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INITIAL BOUNDARY VALUE PROBLEM FOR A INHOMOGENEOUS PSEUDO-PARABOLIC EQUATION

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ABSTRACT. This paper deals with the global existence and blow-up of solutions to a inhomogeneous pseudo-parabolic equation with initial value u_0 in the Sobolev space $H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$ is an integer) is a bounded domain. By using the mountain-pass level d (see (14)), the energy functional J (see (12)) and Nehari function I (see (13)), we decompose the space $H_0^1(\Omega)$ into five parts, and in each part, we show the solutions exist globally or blow up in finite time. Furthermore, we study the decay rates for the global solutions and lifespan (i.e., the upper bound of blow-up time) of the blow-up solutions. Moreover, we give a blow-up result which does not depend on d. By using this theorem, we prove the solution can blow up at arbitrary energy level, i.e. for any $M \in \mathbb{R}$, there exists $u_0 \in H_0^1(\Omega)$ satisfying $J(u_0) = M$ such that the corresponding solution blows up in finite time.

1. INTRODUCTION

In this paper, we consider the following initial-boundary value problem

$$\begin{cases} u_t - \Delta u_t - \Delta u = |x|^\sigma |u|^{p-1} u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1}$$

and its corresponding steady-state problem

$$\begin{cases} -\Delta u = |x|^\sigma |u|^{p-1} u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$ is an integer) is a bounded domain with boundary $\partial\Omega$ and $u_0 \in H_0^1(\Omega)$; the parameters p and σ satisfy

$$\begin{aligned} &1 < p < \begin{cases} \infty, & n = 1,2; \\ \frac{n+2}{n-2}, & n \geq 3, \end{cases} \\ &\sigma > \begin{cases} -n, & n = 1,2; \\ \frac{(p+1)(n-2)}{2} - n, & n \geq 3. \end{cases} \end{aligned} \tag{3}$$

(1) was called homogeneous (inhomogeneous) pseudo-parabolic equation when $\sigma = 0$ ($\sigma \neq 0$). The concept "pseudo-parabolic" was proposed by Showalter and Ting in 1970 in the paper [20], where the linear case was considered. Pseudo-parabolic equations describe a variety of important physical processes, such as the

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seepage of homogeneous fluids through a fissured rock [1], the unidirectional propagation of nonlinear, dispersive, long waves [2, 23], and the aggregation of populations [17].

The homogeneous problem, i.e. $\sigma = 0$, was studied in [3, 4, 5, 7, 9, 10, 13, 15, 16, 21, 24, 25, 26, 27, 28, 29]. Especially, **for the Cauchy problem (i.e. $\Omega = \mathbb{R}^n$ and there is no boundary condition)**, Cao et al. [4] showed the critical Fujita exponent p_c (which was firstly introduced by Fujita in [8]) is $1+2/n$, i.e. if $1 < p \leq p_c$, then any nontrivial solution blows up in finite time, while global solutions exist if $p > p_c$. In [28], Yang et al. proved that for $p > p_c$, there is a secondary critical exponent $\alpha_c = 2/(p-1)$ such that the solution blows up in finite time for u_0 behaving like $|x|^{-\alpha}$ at $|x| \to \infty$ if $\alpha = (0, \alpha_c)$; and there are global solutions for for u_0 behaving like $|x|^{-\alpha}$ at $|x| \to \infty$ if $\alpha = (\alpha_c, n)$. **For the zero Dirichlet boundary problem in a bounded domain** Ω, in [13, 25, 26], the authors studied the properties of global existence and blow-up by potential well method (which was firstly introduced by Sattinger [19] and Payne and Sattinger [18], then developed by Liu and Zhao in [14]), and they showed the global existence, blow-up and asymptotic behavior of solutions with initial energy at subcritical, critical and supercritical energy level. The results of [13, 25, 26] were extended by Luo [15] and Xu and Zhou [24] by studying the lifespan (i.e. the upper bound of the blow-up time) of the blowing-up solutions. Recently, Xu et al. [27] and Han [9] extended the previous studies by considering the problem with general nonlinearity.

Li and Du [12] studied the **Cauchy problem** of equation in (1) with $\sigma > 0$. They got the critical Fujita exponent (p_c) and second critical exponent (α_c) by the integral representation and comparison principle. The main results obtained in [12] are as follows:

(1) If $1 < p \leq p_c := 1 + (2+\sigma)/n$, then every nontrivial solution blows up in finite time.

(2) If $p > p_c$, the distribution of the initial data has effect on the blow-up phenomena. More precisely, if $u_0 \in \Phi_\alpha$ and $0 < \alpha < \alpha_c := (2+\sigma)/(p-1)$ or u_0 is large enough, then the solution blows up in finite time; if $u_0 = \mu\phi(x)$, $\phi \in \Phi^\alpha$ with $\alpha_c < \alpha < n$, $0 < \mu < \mu_1$, then the solution exists globally, where μ_1 is some positive constant,

$$\Phi_\alpha := \left\{ \xi(x) \in BC(\mathbb{R}^n) : \xi(x) \geq 0,\ \liminf_{|x|\uparrow\infty} |x|^\alpha \xi(x) > 0 \right\},$$

and

$$\Phi^\alpha := \left\{ \xi(x) \in BC(\mathbb{R}^n) : \xi(x) \geq 0,\ \limsup_{|x|\uparrow\infty} |x|^\alpha \xi(x) < \infty \right\}.$$

Here $BC(\mathbb{R}^n)$ is the set of bounded continuous functions in $\mathbb{R}^n$.

In view of the above introductions, we find that

(1) for Cauchy problem in $\mathbb{R}^n$, only the case $\sigma \geq 0$ was studied;

(2) for zero Dirichlet problem in a bounded domain Ω, only the case $\sigma = 0$ was studied.

The difficulty of allowing σ to be less than 0 is the term $|x|^\sigma$ become infinity at $x = 0$. In this paper, we consider the problem in a bounded domain Ω with zero Dirichlet boundary condition, i.e. problem (1), and the parameters satisfies (3), which allows σ to be less than 0. To overcome the singularity of $|x|^\sigma$ at $x = 0$, we use potential well method by introducing the $|x|^\sigma$ weighed-$L^{p+1}(\Omega)$ space and assume there is a lower bound of σ, i.e,

$$\sigma > \underbrace{\frac{(p+1)(n-2)}{2} - n}_{<0 \text{ if } n\geq 3}$$

for $n \geq 3$.

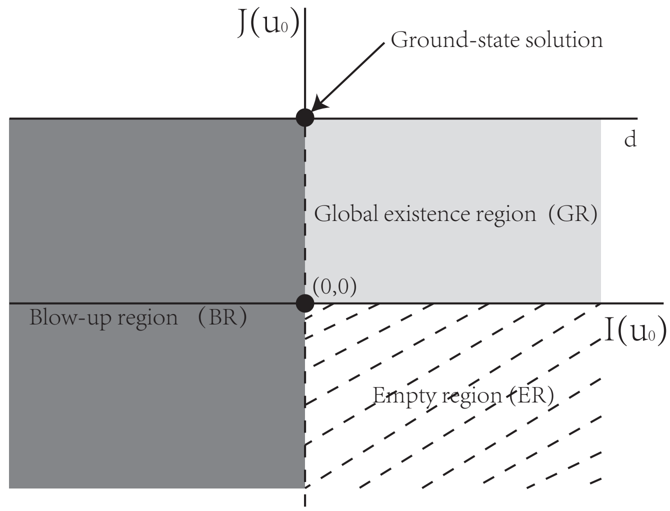

FIGURE 1. The results for $J(u_0) \leq d$.

The main results of this paper can be summarized as follows: Let J and I be the functionals given in (12) and (13), respectively; d be the mountain-pass level given in (14); S_ρ and S^ρ be the sets defined in (20).

(1) (the case $J(u_0) \leq d$, see Fig. 1) If $u_0 \in H_0^1(\Omega)$ such that $(I(u_0), J(u_0))$ is in the dark gray region (BR), then the solution blows up in finite time; if $u_0 \in H_0^1(\Omega)$ such that $(I(u_0), J(u_0))$ is in the light gray region (GR), then the solution exists globally; if $u_0 \in H_0^1(\Omega)$ such that $(I(u_0), J(u_0)) = (0, d)$, then u_0 is a ground-state solution and (1) admits a global solution $u \equiv u_0$; there is no $u_0 \in H_0^1(\Omega)$ such that $(I(u_0), J(u_0))$ is in the dotted part (ER).

(2) (the case $J(u_0) > d$) If $u_0 \in S_\rho$ for some $\rho \geq J(u_0) > d$, then the solution exists globally and goes to 0 in $H_0^1(\Omega)$ as times goes to infinity; if $u_0 \in S^\rho$ for some $\rho \geq J(u_0) > d$, then the solution blows up in finite time.

(3) (arbitrary initial energy level) For any $M \in \mathbb{R}$, there exits a $u_0 \in H_0^1(\Omega))$ satisfying $J(u_0) = M$ such that the corresponding solution blows up in finite time.

(4) Moreover, under suitable assumptions, we show the exponential decay of global solutions and lifespan (i.e. the upper bound of blow-up time) of the blowing-up solutions.

The organizations of the remain part of this paper are as follows. In Section 2, we introduce the notations used in this paper and the main results of this paper; in Section 3, we give some preliminaries which will be used in the proofs; in Section 4, we give the proofs of the main results.

2. Notations and main results

Throughout this paper we denote the norm of $L^\gamma(\Omega)$ for $1 \le \gamma \le \infty$ by $\|\cdot\|_{L^\gamma}$. That is, for any $\phi \in L^\gamma(\Omega)$,

$$\|\phi\|_{L^\gamma} = \begin{cases} \left(\int_\Omega |\phi(x)|^\gamma dx\right)^{\frac{1}{\gamma}}, & \text{if } 1 \le \gamma < \infty; \\ \operatorname{esssup}_{x\in\Omega}|\phi(x)|, & \text{if } \gamma = \infty. \end{cases}$$

We denote the $|x|^\sigma$-weighted $L^{p+1}(\Omega)$ space by $L^{p+1}_\sigma(\Omega)$, which is defined as

$$L^{p+1}_\sigma(\Omega) := \left\{\phi : \phi \text{ is measurable on } \Omega \text{ and } \|u\|_{L^{p+1}_\sigma} < \infty\right\}, \tag{4}$$

where

$$\|\phi\|_{L^{p+1}_\sigma} := \left(\int_\Omega |x|^\sigma |\phi(x)|^{p+1} dx\right)^{\frac{1}{p+1}}, \ \phi \in L^{p+1}_\sigma(\Omega). \tag{5}$$

By standard arguments as the space $L^{p+1}(\Omega)$, one can see $L^{p+1}_\sigma(\Omega)$ is a Banach space with the norm $\|\cdot\|_{L^{p+1}_\sigma}$.

We denote the inner product of $H^1_0(\Omega)$ by $(\cdot,\cdot)_{H^1_0}$, i.e.,

$$(\phi,\varphi)_{H^1_0} := \int_\Omega (\nabla\phi(x)\cdot\nabla\varphi(x) + \phi(x)\varphi(x))\, dx, \ \phi, \ \varphi \in H^1_0(\Omega). \tag{6}$$

The norm of $H^1_0(\Omega)$ is denoted by $\|\cdot\|_{H^1_0}$, i.e.,

$$\|\phi\|_{H^1_0} := \sqrt{(\phi,\phi)_{H^1_0}} = \sqrt{\|\nabla\phi\|^2_{L^2} + \|\phi\|^2_{L^2}}, \ \phi \in H^1_0(\Omega). \tag{7}$$

An equivalent norm of $H^1_0(\Omega)$ is $\|\nabla(\cdot)\|_{L^2}$, and by Poincaré's inequality, we have

$$\|\nabla\phi\|_{L^2} \le \|\phi\|_{H^1_0} \le \sqrt{\frac{\lambda_1+1}{\lambda_1}}\|\nabla\phi\|_{L^2}, \ \phi \in H^1_0(\Omega), \tag{8}$$

where λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition, i.e,

$$\lambda_1 = \inf_{\phi\in H^1_0(\Omega)} \frac{\|\nabla\phi\|^2_{L^2}}{\|\phi\|^2_{L^2}}. \tag{9}$$

Moreover, by Theorem 3.2, we have

(10) for p and σ satisfying (4), $H^1_0(\Omega) \hookrightarrow L^{p+1}_\sigma(\Omega)$ continuously and compactly.

Then we let $C_{p\sigma}$ as the optimal constant of the embedding $H^1_0(\Omega) \hookrightarrow L^{p+1}_\sigma(\Omega)$, i.e.,

$$C_{p\sigma} = \sup_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\|\phi\|_{L^{p+1}_\sigma}}{\|\nabla\phi\|_{L^2}}. \tag{11}$$

We define two functionals J and I on $H^1_0(\Omega)$ by

$$J(\phi) := \frac{1}{2}\|\nabla\phi\|^2_{L^2} - \frac{1}{p+1}\|\phi\|^{p+1}_{L^{p+1}_\sigma} \tag{12}$$

and

$$I(\phi) := \|\nabla\phi\|^2_{L^2} - \|\phi\|^{p+1}_{L^{p+1}_\sigma}. \tag{13}$$

By (3) and (10), we know that J and I are well-defined on $H^1_0(\Omega)$.

We denote the mountain-pass level d by

$$d := \inf_{\phi\in N} J(\phi), \tag{14}$$

where N is the Nehari manifold, which is defined as

$$N := \{\phi \in H_0^1(\Omega) \setminus \{0\} : I(\phi) = 0\}. \tag{15}$$

By Theorem 3.3, we have

$$d = \frac{p-1}{2(p+1)} C_{p\sigma}^{-\frac{2(p+1)}{p-1}}, \tag{16}$$

where $C_{p\sigma}$ is the positive constant given in (11).

For $\rho \in \mathbb{R}$, we define the sub-level set J^ρ of J as

$$J^\rho = \{\phi \in H_0^1(\Omega) : J(\phi) < \rho\}. \tag{17}$$

Then, we define the set $N^\rho := N \cap J^\rho$. In view of (15), (12), (17), we get

$$N^\rho = \left\{\phi \in N : \|\nabla\phi\|_{L^2}^2 < \frac{2(p+1)\rho}{p-1}\right\},\ \rho > d. \tag{18}$$

For $\rho > d$, we define two constants

$$\lambda_\rho := \inf_{\phi \in N^\rho} \|\phi\|_{H_0^1},\ \Lambda_\rho := \sup_{\phi \in N^\rho} \|\phi\|_{H_0^1} \tag{19}$$

and two sets

$$\begin{aligned} S_\rho :=&\{\phi \in H_0^1(\Omega) : \|\phi\|_{H_0^1} \le \lambda_\rho, I(\phi) > 0\},\\ S^\rho :=&\{\phi \in H_0^1(\Omega) : \|\phi\|_{H_0^1} \ge \Lambda_\rho, I(\phi) < 0\}. \end{aligned} \tag{20}$$

Remark 1. There are two remarks on the above definitions.

(1) By the definitions of N^ρ, λ_ρ and Λ_ρ, it is easy to see λ_ρ is non-increasing with respect to ρ and Λ_ρ is non-decreasing with respect to ρ.

(2) By Theorem 3.4, we have

$$\sqrt{\frac{2(p+1)d}{p-1}} \le \lambda_\rho \le \Lambda_\rho \le \sqrt{\frac{2(p+1)(\lambda_1+1)\rho}{\lambda_1(p-1)}}. \tag{21}$$

Then the sets S_ρ and S^ρ are both nonempty. In fact, for any $\phi \in H_0^1(\Omega)\setminus\{0\}$ and $s > 0$,

$$\|s\phi\|_{H_0^1} \le \sqrt{\frac{2(p+1)d}{p-1}} \Leftrightarrow s \le \delta_1 := \sqrt{\frac{2(p+1)d}{p-1}}\|\phi\|_{H_0^1}^{-1},$$

$$I(s\phi) = s^2\|\nabla\phi\|_{L^2}^2 - s^{p+1}\|\phi\|_{L_\sigma^{p+1}}^{p-1} > 0 \Leftrightarrow s < \delta_2 := \left(\frac{\|\nabla\phi\|_{L^2}^2}{\|\phi\|_{L_\sigma^{p+1}}^{p+1}}\right)^{\frac{1}{p-1}},$$

$$\|s\phi\|_{H_0^1} \ge \sqrt{\frac{2(p+1)(\lambda_1+1)\rho}{\lambda_1(p-1)}} \Leftrightarrow s \ge \delta_3 := \sqrt{\frac{2(p+1)(\lambda_1+1)\rho}{\lambda_1(p-1)}}\|\phi\|_{H_0^1}^{-1},$$

$$I(s\phi) = s^2\|\nabla\phi\|_{L^2}^2 - s^{p+1}\|\phi\|_{L_\sigma^{p+1}}^{p-1} < 0 \Leftrightarrow s > \delta_2.$$

So,

$$\{s\phi : 0 < s < \min\{\delta_1, \delta_2\}\} \subset S_\rho,\ \{s\phi : s > \max\{\delta_2, \delta_3\}\} \subset S^\rho.$$

In this paper we consider weak solutions to problem (1), local existence of which can be obtained by Galerkin's method (see for example [22, Chapter II, Sections 3 and 4]) and a standard limit process and the details are omitted.

Definition 2.1. Assume $u_0 \in H_0^1(\Omega)$ and (3) holds. Let $T > 0$ be a constant. A function $u = u(x,t)$ is called a weak solution of problem (1) on $\Omega \times [0,T]$ if $u(\cdot,t) \in L^\infty(0,T;H_0^1(\Omega))$, $u_t(\cdot,t) \in L^2(0,T;H_0^1(\Omega))$ and the following equality

$$\int_\Omega \left(u_t v + \nabla u_t \cdot \nabla v + \nabla u \cdot \nabla v - |x|^\sigma |u|^{p-1} uv\right) dx = 0 \tag{22}$$

holds for any $v \in H_0^1(\Omega)$ and a.e. $t \in [0,T]$. Moreover,

$$u(\cdot,0) = u_0(\cdot) \text{ in } H_0^1(\Omega). \tag{23}$$

Remark 2. There are some remarks on the above definition.

(1) Since $u(\cdot,t) \in L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow L^2(0,T;H_0^1(\Omega))$, $u_t(\cdot,t) \in L^2(0,T;H_0^1(\Omega))$, we have $u \in H^1(0,T;H_0^1(\Omega))$. According to [6], $u \in C([0,T];H_0^1(\Omega))$, then (23) makes sense. Moreover, by (10), all terms in (22) make sense for $u \in C([0,T];H_0^1(\Omega))$ and $u_t \in L^2(0,T;H_0^1(\Omega))$.

(2) Denote by $T_{\max}$ the maximal existence of u, then $u(\cdot,t) \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];H_0^1(\Omega))$, $u_t(\cdot,t) \in L^2(0,T;H_0^1(\Omega))$ for any $T < T_{\max}$.

(3) Taking $v = u$ in (22), we get

$$\|u(\cdot,t)\|_{H_0^1}^2 = \|u_0\|_{H_0^1}^2 - 2\int_0^t I(u(\cdot,s))ds,\ 0 \le t \le T, \tag{24}$$

where $\|\cdot\|_{H_0^1}$ is defined in (7) and I is defined in (13).

(4) Taking $v = u_t$ in (22), we get

$$J(u(\cdot,t)) = J(u_0) - \int_0^t \|u_s(\cdot,s)\|_{H_0^1}^2 ds,\ 0 \le t \le T, \tag{25}$$

where J is defined in (12).

Definition 2.2. Assume (3) holds. A function $u \in H_0^1(\Omega)$ is called a weak solution of (2) if

$$\int_\Omega \left(\nabla u \cdot \nabla v - |x|^\sigma |u|^{p-1} uv\right) dx = 0 \tag{26}$$

holds for any $v \in H_0^1(\Omega)$.

Remark 3. There are some remarks to the above definition.

(1) By (10), we know all the terms in (26) are well-defined.

(2) If we denote by Φ the set of weak solutions to (2), then by the definitions of J in (12) and N in (15), we have

$$\Phi = \{\phi \in H_0^1(\Omega) : J'(\phi) = 0 \text{ in } H^{-1}(\Omega)\} \subset (N \cup \{0\}), \tag{27}$$

where $J'(\phi) = 0$ in $H^{-1}(\Omega)$ means $\langle J'(\phi), \psi\rangle = 0$ for all $\psi \in H_0^1$ and $\langle\cdot,\cdot\rangle$ means the dual product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

With the set Φ defined above, we can defined the ground-state solution to (2).

Definition 2.3. Assume (3) holds. A function $u \in H_0^1(\Omega)$ is called a ground-state solution of (2) if $u \in \Phi \setminus \{0\}$ and

$$J(u) = \inf_{\phi \in \Phi \setminus \{0\}} J(\phi).$$

With the above preparations, now we can state the main results of this paper. Firstly, we consider the case $J(u_0) \le d$. By the sign of $I(u_0)$, we can classify the discussions into three cases:

(1) $J(u_0) \le d$, $I(u_0) > 0$ (see Theorem 2.4);
(2) $J(u_0) \le d$, $I(u_0) < 0$ (see Theorem 2.5);
(3) $J(u_0) \le d$, $I(u_0) = 0$. In this case, by the definition of d in (14), we have $u_0 = 0$ or $J(u_0) = d$ and $I(u_0) = 0$. In Theorem 2.6, we will show problem (1) admits a global solution $u(\cdot, t) \equiv u_0$.

Theorem 2.4. *Assume (3) holds and $u = u(x,t)$ is a weak solution to (1) with $u_0 \in V$, then u exists globally and*

$$\|\nabla u(\cdot,t)\|_{L^2} \le \sqrt{\frac{2(p+1)J(u_0)}{p-1}},\ 0 \le t < \infty, \tag{28}$$

where

$$V := \{\phi \in H_0^1(\Omega) : J(\phi) \le d,\ I(\phi) > 0\}. \tag{29}$$

In, in addition, $J(u_0) < d$, we have the following decay estimate:

$$\|u(\cdot,t)\|_{H_0^1} \le \|u_0\|_{H_0^1} \exp\left[-\frac{\lambda_1}{\lambda_1+1}\left(1-\left(\frac{J(u_0)}{d}\right)^{\frac{p-1}{2}}\right)t\right]. \tag{30}$$

Remark 4. Since $u_0 \in V$, we have $I(u_0) > 0$. Then it follows from the definitions of J in (12) and I in (13) that

$$J(u_0) > \frac{p-1}{2(p+1)}\|\nabla u_0\|_{L^2}^2 > 0.$$

So the equality (28) makes sense.

Theorem 2.5. *Assume (3) holds and $u = u(x,t)$ is a weak solution to (1) with $u_0 \in W$. Then $T_{\max} < \infty$ and u blows up in finite time in the sense of*

$$\lim_{t\uparrow T_{\max}} \int_0^t \|u(\cdot,s)\|_{H_0^1}^2 ds = \infty,$$

where

$$W := \{\phi \in H_0^1(\Omega) : J(\phi) \le d,\ I(\phi) < 0\} \tag{31}$$

and $T_{\max}$ is the maximal existence time of u. If, in addition, $J(u_0) < d$, then

$$T_{\max} \le \frac{4p\|u_0\|_{H_0^1}^2}{(p-1)^2(p+1)(d-J(u_0))}. \tag{32}$$

Remark 5. There are two remarks.

(1) If $J(\phi) < 0$, then we can easily get from the definitions of J and I in (12) and (13) respectively that $I(\phi) < 0$. So we have $\phi \in W$ if $J(\phi) < 0$.
(2) The sets V and W defined in (29) and (31) respectively are both nonempty. In fact for any $\phi \in H_0^1(\Omega) \setminus \{0\}$, we let

$$f(s) = J(s\phi) = \frac{s^2}{2}\|\nabla\phi\|_{L^2}^2 - \frac{s^{p+1}}{p+1}\|\phi\|_{L_\sigma^{p+1}}^{p+1},$$
$$g(s) = I(s\phi) = s^2\|\nabla\phi\|_{L^2}^2 - s^{p+1}\|\phi\|_{L_\sigma^{p+1}}^{p+1}.$$

Then (see Fig. 2)

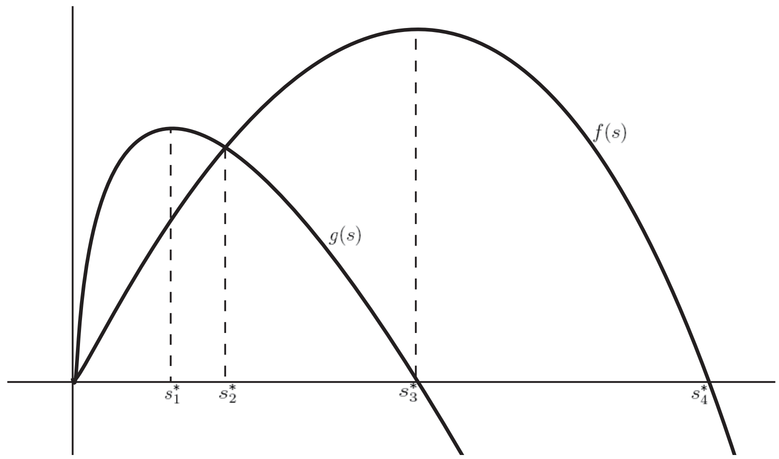

FIGURE 2. The graphs of f and g.

(a) $f(0) = f(s_4^*) = 0$, $f(s)$ is strictly increasing for $s \in (0, s_3^*)$, strictly decreasing for $s \in (s_3^*, \infty)$, $\lim_{s\uparrow\infty} f(s) = -\infty$, and

$$\max_{s\in[0,\infty)} f(s) = f(s_3^*) = \frac{p-1}{2(p+1)} \left(\frac{\|\nabla\phi\|_{L^2}}{\|\phi\|_{L^{p+1}_\sigma}} \right)^{\frac{2(p+1)}{p-1}} \leq \underbrace{d}_{\text{By (14) since } s_3^*\phi\in N}, \tag{33}$$

(b) $g(0) = g(s_3^*) = 0$, $g(s)$ is strictly increasing for $s \in (0, s_1^*)$, strictly decreasing for $s \in (s_1^*, \infty)$, $\lim_{s\uparrow\infty} g(s) = -\infty$, and

$$\max_{s\in[0,\infty)} g(s) = g(s_1^*) = \frac{p-1}{p+1} \left(\frac{2}{p+1} \right)^{\frac{2}{p-1}} \left(\frac{\|\nabla\phi\|_{L^2}}{\|\phi\|_{L^{p+1}_\sigma}} \right)^{\frac{2(p+1)}{p-1}},$$

(c) $f(s) < g(s)$ for $0 < s < s_2^*$, $f(s) > g(s)$ for $s > s_2^*$, and

$$f(s_2^*) = g(s_2^*) = \frac{p-1}{2p} \left(\frac{p+1}{2p} \right)^{\frac{2}{p-1}} \left(\frac{\|\nabla\phi\|_{L^2}}{\|\phi\|_{L^{p+1}_\sigma}} \right)^{\frac{2(p+1)}{p-1}},$$

where

$$s_1^* := \left(\frac{2\|\nabla\phi\|_{L^2}^2}{(p+1)\|\phi\|_{L^{p+1}_\sigma}^{p+1}} \right)^{\frac{1}{p-1}} < s_2^* := \left(\frac{(p+1)\|\nabla\phi\|_{L^2}^2}{2p\|\phi\|_{L^{p+1}_\sigma}^{p+1}} \right)^{\frac{1}{p-1}}$$
$$< s_3^* := \left(\frac{\|\nabla\phi\|_{L^2}^2}{\|\phi\|_{L^{p+1}_\sigma}^{p+1}} \right)^{\frac{1}{p-1}} < s_4^* := \left(\frac{(p+1)\|\nabla\phi\|_{L^2}^2}{2\|\phi\|_{L^{p+1}_\sigma}^{p+1}} \right)^{\frac{1}{p-1}}.$$

So, $\{s\phi : s_1^* < s < s_3^*\} \subset V$, $\{s\phi : s_3^* < s < \infty\} \subset W$.

Theorem 2.6. *Assume* (3) *holds and* $u = u(x,t)$ *is a weak solution to* (1) *with* $u_0 \in G$*. Then problem* (1) *admits a global solution* $u(\cdot,t) \equiv u_0(\cdot)$*, where*

$$G := \{\phi \in H_0^1(\Omega) : J(\phi) = d,\ I(\phi) = 0\}. \tag{34}$$

Remark 6. There are two remarks on the above theorem.

(1) Unlike Remark 5, it is not easy to show $G \neq \emptyset$. In fact, if we use the arguments as in Remark 5, we only have $J(s_3^*\phi) \leq d$ and $I(s_3^*\phi) = 0$ (see Fig. 2 and (33)). In Theorem 2.7, we will use minimizing sequence argument to show $G \neq \emptyset$.

(2) To prove the above Theorem, we only need to show G is the set of the ground-state solution of (2), which is done in Theorem 2.7.

Theorem 2.7. *Assume* (3) *holds and let* G *be the set defined in* (34)*, then* $G \neq \emptyset$ *and* G *is the set of the ground-state solution of* (2)*.*

Secondly, we consider the case $J(u_0) > d$, and we have the following theorem.

Theorem 2.8. *Assume* (3) *holds and the initial value* $u_0 \in H_0^1(\Omega)$ *satisfying* $J(u_0) > d$*.*

(*i*)**:** *If* $u_0 \in S_\rho$ *with* $\rho \geq J(u_0)$*, then problem* (1) *admits a global weak solution* $u = u(x,t)$ *and* $\|u(\cdot,t)\|_{H_0^1} \downarrow 0$ *as* $t \uparrow \infty$*.*

(*ii*)**:** *If* $u_0 \in S^\rho$ *with* $\rho \geq J(u_0)$*, then the weak solution* $u = u(x,t)$ *of problem* (1) *blows up in finite time.*

Here S_ρ *and* S^ρ *are the two sets defined in* (20)*.*

Next, we show the solution of the problem (1) can blow up at arbitrary initial energy level (Theorem 2.10). To this end, we firstly introduce the following theorem.

Theorem 2.9. *Assume* (3) *holds and* $u = u(x,t)$ *is a weak solution to* (1) *with* $u_0 \in \hat{W}$*. Then*

$$T_{\max} \leq \frac{8p\|u_0\|_{H_0^1}^2}{(p-1)^2\left(\frac{\lambda_1(p-1)}{\lambda_1+1}\|u_0\|_{H_0^1}^2 - 2(p+1)J(u_0)\right)} \tag{35}$$

and u *blows up in finite time in the sense of*

$$\lim_{t\uparrow T_{\max}} \int_0^t \|u(\cdot,s)\|_{H_0^1}^2 ds = \infty,$$

where

$$\hat{W} := \left\{\phi \in H_0^1(\Omega) : J(\phi) < \frac{\lambda_1(p-1)}{2(\lambda_1+1)(p+1)}\|\phi\|_{H_0^1}^2\right\}. \tag{36}$$

and $T_{\max}$ *is the maximal existence time of* u*.*

By using the above theorem, we get the following theorem.

Theorem 2.10. *For any* $M \in \mathbb{R}$*, there exists* $u_0 \in H_0^1(\Omega)$ *satisfying* $J(u_0) = M$ *such that the corresponding weak solution* $u = u(x,t)$ *of problem* (1) *blows up in finite time.*

3. Preliminaries

The following lemma can be found in [11].

Lemma 3.1. *Suppose that* $0 < T \le \infty$ *and suppose a nonnegative function* $F(t) \in C^2[0,T)$ *satisfies*

$$F''(t)F(t) - (1+\gamma)\left(F'(t)\right)^2 \ge 0$$

for some constant $\gamma > 0$. *If* $F(0) > 0$, $F'(0) > 0$, *then*

$$T \le \frac{F(0)}{\gamma F'(0)} < \infty$$

and $F(t) \uparrow \infty$ *as* $t \uparrow T$.

Theorem 3.2. *Assume* p *and* σ *satisfy* (3). *Then* $H_0^1(\Omega) \hookrightarrow L_\sigma^{p+1}(\Omega)$ *continuously and compactly.*

Proof. Since $\Omega \subset \mathbb{R}^n$ is a bounded domain, there exists a ball $B(0,R) := \{x \in \mathbb{R}^n : |x| = \sqrt{x_1^2 + \cdots x_n^2} < R\} \supset \Omega$.

We divide the proof into three cases. We will use the notation $a \lesssim b$ which means there exits a positive constant C such that $a \le Cb$.

Case 1. $\sigma \ge 0$. By the assumption on p in (3), one can see

$$H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega) \text{ continuously and compactly.} \tag{37}$$

Then we have, for any $u \in H_0^1(\Omega)$,

$$\|u\|_{L_\sigma^{p+1}}^{p+1} = \int_\Omega |x|^\sigma |u|^{p+1} dx \le R^\sigma \|u\|_{L^{p+1}}^{p+1} \lesssim \|u\|_{H_0^1}^{p+1},$$

which, together with (37), implies $H_0^1(\Omega) \hookrightarrow L_\sigma^{p+1}(\Omega)$ continuously and compactly.

Case 2. $-n < \sigma < 0$ **and** $n = 1$ **or** 2. We can choose $r \in \left(1, -\frac{n}{\sigma}\right)$. Then by Hölder's inequality and

$$H_0^1(\Omega) \hookrightarrow L^{\frac{(p+1)r}{r-1}}(\Omega) \text{ continuously and compactly,} \tag{38}$$

for any $u \in H_0^1(\Omega)$, we have

$$\begin{aligned}
\|u\|_{L_\sigma^{p+1}}^{p+1} &= \int_\Omega |x|^\sigma |u|^{p+1} dx \\
&\le \left(\int_{B(0,R)} |x|^{\sigma r} dx\right)^{\frac{1}{r}} \left(\int_\Omega |u|^{\frac{(p+1)r}{r-1}} dx\right)^{\frac{r-1}{r}} \\
&\le \begin{cases} \left(\frac{2}{\sigma r+1} R^{\sigma r+1}\right)^{\frac{1}{r}} \|u\|_{L^{\frac{(p+1)r}{r-1}}}^{p+1} \lesssim \|u\|_{H_0^1}^{p+1}, & n = 1; \\ \left(\frac{2\pi}{\sigma r+2} R^{\sigma r+2}\right)^{\frac{1}{r}} \|u\|_{L^{\frac{(p+1)r}{r-1}}}^{p+1} \lesssim \|u\|_{H_0^1}^{p+1}, & n = 2, \end{cases}
\end{aligned}$$

which, together with (38), implies $H_0^1(\Omega) \hookrightarrow L_\sigma^{p+1}(\Omega)$ continuously and compactly.

Case 3. $\frac{(p+1)(n-2)}{2} - n < \sigma < 0$ **and** $n \ge 3$. Then there exists a constant $r > 1$ such that

$$-\frac{\sigma}{n} < \frac{1}{r} < 1 - \frac{(p+1)(n-2)}{2n}.$$

By the second inequality of the above inequalities, we have

$$\frac{(p+1)r}{r-1} = \frac{p+1}{1-\frac{1}{r}} < \frac{p+1}{\frac{(p+1)(n-2)}{2n}} = \frac{2n}{n-2}.$$

So,

$$H_0^1(\Omega) \hookrightarrow L^{\frac{(p+1)r}{r-1}}(\Omega) \text{ continuously and compactly.} \tag{39}$$

Then by Hölder's inequality, for any $u \in H_0^1(\Omega)$, we have

$$\begin{aligned}\|u\|_{L_\sigma^{p+1}}^{p+1} &= \int_\Omega |x|^\sigma |u|^{p+1} dx \\ &\leq \left(\int_{B(0,R)} |x|^{\sigma r} dx \right)^{\frac{1}{r}} \left(\int_\Omega |u|^{\frac{(p+1)r}{r-1}} dx \right)^{\frac{r-1}{r}} \\ &\leq \left(\frac{\omega_{n-1}}{\sigma r + n} R^{\sigma r + n} \right)^{\frac{1}{r}} \|u\|_{L^{\frac{(p+1)r}{r-1}}}^{p+1} \lesssim \|u\|_{H_0^1}^{p+1},\end{aligned}$$

which, together with (39), implies $H_0^1(\Omega) \hookrightarrow L_\sigma^{p+1}(\Omega)$ continuously and compactly. Here ω_{n-1} denotes the surface area of the unit ball in $\mathbb{R}^n$. □

Theorem 3.3. *Assume p and σ satisfy (3). Let d be the constant defined in (14), then*

$$d = \frac{p-1}{2(p+1)} C_{p\sigma}^{\frac{2(p+1)}{p-1}},$$

where $C_{p\sigma}$ is the positive constant defined in (11).

Proof. Firstly, we show

$$\inf_{\phi \in N} J(\phi) = \min_{\phi \in H_0^1(\Omega) \setminus \{0\}} J(s_\phi^* \phi), \tag{40}$$

where N is the set defined in (15) and

$$s_\phi^* := \left(\frac{\|\nabla \phi\|_{L^2}^2}{\|\phi\|_{L_\sigma^{p+1}}^{p+1}} \right)^{\frac{1}{p-1}}. \tag{41}$$

By the definition of N in (15) and s_ϕ^* in (41), one can easily see that $s_\phi^* = 1$ if $\phi \in N$ and $s_\phi^* \phi \in N$ for any $\phi \in H_0^1(\Omega) \setminus \{0\}$.

On one hand, since $N \subset H_0^1(\Omega)$ and $s_\phi^* = 1$ for $\phi \in N$, we have

$$\min_{\phi \in H_0^1(\Omega) \setminus \{0\}} J(s_\phi^* \phi) \leq \min_{\phi \in N} J(s_\phi^* \phi) = \min_{\phi \in N} J(\phi).$$

On the other hand, since $\{s_\phi^* \phi : \phi \in H_0^1(\Omega) \setminus \{0\}\} \subset N$, we have

$$\inf_{\phi \in N} J(\phi) \leq \inf_{\phi \in H_0^1(\Omega) \setminus \{0\}} J(s_\phi^* \phi).$$

Then (40) follows from the above two inequalities.

By (40), the definition of d in (14), the definition of J in (12), and the definition of $C_{p\sigma}$ in (11), we have

$$\begin{aligned}d &= \min_{\phi \in H_0^1(\Omega) \setminus \{0\}} J(s_\phi^* \phi) \\ &= \frac{p-1}{2(p+1)} \min_{\phi \in H_0^1(\Omega) \setminus \{0\}} \left(\frac{\|\nabla \phi\|_{L^2}}{\|\phi\|_{L_\sigma^{p+1}}} \right)^{\frac{2(p+1)}{p-1}} \\ &= \frac{p-1}{2(p+1)} C_{p\sigma}^{-\frac{2(p+1)}{p-1}}.\end{aligned}$$

□

Theorem 3.4. *Assume* (3) *holds. Let* λ_ρ *and* Λ_ρ *be the two constants defined in* (19). *Here* $\rho > d$ *is a constant. Then*

$$\sqrt{\frac{2(p+1)d}{p-1}} \le \lambda_\rho \le \Lambda_\rho \le \sqrt{\frac{2(p+1)(\lambda_1+1)\rho}{\lambda_1(p-1)}}. \tag{42}$$

Proof. Let $\rho > d$ and N^ρ be the set defined in (18). By the definitions of λ_ρ and Λ_ρ in (19), it is obvious that

$$\lambda_\rho \le \Lambda_\rho. \tag{43}$$

Since $N^\rho \subset N$, it follows from the definitions of d, J, I in (14), (12), (13), respectively, that

$$\begin{aligned} d &= \inf_{\phi\in N} J(\phi) \\ &= \frac{p-1}{2(p+1)} \inf_{\phi\in N} \|\nabla\phi\|_{L^2}^2 \\ &\le \frac{p-1}{2(p+1)} \inf_{\phi\in N^\rho} \|\phi\|_{H_0^1}^2 \\ &= \frac{p-1}{2(p+1)} \lambda_\rho^2, \end{aligned}$$

which implies

$$\lambda_\rho \ge \sqrt{\frac{2(p+1)d}{p-1}}.$$

On the other hand, by (8) and (18), we have

$$\begin{aligned} \Lambda_\rho &= \sup_{\phi\in N^\rho} \|\phi\|_{H_0^1} \\ &\le \sqrt{\frac{\lambda_1+1}{\lambda_1}} \sup_{\phi\in N^\rho} \|\nabla\phi\|_{L^2} \\ &\le \sqrt{\frac{\lambda_1+1}{\lambda_1}} \sqrt{\frac{2(p+1)\rho}{p-1}}. \end{aligned}$$

Combining the above two inequalities with (43), we get (42), the proof is complete. □

Theorem 3.5. *Assume* (3) *holds and* $u = u(x,t)$ *is a weak solution to* (1). *Then the sets* W *and* V, *defined in* (31) *and* (29) *respectively, are both variant for* u, *i.e.,* $u(\cdot,t) \in W$ $(u(\cdot,t) \in V)$ *for* $0 \le t < T_{\max}$ *when* $u_0 \in W$ $(u_0 \in V)$, *where* $T_{\max}$ *is the maximal existence time of* u.

Proof. We only prove the invariance of W since the proof of the invariance of V is similar.

For any $\phi \in W$, since $I(\phi) < 0$, it follows from the definition of I (see (13)) and (11) that

$$\|\nabla\phi\|_{L^2}^2 < \|\phi\|_{L_\sigma^{p+1}}^{p+1} \le C_{p\sigma}^{p+1} \|\nabla\phi\|_{L^2}^{p+1},$$

which implies

$$\|\nabla\phi\|_{L^2} > C_{p\sigma}^{-\frac{p+1}{p-1}}. \tag{44}$$

Let $u(x,t)$ be the weak solution of problem (1) with $u_0 \in W$. Since $I(u_0) < 0$ and $u \in C([0,T_{\max}), H_0^1(\Omega))$, there exists a constant $\varepsilon > 0$ small enough such that

$$I(u(\cdot,t)) < 0,\ t \in [0,\varepsilon]. \tag{45}$$

Then by (24), $\frac{d}{dt}\|u(\cdot,t)\|_{H_0^1}^2 > 0$ for $t \in [0,\varepsilon]$, and then by (25) and $J(u_0) \le d$, we get

$$J(u(\cdot,t)) < d \text{ for } t \in (0,\varepsilon]. \tag{46}$$

We argument by contradiction. Since $u(\cdot,t) \in C([0,T_{\max}), H_0^1(\Omega))$, if the conclusion is not true, then there exists a $t_0 \in (0,T_{\max})$ such that $u(\cdot,t) \in W$ for $0 \le t < t_0$, but $I(u(\cdot,t_0)) = 0$ and

$$J(u(\cdot,t_0)) < d \tag{47}$$

(note (25) and (46), $J(u(\cdot,t_0)) = d$ cannot happen). By (44), we have $u(\cdot,t) \in C([0,T_{\max}), H_0^1(\Omega))$ and $u(\cdot,t_0) \in \overline{W}$, then

$$\|\nabla u(\cdot,t_0)\|_{L^2} \ge C_{p\sigma}^{-\frac{p+1}{p-1}} > 0,$$

which, together with $I(u(\cdot,t_0)) = 0$, implies $u(\cdot,t_0) \in N$. Then it follows from the definition of d in (14) that

$$J(u(\cdot,t_0)) \ge d,$$

which contradicts (47). So the conclusion holds. $\square$

Theorem 3.6. *Assume (3) holds and $u = u(x,t)$ is a weak solution to (1) with $u_0 \in W$. Then*

$$\|\nabla u(\cdot,t)\|_{L^2}^2 \ge \frac{2(p+1)}{p-1}d,\ 0 \le t < T_{\max}, \tag{48}$$

where W is defined in (31) and $T_{\max}$ is the maximal existence time of u.

Proof. Let $N^- := \{\phi \in H_0^1(\Omega) : I(\phi) < 0\}$. Then by Theorem 3.5, $u(\cdot,t) \in N^-$ for $0 \le t < T_{\max}$.

By the proof in Theorem 3.3,

$$\begin{aligned} d &= \min_{\phi \in H_0^1(\Omega)\setminus\{0\}} J(s_\phi^*\phi) \\ &\le \min_{\phi \in N^-} J(s_\phi^*\phi) \\ &\le J(s_u^* u(\cdot,t)) \\ &= \frac{(s_u^*)^2}{2}\|\nabla u(\cdot,t)\|_{L^2}^2 - \frac{(s_u^*)^{p+1}}{p+1}\|u(\cdot,t)\|_{L_\sigma^{p+1}}^{p+1} \\ &\le \left(\frac{(s_u^*)^2}{2} - \frac{(s_u^*)^{p+1}}{p+1}\right)\|\nabla u(\cdot,t)\|_{L^2}^2, \end{aligned}$$

where we have used $I(u(\cdot,t)) < 0$ in the last inequality. Since $I(u(\cdot,t)) < 0$, we get from (41) that

$$s_u^* = \left(\frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L_\sigma^{p+1}}^{p+1}}\right)^{\frac{1}{p-1}} \in (0,1).$$

Then

$$\begin{aligned} d &\le \max_{0\le s\le 1}\left(\frac{s^2}{2}-\frac{s^{p+1}}{p+1}\right)\|\nabla u(\cdot,t)\|_{L^2}^2 \\ &= \left(\frac{s^2}{2}-\frac{s^{p+1}}{p+1}\right)_{s=1}\|\nabla u(\cdot,t)\|_{L^2}^2 \\ &= \frac{p-1}{2(p+1)}\|\nabla u(\cdot,t)\|_{L^2}^2, \end{aligned}$$

and (48) follows from the above inequality. □

Theorem 3.7. *Assume* (3) *holds and* $u = u(x,t)$ *is a weak solution to* (1) *with* $u_0 \in \hat{W}$*. Then* $I(u(\cdot,t)) < 0$ *for* $0 \le t < T_{\max}$*, where* $T_{\max}$ *is the maximal existence time of* u *and* $\hat{W}$ *is defined in* (36)

Proof. Firstly, we show $I(u_0) < 0$. In fact, by the definition of J in (12), $u_0 \in \hat{W}$, and (8), we get

$$\begin{aligned} \frac{1}{2}\|\nabla u_0\|_{L^2}^2 - \frac{1}{p+1}\|u_0\|_{L^{p+1}_\sigma}^{p+1} &= J(u_0) \\ &< \frac{\lambda_1(p-1)}{2(\lambda_1+1)(p+1)}\|u_0\|_{H_0^1}^2 \\ &\le \frac{p-1}{2(p+1)}\|\nabla u_0\|_{L^2}^2, \end{aligned}$$

which implies

$$I(u_0) = \|\nabla u_0\|_{L^2}^2 - \|u_0\|_{L^{p+1}_\sigma}^{p+1} < 0.$$

Secondly, we prove $I(u(\cdot,t)) < 0$ for $0 < t < T_{\max}$. In fact, if it is not true, in view of $u \in C([0,T_{\max}), H_0^1(\Omega))$, there must exist a $t_0 \in (0,T_{\max})$ such that $I(u(\cdot,t)) < 0$ for $t \in [0,t_0)$ but $I(u(\cdot,t_0)) = 0$. Then by (24), we get $\|u(\cdot,t_0)\|_{H_0^1}^2 > \|u_0\|_{H_0^1}^2$, which, together with $u_0 \in \hat{W}$ and (8), implies

$$\begin{aligned} J(u_0) &< \frac{\lambda_1(p-1)}{2(\lambda_1+1)(p+1)}\|u_0\|_{H_0^1}^2 \\ &< \frac{\lambda_1(p-1)}{2(\lambda_1+1)(p+1)}\|u(\cdot,t_0)\|_{H_0^1}^2 \\ &\le \frac{p-1}{2(p+1)}\|\nabla u(\cdot,t_0)\|_{L^2}^2. \end{aligned} \tag{49}$$

On the other hand, by (24), (12), (13) and $I(u(\cdot,t_0)) = 0$, we get

$$J(u_0) \ge J(u(\cdot,t_0)) = \frac{p-1}{2(p+1)}\|\nabla u(\cdot,t_0)\|_{L^2}^2,$$

which contradicts (49). The proof is complete. □

4. Proofs of the main results

Proof of Theorem 2.4. Let $u = u(x,t)$ be a weak solution to (1) with $u_0 \in V$ and $T_{\max}$ be its maximal existence time. By Theorem 3.5, $u(\cdot,t) \in V$ for $0 \le t < T_{\max}$, which implies $I(u(\cdot,t)) > 0$ for $0 \le t < T_{\max}$. Then it follows from (25), (12) and (13) that

$$J(u_0) \ge J(u(\cdot,t)) \ge \frac{p-1}{2(p+1)}\|\nabla u(\cdot,t)\|_{L^2}^2,\ 0 \le t < T_{\max},$$

which implies u exists globally (i.e. $T_{\max} = \infty$) and

$$\|\nabla u(\cdot,t)\|_{L^2} \le \sqrt{\frac{2(p+1)J(u_0)}{p-1}},\ 0 \le t < \infty. \tag{50}$$

Next, we prove $\|u(\cdot,t)\|_{H_0^1}$ decays exponentially, if in addition, $J(u_0) < d$. By (24), (13), (11), (50), (16) we have

$$\begin{aligned}\frac{d}{dt}\left(\|u(\cdot,t)\|_{H_0^1}^2\right) =& -2I(u(\cdot,t)) = -2\left(\|\nabla u(\cdot,t)\|_{L^2}^2 - \|u(\cdot,t)\|_{L_\sigma^{p+1}}^{p+1}\right)\\ \le& -2\left(1 - C_{p\sigma}^{p+1}\|\nabla u(\cdot,t)\|_{L^2}^{p-1}\right)\|\nabla u(\cdot,t)\|_{L^2}^2\\ \le& -2\left(1 - C_{p\sigma}^{p+1}\left(\sqrt{\frac{2(p+1)J(u_0)}{p-1}}\right)^{p-1}\right)\|\nabla u(\cdot,t)\|_{L^2}^2\\ =& -2\left(1 - \left(\frac{J(u_0)}{d}\right)^{\frac{p-1}{2}}\right)\|\nabla u(\cdot,t)\|_{L^2}^2\\ \le& -\frac{2\lambda_1}{\lambda_1+1}\left(1 - \left(\frac{J(u_0)}{d}\right)^{\frac{p-1}{2}}\right)\|u(\cdot,t)\|_{H_0^1}^2,\end{aligned}$$

which leads to

$$\|u(\cdot,t)\|_{H_0^1}^2 \le \|u_0\|_{H_0^1}^2 \exp\left[-\frac{2\lambda_1}{\lambda_1+1}\left(1 - \left(\frac{J(u_0)}{d}\right)^{\frac{p-1}{2}}\right)t\right].$$

The proof is complete. □

Proof of Theorem 2.5. Let $u = u(x,t)$ be a weak solution to (1) with $u_0 \in W$ and $T_{\max}$ be its maximal existence time.

Firstly, we consider the case $J(u_0) < d$ and $I(u_0) < 0$. By Theorem 3.5, $u(\cdot,t) \in W$ for $0 \le t < T_{\max}$. Let

$$\xi(t) := \left(\int_0^t \|u(\cdot,s)\|_{H_0^1}^2 ds\right)^{\frac{1}{2}},\ \eta(t) := \left(\int_0^t \|u_s(\cdot,s)\|_{H_0^1}^2 ds\right)^{\frac{1}{2}},\ 0 \le t < T_{\max}. \tag{51}$$

For any $T^* \in (0, T_{\max})$, $\beta > 0$ and $\alpha > 0$, we let

$$F(t) := \xi^2(t) + (T^* - t)\|u_0\|_{H_0^1}^2 + \beta(t+\alpha)^2,\ 0 \le t \le T^*. \tag{52}$$

Then

$$F(0) = T^*\|u_0\|_{H_0^1}^2 + \beta\alpha^2 > 0, \tag{53}$$

$$\begin{aligned}F'(t) &= \|u(\cdot,t)\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2 + 2\beta(t+\alpha)\\ &= 2\left(\frac{1}{2}\int_0^t \frac{d}{ds}\|u(\cdot,s)\|_{H_0^1}^2 ds + \beta(t+\alpha)\right),\ 0 \le t \le T^*,\end{aligned} \tag{54}$$

and (by (24), (12), (13), (48), (25))

$$\begin{aligned}F''(t) =& -2I(u(\cdot,t)) + 2\beta\\ =&(p-1)\|\nabla u(\cdot,t)\|_{L^2}^2 - 2(p+1)J(u(\cdot,t)) + 2\beta\\ \ge& 2(p+1)(d - J(u_0)) + 2(p+1)\eta^2(t) + 2\beta,\ 0 \le t \le T^*.\end{aligned} \tag{55}$$

Since $I(u(\cdot,t))<0$, it follow from (24) and the first equality of (54) that

$$F'(t)\geq 2\beta(t+\alpha).$$

Then

$$F(t)=F(0)+\int_0^t F'(s)ds\geq T^*\|u_0\|_{H_0^1}^2+\beta\alpha^2+2\alpha\beta t+\beta t^2,\ 0\leq t\leq T^*. \tag{56}$$

By (6), Schwartz's inequality and Hölder's inequality, we have

$$\begin{aligned}\frac{1}{2}\int_0^t\frac{d}{ds}\|u(\cdot,s)\|_{H_0^1}^2ds&=\int_0^t(u(\cdot,s),u_s(\cdot,s))_{H_0^1}ds\\&\leq\int_0^t\|u(\cdot,s)\|_{H_0^1}\|u_s(\cdot,s)\|_{H_0^1}ds\leq\xi(t)\eta(t),\ 0\leq t\leq T^*,\end{aligned}$$

which, together with the definition of $F(t)$, implies

$$\begin{aligned}&\left(F(t)-(T^*-t)\|u_0\|_{H_0^1}^2\right)\left(\eta^2(t)+\beta\right)\\&=\left(\xi^2(t)+\beta(t+\alpha)^2\right)\left(\eta^2(t)+\beta\right)\\&=\xi^2(t)\eta^2(t)+\beta\xi^2(t)+\beta(t+\alpha)^2\eta^2(t)+\beta^2(t+\alpha)^2\\&\geq\xi^2(t)\eta^2(t)+2\xi(t)\eta(t)\beta(t+\alpha)+\beta^2(t+\alpha)^2\\&\geq(\xi(t)\eta(t)+\beta(t+\alpha))^2\\&\geq\left(\frac{1}{2}\int_0^t\frac{d}{ds}\|u(\cdot,s)\|_{H_0^1}^2ds+\beta(t+\alpha)\right)^2,\ 0\leq t\leq T^*.\end{aligned}$$

Then it follows from (54) and the above inequality that

$$\begin{aligned}(F'(t))^2&=4\left(\frac{1}{2}\int_0^t\frac{d}{ds}\|u(s)\|_{H_0^1}^2ds+\beta(t+\alpha)\right)^2\\&\leq 4F(t)\left(\eta^2(t)+\beta\right),\ 0\leq t\leq T^*.\end{aligned} \tag{57}$$

In view of (55), (56), and (57), we have

$$F(t)F''(t)-\frac{p+1}{2}(F'(t))^2\geq F(t)\left(2(p+1)(d-J(u_0))-2p\beta\right),\ 0\leq t\leq T^*.$$

If we take β small enough such that

$$0<\beta\leq\frac{p+1}{p}(d-J(u_0)), \tag{58}$$

then $F(t)F''(t)-\frac{p+1}{2}(F'(t))^2\geq 0$. Then, it follows from Lemma 3.1 that

$$T^*\leq\frac{F(0)}{\left(\frac{p+1}{2}-1\right)F'(0)}=\frac{T^*\|u_0\|_{H_0^1}^2+\beta\alpha^2}{(p-1)\alpha\beta}.$$

Then for

$$\alpha\in\left(\frac{\|u_0\|_{H_0^1}^2}{(p-1)\beta},\infty\right), \tag{59}$$

we get

$$T^*\leq\frac{\beta\alpha^2}{(p-1)\alpha\beta-\|u_0\|_{H_0^1}^2}.$$

Minimizing the above inequality for α satisfying (59), we get

$$T^* \le \frac{\beta\alpha^2}{(p-1)\alpha\beta - \|u_0\|^2_{H_0^1}}\Bigg|_{\alpha=\frac{2\|u_0\|^2_{H_0^1}}{(p-1)\beta}} = \frac{4\|u_0\|^2_{H_0^1}}{(p-1)^2\beta}.$$

Minimizing the above inequality for β satisfying (58), we get

$$T^* \le \frac{4p\|u_0\|^2_{H_0^1}}{(p-1)^2(p+1)(d-J(u_0))}.$$

By the arbitrariness of $T^* < T_{\max}$ it follows that

$$T_{\max} \le \frac{4p\|u_0\|^2_{H_0^1}}{(p-1)^2(p+1)(d-J(u_0))}.$$

Secondly, we consider the case $J(u_0) = d$ and $I(u_0) < 0$. By the proof of Theorem 3.5, there exists a $t_0 > 0$ small enough such that $J(u(\cdot, t_0)) < d$ and $I(u(\cdot, t_0)) < 0$. Then it follows from the above proof that u will blow up in finite time. The proof is complete. □

Proof of Theorems 2.6 and 2.7. Since Theorem 2.6 follows from Theorem 2.7 directly, we only need to prove Theorem 2.7.

Firstly, we show $G \ne \emptyset$. By the definition of d in (14), we get

$$d = \inf_{\phi\in N} J(\phi) = \frac{p-1}{2(p+1)} \inf_{\phi\in N} \|\nabla\phi\|^2_{L^2}.$$

Then a minimizing sequence $\{\phi_k\}_{k=1}^{\infty} \subset N$ exists such that

$$\lim_{k\uparrow\infty} J(\phi_k) = \frac{p-1}{2(p+1)} \lim_{k\uparrow\infty} \|\nabla\phi_k\|^2_{L^2} = d, \tag{60}$$

which implies $\{\phi_k\}_{k=1}^{\infty}$ is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is reflexive and $H_0^1(\Omega) \hookrightarrow L_\sigma^{p+1}$ continuously and compactly (see (10)), there exists $\varphi \in H_0^1(\Omega)$ such that

(1) $\phi_k \rightharpoonup \varphi$ in $H_0^1(\Omega)$ weakly;
(2) $\phi_k \to \varphi$ in $L_\sigma^{p+1}(\Omega)$ strongly.

Now, in view of $\|\nabla(\cdot)\|_{L^2}$ is weakly lower continuous in $H_0^1(\Omega)$, taking $\liminf_{k\uparrow\infty}$ in the equality $\|\nabla\phi_k\|^2_{L^2} = \|\phi_k\|^{p+1}_{L_\sigma^{p+1}}$ (sine $\phi_k \in N$), we get

$$\|\nabla\varphi\|^2_{L^2} \le \|\varphi\|^{p+1}_{L_\sigma^{p+1}}. \tag{61}$$

We claim

$$\|\nabla\varphi\|^2_{L^2} = \|\varphi\|^{p+1}_{L_\sigma^{p+1}} \text{ i.e. } I(\varphi) = 0. \tag{62}$$

In fact, if the claim is not true, then by (61),

$$\|\nabla\varphi\|^2_{L^2} < \|\varphi\|^{p+1}_{L_\sigma^{p+1}}.$$

By the proof of Theorem 3.3, we know that $s^*_\varphi\varphi \in N$, which, together with the definition of d in (14), implies

$$J(s^*_\varphi\varphi) \ge d, \tag{63}$$

where

$$s^*_\varphi := \left(\frac{\|\nabla\varphi\|^2_{L^2}}{\|\varphi\|^{p+1}_{L_\sigma^{p+1}}}\right)^{\frac{1}{p-1}} \in (0,1).$$

On the other hand, since $s_\varphi^*\varphi\in N$, we get from the definitions of J in (12) and I in (13), $s_\varphi^*\in(0,1)$, $\|\nabla(\cdot)\|_{L^2}$ is weakly lower continuous in $H_0^1(\Omega)$, (60) that

$$\begin{aligned} J(s_\varphi^*\varphi) &= \frac{p-1}{2(p+1)}(s_\varphi^*)^2\|\nabla\varphi\|_{L^2}^2 \\ &< \frac{p-1}{2(p+1)}\|\nabla\varphi\|_{L^2}^2 \\ &\le \frac{p-1}{2(p+1)}\liminf_{k\uparrow\infty}\|\nabla\phi_k\|_{L^2}^2 \\ &= d, \end{aligned}$$

which contradicts to (63). So the claim is true, i.e.

$$\lim_{k\uparrow\infty}\|\nabla\phi_k\|_{L^2}^2=\|\varphi\|_{L_\sigma^{p+1}}^{p+1},$$

which, together with $H_0^1(\Omega)$ is uniformly convex and $\phi_k\rightharpoonup\varphi$ in $H_0^1(\Omega)$ weakly, implies $\phi_k\to\varphi$ strongly in $H_0^1(\Omega)$. Then by (60), $J(\varphi)=d$, which, together with (62) and the definition of G in (34), implies $\varphi\in G$, i.e., $G\neq\emptyset$.

Second, we prove $G\subset\Phi$, where Φ is the set defined in (27). For any $\varphi\in G$, we need to show $\varphi\in\Phi$, i.e. φ satisfies (26). Fix any $v\in H_0^1(\Omega)$ and $s\in(-\varepsilon,\varepsilon)$, where $\varepsilon>0$ is a small constant such that $\|\varphi+sv\|_{L_\sigma^{p+1}}^{p+1}>0$ for $s\in(-\varepsilon,\varepsilon)$. Let

$$\tau(s):=\left(\frac{\|\nabla(\varphi+sv)\|_{L^2}^2}{\|\varphi+sv\|_{L_\sigma^{p+1}}^{p+1}}\right)^{\frac{1}{p-1}},\ s\in(-\varepsilon,\varepsilon). \tag{64}$$

Then $I(\tau(s)(\varphi+sv))=0$. So by the definition of N in (15), the set

$$A:=\{\tau(s)(\varphi+sv):s\in(-\varepsilon,\varepsilon)\}$$

is a curve on N, which passes φ when $s=0$. The function $\tau(s)$ is differentiable and

$$\tau'(s)=\frac{1}{p-1}\left(\frac{\|\nabla(\varphi+sv)\|_{L^2}^2}{\|\varphi+sv\|_{L_\sigma^{p+1}}^{p+1}}\right)^{\frac{2-p}{p-1}}\frac{\xi-\eta}{\|\varphi+sv\|_{L_\sigma^{p+1}}^{2(p+1)}},$$

where

$$\begin{aligned} \xi &:= 2\int_\Omega\nabla(\varphi+sv)\cdot\nabla v dx\|\varphi+sv\|_{L_\sigma^{p+1}}^{p+1}, \\ \eta &:= (p+1)\int_\Omega|x|^\sigma|\varphi+sv|^{p-1}(\varphi+sv)vdx\|\nabla(\varphi+sv)\|_{L^2}^2. \end{aligned}$$

Since (62), we get $\tau(0)=1$ and

$$\tau'(0)=\frac{1}{(p-1)\|\varphi\|_{L_\sigma^{p+1}}^{p+1}}\left(2\int_\Omega\nabla\varphi\nabla vdx-(p+1)\int_\Omega|x|^\sigma|\varphi|^{p-1}\varphi vdx\right). \tag{65}$$

Let

$$\varrho(s):=J(\tau(s)(\varphi+sv))=\frac{\tau^2(s)}{2}\|\nabla(\varphi+sv)\|_{L^2}^2-\frac{\tau^{p+1}(s)}{p+1}\|\varphi+sv\|_{L_\sigma^{p+1}}^{p+1},\ s\in(-\varepsilon,\varepsilon).$$

Since $\tau(s)(\varphi+sv)\in N$ for $s\in(-\varepsilon,\varepsilon)$, $\tau(s)(\varphi+sv)|_{s=0}=\varphi$, $\varrho(0)=J(\varphi)=d$, it follows from the definition of d that $\varrho(s)$ $(s\in(-\varepsilon,\varepsilon))$ achieves its minimum at

$s = 0$, then $\varrho'(0) = 0$. So,

$$\begin{aligned}0 =&\varrho'(0) = \tau(s)\tau'(s)\|\nabla(\varphi+sv)\|_{L^2}^2 + \tau^2(s)\int_\Omega \nabla(\varphi+sv)\cdot\nabla v dx\Big|_{s=0}\\ &-\tau^p(s)\tau'(s)\|\varphi+sv\|_{L_\sigma^{p+1}}^{p+1} - \tau^{p+1}(s)\int_\Omega |x|^\sigma|\varphi+sv|^{p-1}(\varphi+sv)vdx\Big|_{s=0}\\ =&\int_\Omega \nabla\varphi\cdot\nabla v dx - \int_\Omega |x|^\sigma|\varphi|^{p-1}\varphi v dx.\end{aligned}$$

So, $\varphi \in \Phi$, i.e. $G \subset \Phi$. Moreover, we have $G \subset (\Phi \setminus \{0\})$ since $\varphi \neq 0$ for any $\varphi \in G$.

Finally, in view of Definition 2.3 and $J(\varphi) = d$ ($\forall \varphi \in G$), to complete the proof, we only need to show

$$d = \inf_{\phi\in\Phi\setminus\{0\}} J(\phi). \tag{66}$$

In fact, by the above proof and (27), we have $G \subset \Phi \setminus \{0\} \subset N$. Then, in view of the definition of d in (14), i.e.,

$$d = \inf_{\phi\in N} J(\phi)$$

and $J(\varphi) = d$ for any $\varphi \in G$, we get (66). The proof is complete. □

Proof of Theorem 2.8. Let $u = u(x,t)$ be the solution of problem (1) with initial value u_0 satisfying $J(u_0) > d$. We denote by $T_{\max}$ the maximal existence of u. If u is global, i.e. $T_{\max} = \infty$, we denote by

$$\omega(u_0) = \cap_{t\geq 0}\overline{\{u(\cdot,s) : s \geq t\}}^{H_0^1(\Omega)}$$

the ω-limit set of u_0.

(i) Assume $u_0 \in S_\rho = \{\phi \in H_0^1(\Omega) : \|\phi\|_{H_0^1} \leq \lambda_\rho, I(\phi) > 0\}$ (see (20)) with $\rho \geq J(u_0)$. Without loss of generality, we assume $u(\cdot,t) \neq 0$ for $0 \leq t < T_{\max}$. In fact it there exists a t_0 such that $u(\cdot,t_0) = 0$, then it is easy to see the function v defined as

$$v(x,t) = \begin{cases} u(x,t), & \text{if } 0 \leq t \leq t_0;\\ 0, & \text{if } t > t_0\end{cases}$$

is a global weak solution of problem (1), and the proof is complete.

We claim that

$$I(u(\cdot,t)) > 0,\ 0 \leq t < T_{\max}. \tag{67}$$

Since $I(u_0) > 0$, if the claim is not true, there exists a $t_0 \in (0, T_{\max})$ such that

$$I(u(\cdot,t)) > 0,\ 0 \leq t < t_0 \tag{68}$$

and

$$I(u(\cdot,t_0)) = 0, \tag{69}$$

which together with the definition of N in (15) and the assumption that $u(\cdot,t) \neq 0$ for $0 \leq t < T_{\max}$, implies $u(\cdot,t_0) \in N$. Moreover, by using (68), similar to the proof of (46), we have $J(u(\cdot,t_0)) < J(u_0)$, i.e. $u(\cdot,t_0) \in J^{J(u_0)}$ (see (17)). Then $u(\cdot,t_0) \in N^{J(u_0)}$ (since $N^{J(u_0)} = N \cap J^{J(u_0)}$) and then $\|u(\cdot,t_0)\|_{H_0^1} \geq \lambda_{J(u_0)}$ (see (19)). By monotonicity (see Remark 1) and $\rho \geq J(u_0)$, we get

$$\|u(\cdot,t_0)\|_{H_0^1} \geq \lambda_\rho. \tag{70}$$

On the other hand, it follows from (24), (68) and $u_0 \in S_\rho$ that

$$\|u(\cdot,t)\|_{H_0^1} < \|u_0\|_{H_0^1} \le \lambda_\rho,$$

which contradicts (70). So (67) is true. Then by (24) again, we get

$$\|u(\cdot,t)\|_{H_0^1} \le \|u_0\|_{H_0^1},\ 0 \le t < T_{\max},$$

which implies u exists globally, i.e. $T_{\max} = \infty$.

By (24) and (67), $\|u(\cdot,t)\|_{H_0^1}$ is strictly decreasing for $0 \le t < \infty$, so a constant $c \in [0, \|u_0\|_{H_0^1})$ exists such that

$$\lim_{t\uparrow\infty} \|u(\cdot,t)\|_{H_0^1} = c.$$

Taking $t \uparrow \infty$ in (24), we get

$$\int_0^\infty I(u(\cdot,s))ds \le \frac{1}{2}\left(\|u_0\|_{H_0^1}^2 - c\right) < \infty.$$

Note that $I(u(\cdot,s)) > 0$ for $0 \le s < \infty$, so, for any sequence $\{t_n\}$ satisfying $t_n \uparrow \infty$ as $n \uparrow \infty$, if the limit $\lim_{n\uparrow\infty} I(u(\cdot,t_n))$ exists, it must hold

$$\lim_{n\uparrow\infty} I(u(\cdot,t_n)) = 0. \tag{71}$$

Let ω be an arbitrary element in $\omega(u_0)$. Then there exists a sequence $\{t_n\}$ satisfying $t_n \uparrow \infty$ as $n \uparrow \infty$ such that

$$u(\cdot,t_n) \to \omega \text{ in } H_0^1(\Omega) \text{ as } n \uparrow \infty. \tag{72}$$

Then by (71), we get

$$I(\omega) = \lim_{n\uparrow\infty} I(u(\cdot,t_n)) = 0. \tag{73}$$

As the above, one can easily see

$$\|\omega\|_{H_0^1} < \lambda_\rho \le \lambda_{J(u_0)},\ \underbrace{J(\omega) < J(u_0)}_{\Rightarrow \omega \in J^{J(u_0)}},$$

which implies $\omega \notin N^{J(u_0)}$. In fact, if $\omega \in N^{J(u_0)}$, by (19), $\lambda_{J(u_0)} \le \|\omega\|_{H_0^1}$, a contradiction. Since $N^{J(u_0)} = N \cap J^{u_0}$ and $\omega \in J^{J(u_0)}$, we get $\omega \notin N$. Therefore, by the definition of N in (15) and (73), $\omega = 0$, then it follows from $\|u(\cdot,t)\|_{H_0^1}$ is strictly decreasing and (72) that

$$\lim_{t\uparrow\infty} \|u(\cdot,t)\|_{H_0^1} = \lim_{n\uparrow\infty} \|u(\cdot,t_n)\|_{H_0^1} = \|\omega\|_{H_0^1} = 0.$$

(ii) Assume $u_0 \in S^\rho = \{\phi \in H_0^1(\Omega) : \|\phi\|_{H_0^1} \ge \Lambda_\rho, I(\phi) < 0\}$ (see (20)) with $\rho \ge J(u_0)$. We claim that

$$I(u(\cdot,t)) < 0,\ 0 \le t < T_{\max}. \tag{74}$$

Since $I(u_0) < 0$, if the claim is not true, there exists a $t_0 \in (0, T_{\max})$ such that

$$I(u(\cdot,t)) < 0,\ 0 \le t < t_0 \tag{75}$$

and

$$I(u(\cdot,t_0)) = 0. \tag{76}$$

Since (75), by (44) and $u \in C([0,T_{\max}), H_0^1(\Omega))$, we get

$$\|\nabla u(\cdot,t_0)\|_{L^2} \ge C_{p\sigma}^{-\frac{p+1}{p-1}},$$

which, together with the definition of N in (15), implies $u(\cdot,t_0) \in N$. Moreover, by using (75), similar to the proof of (46), we have $J(u(\cdot,t_0)) < J(u_0)$, i.e. $u(\cdot,t_0) \in J^{J(u_0)}$ (see (17)). Then $u(\cdot,t_0) \in N^{J(u_0)}$ (since $N^{J(u_0)} = N \cap J^{J(u_0)}$) and then $\|u(\cdot,t_0)\|_{H_0^1} \le \Lambda_{J(u_0)}$ (see (19)). By monotonicity (see Remark 1) and $\rho \ge J(u_0)$, we get

$$\|u(\cdot,t_0)\|_{H_0^1} \le \Lambda_\rho. \tag{77}$$

On the other hand, it follows from (24), (75) and $u_0 \in S^\rho$ that

$$\|u(\cdot,t)\|_{H_0^1} > \|u_0\|_{H_0^1} \ge \Lambda_\rho,$$

which contradicts (77). So (74) is true.

Suppose by contradiction that u does not blow up in finite time, i.e. $T_{\max} = \infty$. By (24) and (74), $\|u(\cdot,t)\|_{H_0^1}$ is strictly increasing for $0 \le t < \infty$. If the limit $\lim_{t\uparrow\infty} \|u(t)\|_{H_0^1}$ exists, i.e. there exists a constant $\tilde{c} \in [\|u_0\|_{H_0^1}, \infty)$ such that

$$\lim_{t\uparrow\infty} \|u(\cdot,t)\|_{H_0^1} = \tilde{c},$$

Taking $t \uparrow \infty$ in (24), we get

$$-\int_0^\infty I(u(\cdot,s))ds \le \frac{1}{2}\left(\tilde{c} - \|u_0\|_{H_0^1}^2\right) < \infty.$$

Note $-I(u(\cdot,s)) > 0$ for $0 \le s < \infty$, so, for any sequence $\{t_n\}$ satisfying $t_n \uparrow \infty$ as $n \uparrow \infty$, if the limit $\lim_{n\uparrow\infty} I(u(\cdot,t_n))$ exists, it must hold

$$\lim_{n\uparrow\infty} I(u(\cdot,t_n)) = 0. \tag{78}$$

Let ω be an arbitrary element in $\omega(u_0)$. Then there exists a sequence $\{t_n\}$ satisfying $t_n \uparrow \infty$ as $n \uparrow \infty$ such that

$$u(\cdot,t_n) \to \omega \text{ in } H_0^1(\Omega) \text{ as } n \uparrow \infty. \tag{79}$$

Since $\|u(\cdot,t)\|_{H_0^1}$ is strictly increasing, $\lim_{t\uparrow\infty} \|u(\cdot,t)\|_{H_0^1}$ exists and

$$\lim_{t\uparrow\infty} \|u(\cdot,t)\|_{H_0^1} = \lim_{n\uparrow\infty} \|u(\cdot,t_n)\|_{H_0^1} = \|\omega\|_{H_0^1}.$$

Then by (78), we get

$$I(\omega) = \lim_{n\uparrow\infty} I(u(\cdot,t_n)) = 0. \tag{80}$$

By (24), (25) and (74), one can easily see

$$\|\omega\|_{H_0^1} > \|u_0\|_{H_0^1} \ge \Lambda_\rho \ge \Lambda_{J(u_0)}, \ \underbrace{J(\omega) < J(u_0)}_{\Rightarrow \omega \in J^{J(u_0)}},$$

which implies $\omega \notin N^{J(u_0)}$. In fact, if $\omega \in N^{J(u_0)}$, by (19), $\Lambda_{J(u_0)} \ge \|\omega\|_{H_0^1}$, a contradiction. Since $N^{J(u_0)} = N \cap J^{u_0}$ and $\omega \in J^{J(u_0)}$, we get $\omega \notin N$. Therefore, by the definition of N in (15) and (80), $\omega = 0$. However, this contradicts $\|\omega\|_{H_0^1} > \Lambda_{J(u_0)} > 0$. So u blows up in finite time. The proof is complete. □

Proof of Theorem 2.9. Let $u = u(x,t)$ be a weak solution of (1) with $u_0 \in \hat{W}$ and $T_{\max}$ be its maximal existence time, where $\hat{W}$ is defined in (36). By Theorem 3.7, we know that $I(u(\cdot,t)) < 0$ for $0 \le t < T_{\max}$. Then by (8) and (24), we get

$$\|\nabla u(\cdot,t)\|_{L^2}^2 \ge \frac{\lambda_1}{\lambda_1+1}\|u(\cdot,t)\|_{H_0^1}^2 \ge \frac{\lambda_1}{\lambda_1+1}\|u_0\|_{H_0^1}^2, \ 0 \le t < T_{\max}. \tag{81}$$

The remain proofs are similar to the proof of Theorem 2.9. For any $T^* \in (0, T_{\max})$, $\beta > 0$ and $\alpha > 0$, we consider the functional $F(t)$ again (see (52)). We also have (53), (54), but there are some differences in (55), in fact, by (81) and (25), we have

$$\begin{aligned} F''(t) =& -2I(u(\cdot,t)) + 2\beta \\ =&(p-1)\|\nabla u(\cdot,t)\|_{L^2}^2 - 2(p+1)J(u(\cdot,t)) + 2\beta \\ \geq& \frac{\lambda_1(p-1)}{\lambda_1+1}\|u_0\|_{H_0^1}^2 - 2(p+1)J(u_0) + 2(p+1)\eta^2(t) + 2\beta,\ 0 \leq t \leq T^*. \end{aligned} \tag{82}$$

We also have (56) and (57). Then it follows from (56), (57) and (82) that

$$\begin{aligned} &F(t)F''(t) - \frac{p+1}{2}(F'(t))^2 \\ \geq& F(t)\left(\frac{\lambda_1(p-1)}{\lambda_1+1}\|u_0\|_{H_0^1}^2 - 2(p+1)J(u_0) - 2p\beta\right),\ 0 \leq t \leq T^*. \end{aligned}$$

If we take β small enough such that

$$0 < \beta \leq \frac{1}{2p}\left(\frac{\lambda_1(p-1)}{\lambda_1+1}\|u_0\|_{H_0^1}^2 - 2(p+1)J(u_0)\right), \tag{83}$$

then $F(t)F''(t) - \frac{p+1}{2}(F'(t))^2 \geq 0$. Then, it follows from Lemma 3.1 that

$$T^* \leq \frac{F(0)}{\left(\frac{p+1}{2}-1\right)F'(0)} = \frac{T^*\|u_0\|_{H_0^1}^2 + \beta\alpha^2}{(p-1)\alpha\beta}.$$

Then for

$$\alpha \in \left(\frac{\|u_0\|_{H_0^1}^2}{(p-1)\beta}, \infty\right), \tag{84}$$

we get

$$T^* \leq \frac{\beta\alpha^2}{(p-1)\alpha\beta - \|u_0\|_{H_0^1}^2}.$$

Minimizing the above inequality for α satisfying (84), we get

$$T^* \leq \left.\frac{\beta\alpha^2}{(p-1)\alpha\beta - \|u_0\|_{H_0^1}^2}\right|_{\alpha = \frac{2\|u_0\|_{H_0^1}^2}{(p-1)\beta}} = \frac{4\|u_0\|_{H_0^1}^2}{(p-1)^2\beta}.$$

Minimizing the above inequality for β satisfying (58), we get

$$T^* \leq \frac{8p\|u_0\|_{H_0^1}^2}{(p-1)^2\left(\frac{\lambda_1(p-1)}{\lambda_1+1}\|u_0\|_{H_0^1}^2 - 2(p+1)J(u_0)\right)}.$$

By the arbitrariness of $T^* < T_{\max}$ it follows that

$$T_{\max} \leq \frac{8p\|u_0\|_{H_0^1}^2}{(p-1)^2\left(\frac{\lambda_1(p-1)}{\lambda_1+1}\|u_0\|_{H_0^1}^2 - 2(p+1)J(u_0)\right)}.$$

□

Proof of Theorem 2.10. For any $M \in \mathbb{R}$, let $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ be two arbitrary disjoint open domains. Let $\psi \in H_0^1(\Omega_2) \setminus \{0\}$, extending ψ to Ω by letting $\psi = 0$ in $\Omega \setminus \Omega_2$, then $\psi \in H_0^1(\Omega)$. We choose α large enough such that

$$\|\alpha\psi\|_{H_0^1}^2 > \frac{2(\lambda_1+1)(p+1)}{\lambda_1(p-1)} M. \tag{85}$$

For such α and ψ, we take a $\phi \in H_0^1(\Omega_1) \setminus \{0\}$ (which is extended to Ω by letting $\phi = 0$ in $\Omega \setminus \Omega_1$ i.e. $\phi \in H_0^1(\Omega)$) such that

$$J(s_3^*\phi) \geq M - J(\alpha\psi), \tag{86}$$

where (see Remark 5)

$$s_3^* := \left(\frac{\|\nabla\phi\|_{L^2}^2}{\|\phi\|_{L_\sigma^{p+1}}^{p+1}} \right)^{\frac{1}{p-1}},$$

which can be done since

$$J(s_3^*\phi) = \frac{p-1}{2(p+1)} \left(\frac{\|\nabla\phi\|_{L^2}}{\|\phi\|_{L_\sigma^{p+1}}} \right)^{\frac{2(p+1)}{p-1}}$$

and ϕ can be chosen such that $\|\nabla\phi\|_{L^2} \gg \|\phi\|_{L_\sigma^{p+1}}$.

By Remark 5 again,

$$J(\{s\phi : 0 \leq s < \infty\}) = (-\infty, J(s_3^*\phi)]. \tag{87}$$

By (87) and (86), we can choose $s \in [0,\infty)$ such that $v := s\phi$ satisfies $J(v) = M - J(\alpha\psi)$. Letting $u_0 := v + \alpha\psi \in H_0^1(\Omega)$, since Ω_1 and Ω_2 are disjoint, we get

$$J(u_0) = J(v) + J(\alpha\psi) = M$$

and (note (85))

$$\begin{aligned} J(u_0) = M &< \frac{\lambda_1(p-1)}{2(\lambda_1+1)(p+1)} \|\alpha\psi\|_{H_0^1}^2 \\ &\leq \frac{\lambda_1(p-1)}{2(\lambda_1+1)(p+1)} \left(\|\alpha\psi\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 \right) \\ &= \frac{\lambda_1(p-1)}{2(\lambda_1+1)(p+1)} \|u_0\|_{H_0^1}^2. \end{aligned}$$

Let $u = u(x,t)$ be the weak solution of problem (1) with initial value u_0 given above. Then by Theorem 2.9, u blows up in finite time. □

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