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EXISTENCE OF BEST PROXIMITY POINTS SATISFYING TWO CONSTRAINT INEQUALITIES

DURAISAMY BALRAJ, MUTHAIAH MARUDAI, ZORAN D. MITROVIC, OZGUR EGE*
AND VEERARAGHAVAN PIRAMANANTHAM

Abstract. In this paper, we prove the existence of best proximity point and coupled best proximity point on metric spaces with partial order for weak proximal contraction mappings such that these critical points satisfy some constraint inequalities.

1. Introduction and preliminaries

Most of the mathematical problems in ordinary differential equation, partial differential equation, game theory, operation research etc. which appear in the nature of applicability, will have the solution which satisfies the objective in which some constraints are binded together with the problem. So, it is quite natural to find out the fixed point of an operator satisfying the constraints which are coupled with the objective.

Over a century, after the famous Banach contraction principle, many fixed point results were given by mathematicians proving the existence of fixed point for weak contraction mappings on metric spaces [6], [2], [3], [4], [5], [9]. After all these developments in the literature of fixed point theory, Lakshmikantham and Ciric [11] in 2009 introduced the concept of coupled fixed point which has wide range of applicability in partial differential equations.

Later, Choudhury and Maity [4] introduced the cyclic coupled fixed point and gave the existence of strong coupled fixed point. All these mappings were generalized by various authors [6],[8],[10],[1] by the concept called proximity point which are also known to be generalized fixed point. Our results will also be hammering away at establishing the existence of proximity points on metric space with partial order.

Let (X, d) be a metric space endowed with two partial orders $\preceq_1$ and $\preceq_2$ and $P, Q, R, S, T : X \to X$ be five self-operators. Recently, Samet and Jleli [7] have contemplated on the existence of a point $x \in X$ which satisfies the following.

$$(1) \qquad \begin{cases} x = Tx \\ Px \preceq_1 Qx \\ Rx \preceq_2 Sx. \end{cases}$$

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* Corresponding author: ozgur.ege@ege.edu.tr.

Definition 1.1. Let $\preceq$ be a partial order on complete metric space X. Then, $\preceq$ is d-regular if $\{a_n\}, \{b_n\}$ are sequences in X, we have

$$\lim_{n\to\infty} d(a_n, a) = \lim_{n\to\infty} d(b_n, b) = 0$$

with $a_n \preceq b_n$ for all $n \in \mathbb{N}$, then $a \preceq b$, where $(a, b) \in X \times X$.

Definition 1.2. Let X be a nonempty set endowed with two partial orders $\preceq_2$ and $\preceq_1$. Let $T, P, Q, R, S : X \to X$ be given operators. We say that the operator T is $(P, Q, R, S, \preceq_2, \preceq_1)$-stable, if the following condition is satisfied for $x \in X$:

$$Px \preceq_1 Q \text{ implies } R(T(x)) \preceq_2 S(T(x))$$

Definition 1.3. Let Θ be the set all functions $\theta : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(1) θ is a lower semi-continuous function,
(2) $\theta^{-1}(0) = 0$.

Theorem 1.4. [7] *Let (X, d) be a complete metric space endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $P, Q, R, S, T : X \to X$ be given operators. Suppose that the following conditions are satisfied:*

- *(i) $\preceq_i$ is d-regular, for $i = 1, 2$,*
- *(ii) P, Q, R and S are continuous,*
- *(iii) There exists $x_0 \in X$ such that $Px_0 \preceq_1 Qx_0$,*
- *(iv) T is $(P, Q, R, S, \preceq_1, \preceq_2)$-stable,*
- *(v) T is $(R, S, P, Q, \preceq_2, \preceq_1)$-stable,*
- *(vi) If $Px \preceq_1 Qx, Ry \preceq_2 Sy$ implies $d(Tx, Ty) \leq d(x, y) - \theta(d(x, y))$, where $\theta \in \Theta$.*

Then, there exists a point $x \in X$ satisfying (1).

They also raised the question of the existence of best proximity point together with constraint inequalities.

In the sequel, let (X, d) be a complete metric space with two nonempty closed subsets A and B endowed with partial orders $\preceq_1$ on A and $\preceq_2$ on B. Let us consider the operators P, Q which are self maps on A_0 and R, S be self maps on B_0, where

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(y, x) = d(B, A) \text{ for some } x \in A\}.$$

Let T be an operator from A_0 to B_0.

In this paper, our intend is to give the existence of best proximity points and coupled best proximity points satisfying two constraint inequalities. One of our result also generalizes a result in [4].

2. Existence of best proximity point satisfying two constraint inequalities

In this section, our aim is to evince that there exists a element $x \in A$ such that x is a proximity point of an operator T and also satisfies two constraint inequalities.

Definition 2.1. An operator $T : A \to B$ is said to be $(T, P, Q, R, S, \preceq_1, \preceq_2)$-stable if for $x \in A_0$,

$$Px \preceq_1 Qx \text{ implies } R(Tx) \preceq_2 S(Tx),$$

and T is said to be $(T, P, Q, R, S, \preceq_1, \preceq_2)$ P-stable if for $y \in B_0$,

$$Ry \preceq_2 Sy \text{ implies } Pu \preceq_1 Qu,$$

where $u \in A_0$ satisfies $d(u, y) = d(A, B)$.

Example 2.2. Let $A = \{(0, x) : x \in [0, 1]\}$ and $B = \{(1, x) : x \in [0, 1]\}$ be two subsets of $\mathbb{R}^2$ with dictionary order on A and B. Let $P(0, x) = (0, |x - 1|)$ and $Q(0, x) = (0, (x - 1)^2)$ be self-maps on A and $R(1, x) = (1, x)$ and $S(1, x) = (1, x^2)$ on B. Define $T : A \to B$ as

$$T(0,x) = \begin{cases} (1, x), & x \in [0, \frac{1}{2}) \\ (1, 1 - x), & x \in [\frac{1}{2}, 1]. \end{cases}$$

Then T is both $(T, P, Q, R, S, \preceq_1, \preceq_2)$-stable and $(T, P, Q, R, S, \preceq_1, \preceq_2)$ P-stable.

Let Φ be the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that:

(1) φ is continuous and non-decreasing,
(2) $\varphi(t) = 0$ if and only if $t = 0$ and $\varphi(t) < t$ for all $t \in [0, \infty)$,
(3) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in [0, \infty)$.

Definition 2.3. Let $T : A \to B$ be any mapping. Then T is said to be C-proximal φ-contraction if there exist $x, y \in A_0$ such that

$$Px \preceq_1 Qx, R(Ty) \preceq_2 S(Ty) \text{ implies } d(u, v) \leq \varphi(d(x, y)),$$

where $\varphi \in \Phi$ and $d(u, Tx) = d(A, B) = d(v, Ty)$.

Theorem 2.4. *Let A and B be two nonempty closed subsets of a complete metric space (X, d) endowed with partial orders $\preceq_1$ on A_0 and $\preceq_2$ on B_0. Let P, Q be self-maps on A_0 and R, S be self-maps on B_0. Let $T : A \to B$ be a C-proximal φ-contraction. Suppose*

(i) $\{\preceq_i, i = 1, 2\}$ is d-regular on A_0 and B_0,
(ii) P, Q, R, S are continuous,
(iii) there exists $x_0 \in A_0$ such that $Px_0 \preceq_1 Qx_0$,
(iv) T is $(T, P, Q, R, S, \preceq_1, \preceq_2)$-stable and $(T, P, Q, R, S, \preceq_1, \preceq_2)$ P-stable.

Then there exists a point $x \in A_0$ satisfying the followings:

(C_1) $d(x, Tx) = d(A, B)$,
(C_2) $Px \preceq_1 Qx$,
(C_3) $R(Tx) \preceq_2 S(Tx)$.

Proof. Let x_0 be an element in A_0 such that $Px_0 \preceq_1 Qx_0$. Using (iii), we have $Tx_0 \in B_0$ satisfying

$$R(Tx_0) \preceq_2 S(Tx_0).$$

Since $Tx_0 \in B_0$, there exists an element $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Hence, again using (iii) we get

$$Px_1 \preceq_1 Qx_1.$$

Similarly, we can construct sequences $\{x_n\}$ and $\{Tx_n\}$ in A_0 and B_0, respectively satisfying

$$Px_n \preceq_1 Qx_n \quad \text{and} \quad R(Tx_n) \preceq_2 S(Tx_n).$$

Since T is a C-proximal φ-contraction, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \varphi(d(x_{n-1}, x_n)) \\ &\vdots \\ &\leq \varphi^n(d(x_0, x_1)). \end{aligned}$$

Therefore, $d(x_n, x_{n+1})$ is decreasing and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ by the definition of φ. We claim that $\{x_n\}$ is a Cauchy sequence. Given that $\epsilon > 0$, there exists $N(\epsilon) \in \mathbf{N}$ such that

$$d(x_n, x_{n+1}) < \epsilon - \varphi(\epsilon), \text{ for all } n \geq N(\epsilon).$$

We prove that $\{x_n\}$ is a Cauchy sequence by the method of induction. Fix $m \geq N(\epsilon)$ and assume that $d(x_m, x_i) < \epsilon$, for all $i = m+1, \ldots, n$. Therefore,

$$\begin{aligned} d(x_m, x_{n+1}) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) \\ &< \epsilon - \varphi(\epsilon) + \varphi(d(x_m, x_n)) \\ &\leq \epsilon - \varphi(\epsilon) + \varphi(\epsilon) \\ &= \epsilon. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence and hence converges to some element $x \in A_0$. Let $x^* \in A$ such that $d(x^*, Tx) = d(A, B)$. We have

$$\begin{aligned} d(x_{n+1}, Tx) &\leq d(x_{n+1}, x^*) + d(A, B) \\ &\leq \varphi d(x_n, x) + d(A, B) \\ &< d(x_n, x) + d(A, B). \end{aligned}$$

As a result, $d(x_{n+1}, Tx) \to d(A, B)$ as $n \to \infty$ and hence $d(x, Tx) = d(A, B)$.

Since P and Q are continuous maps and $\preceq_1$ is d-regular, we obtain $Px \preceq_1 Qx$. Now, using conditions (i) and (ii), we get $R(Tx) \preceq_2 S(Tx)$. Thus x is an proximity point of T and satisfies the constraint inequalities. □

Example 2.5. Let $A = \{(0, x) : x \in [0, 1]\}$ and $B = \{(1, x) : x \in [0, 1]\}$ be two subsets of $\mathbb{R}^2$ with dictionary order on A and B. Let $P(0, x) = (0, |x-1|)$ and $Q(0, x) = (0, (x-1)^2)$ be self-maps on A and $R(1, x) = (1, x)$ and $S(1, x) = (1, x^2)$ on B. Define $T : A \to B$ as $T(0, x) = (1, \frac{x}{2})$. Let $\varphi \in \Phi$ be defined as $\varphi(r) = \frac{r}{2}$ for all $r \in \mathbb{R}$, then T satisfies all conditions in Theorem 2.4. Here $(0, 0)$ is the required point on A.

Corollary 1. *Let (X, d) be a complete metric space with two partial orders $\preceq_1$ and $\preceq_2$ to itself. Let T, Q, R, S are self-maps on X satisfying that*

(i) there exists $\varphi \in \Phi$ such that

$$Px \preceq_1 Qx, Ry \preceq_2 Sy \text{ implies } d(Tx, Ty) \leq \varphi(d(x, y)),$$

(ii) $\{\preceq_i, i = 1, 2\}$ is d-regular,
(iii) P, Q, R, S are continuous,
(iv) there exists $x_0 \in A_0$ such that $Px_0 \preceq_1 Qx_0$,
(v) T is $(T, P, Q, R, S, \preceq_1, \preceq_2)$-stable and $(T, P, Q, R, S, \preceq_1, \preceq_2)$ P-stable.

Then there exists a point $x \in X$ such that $Tx = x$.

Proof. The proof is the same as Theorem 2.4 with $A = B = X$. □

3. Best proximity points for cyclic coupled mapping

In this section, we give the existence of proximity points for coupled maps of cyclic type with respect to A and B. In the next section, we prove the same with proximity point also satisfying two constraints. The result of this section generalizes the result of [4].

Definition 3.1. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $F : X \times X \to X$ is called coupled proximal mapping on A and B if $F : A \times B \to B$ and $F : B \times A \to A$ satisfy the inequality

$$d(F(y_1, x_1), u)) \leq k[d(y_1, F(y_1, x_1)) + d(x_2, F(x_2, y_2))] - 2k\ d(A, B),$$

for some $k \in (0, \frac{1}{2})$, where $x_1, x_2, u \in A_0, y_1, y_2 \in B_0$ and $d(u, F(x_2, y_2)) = d(A, B)$.

Definition 3.2. Let A and B be two closed subsets of a metric space (X, d). Then A and B is said to satisfy the P-property if for $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ satisfies

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B) \text{ implies } d(x_1, x_2) = d(y_1, y_2).$$

Definition 3.3. Let (X, d) be a metric space. An element $(x, y) \in X \times X$ is said to be strong coupled proximal point if

$$d(x, F(x, y)) = d(y, F(y, x)) = d(x, y) = d(A, B).$$

Remark 1. (1) We have $x = F(y, x)$ and $y = F(x, y)$ if A and B satisfy the P-property.
(2) If $d(A, B) = 0$, then strong coupled proximal point is reduced to strong coupled fixed point, i.e., $F(x, x) = x$.

Theorem 3.4. *Let (X, d) be a complete metric space and A, B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be coupled proximal mapping on A and B. Then F has strong coupled proximal point if A and B satisfy the P-property.*

Proof. Let $x_0 \in A, y_0 \in B$ be any two arbitrary elements of X. Let two sequences $\{x_n\}$ and $\{y_n\}$ defined as $F(x_n, y_n) = y_{n+1}$, $F(y_n, x_n) = x_{n+1}$ and u_n defined by $d(u_n, y_n) = d(A, B)$. By the definition of F, we have

$$\begin{aligned} d(x_n, u_{n-1}) =& k[d(y_{n-1}, F(y_{n-1}, x_{n-1}) + d(x_{n-2}, F(x_{n-2}, y_{n-2}))] - 2k\ d(A, B) \\ \leq& k[d(y_{n-1}, x_n) + d(x_{n-2}, y_{n-1})] - 2k\ d(A, B) \\ \leq& k[d(u_{n-1}, x_n) + d(u_{n-1}, y_{n-1}) + d(x_{n-2}, u_{n-1}) + d(u_{n-1}, y_{n-1})] \\ &- 2k\ d(A, B) \\ \leq& k[d(u_{n-1}, x_n) + d(x_{n-2}, u_{n-1})] \end{aligned}$$

and hence, $d(x_n, u_{n-1}) \leq c\, d(x_{n-2}, u_{n-1})$ where $c = \frac{k}{1-k}$. Moreover, we obtain

$$\begin{aligned} d(x_n, u_{n+1}) =& k[d(y_{n-1}, F(y_{n-1}, x_{n-1}) + d(x_n, F(x_n, y_n))] - 2k\ d(A, B) \\ \leq& k[d(y_{n-1}, x_n) + d(x_n, y_{n+1})] - 2k\ d(A, B) \\ \leq& k[d(u_{n-1}, x_n) + d(u_{n-1}, y_{n-1}) + d(x_n, u_{n+1}) + d(u_{n+1}, y_{n+1})] \\ &- 2k\ d(A, B) \\ \leq& k[d(u_{n-1}, x_n) + d(x_n, u_{n+1})] \end{aligned}$$

and so, $d(x_n, u_{n+1}) \leq c\, d(x_n, u_{n-1})$ where $c = \frac{k}{1-k}$. Now, by using the above inequalities, we have the following:

$$\begin{aligned} d(x_n, u_{n-1}) &\leq cd(x_{n-2}, u_{n-1}) \\ &\leq c^2 d(x_{n-2}, u_{n-3}) \\ &\vdots \\ &\leq c^n M, \end{aligned}$$

where $M = \max\{d(x_0, u_1), d(u_0, x_1)\}$. Define a sequence $\{z_n\}$ by

$$\begin{cases} z_{2n} = u_n, & n > 0 \\ z_{2n-1} = x_n, & n > 0. \end{cases}$$

Since

$$\sum_{n=1}^{\infty} d(z_n, z_{n-1}) \leq [1 + c + c^2 + c^3 + \ldots]M$$
$$= \frac{1}{1-c}M,$$

$\{z_n\}$ is a Cauchy sequence. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences which converge to $x \in A$ and $y \in B$, respectively. Then $d(x, y) = d(A, B)$ by the continuity of d and P-property.

Let $d(u', F(x, y)) = d(A, B)$, we have

$$\begin{aligned} d(x, u') &= \lim_{n\to\infty} d(x_n, u') \\ &\leq \lim_{n\to\infty} d(F(y_{n-1}, x_{n-1}), u') \\ &\leq \lim_{n\to\infty} k[d(y_{n-1}, F(y_{n-1}, x_{n-1}) + d(x, F(x, y)] - 2k\ d(A, B) \\ &\leq \lim_{n\to\infty} k[d(u_{n-1}, x_{n+1}) + d(x, u')]. \end{aligned}$$

Therefore, $d(x, u') = 0$ as $n \to \infty$ which implies $d(x, F(x, y)) = d(A, B)$. Similarly, we can prove that $d(y, F(y, x)) = d(A, B)$ which concludes that (x, y) is the strong coupled proximal point of F. □

Example 3.5. Consider $A = \{[-2.5, a] : a \in [-1, 0]\}$ and $B = \{[2.5, b] : b \in [-1, 0]\}$ on $\mathbf{R}^2$ under 1−norm with $d(A, B) = 5$. These sets satisfy the P-property. Define

$$F(a', b') = \begin{cases} (2.5, \frac{ab}{3}) \text{ if } (a', b') \in A \times B, \text{ where } a' = (-2.5, a) \text{ and } b' = (2.5, b), \\ (-2.5, \frac{ab}{3}) \text{ if } (a', b') \in B \times A, \text{ where } a' = (2.5, a) \text{ and } b' = (-2.5, b). \end{cases}$$

Let $x' = (-2.5, x)$ and $v' = (-2.5, v)$ be the elements of A_0 and $y' = (2.5, y)$, $u' = (2.5, u)$ be the elements of B_0. Now,

$$\begin{aligned} d(F(x', y'), w) &= d((2.5, \frac{xy}{3}), (2.5, \frac{uv}{3})) \quad (\text{where } d(w, F(u', v')) = d(A, B)) \\ &= \mid \frac{xy}{3} - \frac{uv}{3} \mid \\ &\leq \frac{1}{3}[\mid x \mid + \mid u \mid] \\ &\leq \frac{1}{3}\left[\mid x - \frac{xy}{3} \mid + \mid u - \frac{uv}{3} \mid\right] \\ &\leq \frac{1}{3}\left[\mid x - \frac{xy}{3} \mid + \mid 5 \mid - \mid 5 \mid + \mid u - \frac{uv}{3} \mid + \mid 5 \mid - \mid 5 \mid\right] \\ &\leq \frac{1}{3}\left[\left[\mid x - \frac{xy}{3} \mid\right] + \mid 5 \mid + \left[\mid u - \frac{uv}{3} \mid\right] + \mid 5 \mid\right] - 2(\frac{5}{3}) \\ &\leq \frac{1}{3}\left[\left[\mid x - \frac{xy}{3} \mid\right] + \mid 5 \mid + \left[\mid u - \frac{uv}{3} \mid + \mid 5 \mid\right]\right] - \frac{2}{3}\ d(A, B) \\ &\leq \frac{1}{3}\left[d(x', F(x', y,)) + d(u', F(u', v'))\right] - \frac{2}{3}\ d(A, B). \end{aligned}$$

Hence, F satisfies all conditions of Theorem 3.4 and

$$((-2.5,0),(2.5,0)) \in \mathbf{R}^2 \times \mathbf{R}^2$$

is the coupled proximity pair.

4. Existence of coupled best proximity point satisfying two constraint inequalities

This section apart from the previous section shows that two points (x_0, y_0) satisfying the inequalities are enough for the existence of coupled best proximity instead of an arbitrary pair (x, y). Let (X, d) be a complete metric space with A and B as mentioned above. Define $F : X \times X \to X$ be a coupled mapping with respect to A and B. Let P, Q, R, S are the same as pre-defined. Our intend is to find $(x, y) \in A \times B$ such that

(C_1) $d(x, y) = d(A, B)$, where $y = F(x, y)$,
(C_2) $Px \preceq_1 Qx$,
(C_3) $Ry \preceq_2 Sy$.

Definition 4.1. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $F : X \times X \to X$ is called C-coupled proximal mapping on A and B if $F : A \times B \to B$ and $F : B \times A \to A$ satisfy the following:
$P(F(y_1, x_1)) \preceq_1 Q(F(y_1, x_1))$ and $R(F(x_2, y_2)) \preceq_2 S(F(x_2, y_2))$ implies

$$d(F(y_1, x_1), u)) \leq k[d(y_1, F(y_1, x_1)) + d(x_2, F(x_2, y_2))] - 2kd(A, B),$$

where $x_1, x_2 \in A_0, y_1, y_2 \in B_0$ and $d(u, F(x_2, y_2)) = d(A, B)$ for some $k \in (0, \frac{1}{2})$.

Definition 4.2. Let A and B be two subsets of X with partial orders $\preceq_1$ and $\preceq_2$. Let $F : X \times X \to X$ be a mapping satisfying that

(C_1) $F : A \times B \to B$,
(C_2) $F : B \times A \to A$.

Then F is said to be $(F, P, Q, R, S, \preceq_1, \preceq_2)$ coupled stable if $x \in A$ and $y \in B$, then

$$R(F(x, y)) \preceq_2 S(F(x, y)) \text{ and } P(F(y, x)) \preceq_1 Q(F(y, x)),$$

whenever,

$$Px \preceq_1 Qx \text{ and } Ry \preceq_2 Sy.$$

Theorem 4.3. *Let (X, d) be a complete metric space and A, B be two nonempty closed subsets of X. Let P, Q be self-mappings on A and R, S be self-mappings on B. Let $F : X \times X \to X$ be C-coupled proximal mapping with respect to A and B. Suppose that*

(i) $\{\preceq_i, i = 1, 2\}$ *is d-regular on A_0 and B_0,*
(ii) P, Q, R, S *are continuous,*
(iii) there exist $x_0 \in A$ and $y_0 \in B$ such that $Px_0 \preceq_1 Qx_0$ and $Ry_0 \preceq_1 Sy_0$,
(iv) F is $(F, P, Q, R, S, \preceq_1, \preceq_2)$ coupled stable,
(v) A and B satisfy the P-property.

Then there exists a point $(x, y) \in A \times B$ which satisfies

$$\begin{cases} d(x, y) = d(A, B), \text{ where } y = F(x, y), \\ Px \preceq_1 Qx, \\ Ry \preceq_2 Sy. \end{cases} \tag{2}$$

Proof. Let $x_0 \in A, y_0 \in B$ such that $Px_0 \preceq_1 Qx_0$ and $Ry_0 \preceq_2 Sy_0$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences defined by $F(x_n, y_n) = y_{n+1}$, $F(y_n, x_n) = x_{n+1}$ and u_n be defined as $d(u_n, y_n) = d(A, B)$. Since F is $(F, P, Q, R, S, \preceq_1, \preceq_2)$ coupled stable, each of $\{x_n\}$ and $\{y_n\}$ satisfies

$$Px_n \preceq_1 Qx_n \, and \, Ry_n \preceq_2 Sy_n.$$

From the definition of F, we have

$$\begin{aligned} d(x_n, u_{n-1}) =& k[d(y_{n-1}, F(y_{n-1}, x_{n-1}) + d(x_{n-2}, F(x_{n-2}, y_{n-2}))] - 2k\ d(A, B) \\ \leq& k[d(y_{n-1}, x_n) + d(x_{n-2}, y_{n-1})] - 2k\ d(A, B) \\ \leq& k[d(u_{n-1}, x_n) + d(u_{n-1}, y_{n-1}) + d(x_{n-2}, u_{n-1}) + d(u_{n-1}, y_{n-1})] \\ & - 2k\ d(A, B) \\ \leq& k[d(u_{n-1}, x_n) + d(x_{n-2}, u_{n-1})] \end{aligned}$$

and so, $d(x_n, u_{n-1}) \leq c\, d(x_{n-2}, u_{n-1})$, where $c = \frac{k}{1-k}$. On the other hand,

$$\begin{aligned} d(x_n, u_{n+1}) =& k[d(y_{n-1}, F(y_{n-1}, x_{n-1}) + d(x_n, F(x_n, y_n))] - 2kd(A, B) \\ \leq& k[d(y_{n-1}, x_n) + d(x_n, y_{n+1})] - 2kd(A, B) \\ \leq& k[d(u_{n-1}, x_n) + d(u_{n-1}, y_{n-1}) + d(x_n, u_{n+1}) + d(u_{n+1}, y_{n+1})] \\ & - 2kd(A, B) \\ \leq& k[d(u_{n-1}, x_n) + d(x_n, u_{n+1})] \end{aligned}$$

and hence, $d(x_n, u_{n+1}) \leq c\, d(x_n, u_{n-1})$, where $c = \frac{k}{1-k}$. Using the above inequalities, we obtain

$$\begin{aligned} d(x_n, u_{n-1}) &\leq cd(x_{n-2}, u_{n-1}) \\ &\leq c^2 d(x_{n-2}, u_{n-3}) \\ &\vdots \\ &\leq c^n M, \end{aligned}$$

where $M = \max\{d(x_0, u_1), d(u_0, x_1)\}$. Define a sequence $\{z_n\}$ by

$$\begin{cases} z_{2n} = u_n, & n > 0 \\ z_{2n-1} = x_n, & n > 0. \end{cases}$$

By the following fact that

$$\begin{aligned} \sum_{n=1}^{\infty} d(z_n, z_{n-1}) &\leq [1 + c + c^2 + c^3 + \cdots]M \\ &= \frac{1}{1-c} M, \end{aligned}$$

$\{z_n\}$ is a Cauchy sequence. Then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences which converge to $x \in A$ and $y \in B$, respectively, such that $d(x, y) = d(A, B)$ by the continuity of d and the P-property. Let $d(u', F(x, y)) = d(A, B)$. Then we obtain

$$\begin{aligned} d(x, u') &= \lim_{n\to\infty} d(x_n, u') \\ &\leq \lim_{n\to\infty} d(F(y_{n-1}, x_{n-1}), u') \\ &\leq \lim_{n\to\infty} k[d(y_{n-1}, F(y_{n-1}, x_{n-1}) + d(x, F(x, y)] - 2kd(A, B) \\ &\leq \lim_{n\to\infty} k[d(u_{n-1}, x_{n+1}) + d(x, u')]. \end{aligned}$$

Thus, as $n \to \infty$, $d(x, u') = 0$, we conclude that $d(x, F(x, y)) \le d(A, B)$, i.e., $d(x, F(x, y)) = d(A, B)$. Since P, Q are continuous maps and $\preceq_1$ is d-regular, we have $Px \preceq_1 Qx$. From the conditions (i) and (ii), we get that $Ry \preceq_2 Sy$. Thus (x, y) satisfies (2). □

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Duraisamy Balraj, Department of Mathematics, Bharathidasan University, Trichirapalli, Tamilnadu, India
Email address: duraisamybalraj@gmail.com

Muthaiah Marudai, Department of Mathematics, Bharathidasan University, Trichirapalli, Tamilnadu, India
Email address: mmarudai@yahoo.co.in

Zoran D. Mitrovic, University of Banja Luka, Faculty of Electrical Engineering, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina
Email address: zoran.mitrovic@etf.unibl.org

Ozgur Ege, Department of Mathematics, Ege University, Bornova, 35100, Izmir, Turkey
Email address: ozgur.ege@ege.edu.tr

Veeraraghavan Piramanantham, Department of Mathematics, Bharathidasan University, Trichirapalli, Tamilnadu, India
Email address: piramanantham@gmail.com