m^2)}{(l^2+12m^2)^3} = \sum_{\substack{l,m\in\mathbb{Z}\\ m \text{ even}}}{}' \frac{2(l^2-3m^2)}{(l^2+3m^2)^3} = \sum_{(0,0)}{}' \frac{2(l^2-3m^2)}{(l^2+3m^2)^3} + \sum_{(1,0)}{}' \frac{2(l^2-3m^2)}{(l^2+3m^2)^3}$$
$$= \frac{9}{4} L(f,3) + 2L(f\otimes\chi_{-4},3).$$

Applying the functional equation (the sign can be settled by finite precision numerical computation, it is '+' for our case)

$$\left(\frac{\sqrt{N}}{2\pi}\right)^{s} \Gamma(s)L(f,s) = \pm\left(\frac{\sqrt{N}}{2\pi}\right)^{3-s} \Gamma(3-s)L(f,3-s), \tag{3.1}$$

we obtain

$$\begin{aligned} &\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A_M+16B_M)\\ =&\frac{16\sqrt{3}}{\pi^3}\left(\frac{9}{4}\frac{\pi^3}{6\sqrt{3}}L'(f,0)+\frac{2\pi^3}{48\sqrt{3}}L'(f\otimes\chi_{-4},0)\right)\\ =&6L'(f,0)+\frac{2}{3}L'(f\otimes\chi_{-4},0). \end{aligned}$$

3.1.2. *The part of the Dirichlet L-Series.* Now let us deal with the Dirichlet L-function part. It is clear that

$$\sum_{m\equiv l(2)}{}' \frac{1}{(l^2+3m^2)^2} = \sum_{(0,0)}{}' \frac{1}{(l^2+3m^2)^2} + \sum_{(1,1)}{}' \frac{1}{(l^2+3m^2)^2}.$$

One can easily check that

$$\begin{aligned} &\{I\subset O_K\}\\ =&\left\{\alpha O_K \,\middle|\, \alpha=\frac{x+y\sqrt{-3}}{2},\ \begin{matrix} x\equiv 1 \pmod 2\\ y\equiv 1 \pmod 2\end{matrix}\right\} \sqcup \left\{\alpha O_K \,\middle|\, \alpha = x+y\sqrt{-3},\ \begin{matrix} x\in\mathbb{Z}\\ y\in\mathbb{Z}\end{matrix}\right\}, \end{aligned}$$

where $\sqcup$ means disjoint union.

Thus we have

$$6\sum_{I\subset O_K}\frac{1}{N(I)^2} = \sum_{(1,1)}\frac{16}{(x^2+3y^2)^2} + \sum_{x,y\in\mathbb{Z}}\frac{1}{(x^2+y^2)^2},$$

which implies

$$X = \frac{1}{16}(6\zeta_K(2)-C).$$

Here we used the fact that $|U(K)|=6$.

By Lemma 2.3, we know that the following results hold

$$C = 2(1+2^{1-2\cdot 2})\zeta_K(2) = \frac{9}{4}\zeta_K(2). \tag{3.2}$$

Hence,

$$X = \frac{1}{16}\cdot\frac{15}{4}\zeta_K(2) = \frac{15}{64}\cdot L_1(2)L_{-3}(2). \tag{3.3}$$

Therefore,

$$
\begin{aligned}
&\frac{2\mathrm{Im}(\tau)}{\pi^3}\left(-A_D+16B_D\right)\\
=&\frac{2\frac{\sqrt{3}}{2}}{\pi^3}\left(-\sum_{m\equiv l(2)}{}' \frac{16}{(l^2+3m^2)^2}+16\sum_{l,m\in\mathbb{Z}}{}' \frac{1}{(l^2+12m^2)^2}\right)\\
=&\frac{16\sqrt{3}}{\pi^3}\left(-\left(\sum_{(0,0)}{}' \frac{1}{(l^2+3m^2)^2}+\sum_{(1,1)}{}' \frac{1}{(l^2+3m^2)^2}\right)+\sum_{l,m\in\mathbb{Z}}{}' \frac{1}{(l^2+12m^2)^2}\right)\\
=&\frac{16\sqrt{3}}{\pi^3}\left(-\frac{9}{64}\cdot\frac{\pi^2}{6}L_{-3}(2)-\frac{15}{64}\cdot\frac{\pi^2}{6}L_{-3}(2)+\frac{69}{64}\cdot\frac{\pi^2}{6}L_{-3}(2)+\frac{\sqrt{3}}{18}\pi^2L_{-4}(2)\right)\\
=&\frac{16\sqrt{3}}{6\pi}\left(\frac{45}{64}L_{-3}(2)+\frac{\sqrt{3}}{3}L_{-4}(2)\right)=\frac{8\sqrt{3}}{3\pi}\left(\frac{45}{64}\frac{4\pi}{3\sqrt{3}}d_3+\frac{\sqrt{3}}{3}\frac{\pi}{2}d_4\right)=\frac{5}{2}d_3+\frac{4}{3}d_4.
\end{aligned}
$$

Here, we used the functional equation

$$
\left(\frac{\pi}{k}\right)^{-\frac{2-s}{2}}\Gamma\left(\frac{2-s}{2}\right)L(\chi_{-k},1-s)=\left(\frac{\pi}{k}\right)^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(\chi_{-k},s). \tag{3.4}
$$

In conclusion,

$$
\begin{aligned}
s_2(q(\tau))=&\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A+16B)=\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A_M+16B_M-A_D+16B_D)\\
=&6L'(f,0)+\frac{2}{3}L'(f\otimes\chi_{-4},0)+\frac{5}{2}d_3+\frac{4}{3}d_4=\frac{1}{6}\Big(4M_{12\otimes(-4)}+36M_{12}+15d_3+8d_4\Big).
\end{aligned}
$$

3.2. $k=-104+60\sqrt{3}$**.** In this subsection, we still denote $\mathbb{Q}(\sqrt{-3})$ by K. By Lemma 2.1, we have

$$
s_2\left(q\left(\frac{3+\sqrt{-3}}{6}\right)\right)=\mathfrak{f}\left(\frac{3+\sqrt{-3}}{6}\right)^{24}=-104+60\sqrt{3}.
$$

So in this case,

$$
\tau=\frac{3+\sqrt{-3}}{6},\ \mathrm{Im}(\tau)=\frac{\sqrt{3}}{6},\ \mathrm{Re}(\tau)=\frac{1}{2}.
$$

To ease the notation, we set

$$
\begin{aligned}
A_M &= \sum_{m\equiv l(2)}{}' \frac{288(3l^2-m^2)}{(3l^2+m^2)^3}, & B_M &= \sum_{l,m\in\mathbb{Z}}{}' \frac{18(3l^2-4m^2)}{(3l^2+4m^2)^3},\\
A_D &= \sum_{m\equiv l(2)}{}' \frac{12^2}{(3l^2+m^2)^2}, & B_D &= \sum_{l,m\in\mathbb{Z}}{}' \frac{3^2}{(3l^2+4m^2)^2},\\
C &= \sum_{l,m\in\mathbb{Z}}{}' \frac{1}{(3l^2+m^2)^2}, & X &= \sum_{(1,1)}{}' \frac{1}{(3l^2+m^2)^2},\\
D &= \sum_{\substack{l,m\in\mathbb{Z}\\ l\text{ even}}}{}' \frac{1}{(3l^2+m^2)^2}, & Y &= \sum_{(1,0)}{}' \frac{1}{(3l^2+m^2)^2}.
\end{aligned}
$$

We need to calculate the values of A and B in Samart's formula, where

$$A = \sum_{m,n\in\mathbb{Z}}' \left(\frac{12^3(2n+m)^2}{(3(2n+m)^2+m^2)^3} - \frac{12^2}{(3(2n+m)^2+m^2)^2} \right)$$
$$= \sum_{m\equiv l(2)}' \left(\frac{12^3 l^2}{(3l^2+m^2)^3} - \frac{12^2}{(3l^2+m^2)^2} \right) \quad (l = 2n+m)$$
$$= \sum_{m\equiv l(2)}' \frac{288(3l^2-m^2)}{(3l^2+m^2)^3} + \sum_{m\equiv l(2)}' \frac{12^2}{(3l^2+m^2)^2} = A_M + A_D,$$

and

$$B = \sum_{m,n\in\mathbb{Z}}' \left(\frac{4(4m\frac{1}{2}+n)^2}{\left(n^2+4mn+\frac{16m^2}{3}\right)^3} - \frac{1}{\left(n^2+4mn+\frac{16m^2}{3}\right)^2} \right)$$
$$= \sum_{m,n\in\mathbb{Z}}' \left(\frac{4(n+2m)^2}{\left((n+2m)^2+\frac{4m^2}{3}\right)^3} - \frac{1}{\left((n+2m)^2+\frac{4m^2}{3}\right)^2} \right)$$
$$= \sum_{l,m\in\mathbb{Z}}' \left(\frac{3^3\cdot 4l^2}{(3l^2+4m^2)^3} - \frac{3^2}{(3l^2+4m^2)^2} \right) \quad (l = n+2m)$$
$$= \sum_{l,m\in\mathbb{Z}}' \frac{18(3l^2-4m^2)}{(3l^2+4m^2)^3} + \sum_{l,m\in\mathbb{Z}}' \frac{3^2}{(3l^2+4m^2)^2} = B_M + B_D.$$

3.2.1. *The part of the L-Series of a modular form.* From the computation in the last section, we know that $A_M = 0$. Define $f(\tau)$ just in the same way as in the last section. We obtain

$$\sum_{l,m\in\mathbb{Z}}' \frac{(4m^2-3l^2)}{(3l^2+4m^2)^3} = \sum_{\substack{l,m\in\mathbb{Z}\\ m \text{ even}}}' \frac{(m^2-3l^2)}{(3l^2+m^2)^3} = \sum_{(0,0)}' \frac{(m^2-3l^2)}{(m^2+3l^2)^3} + \sum_{(1,0)} \frac{(m^2-3l^2)}{(m^2+3l^2)^3}$$
$$=2\sum_{l,m\in\mathbb{Z}}' \frac{4(m^2-3l^2)}{4^3\cdot 2\,(m^2+3l^2)^3} + 2\sum_{(1,0)} \frac{(m^2-3l^2)}{2\,(m^2+3l^2)^3}$$
$$=\frac{1}{8}L(f,3) + 2\cdot\frac{1}{2}(L(f,3) - L(f\otimes\chi_{-4},3)) = \frac{9}{8}L(f,3) - L(f\otimes\chi_{-4},3)$$
$$=\frac{9}{8}\frac{\pi^3}{6\sqrt{3}}L'(f,0) - \frac{\pi^3}{48\sqrt{3}}L'(f\otimes\chi_{-4},0).$$

Thus,

$$\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A_M+16B_M) = \frac{16\sqrt{3}}{3\pi^3}\sum_{l,m\in\mathbb{Z}}' \frac{18(3l^2-4m^2)}{(3l^2+4m^2)^3}$$
$$= -\frac{96\sqrt{3}}{\pi^3}\left(\frac{9}{8}\frac{\pi^3}{6\sqrt{3}}L'(f,0) - \frac{\pi^3}{48\sqrt{3}}L'(f\otimes\chi_{-4},0)\right) = -18L'(f,0) + 2L'(f\otimes\chi_{-4},0).$$

3.2.2. *The part of the Dirichlet L-Series.* We now treat the part involving the Dirichlet L-Series. By Lemma 2.3, we have

$$D = \sum_{l,m\in\mathbb{Z}}' \frac{1}{(l^2+12m^2)^2} = (1+2^{-4}+2^{-6})L_1(2)L_{-3}(2) + L_{-4}(2)L_{12}(2)$$

$$= \frac{69}{64}\cdot\frac{\pi^2}{6}L_{-3}(2)+\frac{\sqrt{3}}{18}\pi^2 L_{-4}(2), \tag{3.5}$$

where we used Remark 2.4 that $L_{12}(2)=\frac{\sqrt{3}}{18}\pi^2$. It is clear that $Y=C-X-D$. By identities (2)-(5), we do the following calculation

$$\begin{aligned}
&\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A_D+16B_D)\\
=&\frac{12^2\sqrt{3}}{3\pi^3}\left(-\sum_{m\equiv l(2)}{}' \frac{1}{(m^2+3l^2)^2}+\sum_{l,m\in\mathbb{Z}}{}' \frac{1}{(4m^2+3l^2)^2}\right)\\
=&\frac{12^2\sqrt{3}}{3\pi^3}\left(-\sum_{(1,1)}{}' \frac{1}{(m^2+3l^2)^2}+\sum_{(1,0)} \frac{1}{(m^2+3l^2)^2}\right)=\frac{12^2\sqrt{3}}{3\pi^3}(-X+Y)\\
=&\frac{12^2\sqrt{3}}{3\pi^3}(C-2X-D)=\frac{12^2\sqrt{3}}{3\pi^3}\left(\frac{45}{64}\frac{\pi^2}{6}L_{-3}(2)-\frac{\sqrt{3}}{18}\pi^2 L_{-4}(2)\right)\\
=&\frac{12^2\sqrt{3}}{3\pi^3}\left(\frac{45}{64}\frac{\pi^2}{6}\frac{4\pi}{3\sqrt{3}}d_3-\frac{\sqrt{3}}{18}\pi^2\frac{\pi}{2}d_4\right)=\frac{15}{2}d_3-4d_4.
\end{aligned}$$

Thus

$$\begin{aligned}
s_2(q(\tau))&=\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A+16B)=\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A_M+16B_M-A_D+16B_D)\\
&=-18L'(f,0)+2L'(f\otimes\chi_{-4},0)+\frac{15}{2}d_3-4d_4\\
&=\frac{1}{2}\left(4L'(f\otimes\chi_{-4},0)-36L'(f,0)+15d_3-8d_4\right).
\end{aligned}$$

4. Computation of $s_2(k)$ for $k=4096,-2024\pm765\sqrt{7}$

In this section, let $K=\mathbb{Q}(\sqrt{-7})$. Then $Cl(O_K)=1\Rightarrow O_K$ is PID, $U(K)=\{\pm1\}$ and 2 split completely in K. Let

$$w=\frac{1+\sqrt{-7}}{2}$$

and

$$\wp=(w).$$

Hence,

$$2O_K=\wp\bar{\wp}.$$

On the other hand, let us do the following settings

$$C=\sum_{I\subset O_K}{}' \frac{1}{N(I)^2}=\zeta_K(2),\quad F=\sum_{l,m\in\mathbb{Z}}{}' \frac{1}{(l^2+7m^2)^2},$$

where I denotes the ideals in O_K. In addition we set

$$\star=\left(\sum_{l,m\in\mathbb{Z}}{}'+2\sum_{(0,0)}{}'-\sum_{m\equiv l(2)}{}'\right)\frac{1}{(l^2+7m^2)^2}.$$

4.1. $k = 4096$**.** By Lemma 2.1, we know that

$$s_2\left(q\left(\frac{\sqrt{-7}}{2}\right)\right) = \mathfrak{f}(\sqrt{-7})^{24} = 4096.$$

So in this case,

$$\tau = \frac{\sqrt{-7}}{2},\ \mathrm{Im}(\tau) = \frac{\sqrt{7}}{2},\ \mathrm{Re}(\tau) = 0.$$

To ease the notation, we set

$$\begin{aligned} A_M &= \sum_{\substack{l,m\in\mathbb{Z}\\ l \text{ even}}}{}' \tfrac{32(l^2-7m^2)}{(l^2+7m^2)^3}, & B_M &= \sum_{m,n\in\mathbb{Z}}{}' \tfrac{2(n^2-28m^2)}{(n^2+28m^2)^3}, \\ A_D &= \sum_{\substack{l,m\in\mathbb{Z}\\ l \text{ even}}}{}' \tfrac{16}{(l^2+7m^2)^2}, & B_D &= \sum_{m,n\in\mathbb{Z}}{}' \tfrac{1}{(n^2+28m^2)^2}, \end{aligned}$$

We have

$$\begin{aligned} A &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{256n^2}{(7m^2+4n^2)^3} - \frac{16}{(7m^2+4n^2)^2}\right) \\ &= \sum_{\substack{l,m\in\mathbb{Z}\\ l \text{ even}}}{}' \frac{32(l^2-7m^2)}{(l^2+7m^2)^3} + \sum_{\substack{l,m\in\mathbb{Z}\\ l \text{ even}}}{}' \frac{16}{(l^2+7m^2)^2} = A_M + A_D, \end{aligned}$$

and

$$\begin{aligned} B &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4n^2}{(28m^2+n^2)^3} - \frac{1}{(28m^2+n^2)^2}\right) \\ &= \sum_{m,n\in\mathbb{Z}}{}' \frac{2(n^2-28m^2)}{(n^2+28m^2)^3} + \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(n^2+28m^2)^2} = B_M + B_D. \end{aligned}$$

4.1.1. *The part of the L-Series of a modular form.* Let $g_7 = \eta(\tau)^3\eta(7\tau)^3 = \sum_{m,n\in\mathbb{N}} \chi_{-4}(mn)mnq^{\frac{m^2+7n^2}{8}}$. We have following result.

Theorem 4.1. *Let*

$$g = \sum_{x-y\equiv 1(2)} \left(\frac{x^2-7y^2}{2}\right) q^{x^2+7y^2}.$$

Then,

$$g \otimes \chi_{-4} = g_7 \otimes \chi_{-4}$$

Proof. Let $\Lambda' = (2)$ and define $\phi(\mathfrak{a}) = \alpha^2$ for a generator of an integral ideal $\mathfrak{a}$ which satisfies $\alpha \equiv 1(\Lambda')$.

In fact, if (α) is coprime to Λ', then $\alpha \equiv 1(\Lambda')$. To see this, consider the norm of α, it is odd (since $(\alpha) + \Lambda' = O_K$) and moreover,

$$\begin{aligned} N(\alpha) = N\left(x + y\frac{1+\sqrt{-7}}{2}\right) &= \left(x + \frac{y}{2} + \frac{\sqrt{-7}}{2}y\right)\left(x + \frac{y}{2} - \frac{\sqrt{-7}}{2}y\right) \\ &= \left(x+\frac{y}{2}\right)^2 - \frac{-7}{4}y^2 = \left(x+\frac{y}{2}\right)^2 + \frac{7}{4}y^2 = x^2 + xy + 2y^2 \equiv x^2 + xy(\Lambda'). \end{aligned}$$

Hence,

$$(x, y) \equiv (1, 0) \pmod 2.$$

Thus if (α) is coprime to Λ', then $\alpha \equiv 1(\Lambda')$. It is clear that ϕ is multiplicative and satisfies $\phi(\alpha O_K) = \alpha^2$. Therefore, ϕ is a Hecke charactor which satisfies the condition of Theorem 1.31 in [9]. So

$$g(\tau) = \sum_{\mathfrak{a}} \phi(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_n \left(\sum_{\substack{\mathfrak{a} \\ N(\mathfrak{a})=n}} \phi(\mathfrak{a}) q^n \right)$$

is a newform in $S_3(\Gamma_0(28), \chi_{-7})$.

As y is even, we can rewrite the norm as

$$\begin{aligned} N(\alpha) &= N(x + y'(1+\sqrt{-7})) \quad (y' = y/2) \\ &= N(x + y' + y'\sqrt{-7}) \\ &= N(x'' + y''\sqrt{-7}) \quad (x'' = x + y', y'' = y'). \end{aligned}$$

Here, x'', y'' satisfy that $(x'', y'') \in \mathbb{Z}$ and $x'' - y'' \equiv 1 \pmod 2$ since $x = x'' - y''$ should be odd from above.

Note that $\alpha = x + y\sqrt{-7}$ and $\beta = x - y\sqrt{-7}$ share the same norm value, so we have:

$$\begin{aligned} g(\tau) &= \frac{1}{2} \sum_{(\alpha, \Lambda')=1} (\phi(\alpha) + \phi(\beta)) q^{N(\alpha)} \\ &= \frac{1}{2} \cdot \frac{1}{2} \sum_{x-y\equiv 1(2)} \left(x^2 + y^2(-7) + 2xy\sqrt{-7} + x^2 + y^2(-7) - 2xy\sqrt{-7} \right) q^{x^2+7y^2} \\ &= \sum_{x-y\equiv 1(2)} \left(\frac{x^2 - 7y^2}{2} \right) q^{x^2+7y^2}. \end{aligned}$$

So one can deduce that $g \otimes \chi_{-4} = g_7 \otimes \chi_{-4}$ whose level divides 112 by considering the Sturm Bound (with the help of SageMath[13]). Indeed, the Sturm Bound for our case is at most 48. In addition,

$$\begin{aligned} g_7 = &q - 3q^2 + 5q^4 - 7q^7 - 3q^8 + 9q^9 - 6q^{11} + 21q^{14} - 11q^{16} - 27q^{18} + 18q^{22} \\ &+ 18q^{23} + 25q^{25} - 35q^{28} - 54q^{29} + 45q^{32} + 45q^{36} - 38q^{37} + 58q^{43} \\ &- 30q^{44} - 54q^{46} + 49q^{49} + O(q^{50}), \end{aligned}$$

$$g = q - 7q^7 + 9q^9 - 6q^{11} + 18q^{23} + 25q^{25} - 54q^{29} - 38q^{37} + 58q^{43} + 49q^{49} + O(q^{50}),$$

and

$$\begin{aligned} &g \otimes \chi_{-4} = g_7 \otimes \chi_{-4} \\ &= q + 7q^7 + 9q^9 + 6q^{11} - 18q^{23} + 25q^{25} - 54q^{29} - 38q^{37} - 58q^{43} + 49q^{49} + O(q^{50}). \end{aligned}$$

□

Since

$$\begin{aligned} -A_M + 16B_M = &-\sideset{}{'}\sum_{\substack{l,m\in\mathbb{Z} \\ l \text{ even}}} \frac{32(l^2 - 7m^2)}{(l^2 + 7m^2)^3} + \sideset{}{'}\sum_{\substack{l,m\in\mathbb{Z} \\ m \text{ even}}} \frac{32(l^2 - 7m^2)}{(l^2 + 7m^2)^3} \\ =32 &\left(-\sideset{}{'}\sum_{(0,0)} - \sum_{(0,1)} + \sideset{}{'}\sum_{(0,0)} + \sum_{(1,0)} \right) = 32 \left(\sum_{(1,0)} - \sum_{(0,1)} \right) = 64L(g_7 \otimes \chi_{-4}, 3), \end{aligned}$$

one can get

$$\begin{aligned}&\frac{2\mathrm{Im}(\sqrt{-7}/2)}{\pi^3}(-A_M+16B_M)=\frac{\sqrt{7}}{\pi^3}(-A_M+16B_M)\\=&\frac{\sqrt{7}}{\pi^3}64L(g_7\otimes\chi_{-4},3)=\frac{\sqrt{7}}{\pi^3}\cdot 64\cdot\frac{\pi^3}{112\sqrt{7}}L'(g_7\otimes\chi_{-4},0)=\frac{4}{7}L'(g_7\otimes\chi_{-4},0).\end{aligned}$$

Here we applied the functional equation (1) to $L(g_7\otimes\chi_{-4},3)$.

4.1.2. *The part of the Dirichlet L-Series.* Now let us deal with the Dirichlet L-Series part, or more exactly, the computation of the term

$$\sum_{\substack{l,m\in\mathbb{Z}\\ m\text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}-\sum_{\substack{l,m\in\mathbb{Z}\\ l\text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}.$$

Firstly, by

$$\sum_{m\equiv l(2)}{}' = \sum_{m,l\in\mathbb{Z}}{}' - \sum_{\substack{l,m\in\mathbb{Z}\\ m\text{ even}}}{}' - \sum_{\substack{l,m\in\mathbb{Z}\\ l\text{ even}}}{}' + 2\sum_{(0,0)}{}',$$

we obtain

$$\star = \sum_{\substack{l,m\in\mathbb{Z}\\ m\text{ even}}}{}' + \sum_{\substack{l,m\in\mathbb{Z}\\ l\text{ even}}}{}'.$$

Thus,

$$\sum_{\substack{l,m\in\mathbb{Z}\\ m\text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}-\sum_{\substack{l,m\in\mathbb{Z}\\ l\text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}=2\sum_{\substack{l,m\in\mathbb{Z}\\ m\text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}-\star.$$

By Lemma 2.3,

$$\sum_{\substack{l,m\in\mathbb{Z}\\ m\text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}=\sum_{x,y\in\mathbb{Z}}{}' \frac{1}{(x^2+28y^2)^2}=\frac{41}{64}L_1(2)L_{-7}(2)+L_{-4}(2)L_{28}(2).$$

On the other hand, we have

$$\star=\sum_{l,m\in\mathbb{Z}}{}' + 2\sum_{(0,0)}{}' - \sum_{(0,0)}{}' - \sum_{(1,1)}{}' = \sum_{l,m\in\mathbb{Z}}{}' + \sum_{(0,0)}{}' - \sum_{(1,1)}{}'.$$

In addition, one can easily check that

$$\begin{aligned}&\{I\subset O_K\}\\=&\left\{\alpha O_K\,\middle|\,\alpha=\frac{x+y\sqrt{-7}}{2},\ \begin{matrix}x\equiv 1\pmod 2\\ y\equiv 1\pmod 2\end{matrix}\right\}\sqcup\left\{\alpha O_K\,\middle|\,\alpha=x+y\sqrt{-7},\ \begin{matrix}x\in\mathbb{Z}\\ y\in\mathbb{Z}\end{matrix}\right\}\end{aligned}$$

which implies

$$16\sum_{(1,1)}\frac{1}{(l^2+7m^2)^2}=2\zeta_K(2)-F.$$

By Lemma 2.3,

$$\star=\frac{41}{32}\zeta_K(2)=\frac{41}{32}L_1(2)L_{-7}(2),$$

Therefore,

$$16\left(\sum_{\substack{l,m\in\mathbb{Z}\\ m\text{ even}}}{}' \frac{1}{(l^2+7m^2)^2} - \sum_{\substack{l,m\in\mathbb{Z}\\ l\text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}\right) = 16\left(2\sum_{\substack{l,m\in\mathbb{Z}\\ m\text{ even}}}{}' - \star\right)$$
$$=32\left(\frac{41}{64}L_1(2)L_{-7}(2)+L_{-4}(2)L_{28}(2)\right)-16\cdot\frac{41}{32}L_1(2)L_{-7}(2)$$
$$=32L_{-4}(2)L_{28}(2).$$

Since $L_{28}(2)=2\pi^2/(7\sqrt{7})$ and by the functional equation (5) we have

$$L_{-4}(2)=\frac{\pi}{2}L'_{-4}(-1)=\frac{\pi}{2}d_4.$$

Hence,

$$\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A_D+16B_D)=\frac{\sqrt{7}}{\pi^3}32L_{-4}(2)L_{28}(2)=\frac{\sqrt{7}}{\pi^3}\cdot 32\cdot\frac{\pi}{2}\cdot d_4\cdot\frac{2\pi^2}{7\sqrt{7}}=\frac{32}{7}d_4.$$

Finally, we obtain

$$\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A+16B)=\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A_M-A_D+16B_M+16B_D)=\frac{4}{7}(L'(g_7\otimes\chi_{-4},0)+8d_4),$$

as desired.

4.2. $k=-2024-765\sqrt{7}$**.** To ease the notation, in this subsection we set

$$A_M = \sum_{\substack{l,m\in\mathbb{Z}\\ l\text{ even}}}{}' \frac{32(l^2-7m^2)}{(l^2+7m^2)^3},\quad B_M = \sum_{m,n\in\mathbb{Z}}{}' \frac{2(n^2-28m^2)}{(n^2+28m^2)^3},$$
$$A_D = \sum_{\substack{l,m\in\mathbb{Z}\\ l\text{ even}}}{}' \frac{16}{(l^2+7m^2)^2},\quad B_D = \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(n^2+28m^2)^2}.$$

Following Samart's method, we will calculate A and B in the statement of Lemma 2.2 for our case, where $\tau=\frac{1+\sqrt{-7}}{2}$ by Lemma 2.1(2).

$$A=\sum_{m,n\in\mathbb{Z}}{}'\left(\frac{64(2n+m)^2}{((2n+m)^2+7m^2)^3}-\frac{16}{((2n+m)^2+7m^2)^2}\right)$$
$$=\sum_{m\equiv l(2)}{}'\left(\frac{64l^2}{(l^2+7m^2)^3}-\frac{16}{(l^2+7m^2)^2}\right)\quad(\text{Let } l=2n+m)$$
$$=\sum_{m\equiv l(2)}{}'\frac{32(l^2-7m^2)}{(l^2+7m^2)^3}+\sum_{m\equiv l(2)}{}'\frac{16}{(l^2+7m^2)^2}=A_D+A_M,$$

and

$$B=\sum_{m,n\in\mathbb{Z}}{}'\left(\frac{4(n+2m)^2}{((n+2m)^2+28m^2)^3}-\frac{1}{((n+2m)^2+28m^2)^2}\right)$$
$$=\sum_{l,m\in\mathbb{Z}}{}'\left(\frac{4l^2}{(l^2+28m^2)^3}-\frac{1}{(l^2+28m^2)^2}\right)\quad(\text{Let } l=n+2m)$$
$$=\sum_{l,m\in\mathbb{Z}}{}'\frac{2(l^2-28m^2)}{(l^2+28m^2)^3}+\sum_{l,m\in\mathbb{Z}}{}'\frac{1}{(l^2+28m^2)^2}=B_M+B_D.$$

4.2.1. *The part of the L-Series of a modular form.* To prove this identity, we need to find another expression for g_7 defined in last subsection.

Theorem 4.2. *We have*

$$g_7 = \sum_{x \equiv y(2)} \left(\frac{x^2 - 7y^2}{8} \right) q^{\frac{x^2+7y^2}{4}} \in S_3(\Gamma_0(7), \chi_{-7}).$$

Proof. Take modulus $\Lambda = (1)$ and define $\phi(\mathfrak{a}) = \alpha^2$ for a generator α of $\mathfrak{a}$ which satisfies $\alpha \equiv 1(\Lambda)$. It is clear that ϕ is multiplicative and satisfies $\phi(\alpha O_L) = \alpha^2$. Therefore, ϕ is a Hecke character which satisfies the conditions of Theorem 1.31 in [9]. So

$$G_7(\tau) = \sum_{\mathfrak{a}} \phi(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_n \left(\sum_{\substack{\mathfrak{a} \\ N(\mathfrak{a})=n}} \phi(\mathfrak{a}) q^n \right)$$

is a newform in $S_3(\Gamma_0(7), \chi_{-7})$. Here, the sum in the definition of $G_7(\tau)$ is over all ideals in O_K.

Rewrite $\alpha = x + y\frac{1+\sqrt{-7}}{2}$ as $\alpha = \frac{2x+y+y\sqrt{-7}}{2} = \frac{x'+y'\sqrt{-7}}{2}$, $x, y \in \mathbb{Z}$ and $x' = 2x+y$, $y' = y$. Then we have $x' \equiv y' (\text{mod } 2)$.

Note that $\alpha = x + y\sqrt{-7}$ and $\beta = x - y\sqrt{-7}$ share the same norm value, so we have

$$\begin{aligned}
G_7(\tau) &= \frac{1}{2} \sum_{I=(\alpha)} (\phi(\alpha) + \phi(\beta)) q^{N(\alpha)} \\
&= \frac{1}{2} \cdot \frac{1}{2} \sum_{x \equiv y(2)} \left(\left(\frac{x + y\sqrt{-7}}{2} \right)^2 + \left(\frac{x - y\sqrt{-7}}{2} \right)^2 \right) q^{\frac{x^2+7y^2}{4}} \\
&= \frac{1}{2} \cdot \frac{1}{2} \sum_{x \equiv y(2)} \left(\frac{x^2 + y^2(-7) + 2xy\sqrt{-7}}{4} + \frac{x^2 + y^2(-7) - 2xy\sqrt{-7}}{4} \right) q^{\frac{x^2+7y^2}{4}} \\
&= \sum_{x \equiv y(2)} \left(\frac{x^2 - 7y^2}{8} \right) q^{\frac{x^2+7y^2}{4}}.
\end{aligned}$$

Finally one can show that $g_7 = G_7$ with the help of SageMath [13] and Sturm bound. □

Theorem 4.3. *With the notation above, the following identity holds:*

$$3L(g_7, 3) = -2\sum_{(1,1)}{}' \frac{(l^2 - 7m^2)}{(l^2 + 7m^2)^3} + \sum_{(1,0)}{}' \frac{(l^2 - 7m^2)}{(l^2 + 7m^2)^3} + \sum_{(0,1)}{}' \frac{(l^2 - 7m^2)}{(l^2 + 7m^2)^3}.$$

Proof. Firstly, by the fact that,

$$\sum_{(0,0)} = \sum_{4|l,2||m} + \sum_{2||l,4|m} + \sum_{2||l,2||m} + \sum_{4|l,4|m},$$

we have

$$\begin{aligned}
&L(g_7,3)\\
=&8\left(\sum_{(1,1)}\frac{l^2-7m^2}{(l^2+7m^2)^3}+\sum_{(0,0)}\frac{l^2-7m^2}{(l^2+7m^2)^3}\right)\\
=&8\left(\sum_{(1,1)}\frac{l^2-7m^2}{(l^2+7m^2)^3}+\left(\frac{1}{2^4}+\frac{1}{2^8}+\cdots\right)\left(\sum\nolimits_{(1,1)}+\sum\nolimits_{(1,0)}+\sum\nolimits_{(0,1)}\right)\right)\\
=&\frac{128}{15}\sum\nolimits_{(1,1)}+\frac{8}{15}\sum\nolimits_{(1,0)}+\frac{8}{15}\sum\nolimits_{(0,1)}.
\end{aligned}\tag{4.1}$$

Secondly, we have

$$\left\{I\underset{\text{ideal}}{\subseteq}O_K\right\}=S_1\sqcup S_2\sqcup S_3\sqcup S_4,$$

where

$$\begin{aligned}
S_1=&\{(\alpha)|\alpha\in O_K,((2),(\alpha))=1\},\\
S_2=&\{(\alpha)|\alpha\in O_K,\wp\mid(\alpha),\bar{\wp}\nmid(\alpha)\},\\
S_3=&\{(\alpha)|\alpha\in O_K,\bar{\wp}\mid(\alpha),\wp\nmid(\alpha)\},\\
S_4=&\{(\alpha)|\alpha\in O_K,2|(\alpha)\}.
\end{aligned}$$

And it is clear that:

$$\begin{aligned}
S_2=&\{(\alpha)|\alpha\in O_K,\alpha=w^j\beta,j\geq 1,(\beta)\in S_1\},\\
S_3=&\{(\alpha)|\alpha\in O_K,\alpha=\bar{w}^j\beta,j\geq 1,(\beta)\in S_1\},\\
S_4=&S_{4,1}\sqcup S_{4,2}\sqcup S_{4,3}.
\end{aligned}$$

where

$$\begin{aligned}
S_{4,1}=&\{(\alpha)|\alpha\in O_K,\alpha=2^j\beta,j\geq 1,(\beta)\in S_1\},\\
S_{4,2}=&\{(\alpha)|\alpha\in O_K,\alpha=2^jw^k\beta,\text{where }j,k\geq 1,(\beta)\in S_1\},\\
S_{4,3}=&\{(\alpha)|\alpha\in O_K,\alpha=2^j\bar{w}^k\beta,\text{where }j,k\geq 1,(\beta)\in S_1\}.
\end{aligned}$$

Write g_7 as following form

$$g_7=\sum_{I=(\alpha)}\alpha^2q^{N(\alpha)},$$

i.e.,

$$g_7=\sum_{\substack{I=(\alpha)\\ I\in S_1}}\alpha^2q^{N(\alpha)}+\sum_{\substack{I=(\alpha)\\ I\in S_2}}\alpha^2q^{N(\alpha)}+\sum_{\substack{I=(\alpha)\\ I\in S_3}}\alpha^2q^{N(\alpha)}+\sum_{\substack{I=(\alpha)\\ I\in S_4}}\alpha^2q^{N(\alpha)}.$$

We can find that

$$\begin{aligned}
\sum_{\substack{I=(\alpha)\\ I\in S_1}}\alpha^2q^{N(\alpha)}=&g,\\
\sum_{\substack{I=(\alpha)\\ I\in S_2}}\alpha^2q^{N(\alpha)}=&\sum_{\substack{I=(\alpha)\\ \alpha\in S_1}}\sum_{j=1}^{\infty}(w^j\alpha)^2q^{N(w^j\alpha)}=\sum_{j=1}^{\infty}w^{2j}\sum_{\substack{I=(\alpha)\\ \alpha\in S_1}}\alpha^2q^{2^jN(\alpha)}
\end{aligned}$$

$$=\sum_{j=1}^{\infty} w^{2j} g(2^j\tau),$$

$$\sum_{\substack{I=(\alpha)\\ I\in S_3}} \alpha^2 q^{N(\alpha)} = \sum_{\substack{I=(\alpha)\\ \alpha\in S_1}} \sum_{j=1}^{\infty} (\bar{w}^j\alpha)^2 q^{N(\bar{w}^j\alpha)} = \sum_{j=1}^{\infty} \bar{w}^{2j} \sum_{\substack{I=(\alpha)\\ \alpha\in S_1}} \alpha^2 q^{2^j N(\alpha)}$$

$$=\sum_{j=1}^{\infty} \bar{w}^{2j} g(2^j\tau).$$

In addition,

$$\sum_{\substack{I=(\alpha)\\ I\in S_4}} \alpha^2 q^{N(\alpha)} = \sum_{\substack{I=(\alpha)\\ I\in S_{4,1}}} \alpha^2 q^{N(\alpha)} + \sum_{\substack{I=(\alpha)\\ I\in S_{4,2}}} \alpha^2 q^{N(\alpha)} + \sum_{\substack{I=(\alpha)\\ I\in S_{4,3}}} \alpha^2 q^{N(\alpha)},$$

where

$$\sum_{\substack{I=(\alpha)\\ I\in S_{4,1}}} \alpha^2 q^{N(\alpha)} = \sum_{\substack{I=(\alpha)\\ \alpha\in S_1}} \sum_{j=1}^{\infty} (2^j\alpha)^2 q^{N(2^j\alpha)} = \sum_{j=1}^{\infty} 2^{2j} \sum_{\substack{I=(\alpha)\\ \alpha\in S_1}} \alpha^2 q^{2^{2j} N(\alpha)}$$

$$=\sum_{j=1}^{\infty} 2^{2j} g(2^{2j}\tau),$$

$$\sum_{\substack{I=(\alpha)\\ I\in S_{4,2}}} \alpha^2 q^{N(\alpha)} = \sum_{\substack{I=(\alpha)\\ \alpha\in S_1}} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (2^k w^j\alpha)^2 q^{N(2^k w^j\alpha)} = \sum_{k=1}^{\infty} 2^{2k} \sum_{j=1}^{\infty} w^{2j} \sum_{\substack{I=(\alpha)\\ \alpha\in S_1}} \alpha^2 q^{2^{2k}2^j N(\alpha)},$$

$$\sum_{\substack{I=(\alpha)\\ I\in S_{4,3}}} \alpha^2 q^{N(\alpha)} = \sum_{\substack{I=(\alpha)\\ \alpha\in S_1}} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (2^k \bar{w}^j\alpha)^2 q^{N(2^k \bar{w}^j\alpha)} = \sum_{k=1}^{\infty} 2^{2k} \sum_{j=1}^{\infty} \bar{w}^{2j} \sum_{\substack{I=(\alpha)\\ \alpha\in S_1}} \alpha^2 q^{2^{2k}2^j N(\alpha)}.$$

Let $h(\tau) = \sum_{j=1}^{\infty}(w^{2j}+\bar{w}^{2j})g(2^j\tau)$. Then

$$g_7 = g(\tau) + h(\tau) + \sum_{j=1}^{\infty} 2^{2j} g(2^{2j}\tau) + \sum_{k=1}^{\infty} 2^{2k} h(2^{2k}\tau).$$

Note that

$$L(h,3) = \sum_{j=1}^{\infty} \frac{(w^{2j}+\bar{w}^{2j})}{2^{3j}} L(g,3).$$

Therefore,

$$\begin{aligned} L(g_7,3) =& L(g,3)\left(1+\sum_{j=1}^{\infty}\frac{w^{2j}+\bar{w}^{2j}}{2^{3j}}\right)\left(1+\frac{1}{2^4}+\frac{1}{2^8}+\cdots\right)\\ =&\frac{16}{15}L(g,3)\left(1+\sum_{j=1}^{\infty}\left(\left(\frac{w^2}{8}\right)^j+\left(\frac{\bar{w}^2}{8}\right)^j\right)\right)\\ =&\frac{16}{15}L(g,3)\left(1+\frac{w^2/8}{1-w^2/8}+\frac{\bar{w}^2/8}{1-\bar{w}^2/8}\right)\\ =&\frac{16}{23}L(g,3) = \frac{8}{23}\left(\sum\nolimits_{(1,0)}+\sum\nolimits_{(0,1)}\right). \end{aligned}$$

Combining the relation (6), we have

$$\sum\nolimits_{(1,1)} = -\frac{1}{46}\left(\sum\nolimits_{(1,0)} + \sum\nolimits_{(0,1)}\right).$$

Thus

$$-\frac{2}{3}\sum\nolimits_{(1,1)} + \frac{1}{3}\left(\sum\nolimits_{(1,0)} + \sum\nolimits_{(0,1)}\right) = \frac{8}{23}\left(\sum\nolimits_{(1,0)} + \sum\nolimits_{(1,1)}\right) = L(g_7, 3),$$

as desired. □

By functional equations, one can show that

$$L'(g_7, 0) = \frac{7\sqrt{7}}{4\pi^3} L(g_7, 3),$$

$$L'(g_7 \otimes \chi_{-4}, 0) = \frac{112\sqrt{7}}{\pi^3} L(g_7 \otimes \chi_{-4}, 3).$$

In addition

$$\begin{aligned} \frac{1}{14}(4M_{7\otimes(-4)} + 384M_7) =& \frac{4}{14}\cdot\frac{112\sqrt{7}}{\pi^3} L(g_7 \otimes \chi_{-4}, 3) + \frac{384}{14}\cdot\frac{7\sqrt{7}}{4\pi^3} L(g_7, 3) \\ =& \frac{\sqrt{7}}{\pi^3}\Big(32L(g_7 \otimes \chi_{-4}, 3) + 48L(g_7, 3)\Big) \\ =& \frac{16\sqrt{7}}{\pi^3}\Big(2L(g_7 \otimes \chi_{-4}, 3) + 3L(g_7, 3)\Big). \end{aligned} \tag{4.2}$$

On the other hand,

$$\begin{aligned} &\frac{\sqrt{7}}{\pi^3}(-A_M + 16B_M) \\ =&\frac{16\sqrt{7}}{\pi^3}\left(-\sum_{m\equiv l(2)}{}' \frac{2(l^2-7m^2)}{(l^2+7m^2)^3} + \sum_{\substack{l,m\in\mathbb{Z}\\ m \text{ even}}}{}' \frac{2(l^2-7m^2)}{(l^2+7m^2)^3}\right) \\ =&\frac{32\sqrt{7}}{\pi^3}\left(-\sum_{m\equiv l(2)}{}' + \sum_{\substack{l,m\in\mathbb{Z}\\ m \text{ even}}}{}'\right) \\ =&\frac{32\sqrt{7}}{\pi^3}\left(-\sum_{(1,1)}{}' + \sum_{(1,0)}{}'\right) \\ =&\frac{32\sqrt{7}}{\pi^3}\left(-\sum_{(1,1)}{}' + \frac{1}{2}\sum_{(1,0)}{}' + \frac{1}{2}\sum_{(0,1)}{}' + \frac{1}{2}\left(\sum_{(1,0)}{}' - \sum_{(0,1)}{}'\right)\right) \\ =&\frac{32\sqrt{7}}{\pi^3}\left(-\sum_{(1,1)}{}' + \frac{1}{2}\sum_{(1,0)}{}' + \frac{1}{2}\sum_{(0,1)}{}' + L(g\otimes\chi_{-4}, 3)\right). \end{aligned} \tag{4.3}$$

Comparing (7) and (8) and by Theorem 4.2,4.3, we are done.

4.2.2. *The part of the Dirichlet L-Series.*

$$-A_D + 16B_D = -\sum_{(0,0)}{}' \frac{16}{(l^2+7m^2)^2} - \sum_{(1,1)}{}' \frac{16}{(l^2+7m^2)^2} + \sum_{\substack{l,m\in\mathbb{Z}\\ m \text{ even}}}{}' \frac{16}{(l^2+7m^2)^2}$$
$$=16\left(-\sum_{(0,0)}{}' \frac{1}{(l^2+7m^2)^2} - \sum_{(1,1)}{}' \frac{1}{(l^2+7m^2)^2} + \sum_{\substack{l,m\in\mathbb{Z}\\ m \text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}\right).$$

From last subsection, one can get

$$\begin{aligned}-A_D + 16B_D =&16\left(-\frac{5}{64}C - \frac{3}{64}C + \frac{41}{64}C + L_{-4}(2)L_{28}(2)\right)\\ =&\frac{33}{4}C + L_{-4}(2)L_{28}(2) = \frac{33}{4}L_1(2)L_{-7}(2) + 16L_{-4}(2)L_{28}(2)\\ =&\frac{11}{14}L'(\chi_{-7},-1) + \frac{16}{7}L'(\chi_{-4},-1) = \frac{11}{14}d_7 + \frac{16}{7}d_4,\end{aligned}$$

as desired. Here we used the following facts:

$$\begin{aligned}L_1(2) =&\frac{\pi^2}{6}; L_{28}(2) = \frac{2\pi^2}{7\sqrt{7}}; \quad \text{(Remark 2.4)}\\ L_{-4}(2) =&\frac{\pi}{2}d_4; L_{-7}(2) = \frac{4\pi}{7\sqrt{7}}d_7. \quad \text{(by functional equation)}\end{aligned}$$

4.3. $k = -2024 + 765\sqrt{7}$**.** To ease the notation, we set:

$$\begin{aligned}A_M &= \sum_{m\equiv l(2)}{}' \tfrac{7^2\cdot 2^5\cdot(7l^2-m^2)}{(7l^2+m^2)^3}, & B_M &= \sum_{m,n\in\mathbb{Z}}{}' \tfrac{2\cdot 7^2(7l^2-4m^2)}{(7l^2+4m^2)^3},\\ A_D &= \sum_{m\equiv l(2)}{}' \tfrac{7^2\cdot 16}{(7l^2+m^2)^2}, & B_D &= \sum_{m,n\in\mathbb{Z}}{}' \tfrac{7^2}{(7l^2+4m^2)^2}.\end{aligned}$$

in this subsection.

Following Samart's method, we will calculate A and B in the statement of Lemma 2.2 for our case, where $\tau = \frac{7+\sqrt{-7}}{14}$ by Lemma 2.1(3).

$$\begin{aligned}A =& \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{64(2n+m)^2}{\left((2n+m)^2+\frac{m^2}{7}\right)^3} - \frac{16}{\left((2n+m)^2+\frac{m^2}{7}\right)^2}\right)\\ =& \sum_{m\equiv l(2)}{}' \left(\frac{64l^2}{\left(l^2+\frac{m^2}{7}\right)^3} - \frac{16}{\left(l^2+\frac{m^2}{7}\right)^2}\right) \quad (\text{Let } l = 2n+m)\\ =& \sum_{m\equiv l(2)}{}' \left(\frac{7^2\cdot 2^5\cdot(7l^2-m^2)}{(7l^2+m^2)^3} + \frac{7^2\cdot 2^5\cdot(7l^2+m^2)}{(7l^2+m^2)^3} - \frac{7^2\cdot 16}{(7l^2+m^2)^2}\right)\\ =& \sum_{m\equiv l(2)}{}' \frac{7^2\cdot 2^5\cdot(7l^2-m^2)}{(7l^2+m^2)^3} + \sum_{m\equiv l(2)}{}' \frac{7^2\cdot 16}{(7l^2+m^2)^2} = A_M + A_D,\end{aligned}$$

and

$$\begin{aligned} B &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{7^3\cdot 4(n+2m)^2}{(7(n+2m)^2+4m^2)^3} - \frac{7^2}{(7(n+2m)^2+4m^2)^2} \right) \\ &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{7^3\cdot 4l^2}{(7l^2+4m^2)^3} - \frac{7^2}{(7l^2+4m^2)^2} \right) \quad (\text{Let } l = n+2m,) \\ &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{2\cdot 7^2(7l^2-4m^2)}{(7l^2+4m^2)^3} + \frac{2\cdot 7^2(7l^2+4m^2)}{(7l^2+4m^2)^3} - \frac{7^2}{(7l^2+4m^2)^2} \right) \\ &= \sum_{m,n\in\mathbb{Z}}{}' \frac{2\cdot 7^2(7l^2-4m^2)}{(7l^2+4m^2)^3} + \sum_{m,n\in\mathbb{Z}}{}' \frac{7^2}{(7l^2+4m^2)^2} = B_M + B_D. \end{aligned}$$

4.3.1. *The part of the L-Series of a modular form.* By functional equations, we have

$$L'(g_7,0) = \frac{7\sqrt{7}}{4\pi^3} L(g_7,3),$$

$$L'(g_7\otimes\chi_{-4},0) = \frac{112\sqrt{7}}{\pi^3} L(g_7\otimes\chi_{-4},3).$$

And,

$$\begin{aligned} \frac{1}{2}(4M_{7\otimes(-4)} - 384M_7) &= \frac{4}{2}\cdot\frac{112\sqrt{7}}{\pi^3} L(g_7\otimes\chi_{-4},3) - \frac{384}{2}\cdot\frac{7\sqrt{7}}{4\pi^3} L(g_7,3) \\ &= \frac{\sqrt{7}}{\pi^3}\Big(224L(g_7\otimes\chi_{-4},3) - 336L(g_7,3)\Big) \\ &= \frac{112\sqrt{7}}{\pi^3}\Big(2L(g_7\otimes\chi_{-4},3) - 3L(g_7,3)\Big). \end{aligned} \tag{4.4}$$

On the other hand,

$$\begin{aligned} &\frac{\sqrt{7}}{7\pi^3}\left(-A_M + 16B_M\right) \\ =&\frac{224\sqrt{7}}{\pi^3}\left(\sum_{m\equiv l(2)}{}' \frac{(l^2-7m^2)}{(l^2+7m^2)^3} - \sum_{\substack{l,m\in\mathbb{Z}\\ l \text{ even}}}{}' \frac{(l^2-7m^2)}{(l^2+7m^2)^3} \right) \\ =&\frac{224\sqrt{7}}{\pi^3}\left(\sum_{(1,1)}{}' - \sum_{(0,1)}{}' \right) \\ =&\frac{224\sqrt{7}}{\pi^3}\left(\sum_{(1,1)}{}' - \frac{1}{2}\sum_{(1,0)}{}' - \frac{1}{2}\sum_{(0,1)}{}' + \frac{1}{2}\left(\sum_{(1,0)}{}' - \sum_{(0,1)}{}' \right) \right) \\ =&\frac{224\sqrt{7}}{\pi^3}\left(\sum_{(1,1)}{}' - \frac{1}{2}\sum_{(1,0)}{}' - \frac{1}{2}\sum_{(0,1)}{}' + L(g\otimes\chi_{-4},3) \right). \end{aligned} \tag{4.5}$$

By Theorem 4.2 and Theorem 4.3, comparing (9) and (10), we are done.

4.3.2. *The part of the Dirichlet L-Series.*

$$-A_D+16B_D=-\sum_{(0,0)}{}' \frac{7^2\cdot 16}{(l^2+7m^2)^2}-\sum_{(1,1)}{}' \frac{7^2\cdot 16}{(l^2+7m^2)^2}+\sum_{\substack{l,m\in\mathbb{Z}\\ l \text{ even}}}{}' \frac{7^2\cdot 16}{(l^2+7m^2)^2}$$

$$=7^2\cdot 16\left(-\sum_{(0,0)}{}' \frac{1}{(l^2+7m^2)^2}-\sum_{(1,1)}{}' \frac{1}{(l^2+7m^2)^2}+\sum_{\substack{l,m\in\mathbb{Z}\\ l \text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}\right).$$

From previous computation, we have

$$\sum_{\substack{l,m\in\mathbb{Z}\\ l \text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}=-\sum_{\substack{l,m\in\mathbb{Z}\\ m \text{ even}}}{}' \frac{1}{(l^2+7m^2)^2}+\star.$$

Hence,

$$\begin{aligned}-A_D+16B_D=&7^2\cdot 16\left(-\frac{5}{64}C-\frac{3}{64}C-\frac{41}{64}C-L_{-4}(2)L_{28}(2)+\frac{41}{32}C\right)\\ =&7^2\left(\frac{33}{4}L_1(2)L_{-7}(2)-16L_{-4}(2)L_{28}(2)\right)\\ =&7^2\left(\frac{11}{14}L'(\chi_{-7},-1)-\frac{16}{7}L'(\chi_{-4},-1)\right)=7^2\left(\frac{11}{14}d_7-\frac{16}{7}d_4\right),\end{aligned}$$

as desired. Here we used the following facts:

$$L_1(2)=\frac{\pi^2}{6};L_{28}(2)=\frac{2\pi^2}{7\sqrt{7}};\quad \text{(Remark 2.4.)}$$

$$L_{-4}(2)=\frac{\pi}{2}d_4;L_{-7}(2)=\frac{4\pi}{7\sqrt{7}}d_7\quad \text{(by functional equation)}$$

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