ELECTRONIC RESEARCH ARCHIVE
Volume 28 Number 1, March 2020 Pages 559–566
eISSN: 2688-1594 AIMS (2020)
doi:10.3934/era.2020029

ON SUMS OF FOUR PENTAGONAL NUMBERS WITH COEFFICIENTS

DMITRY KRACHUN AND ZHI-WEI SUN*

Abstract. The pentagonal numbers are the integers given by $p_5(n) = n(3n-1)/2$ $(n = 0, 1, 2, \ldots)$. Let (b, c, d) be one of the triples $(1, 1, 2), (1, 2, 3), (1, 2, 6)$ and $(2, 3, 4)$. We show that each $n = 0, 1, 2, \ldots$ can be written as $w+bx+cy+dz$ with w, x, y, z pentagonal numbers, which was first conjectured by Z.-W. Sun in 2016. In particular, any nonnegative integer is a sum of five pentagonal numbers two of which are equal; this refines a classical result of Cauchy claimed by Fermat.

1. Introduction

For each $m = 3, 4, 5, \ldots$, the *polygonal numbers of order m* are given by

$$p_m(n) = (m-2)\binom{n}{2} + n \quad (n \in \mathbb{N} = \{0, 1, 2, \ldots\}).$$

In particular, those $p_5(n)$ with $n \in \mathbb{N}$ are called *pentagonal numbers.* A famous claim of Fermat states that each $n \in \mathbb{N}$ can be written as a sum of m polygonal numbers of order m. This was proved by Lagrange for $m = 4$ in 1770, by Gauss for $m = 3$ in 1796, and by Cauchy for $m \geq 5$ in 1813. For Cauchy's polygonal number theorem, one may consult Nathanson [6] and [7, Chapter 1, pp. 3-34] for details. In 1830 Legendre refined Cauchy's polygonal number theorem by showing that for any $m = 5, 6, \ldots$ every sufficiently large integer is a sum of five polygonal numbers of order m one of which is 0 or 1 (cf. [7, p. 33]).

In 2016 Sun [9, Conjecture 5.2(ii)] conjectured that each $n \in \mathbb{N}$ can be written as

$$p_5(w) + bp_5(x) + cp_5(y) + dp_5(z) \quad \text{with } w, x, y, z \in \mathbb{N},$$

provided that (b, c, d) is among the following 15 triples:

$$(1, 1, 2), (1, 2, 2), (1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 2, 6), (1, 3, 6),$$
$$(2, 2, 4), (2, 2, 6), (2, 3, 4), (2, 3, 5), (2, 3, 7), (2, 4, 6), (2, 4, 7), (2, 4, 8).$$

In 2017, Meng and Sun [5] confirmed this for $(b, c, d) = (1, 2, 2), (1, 2, 4)$. In this paper we prove the conjecture for

$$(b, c, d) = (1, 1, 2),\ (1, 2, 3),\ (1, 2, 6),\ (2, 3, 4).$$

Received by the editors January, 2020.

2010 *Mathematics Subject Classification.* Primary: 11B13, 11E25; Secondary: 11D85, 11E20, 11P70.

Key words and phrases. Pentagonal numbers, additive bases, ternary quadratic forms.

The work is supported by the NSFC(Natural Science Foundation of China)-RFBR(Russian Foundation for Basic Research) Cooperation and Exchange Program (grants NSFC 11811530072 and RFBR 18-51-53020-GFEN-a). The second author is also supported by the Natural Science Foundation of China (grant no. 11971222).

* Corresponding author: Zhi-Wei Sun.

Theorem 1.1. *Each $n \in \mathbb{N}$ can be written as a sum of five pentagonal numbers two of which are equal, that is, there are $x, y, z, w \in \mathbb{N}$ such that*

$$n = p_5(x) + p_5(y) + p_5(z) + 2p_5(w).$$

Remark 1.1. Clearly, Theorem 1.1 is stronger than the classical result that any nonnegative integer is a sum of five pentagonal numbers. In Feb. 2019 the second author even conjectured that any integer $n > 33066$ is a sum of three pentagonal numbers.

Theorem 1.2. *Any $n \in \mathbb{N}$ can be written as $p_5(w) + 2p_5(x) + 3p_5(y) + 4p_5(z)$ with $w, x, y, z \in \mathbb{N}$.*

Theorem 1.3. *Let $\delta \in \{1, 2\}$. Then any $n \in \mathbb{N}$ can be written as $p_5(w) + p_5(x) + 2p_5(y) + 3\delta p_5(z)$ with $w, x, y, z \in \mathbb{N}$.*

We will prove Theorems 1.1-1.3 in Sections 2-4 respectively. Our proofs use some known results on ternary quadratic forms.

Those $p_5(x) = x(3x-1)/2$ with $x \in \mathbb{Z}$ are called *generalized pentagonal numbers*. Clearly,

$$\{p_5(x) :\ x \in \mathbb{Z}\} = \left\{\frac{n(3n-1)}{2} :\ n \in \mathbb{N}\right\} \bigcup \left\{\frac{n(3n+1)}{2} :\ n \in \mathbb{N}\right\}.$$

Recently, Ju [3] showed that for any positive integers $a_1, \ldots, a_k$ the set

$$\{a_1 p_5(x_1) + \ldots + a_k p_k(x_k) :\ x_1, \ldots, x_k \in \mathbb{Z}\}$$

contains all nonnegative integers whenever it contains the twelve numbers

$$1,\ 3,\ 8,\ 9,\ 11,\ 18,\ 19,\ 25,\ 27,\ 43,\ 98,\ 109.$$

The generalized octagonal numbers are those $p_8(x) = x(3x-2)$ with $x \in \mathbb{Z}$. In 2016, Sun [9] proved that any positive integer can be written as a sum of four generalized octagonal numbers one of which is odd. See also Sun [11] and [10] for representations of nonnegative integers in the form $x(ax+b)/2 + y(cy+d)/2 + z(ez+f)/2$ with x, y, z integers or nonnegative integers, where a, c, e are positive integers and b, d, f are integers with $a+b, c+d, e+f$ all even.

2. Proof of Theorem 1.1

Lemma 2.1. *Any positive even number n not in the set $\{5^{2k+1}m :\ k, m \in \mathbb{N}$ and $m \equiv \pm 2 \pmod 5\}$ can be written as $x^2 + y^2 + z^2 + (x+y+z)^2/2$ with $x, y, z \in \mathbb{Z}$.*

Proof. By Dickson [2, pp. 112-113],

$$\begin{aligned}&\mathbb{N} \setminus \{x^2 + 2y^2 + 10z^2 :\ x, y, z \in \mathbb{Z}\}\\ =&\{8m+7 :\ m \in \mathbb{N}\} \cup \{5^{2k+1}l :\ k, l \in \mathbb{N} \text{ and } l \equiv \pm 1 \pmod 5\}.\end{aligned}$$

Thus $8n = s^2 + 2t^2 + 10z^2$ for some $s, t, z \in \mathbb{Z}$. Clearly, $2 \mid s$ and $t \equiv z \pmod 2$. Without loss of generality, we may assume that $t \not\equiv z \pmod 4$ if $2 \nmid z$. (If $t \equiv z \pmod 4$ with z odd, then $-t \not\equiv z \pmod 4$.) Write $s = 2r$ and $t = 2w + z$ with $r, w \in \mathbb{Z}$. Then $2 \nmid w$ if $2 \nmid z$. Since

$$0 \equiv 8n = s^2 + 2(2w+z)^2 + 10z^2 = (2r)^2 + 12z^2 + 8w(w+z) \pmod{16},$$

both $r-z$ and $w(w+z)$ are even. If $2 \mid z$ then $2 \mid w$. Recall that $2 \nmid w$ if $2 \nmid z$. So $w \equiv z \equiv r \pmod 2$. Now, both $x=(r+w)/2$ and $y=(w-r)/2$ are integers. Observe that

$$\begin{aligned}2n =& r^2+2\left(w+\frac{z}{2}\right)^2+10\left(\frac{z}{2}\right)^2 = r^2+(w+z)^2+w^2+2z^2\\ =&(x-y)^2+(x+y)^2+(x+y+z)^2+2z^2\\ =&2x^2+2y^2+2z^2+(x+y+z)^2\end{aligned}$$

and hence $n=x^2+y^2+z^2+(x+y+z)^2/2$. This ends the proof. □

Lemma 2.2. *Let $n \in \mathbb{N}$. Suppose that there are $B \in \mathbb{N}$ and $x,y,z \in \mathbb{Z}$ such that $3 \mid n+B$ and*

$$\frac{2}{3}(n+B)+B-5B^2 = x^2+y^2+z^2+\frac{(x+y+z)^2}{2} \le B^2.$$

Then $n = p_5(x_0)+p_5(y_0)+p_5(z_0)+2p_5(w_0)$ for some $x_0,y_0,z_0,w_0 \in \mathbb{N}$.

Proof. Clearly, $w=-(x+y+z)/2 \in \mathbb{Z}$. As $|x|,|y|,|z|,|w| \le B$, all the numbers

$$x_0=x+B,\ y_0=y+B,\ z_0=z+B,\ w_0=w+B$$

are nonnegative integers. Observe that

$$\begin{aligned}&p_5(x_0)+p_5(y_0)+p_5(z_0)+2p_5(w_0)\\ =&\frac{3(x_0^2+y_0^2+z_0^2+2w_0^2)-(x_0+y_0+z_0+2w_0)}{2}\\ =&\frac{3(5B^2+x^2+y^2+z^2+2w^2)-5B}{2} = \frac{2n+5B-5B}{2} = n.\end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.1. We can easily verify the desired result for $n=0,\dots,8891$. Below we assume that $n \ge 8892$. If

$$\frac{\sqrt{n}}{3}+\frac{1}{6} \le B \le \sqrt{\frac{2n}{15}}+\frac{1}{6}, \tag{2.1}$$

then

$$\frac{2}{3}(n+B)+B-5B^2 \ge \frac{15(B-1/6)^2}{3}+\frac{5B}{3}-5B^2 = \frac{5}{36} > 0$$

and

$$\begin{aligned}\frac{2}{3}(n+B)+B-5B^2 \le& \frac{2}{3}\left(3\left(B-\frac{1}{6}\right)\right)^2+\frac{5B}{3}-5B^2\\ =&B^2-\frac{1}{3}\left(B-\frac{1}{2}\right) \le B^2.\end{aligned}$$

Case 1. $5 \nmid n$.

As

$$n \ge \left\lceil \frac{3^2}{(\sqrt{2/15}-1/3)^2} \right\rceil = 8892,$$

we have

$$\sqrt{\frac{2n}{15}}+\frac{1}{6}-\left(\sqrt{\frac{n}{3}}+\frac{1}{6}\right) = \left(\sqrt{\frac{2}{15}}-\frac{1}{3}\right)\sqrt{n} \ge 3$$

and hence there is an integer B satisfying (2.1) with $B \equiv -n \pmod 3$. By the above,

$$0 \le \frac{2}{3}(n+B)+B-5B^2 = \frac{2n+5B}{3} - 5B^2 \le B^2.$$

As the even number $\frac{2}{3}(n+B)+B-5B^2$ is not divisible by 5, in light of Lemma 2.1 there are $x,y,z \in \mathbb{Z}$ such that

$$\frac{2}{3}(n+B)+B-5B^2 = x^2+y^2+z^2+\frac{(x+y+z)^2}{2}.$$

Now, by applying Lemma 2.2 we find that $n = p_5(x_0)+p_5(y_0)+p_5(z_0)+2p_5(w_0)$ for some $x_0, y, z_0, w_0 \in \mathbb{N}$.

Case 2. $n = 5q$ for some $q \in \mathbb{N}$.

In this case, we can easily verify the desired result when $8892 \le n \le 222288$. Below we assume that

$$n \ge 222289 = \left\lceil \frac{15^2}{(\sqrt{2/15}-1/3)^2} \right\rceil.$$

Choose $\delta \in \{0, \pm 1\}$ such that $1-q-\delta \not\equiv 0, \pm 2 \pmod 5$. As

$$\sqrt{\frac{2n}{15}}+\frac{1}{6}-\left(\sqrt{\frac{n}{3}}+\frac{1}{6}\right) = \left(\sqrt{\frac{2}{15}}-\frac{1}{3}\right)\sqrt{n} \ge 15,$$

there is an integer B satisfying (2.1) such that $B \equiv -n \pmod 3$ and $(B-1)^2 \equiv \delta \pmod 5$. Note that

$$\frac{2}{3}(n+B)+B-5B^2 = 5\left(\frac{2q+B}{3}-B^2\right)$$

and

$$\frac{2q+B}{3}-B^2 \equiv -\frac{2q+B}{2}-B^2 \equiv 1-q-(B-1)^2 \equiv 1-q-\delta \not\equiv 0, \pm 2 \pmod 5.$$

Thus, by applying Lemmas 2.1 and 2.2 we get that $n = p_5(x_0)+p_5(y_0)+p_5(z_0)+2p_5(w_0)$ for some $x_0, y, z_0, w_0 \in \mathbb{N}$.

In view of the above, we have completed the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

Lemma 3.1. *Let $q \in \mathbb{N}$ with q odd and not squarefree, or $2 \mid q$ and $q \notin \{4^k(16l+6) : k,l \in \mathbb{N}\}$. Then there are $x,y,z \in \mathbb{Z}$ such that*

$$6q = 2x^2+3y^2+4z^2+(2x+3y+4z)^2.$$

Proof. By K. Ono and K. Soundararajan [8], and Dickson [1], the Ramanujan form $x^2+y^2+10z^2$ represents q. Write $q = a^2+b^2+10c^2$ with $a,b,c \in \mathbb{Z}$. Then, for

$$x = a+b+2c,\ y = -b+2c,\ z = -3c,$$

we have

$$2x^2+3y^2+4z^2+(2x+3y+4z)^2 = 6(a^2+b^2+10c^2) = 6q.$$

This concludes the proof. □

Lemma 3.2. *Let $n \in \mathbb{N}$. Suppose that there are $B \in \mathbb{N}$ and $x, y, z \in \mathbb{Z}$ such that*

$$\frac{2n+10B}{3} - 10B^2 = 2x^2 + 3y^2 + 4z^2 + (2x+3y+4z)^2 < (B+1)^2.$$

Then $n = p_5(w_0) + 2p_5(x_0) + 3p_5(y_0) + 4p_5(z_0)$ for some $w_0, x_0, y_0, z_0 \in \mathbb{N}$.

Proof. Set $w = -(2x+3y+4z)$. As $|w|, |x|, |y|, |z| \le B$, all the numbers

$$w_0 = w + B,\ x_0 = x + B,\ y_0 = y + B,\ z_0 = z + B$$

are nonnegative integers. Observe that

$$\begin{aligned} &p_5(w_0) + 2p_5(x_0) + 3p_5(y_0) + 4p_5(z_0) \\ =&\frac{3(w_0^2 + 2x_0^2 + 3y_0^2 + 4z_0^2) - (w_0 + 2x_0 + 3y_0 + 4z_0)}{2} \\ =&\frac{3(10B^2 + w^2 + 2x^2 + 3y^2 + 4z^2) - 10B}{2} = \frac{2n + 10B - 10B}{2} = n. \end{aligned}$$

This ends the proof. □

Proof of Theorem 1.2. We can verify the result for $n = 0, \ldots, 45325137$ directly via a computer. Below we assume that

$$n \ge 45325138 = \left\lceil \frac{(81 - 1/6 + 1/16)^2}{(\sqrt{1/15} - \sqrt{2/33})^2} \right\rceil.$$

Since

$$\sqrt{\frac{n}{15}} + \frac{1}{6} - \left(\sqrt{\frac{2n}{33}} + \frac{1}{16}\right) \ge 81,$$

there is an integer B with

$$\sqrt{\frac{2n}{33}} + \frac{1}{16} \le B \le \sqrt{\frac{n}{15}} + \frac{1}{6}$$

such that

$$B \equiv -9n^3 + 12n^2 - 38n \pmod{81}$$

if n is odd, and $B \equiv 3n - 1 \pmod 8$ and $B \equiv 3n^2 - 2n \pmod 9$ if n is even. Note that

$$\frac{2n+10B}{3} - 10B^2 \ge \frac{30(B - 1/6)^2 + 10B}{3} - 10B^2 = \frac{5}{18} > 0$$

and

$$\begin{aligned} \frac{2n+10B}{3} - 10B^2 \le& \frac{33(B - 1/16)^2 + 10B}{3} - 10B^2 \\ =& B^2 + \frac{47}{24}B + \frac{11}{256} < (B+1)^2. \end{aligned}$$

Let $q = (n + 5B - 15B^2)/9$. When n is odd, we can easily see that q is an odd integer divisible by 9. When n is even, q is an even integer with $q \equiv 4 \pmod 8$, and hence $q \ne 4^k(16l + 6)$ for any $k, l \in \mathbb{N}$. By Lemma 3.1, we can write $6q = (2n + 10B)/3 - 10B^2$ as $2x^2 + 3y^2 + 4z^2 + (2x+3y+4z)^2$ with $x, y, z \in \mathbb{Z}$. Applying Lemma 3.2, we see that $n = p_5(w_0) + 2p_5(x_0) + 3p_5(y_0) + 4p_5(z_0)$ for some $w_0, x_0, y_0, z_0 \in \mathbb{N}$.

The proof of Theorem 1.2 is now complete. □

4. Proof of Theorem 1.3

Lemma 4.1. *Let $q \in \mathbb{N}$ be a multiple of 9 with $7 \nmid q$ or*

$$q \in \{7r : \ r \in \mathbb{Z} \text{ and } r \equiv 1, 2, 4 \pmod 7\}.$$

Then there are $x, y, z \in \mathbb{Z}$ such that

$$6q = x^2 + 2y^2 + 3z^2 + (x + 2y + 3z)^2.$$

Proof. Since $9 \mid q$ and

$$q \notin \{7^{2k+1}l : \ k, l \in \mathbb{N} \text{ and } l \equiv 3, 5, 6 \pmod 7\},$$

by [4, Theorem 2] we can write q as $a^2 + b^2 + 7c^2$ with $a, b, c \in \mathbb{Z}$. For

$$x = 6c, \ y = a - b - c, \ z = b - c,$$

we have

$$x^2 + 2y^2 + 3z^2 + (x + 2y + 3z)^2 = 6(a^2 + b^2 + 7c^2) = 6q.$$

This concludes the proof. □

Lemma 4.2. *Let $n \in \mathbb{N}$ and $\delta \in \{1, 2\}$. Suppose that there are $B \in \mathbb{N}$ and $x, y, z \in \mathbb{Z}$ such that*

$$\frac{2n + (3\delta + 4)B}{3} - (3\delta + 4)B^2 = x^2 + 2y^2 + 3\delta z^2 + (x + 2y + 3\delta z)^2 < (B + 1)^2.$$

Then $n = p_5(w_0) + p_5(x_0) + 2p_5(y_0) + 3\delta p_5(z_0)$ for some $w_0, x_0, y_0, z_0 \in \mathbb{N}$.

Proof. Set $w = -(x + 2y + 3\delta z)$. As $|w|, |x|, |y|, |z| \leq B$, all the numbers

$$w_0 = w + B, \ x_0 = x + B, \ y_0 = y + B, \ z_0 = z + B$$

are nonnegative integers. Observe that

$$\begin{aligned} & p_5(w_0) + p_5(x_0) + 2p_5(y_0) + 3\delta p_5(z_0) \\ = & \frac{3(w_0^2 + x_0^2 + 2y_0^2 + 3\delta z_0^2) - (w_0 + x_0 + 2y_0 + 3\delta z_0)}{2} \\ = & \frac{3((3\delta + 4)B^2 + w^2 + x^2 + 2y^2 + 3\delta z^2) - (3\delta + 4)B}{2} \\ = & \frac{2n + (3\delta + 4)B - (3\delta + 4)B}{2} = n. \end{aligned}$$

This ends the proof. □

Proof of Theorem 1.3 with $\delta = 1$. We can verify the desired result for $n = 0, 1, \ldots, 808834880$ directly via a computer. Below we assume that

$$n \geq 808834881 = \left\lceil \frac{(7 \times 81 + 1/48 - 1/6)^2}{(\sqrt{2/21} - \sqrt{1/12})^2} \right\rceil.$$

Since

$$\sqrt{\frac{2n}{21}} + \frac{1}{6} - \left(\sqrt{\frac{n}{12}} + \frac{1}{48} \right) \geq 7 \times 81,$$

there is an integer B with

$$\sqrt{\frac{n}{12}} + \frac{1}{48} \leq B \leq \sqrt{\frac{2n}{21}} + \frac{1}{6}$$

such that $B \equiv 18n^3+3n^2-35n \pmod{81}$, and $3n/7+1-(B+1)^2 \equiv 3,5,6 \pmod 7$ if $7 \mid n$. Such an integer B exists in view of the Chinese Remainder Theorem and the simple observations

$$0-1^2 \equiv 1-3^2 \equiv 6-0^2 \equiv 6 \pmod 7,$$
$$2-2^2 \equiv 5-0^2 \equiv 5 \pmod 7,\ 3-0^2 \equiv 4-1^2 \equiv 3 \pmod 7.$$

Note that

$$\frac{2n+7B}{3}-7B^2 \geq \frac{21(B-1/6)^2+7B}{3}-7B^2=\frac{7}{36}>0$$

and

$$\frac{2n+7B}{3}-7B^2 \leq \frac{24(B-1/48)^2+7B}{3}-7B^2=B^2+2B+\frac{1}{288}<(B+1)^2.$$

It is easy to see that

$$q:=\frac{1}{6}\left(\frac{2n+7B}{3}-7B^2\right)$$

is an integer divisible by 9. If $n=7n_0$ for some $n_0 \in \mathbb{N}$, then

$$\begin{aligned}\frac{q}{7} =&\frac{1}{6}\left(\frac{2n_0+B}{3}-B^2\right)\\ \equiv&-\left(\frac{9n_0-6B}{3}-B^2\right)=(B+1)^2-(3n_0+1)\equiv 1,2,4 \pmod 7.\end{aligned}$$

By Lemma 4.1, we can write $6q=(2n+7B)/3-7B^2$ as $x^2+2y^2+3z^2+(x+2y+3z)^2$ with $x,y,z \in \mathbb{Z}$. Applying Lemma 4.2 with $\delta=1$, we see that $n=p_5(w_0)+p_5(x_0)+2p_5(y_0)+3p_5(z_0)$ for some $w_0,x_0,y_0,z_0 \in \mathbb{N}$. This completes the proof. □

Lemma 4.3. *Let $q \in \mathbb{N}$ with $q \not\equiv 7 \pmod 8$ or*

$$q \notin \{5^{2k+1}l:\ k,l \in \mathbb{N} \text{ and } l \equiv \pm 1 \pmod 5\}.$$

Then there are $x,y,z \in \mathbb{Z}$ such that

$$6q=x^2+2y^2+6z^2+(x+2y+6z)^2.$$

Proof. By Dickson [2, pp. 112-113], we can write q as $a^2+2b^2+10c^2$ with $a,b,c \in \mathbb{Z}$. For

$$x=2a-b+3c,\ y=-a-b+3c,\ z=-2c,$$

we have

$$x^2+2y^2+6z^2+(x+2y+6z)^2=6(a^2+2b^2+10c^2)=6q.$$

This ends the proof. □

Proof of Theorem 1.3 with $\delta=2$. We can verify the desired result for $n=0,1,\ldots,897099188$ directly via a computer.

Below we assume that

$$n \geq 897099189=\left\lceil\frac{(360+1/16-1/6)^2}{(\sqrt{1/15}-\sqrt{2/33})^2}\right\rceil.$$

Since

$$\sqrt{\frac{n}{15}}+\frac{1}{6}-\left(\sqrt{\frac{2n}{33}}+\frac{1}{16}\right) \geq 5\times 8\times 9,$$

there is an integer B with

$$\sqrt{\frac{2n}{33}}+\frac{1}{16}\le B\le\sqrt{\frac{n}{15}}+\frac{1}{6}$$

such that $B\equiv 3n^2-2n \pmod 9$ and $B\equiv n^2-n-1 \pmod 8$, and $(B-1)^2\not\equiv 2n_0\pm 1, 2n_0-2 \pmod 5$ if $n=5n_0$ with $n_0\in\mathbb{N}$. Then

$$q=\frac{1}{6}\left(\frac{2n+10B}{3}-10B^2\right)=\frac{n+5B-15B^2}{9}\in\mathbb{Z}$$

and $q\not\equiv 7 \pmod 8$. If $n=5n_0$ for some $n_0\in\mathbb{N}$, then

$$\begin{aligned}\frac{q}{5}=&\frac{n_0+B-3B^2}{9}\equiv 3B^2-B-n_0\equiv\frac{B^2-2B}{2}-n_0\\=&\frac{(B-1)^2-2n_0-1}{2}\not\equiv 0,\pm 1 \pmod 5.\end{aligned}$$

As in the proof of Theorem 1.2, we also have

$$0<6q=\frac{2n+10B}{3}-10B^2<(B+1)^2.$$

Now applying Lemma 4.3 and Lemma 4.2 with $\delta=2$, we obtain that $n=p_5(w_0)+p_5(x_0)+2p_5(y_0)+6p_5(z_0)$ for some $w_0,x_0,y_0,z_0\in\mathbb{N}$.

The proof of Theorem 1.3 with $\delta=2$ is now complete. □

References

[1] L. E. Dickson, Quaternary quadratic forms representing all integers, *Amer. J. Math.*, **49** (1927), 39–56. MR 1506600

[2] L. E. Dickson, *Modern Elementary Theory of Numbers*, University of Chicago Press, Chicago, 1939. MR 0000387

[3] J. Ju, Universal sums of generalized pentagonal numbers, *Ramanujan J.*, **51** (2020), 479–494.

[4] I. Kaplansky, The first nontrivial genus of positive definite ternary forms, *Math. Comput.*, **64** (1995), 341–345. MR 1265017

[5] X.-Z. Meng and Z.-W. Sun, Sums of four polygonal numbers with coefficients, *Acta Arith.*, **180** (2017), 229–249. MR 3709643

[6] M. B. Nathanson, A short proof of Cauchy's polygonal theorem, *Proc. Amer. Math. Soc.*, **99** (1987), 22–24. MR 866422

[7] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Grad. Texts in Math., vol. 164, Springer, New York, 1996. MR 1395371

[8] K. Ono and K. Soundararajan, Ramanujan's ternary quadratic form, *Invent. Math.*, **130** (1997), 415–454. MR 1483991

[9] Z.-W. Sun, A result similar to Lagrange's theorem, *J. Number Theory*, **162** (2016), 190–211. MR 3448267

[10] Z.-W. Sun, On universal sums $x(ax+b)/2+y(cy+d)/2+z(ez+f)/2$, *Nanjing Univ. J. Math. Biquarterly*, **35** (2018), 85–199.

[11] Z.-W. Sun, Universal sums of three quadratic polynomials, *Sci. China Math.*, **63** (2020), 501–520. MR 4070776

Dmitry Krachun, St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023, St. Petersburg, Russia
Email address: dmitrykrachun@gmail.com

Zhi-Wei Sun, Department of Mathematics, Nanjing University, Nanjing 210093, China
Email address: zwsun@nju.edu.cn