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FINITE TIME BLOW-UP FOR A WAVE EQUATION WITH DYNAMIC BOUNDARY CONDITION AT CRITICAL AND HIGH ENERGY LEVELS IN CONTROL SYSTEMS

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ABSTRACT. We study the initial boundary value problem of linear homogeneous wave equation with dynamic boundary condition. We aim to prove the finite time blow-up of the solution at critical energy level or high energy level with the nonlinear damping term on boundary in control systems.

1. INTRODUCTION

In this paper, we mainly discuss the initial boundary value problem of linear homogeneous wave equation with dynamic boundary condition

$$u_{tt} - \Delta u = 0 \quad \text{in } (0, \infty) \times \Omega, \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } [0, \infty) \times \Gamma_0, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = -Q(u_t) + f(u) \quad \text{on } [0, \infty) \times \Gamma_1, \tag{1.3}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \quad \text{on } \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a regular, bounded and connected domain with boundary $\partial\Omega = \Gamma_0 \bigcup \Gamma_1$, $\Gamma_0 \bigcap \Gamma_1 = \emptyset$, where Γ_0 and Γ_1 are measurable over $\partial\Omega$ endowed with the $(n-1)-$ dimensional Lebesgue measures $\lambda_{n-1}(\Gamma_0)$ and $\lambda_{n-1}(\Gamma_1)$; Δ is the Laplacian operator with respect to the x; $Q(u_t) = |u_t|^{m-2}u_t,\ f(u) = |u|^{p-2}u,\ m \geq 2,\ p \geq 2$. These properties of Ω, Γ_0 and Γ_1 will be assumed throughout the paper. The initial data are $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$, with the compatibility condition $u_0 = 0$ on Γ_0. We always assume that $\lambda_{n-1}(\Gamma_0) > 0$ and $\lambda_{n-1}(\Gamma_1) > 0$ throughout the paper.

For the wave equation with nonlinear dynamic boundary condition like problem (1.1)-(1.4), arising in the physical models and control systems, there have been many papers dealing with the existence and blow-up of the solution. In [4]-[7], [12]-[18] and [33], the global existence and decay properties of the solution of the problem (1.1)-(1.4) were proved for arbitrarily large initial data when $f \equiv 0$ or $f(x,u)u \leq 0$ if Q and f were under some special assumptions. When $f(x,u)u \geq 0$, which means f is a source term, the situation is quite different. If $f(x,u) = |u|^{p-2}u,\ p > 2$ and $Q \equiv 0$, when the $(n-1)$-dimensional Lebesgue measure $\lambda_{n-1}(\Gamma_0)$ and $\lambda_{n-1}(\Gamma_1)$ are assumed to be positive, the authors of [18] obtained the finite time blow-up when the initial

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energy is negative for problem (1.1)-(1.4). For the same problem above, Levine and Smith [11] proved the global existence of the solution when initial data u_0 and u_1 are very small. Zhang et al. [30] considered the Kirchhoff equation with dynamic boundary condition. They obtained the energy decay and blow-up of a solution with negative and small positive initial energy. Recently, some authors have studied the viscoelastic wave equation with boundary damping and source terms. In [9], Lee et al. proved the global existence and exponential growth solution. In [16], they proved the blow-up result of solutions under suitable conditions of the initial data, and in [17] they studied the existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions. In [25], the author studied problem (1.1)-(1.4) when $Q(u_t) = |u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u$, and showed that the solution of problem (1.1)-(1.4) globally exists in time for arbitrary initial data when $2 \leq p \leq m$, in opposition with the finite time blow-up occurring when $m = 2 < p$. In [2] they proved more general existence and stability results by using a natural tool for the problem–monotone operator theory that contrast with the Schauder fixed point arguments used in [25]. For the same problem with $p > m$, Zhang and Hu [29] obtained the nonexistence of the solution under the energy level $E(0) < d$ when the initial data are in the unstable set. As well known, in the frame of potential well theory, the variational arguments are usually taken by considering different levels of the initial data, see its applications to differential kinds of model equations in[21], [27] and [28], which are very different from the other aspects of the studies on the wave equations [3], [10], [11], [22], [23], [26] and [31]. However, no results were obtained about the finite time blow-up of the solution for problem (1.1)-(1.4) when $Q(u_t) = |u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u,\ p > m$ at critical energy level $E(0) = d$ or high energy level $E(0) > d$. The main purpose of this paper is to get the finite time blow-up result of the solution of the problem (1.1)-(1.4) when $Q(u_t) = |u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u,\ p > m$ for the critical initial data and arbitrarily large initial data. We mainly adapt the method introduced by Vitillaro in [24] to study the solution of problem (1.1)-(1.4) when $Q(u_t) = |u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u,\ p > m$ at critical energy level and the convexity method introduced by Gazzola and Squassina in [8] to get the finite time blow-up of the solution at high energy level.

In Section 2, we introduce some basic setup, notations and some known results of the solution to problem (1.1)-(1.4). In section 3, we prove the blow-up result of the solution when $E(0) = d$. Finally in Section 4, we obtain the blow-up result when $E(0) > d$.

2. Set up and notations

First we denote

$$\|\cdot\| = L^2(\Omega),\ \|\cdot\|_q = L^q(\Omega),\ \|\cdot\|_{q,\Gamma_1} = L^q(\Gamma_1),\ 1 \leq q \leq \infty,$$

and

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) \mid u|_{\Gamma_0} = 0\}, \tag{2.1}$$

($u|_{\Gamma_0}$ is in the sense of trace). We can endow $H^1_{\Gamma_0}(\Omega)$ the equivalent norm $\|u\|^p_{p,\Gamma_1} = \|\nabla u\|$ because of the Poincarè inequality (see [17]) and the fact that $\lambda_{n-1}(\Gamma_0) > 0$. We also define some useful functionals

$$J(u) = \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p}\|u\|^p_{p,\Gamma_1}, \tag{2.2}$$

$$I(u) = \|\nabla u\|^2 - \|u\|^p_{p,\Gamma_1}, \tag{2.3}$$

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p}\|u\|_{p,\Gamma_1}^p. \tag{2.4}$$

All these functionals are defined on $H^1_{\Gamma_1}(\Omega)$. According to the definition of $E(t)$, we have

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{p}\|u_0\|_{p,\Gamma_1}^p.$$

We also use the trace-Sobolev embedding $H^1_{\Gamma_1}(\Omega) \hookrightarrow L^p(\Gamma_1)$ for $2 \leq p < r$ introduced in [1], where

$$r = \begin{cases} \frac{2(n-1)}{n-2}, & if\ n \geq 3; \\ +\infty, & if\ n = 1, 2. \end{cases} \tag{2.5}$$

We also have the embedding inequality

$$\|u\|_{p,\Gamma_1}^p \leq C_* \|\nabla u\|, \tag{2.6}$$

where C_* is the embedding constant. Then we introduce the unstable set V defined by

$$V = \{(u_0, u_1) \in H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \mid I(0) < 0, 0 < E(0) = d\}, \tag{2.7}$$

where d is the mountain pass level, characterized as

$$d = \inf_{u \in H^1_{\Gamma_0}(\Omega), u|\Gamma_1 \neq 0} \left(\sup_{\lambda > 0} J(\lambda u) \right). \tag{2.8}$$

We define

$$\lambda_1 := C_*^{\frac{-p}{p-2}}. \tag{2.9}$$

It has been proved in [25] that $\left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2$ is the potential well depth, that is

$$d = \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2. \tag{2.10}$$

In [25], the author have proved the local and global existence of the solution for problem (1.1)-(1.4). Now we introduce some results in [25] as follows:

Theorem 2.1. *(Local existence of the solution) Suppose that* $m > 1, 2 \leq p < r$ *and* $m > \frac{r}{r+1-p}$. *Then, given initial data* $u_0 \in H^1_{\Gamma_0}(\Omega)$ *and* $u_1 \in L^2(\Omega)$, *there is a* $T > 0$ *and a weak solution* u *of the problem* (1.1)-(1.4) *on* $(0,T) \times \Omega$ *such that* $u \in C([0,T]; H^1_{\Gamma_0}(\Omega) \cap C^1([0,T]; L^2(\Omega)))$ *,*$u_t \in L^m((0,T) \times \Gamma_1)$,

$$E(t) + \int_s^t \|u_\tau(\tau)\|_{m,\Gamma_1}^m d\tau = E(s), \tag{2.11}$$

holds for $0 \leq s \leq t \leq T$.

3. Finite time blow-up for critical initial energy $E(0) = d$

In this section, we mainly show the finite time blow-up of the solution when initial data are at critical level. In order to prove the finite time blow-up of the solution, we first prove some basic lemmas.

Lemma 3.1. *(Invariant manifolds) We suppose that $m > 1$, $2 \leq p < r$ and $m > \frac{r}{r+1-p}$. Let*

$$V' = \{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \mid \|\nabla u_0\| > \lambda_1, 0 < E(0) = d\},$$

then we have $V = V'$.

Proof. First we show that $V \subset V'$. Let $(u_0, u_1) \in V$, then we have $I(0) < 0$ (i.e $\|\nabla u_0\|^2 < \|u_0\|^p_{p,\Gamma_1}$). By using the embedding inequality (2.6), we can obtain that

$$\|\nabla u_0\|^2 < \|u_0\|^p_{p,\Gamma_1} \leq C^p_* \|\nabla u_0\|^p.$$

Hence we have $\|\nabla u_0\| > C_*^{-\frac{p}{p-2}} = \lambda_1$. So we get that $V \subset V'$. Then we show that $V' \subset V$. Let $(u_0, u_1) \in V'$, then we have $\|\nabla u_0\| > \lambda_1$ and $E(0) = d$. Supposing by contradiction that $I(0) \geq 0$, we have

$$\|\nabla u_0\|^2 \geq \|u_0\|^p_{p,\Gamma_1}.$$

Combining

$$E(0) = \frac{1}{2}\|u_1\|^2_2 + \frac{1}{2}\|\nabla u_0\|^2_2 - \frac{1}{p}\|u_0\|^p_{p,\Gamma_1},$$

we can get

$$\frac{1}{2}\|\nabla u_0\|^2_2 - \frac{1}{p}\|u_0\|^p_{p,\Gamma_1} \leq d.$$

Then we obtain that

$$d \geq \left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u_0\|^2.$$

Since $\|\nabla u_0\| > \lambda_1$, it follows that

$$d > \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_1^2 = d,$$

which leads to a contradiction. This completes the proof. □

Lemma 3.2. *(Invariant manifolds and boundness) Suppose that $m > 1$, $2 \leq p < r$ and $m > \frac{r}{r+1-p}$. Let $(u_0, u_1) \in V$ and $u(x,t)$ be the weak solution of the problem (1.1)-(1.4) on $[0, T_{max})$. Then $(u(t,\cdot), u_t(t,\cdot))$ remains inside V for any $[0, T_{max})$. Furthermore, we have*

$$\begin{aligned} &\|\nabla u(t)\|^2 < \|u(t)\|^p_{p,\Gamma_1}, \quad t \in [0, t_{max}),\\ &\|u(t)\|_{p,\Gamma_1} > C_*\lambda_1, \quad t \in [0, t_{max}),\\ &\|\nabla u(t)\| > \lambda_1, \quad t \in [0, t_{max}). \end{aligned}$$

Proof. From $I(0) < 0$ and the continuity of $I(u)$ respecting to t, it follows that there exists a sufficiently small $t_1 > 0$, such that $I(u) < 0$ for $0 < t < t_1$. Combining (2.11), we set that $d_1 = E(t_1)$, then we have

$$0 < d_1 = d - \int_0^{t_1} \|u_\tau(\tau)\|^m_{m,\Gamma_1} d\tau < d.$$

We choose $t = t_1$ as the initial time, by the same proceeding in Theorem 2.3 [12], we can get that $(u(t,\cdot), u_t(t,\cdot))$ remains inside V for any $t \in [0, T_{max})$ (Here we have already known the invariance when $E(0) < d$, so by selecting t_1 as the initial time, we can have $E(t_1) < d$ again, then we can use the invariance conclusion for

$E(t_1) < d$). Furthermore, noting $I(u) < 0$ for all $t \in [0, T_{max})$, according to the definition of $I(u)$, we have

$$\|\nabla u(t)\|^2 < \|u(t)\|^p_{p,\Gamma_1}, \quad t \in [0, T_{max}). \tag{3.1}$$

Moreover, according to Lemma 3.1, we can obtain that

$$\|\nabla u(t)\| > \lambda_1, \quad t \in [0, T_{max}). \tag{3.2}$$

Then, by using (2.9) and (3.1), we have

$$\|u(t)\|_{p,\Gamma_1} > C_*\lambda_1, \quad t \in [0, T_{max}). \tag{3.3}$$

This completes the proof. □

By the similar method in [24], we can prove that in the manifold $V' = \{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \mid \|\nabla u_0\| > \lambda_1, 0 < E(0) = d\}$, there is a constant λ_2 between $\|\nabla u(t)\|$ and λ_1, i.e., there is a $\lambda_2 > \lambda_1$ such that $\|\nabla u(t)\| \geq \lambda_2 > \lambda_1$. This will be given by the following lemma. And this lemma will be used to prove the finite time blow-up for the critical case $E(0) = d$.

Lemma 3.3. *Suppose that $m > 1,\ 2 \leq p < r$ and $m > \frac{r}{r+1-p}$. Let $(u_0, u_1) \in V$ and $u(x,t)$ be the weak solution of the problem* (1.1)-(1.4) *on $[0, T_{max})$. There is a $\lambda_2 > \lambda_1$ such that $\|\nabla u(t)\| \geq \lambda_2 > \lambda_1$.*

Proof. According to Lemma 3.2, we have $I(u) < 0$ for all $t \in [0, T_{max})$. By (2.6), we have

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p}\|u\|^p_{p,\Gamma_1} \geq \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p}\|u\|^p_{p,\Gamma_1} \\ &\geq \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p}C_*^p\|\nabla u\|^p := g(\|\nabla u\|), \end{aligned} \tag{3.4}$$

where $g(\lambda) = \frac{1}{2}\lambda^2 - \frac{1}{p}C_*^p\lambda^p$ for $\lambda \geq 0$. It is easy to see that g takes its maximum at $\lambda = \lambda_1$, with $g(\lambda_1) = d$, being strictly decreasing for $\lambda \geq \lambda_1$, and $g(\lambda) \to -\infty$ as $\lambda \to \infty$. Combining the fact that $E(t)$ is decreasing when $t \in [0, T_{max})$ and $E(0) = d$, we can continue to argue as follows. By the continuity of $\|\nabla u(\cdot)\|$, there are only two possibilities:

(a) there is a $t_0 \geq 0$ such that $E(t_0) < d$ and $\|\nabla u(t_0)\| > \lambda_1$;

(b) there is an $\varepsilon_0 > 0$ such that $E(t) = d$ on $[0, \varepsilon_0)$.

In the first case, we choose t_0 as the initial time. Due to the fact that $E(t)$ and $g(\lambda)$ are both decreasing and continuous, there exists a $\lambda_2 > \lambda_1$ such that $E(t_0) = g(\lambda_2)$. We now claim that

$$\|\nabla u(t_0)\| \geq \lambda_2.$$

Suppose for contradiction that $\|\nabla u(t_0)\| < \lambda_2$ for some $t \in [0, T_{max})$. By using (3.4) and the fact that $g(\lambda)$ is a decreasing function, we have

$$E(t_0) \geq g(\|\nabla u(t_0)\|) > g(\lambda_2) = E(t_0),$$

which leads to a contradiction. We can also obtain that

$$\frac{1}{2}\|\nabla u\|^2 \geq \frac{1}{2}{\lambda_2}^2 > \frac{1}{2}{\lambda_1}^2.$$

We then choose $\lambda_0 = \frac{1}{2}{\lambda_1}^2$ and $E_1 = (1-\frac{p}{2})\lambda_0$. By doing the same process in Theorem 2.3 in [24], we can conclude the proof for the first case.

In the second case, for $t \in [0, \varepsilon_0)$, we have $E(t) = d$. By using (2.11), we have

$$\int_0^t \|u_\tau(\tau)\|_{m,\Gamma_1}^m d\tau = 0 \qquad t \in [0, \varepsilon_0).$$

Due to the fact that $\|u_t(t)\|_{m,\Gamma_1}^m \geq 0$, we have $u_t = 0$ and $u(t) = u_0$ on $[0, \varepsilon_0)$. Suppose for contradiction that $\|\nabla u_0\| < \lambda_2$ for $t \in [0, \varepsilon_0)$. We can obtain that

$$d = E(0) = \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{p}\|u_0\|_{p,\Gamma_1}^p \geq g(\|\nabla u_0\|) > g(\lambda_2) = E(0) = d,$$

which leads to a contradiction. This completes the proof. □

Theorem 3.4. *(Finite time blow-up of solutions for $E(0) = d$) Assume that $1 < m < p$, $m > \frac{r}{r+1-p}$, $2 \leq p < r$. If $(u_0, u_1) \in V'$, then the solution of problem* (1.1)-(1.4) *blows up in finite time.*

Proof. Arguing by contradition, we assume that there exists a global weak solution of problem (1.1)-(1.4). We set $H(t) = d - E(t)$. By Theorem 2.1, $E(t)$ is decreasing about t. So $H(t)$ is an increasing function, then we have

$$H(t) \geq H(0) = d - E(0) = 0, \quad t \geq 0. \tag{3.5}$$

Next, by using the definition of $E(t)$, we have

$$H(t) \leq d - \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{p}\|u(t)\|_{p,\Gamma_1}^p, \quad t \geq 0. \tag{3.6}$$

By using (3.2) and (2.10), we can obtain that

$$d - \frac{1}{2}\|\nabla u(t)\|_2^2 \leq d - \frac{1}{2}\lambda_1^2 = -\frac{1}{p}\lambda_1^2 < 0.$$

Combining (3.6), we have

$$H(t) - \frac{1}{p}\|u(t)\|_{p,\Gamma_1}^p \leq d - \frac{1}{2}\|\nabla u(t)\|_2^2 < 0.$$

Then (3.5) tells

$$H(0) \leq H(t) < \frac{1}{p}\|u(t)\|_{p,\Gamma_1}^p, \quad t \geq 0. \tag{3.7}$$

Next, using the definition of $E(t)$ and (3.6), we have

$$\begin{aligned}
\frac{d}{dt}(u, u_t) &= \|u_t\|^2 - \|\nabla u(t)\|_2^2 + \|u(t)\|_{p,\Gamma_1}^p - \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \\
&= 2\|u_t\|^2 + \left(1 - \frac{2}{p}\right)\|u(t)\|_{p,\Gamma_1}^p - 2E(t) - \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \\
&= 2\|u_t\|^2 + \left(1 - \frac{2}{p}\right)\|u(t)\|_{p,\Gamma_1}^p - 2d + 2H(t) - \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma.
\end{aligned}$$

Then, According to Lemma 3.3, we choose a λ_2, such that $\|\nabla u(t)\| \geq \lambda_2 > \lambda_1$, then by using (3.3), we have

$$\begin{aligned}\frac{d}{dt}(u,u_t) &\geq 2\|u_t\|^2 + \left(1-\frac{2}{p}-2d(C_*\lambda_2)^{-p}\right)\|u(t)\|_{p,\Gamma_1}^p \\ &\quad + 2H(t) - \int_{\Gamma_1}|u_t|^{m-2}u_t u d\sigma \\ &= 2\|u_t\|^2 + C_1\|u(t)\|_{p,\Gamma_1}^p + 2H(t) - \int_{\Gamma_1}|u_t|^{m-2}u_t u d\sigma,\end{aligned} \tag{3.8}$$

where $C_1 = 1-\frac{2}{p}-2d(C_*\lambda_2)^{-p} > 0$. To obtain $\frac{d}{dt}(u,u_t)$, we first estimate the last term in (3.8). By Hölder's inequality, we obtain

$$\left|\int_{\Gamma_1}|u_t|^{m-2}u_t u d\sigma\right| \leq \|u_t\|_{m,\Gamma_1}^{m-1}\|u\|_{p,\Gamma_1} = \|u_t\|_{m,\Gamma_1}^{m-1}\|u\|_{p,\Gamma_1}^{1-\frac{p}{m}}\|u\|_{p,\Gamma_1}^{\frac{p}{m}}, \tag{3.9}$$

in which $\frac{1}{p}+\frac{1}{m}=1$, and then, by (3.7), Hölder's inequality, Young inequality and the fact that $H'(t) = \|u_t\|_{m,\Gamma_1}^m$, we obtain that

$$\begin{aligned}\left|\int_{\Gamma_1}|u_t|^{m-2}u_t u d\sigma\right| &\leq \|u\|_{p,\Gamma_1}^{1-\frac{p}{m}}\|u\|_{p,\Gamma_1}^{\frac{p}{m}}\|u_t\|_{m,\Gamma_1}^{m-1} \\ &\leq C_2 H^{\frac{1}{p}-\frac{1}{m}}(t)\|u\|_{p,\Gamma_1}^{\frac{p}{m}}\|u_t\|_{m,\Gamma_1}^{m-1} \\ &\leq C_3(\varepsilon^m\|u\|_{p,\Gamma_1}^p + \varepsilon^{-m'}\|u_t\|_{m,\Gamma_1}^m)H^{-\bar{\alpha}}(t) \\ &\leq C_3(\varepsilon^m\|u\|_{p,\Gamma_1}^p + \varepsilon^{-m'}H'(t))H^{-\bar{\alpha}}(t),\end{aligned} \tag{3.10}$$

for any $\varepsilon > 0$, where $\bar{\alpha} = \frac{1}{m}-\frac{1}{p} > 0$ and $\frac{1}{m}+\frac{1}{m'}=1$, and we denote C_1, C_2,..., as suitable positive constants. Let $0 < \alpha < \bar{\alpha}$, by (3.5) and (3.10), we have

$$\left|\int_{\Gamma_1}|u_t|^{m-2}u_t u d\sigma\right| \leq C_3(\varepsilon^m H^{-\bar{\alpha}}(0)\|u\|_{p,\Gamma_1}^p + \varepsilon^{-m'}H'(t)H^{-\alpha}(t)H^{\alpha-\bar{\alpha}}(0)). \tag{3.11}$$

Now we introduce an auxiliary function

$$Z(t) = H^{1-\alpha}(t) + \delta\int_\Omega u u_t dx,$$

where δ is a small positive constant which will be decided later. By (3.8) and (3.11), we have

$$\begin{aligned}Z'(t) &\geq (1-\alpha)H^{-\alpha}(t)H'(t) \\ &\quad + \delta\left(2\|u_t\|^2 + C_1\|u\|_{p,\Gamma_1}^p + 2H(t) - \int_{\Gamma_1}|u_t|^{m-2}u_t u d\sigma\right) \\ &\geq (1-\alpha-\delta C_3\varepsilon^{-m'}H^{\alpha-\bar{\alpha}}(0))H^{-\alpha}(t)H'(t) \\ &\quad + \delta(C_1 - C_3\varepsilon^m H^{-\bar{\alpha}}(0))\|u\|_{p,\Gamma_1}^p + 2\delta\|u_t\|^2 + 2\delta H(t).\end{aligned} \tag{3.12}$$

Let $\delta < (1-\alpha)C_3^{-1}\varepsilon^{m'}H^{\bar{\alpha}-\alpha}(0)$, so the first term on the right hand side of (3.12) is positive. Moreover, if we choose ε sufficiently small, we can obtain that

$$C_1 - C_3\varepsilon^m H^{-\bar{\alpha}}(0) \geq \frac{1}{2}C_1,$$

further,

$$Z'(t) \geq \frac{1}{2}C_0\delta\|u\|_{p,\Gamma_1}^p + 2\delta\|u_t\|^2 + 2\delta H(t) \geq C_4\delta(\|u\|_{p,\Gamma_1}^p + \|u_t\|^2 + H(t)). \tag{3.13}$$

Letting δ sufficiently small, we have $Z'(0) > 0$. And noting that $H(t)$ is an increasing function, we have $Z(t) \geq Z(0)$ for $t \geq 0$. Now we set $r = \frac{1}{1-\alpha}$, since $\alpha < \bar{\alpha} < 1$, it is evident that $1 < r < \bar{r} := \frac{1}{1-\bar{\alpha}}$. According to the following inequality

$$|a+b|^r \leq 2^{r-1}(|a|^r + |b|^r) \quad for\ r \geq 1,$$

Young inequality and Cauchy-Schwarz inequality, we have

$$Z^r(t) \leq 2^{r-1}(H(t) + \delta^r \|u_t\|^r \|u\|^r) \leq C_4(H(t) + \|u_t\|^2 + \|u\|^{\frac{1}{\frac{1}{2}-\alpha}}). \tag{3.14}$$

Now by choosing α sufficiently small, we have

$$\|u\|^{\frac{1}{\frac{1}{2}-\alpha}} \leq 1 + \|u\|^2. \tag{3.15}$$

Using Poincaré inequality and combining (3.1), (3.14) and (3.15), we have

$$Z^r(t) \leq C_5(H(t) + \|u_t\|^2 + \|\nabla u\|^2) \leq C_6(\|u\|^p_{p,\Gamma_1} + \|u_t\|^2 + H(t)). \tag{3.16}$$

In turning by (3.13) and (3.16), and the fact that $r > 1$, we obtain that

$$Z'(t) \geq C_7 Z^r(t). \tag{3.17}$$

Solving (3.17), we can obtain that there exist positive constants C_8 and C_9, such that

$$Z^{r-1}(t) \geq \frac{1}{-C_8 t + C_9}.$$

Then we have

$$\lim_{t \to \frac{C_9}{C_8}} Z^{r-1}(t) = \infty.$$

So $Z(t)$ is not global. This completes the proof. □

4. Blow up for high initial energy $E(0) > d$ when $Q(u_t) = 0$

In this section, we mainly discuss the problem (1.1)-(1.4) without damping term, i.e., $Q(u_t) = 0$. We mainly adapt the convexity method introduced in [8].

Lemma 4.1. *Let $\gamma > 0$, $T > 0$ and $h(t)$ be a Lipschitzian function over $[0,T)$. Assume that $h(0) \geq 0$ and $h'(t) + \gamma h(t) > 0$ for all $t \in (0,T)$. Then $h(t) > 0$ for all $t \in (0,T)$.*

Theorem 4.2. *(Finite time blow-up of solutions for $E(0) > d$.) Assume that $Q(u_t) = 0$, $1 < m < p$, $m > \frac{r}{r+1-p}$, $2 \leq p < r$. If*

$$\begin{aligned}
&I(0) < 0,\\
&E(0) > d,\\
&\int_\Omega u_0 u_1 dx \geq 0,\\
&\|u_0\|^2 > \frac{2p}{p-2} E(0),
\end{aligned}$$

then the solution of problem (1.1)-(1.4) blows up in finite time.

Proof. We will prove the result by two steps.

Step I. We first show that

$$I(u(t)) < 0 \textit{ and } \|u\|^2 > \frac{2p}{p-2}E(0), \quad t \geq 0. \tag{4.1}$$

Arguing by contradiction, we suppose by the continuity of $I(t)$ that there exists a first time $t_0 > 0$ such that $I(u(t_0)) = 0$. Then we consider the $L(t)$ function defined by

$$L(t) := \|u\|^2.$$

We have

$$L'(t) = 2\int_\Omega uu_t dx.$$

From the definition of $I(u)$, we obtain that

$$L''(t) = 2\|u_t\|^2 - 2I(u).$$

Noticing that

$$I(u) \leq 0, \quad t \in (0, t_0].$$

As $L''(t) \geq 0$ and $L'(0) = \int_\Omega u_0 u_1 dx \geq 0$ holds for all $t \in (0, t_0]$, by Lemma 4.1, we can obtain that $L'(t) > 0$ of all $t \in (0, t_0)$. So we can know that $L(t)$ is strictly increasing on $(0, t_0]$. Thus,

$$L(t) \geq \|u_0\|^2 > \frac{2p}{p-2}E(0), \quad t \in (0, t_0].$$

As a consequence, we have

$$L(t_0) > \frac{2p}{p-2}E(0).$$

On the other hand, combining the fact that $E(t)$ is a decreasing function, we have

$$E(t_0) \leq E(t) < E(0), \quad t \in (0, t_0].$$

According to the assumption, when $I(t_0) = 0$, we have

$$\|\nabla u(t_0)\|^2 \leq \frac{2p}{p-2}E(0).$$

From $\|u\|_{H^1_{\Gamma_0}(\Omega)} = \|\nabla u\|$ for $u \in H^1_{\Gamma_0}(\Omega)$, we have

$$L(t_0) = \|u(t_0)\|^2 \leq \|u(t_0)\|^2_{H^1_{\Gamma_0}(\Omega)} = \|\nabla u(t_0)\|^2 \leq \frac{2p}{p-2}E(0)$$

which leads to a contradiction. Thus we have proved that

$$I(u) < 0, \quad t \in [0, T_{max}).$$

By the discussion above, we see that $L(t)$ is strictly increasing on $[0, T_{max})$ provided $I(u) < 0$, which implies

$$\|\nabla u\|^2 \geq L(t) > \frac{2p}{p-2}E(0), \quad t \in [0, T_{max}). \tag{4.2}$$

Step II. Now we prove the finite time blow-up of the solution. As discussed above, we assume that u is global first. Then for any $t > 0$, we use Cauchy-Schwarz inequality

$$\|u\|\|u_t\| \geq \int_\Omega uu_t dx$$

to get

$$\begin{aligned} L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 &= 2\|u\|^2(\|u_t\|^2 - I(u)) - \frac{p+2}{4}(2\int_\Omega uu_t dx)^2 \\ &\geq 2\|u\|^2(\|u_t\|^2 - I(u)) - (p+2)\|u\|^2\|u_t\|^2 \\ &\geq \|u\|^2\xi(t), \end{aligned}$$

where

$$\xi(t) = -2pE(0) + (p-2)\|\nabla u\|^2.$$

According to (4.2), we have

$$\|\nabla u\|^2 > \frac{2p}{p-2}E(0).$$

So we can obtain that

$$2pE(0) < (p-2)\|\nabla u\|^2.$$

Then we have $\xi(t) > 0$. Further we have

$$L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 \geq 0.$$

Thus

$$\left(L(t)^{-\alpha}\right)'' = \frac{-\alpha}{L^{\alpha+2}(t)}\left(L(t)L''(t) - (1+\alpha)\left(L'(t)\right)^2\right) < 0,$$

where $\alpha = \frac{p-2}{4} > 0$. Hence, it proves that $L^{-\alpha}(t)$ reaches 0 in finite time, then there exists a $T \in [0, \infty)$ such that

$$\lim_{t\to T} L(t) = \infty.$$

Finally we prove that $L(t)$ is not global. This completes the proof. □

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References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975. MR 0450957

[2] M. M. Cavalcanti, V. N. Domingos Cavalcanti and I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping–source interaction, *J. Differential Equations*, **236** (2007), 407–459. MR 2322019

[3] C. E. Kenig, The method of energy channels for nonlinear wave equations, *Discrete and Continuous Dynamical Systems*, **39** (2019), 6979–6993. MR 4026177

[4] G. Chen, Energy decay estimates and exact boundary value controllabiity for the wave equation in a bounded domin, *J. Math. Pures Appl.*, **58** (1979), 249–273. MR 544253

[5] G. Chen, Control and stabilization for the wave equation in a bounded domain, *SIAM J. Control Optim.*, **17** (1979), 66–81. MR 516857

[6] G. Chen, Control and stabilization for the wave equation in a bounded domain, part II, *SIAM J. Control Optim.*, **19** (1981), 114–122. MR 603084

[7] G. Chen, A note on the boundary stabilization of the wave equation, *SIAM J. Control Optim.*, **19** (1981), 106–113. MR 603083

[8] F. Gazzola and M. Squassina, Global solutions and finite time blow-up for damped semilinear wave equations, *Nonlinear Analysis*, **23** (2006), 185–207. MR 2201151

[9] S. Gerbi and B. Said-Houari, Global existence and exponential growth for a viscoelastic wave equation with dynamic boundary conditions, *Adv. Nonlinear Anal.*, **2** (2013), 163–193. MR 3055533

[10] N. Hoai-Minh, Superlensing using complementary media and reflecting complementary media for electromagnetic waves, *Adv. Nonlinear Anal.*, **7** (2018), 449–467. MR 3871415

[11] E. Iryna, M. Johanna and T. Gerald, Rarefaction waves for the toda equation via nonlinear steepest descent, *Discrete and Continuous Dynamical Systems*, **38** (2018), 2007–2028. MR 3809023

[12] V. Komorkin and E. Zuazua, A direat method for boundary stablization of wave equation, *J. Math. Pures Appl.*, **69** (1990), 33–54. MR 1054123

[13] J. Lagnese, Deacy of solutions of wave equations in a bounded region with boundary dissipation, *Journal of Differential Equations*, **50** (1983), 163–182. MR 719445

[14] J. Lagnese, Note on boundary stabilization of wave equations, *SIAM J. Control Optim.*, **26** (1988), 1250–1256. MR 957663

[15] I. Lasiecka and D. Tataru, Uniform boundary stabilization of wave equation with nonlieary boundary damping, *Differential and Integral Equations*, **6** (1990), 507–533. MR 1202555

[16] M. J. Lee, J. R. Kang and S. H. Park, Blow-up of solution for quasilinear viscoelastic wave equation with boundary nonlinear damping and source terms, *Bound. Value Probl.*, **67** (2019), 11pp. MR 3936288

[17] M. J. Lee and J. Y. Park, Energy decay of solutions of nonlinear viscoelastic problem with the dynamic and acoustic boundary conditions, *Bound. Value Probl.*, **1** (2018), 26pp. MR 3741699

[18] H. A. Levine and J. Serrin, Global nonexistence theorems for quasilinear evolution equations with dissipation, *Arch. Rational Mech. Anal.*, **137** (1997), 341–361. MR 1463799

[19] H. A. Levine and A. Smith, A potential well theory for the wave equation with a nonlinear boundary conditions, *J. Reine angew. Math.*, **374** (1987), 1–23. MR 876218

[20] H. A. Levine and L. E. Payn, Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time, *J. Differential Equations*, **16** (1974), 319–334. MR 470481

[21] W. Lian and R. Z. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, *Adv. Nonlinear Anal.*, **9** (2020), 613–632. MR 3980417

[22] G. Olivier and M. Imen, Theoretical analysis of a water wave model with a nonlocal viscous dispersive term using the diffusive approach, *Adv. Nonlinear Anal.*, **8** (2019), 253–266. MR 3918376

[23] C. Shane and S. Anton, Homogenisation with error estimates of attractors for damped semilinear anisotropic wave equations, *Adv. Nonlinear Anal.*, **9** (2020), 745–787. MR 3995731

[24] E. Vitillaro, Some new results on global nonexistence and blow-up for evolution problems with positive initial energy, *Rend. Istit. Mat. Univ. Trieste*, **31** (2000), 245–275. MR 1800451

[25] E. Vitillaro, Global existence for the wave equation with nonlinear boundary damping and source term, *J. Diffrential Equations*, **186** (2002), 259–298. MR 1941100

[26] B. Vural, N. Emil and O. Ibrahim, Local-in-space blow-up crireria for two-component nonlinear dispersive wave sysytem, *Discrete and Continuous Dynamical Systems*, **39** (2019), 6023–6037. MR 4027022

[27] R. Z. Xu, M. Y. Zhang and S. H. Chen, The initial-boundary value problems for a class of sixth order nonlinear wave equation, *Discrete and Continuous Dynamical Systems*, **37** (2017), 5631–5649. MR 3681953

[28] R. Z. Xu and J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.*, **264** (2013), 2732–2763. MR 3045640

[29] H. W. Zhang and Q. Y. Hu, Asymptotic behavior and nonexistence of wave equation with nonlinear boundary condition, *Commun. Pure Appl. Anal.*, **4** (2005), 861–869. MR 2172726

[30] H. W. Zhang, C. S. Hou and Q. Y. Ho, Energy decay and blow-up of solution for a Kirchhoff equation with dynamic boundary condition, *Bound. Value Probl.*, **2013** (2013), 12pp. MR 3089846

[31] X. Zhao and W. P. Yan, Existence of standing waves for quasi-linear Schrödinger equations on T^n, *Adv. Nonlinear Anal.*, **9** (2020), 978–933. MR 3998217

[32] W. P. Ziemer, *Weakly Differently Functions*, Graduate Text in Mathematicas, Springer, New York, 1989. MR 1014685

[33] E. Zuazua, Uniform stabilization of the wave equations by nonlinear boundary feedback, *SIAM. J. Control Optim.*, **28** (1990), 466–477. MR 1040470

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