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GENERALIZATIONS OF SOME ORDINARY AND EXTREME CONNECTEDNESS PROPERTIES OF TOPOLOGICAL SPACES TO RELATOR SPACES

MUWAFAQ SALIH* AND ÁRPÁD SZÁZ

Dedicated to the memory of János Kurdics
who was the first to note that connectedness is a particular case of well-chainedness

Abstract. Motivated by some ordinary and extreme connectedness properties of topologies, we introduce several reasonable connectedness properties of relators (families of relations). Moreover, we establish some intimate connections among these properties.

More concretely, we investigate relationships among various minimalness (well-chainedness), connectedness, hyper- and ultra-connectedness, door, superset, submaximality and resolvability properties of relators.

Since most generalized topologies and all proper stacks (ascending systems) can be derived from preorder relators, the results obtained greatly extends some standard results on topologies. Moreover, they are also closely related to some former results on well-chained and connected uniformities.

1. Connectedness properties of topologies

By Thron [212, p. 18], topological spaces were first suggested by Tietze [213] and Alexandroff [4]. They were later standardized by Bourbaki [18], Kelley [80] and Engelking [52]. (For some historical facts, see also Folland [56].)

If $\mathcal{T}$ is a family of subsets of a set X such that $\mathcal{T}$ is closed under finite intersections and arbitrary unions, then the family $\mathcal{T}$ is called a *topology* on X, and the ordered pair $X(\mathcal{T}) = (X, \mathcal{T})$ is called a *topological space*.

The members of $\mathcal{T}$ are called the *open subsets* of X. While, the members of $\mathcal{F} = \mathcal{T}^c = \{A^c : \ A \in \mathcal{T}\}$ are called the *closed subsets* of X. And, the members of $\mathcal{T} \cap \mathcal{F}$ are called the *clopen subsets* of X.

Note that $\emptyset \subseteq \mathcal{T}$ such that $\emptyset = \bigcup \emptyset$ and $X = \bigcap \emptyset$. Therefore, we necessarily have $\{\emptyset, X\} \subseteq \mathcal{T}$, and thus also $\{\emptyset, X\} \subseteq \mathcal{F}$. Consequently, $\{\emptyset, X\} \subseteq \mathcal{T} \cap \mathcal{F}$ is always true. That is, $\emptyset$ and X are always clopen subsets of X.

According to [166, 169, 178, 182], the members of the family

$$\mathcal{E} = \big\{ A \subseteq X : \quad \exists \ U \in \mathcal{T} \setminus \{\emptyset\} : \quad U \subseteq A \big\}$$

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* Corresponding author: Muwafaq Salih.

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may be naturally called the *fat subsets* of X.

Hence, it is clear that $\mathcal{E} \neq \emptyset$ if and only if $X \neq \emptyset$. Moreover, $\mathcal{E}$ is a *proper stack* on X in the sense that $\emptyset \notin \mathcal{E}$ and $\mathcal{E}$ is *ascending* in X. That is, if $A \in \mathcal{E}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{E}$ also holds.

Moreover, it can be easily seen that

$$\mathcal{D} = \left\{ A \subseteq X : \quad A^c \notin \mathcal{E} \right\} = \left\{ A \subseteq X : \quad \forall \ B \in \mathcal{E} : \quad A \cap B \neq \emptyset \right\}.$$

Thus, $\mathcal{D}$ is just the family of all *dense subsets* of X.

For instance, if $A \subseteq X$ such that there exists $B \in \mathcal{E}$ such that $A \cap B = \emptyset$, then $B \subseteq A^c$. Hence, by using that $\mathcal{E}$ is ascending, we can infer that $A^c \in \mathcal{E}$. Therefore, $A^c \notin \mathcal{E}$ implies that $A \cap B \neq \emptyset$ for all $B \in \mathcal{E}$.

Now, having in mind the *poset* (partially ordered set) $\mathcal{P}(X)$ of all subsets of X, a topology $\mathcal{T}$ on X may be naturally called *minimal* and *maximal*, instead of indiscrete and discrete, if $\mathcal{T} = \{\emptyset, X\}$ and $\mathcal{T} = \mathcal{P}(X)$, respectively,

Moreover, by the celebrated Riesz-Lennes-Hausdorff definition of connectedness [212, 216], the topology $\mathcal{T}$ may be naturally called *connected* if $\mathcal{T} \cap \mathcal{F} = \{ \emptyset, X \}$. That is, the family of clopen sets is minimal.

On the other hand, by Steen and Seebach [158, p. 29], the topology $\mathcal{T}$ may be naturally called *hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{T} \setminus \{\emptyset\}$. That is, the family $\mathcal{T} \setminus \{\emptyset\}$ has a certain pairwise intersection property.

Hyperconnected topologies were formerly studied by Bourbaki [19, p. 119] and Levine [97] under the names irreducible and dense topologies. It is noteworthy that $\mathcal{T}$ is hyperconnected if and only if $\mathcal{T} \setminus \{\emptyset\} \subseteq \mathcal{D}$, or equivalently $\mathcal{E} \subseteq \mathcal{D}$.

Also by Steen and Seebach [158, p. 29], the topology $\mathcal{T}$ may be naturally called *ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F} \setminus \{\emptyset\}$. Ultraconnected topologies were formerly studied by Levine [95] under the name strongly connected topologies.

Following Kelley [80, p. 76], a topology $\mathcal{T}$ on X may be naturally called a *door topology* if every subset of X is either open or closed. That is, $\mathcal{P}(X) = \mathcal{T} \cup \mathcal{F}$. Thus, unlike a door, a subset of X can be both open and closed.

While, according to Levine [96], a topology $\mathcal{T}$ on X may be naturally called a *superset topology* if if every subset of X which contains a nonvoid member of $\mathcal{T}$ is also in $\mathcal{T}$. That is, $\mathcal{E} \subseteq \mathcal{T}$ in our former notation.

Now, following Dontchev [38], a connected superset topology $\mathcal{T}$ on X may be naturally called *superconnected*. The importance of this notion lies in the fact that a topology $\mathcal{T}$ on X is superconnected if and only if $\mathcal{E} = \mathcal{T} \setminus \{\emptyset\}$.

Moreover, by Bourbaki [18, p. 139] and Hewitt [73], a topology $\mathcal{T}$ on X may be naturally called *submaximal* and *resolvable* if $\mathcal{D} \subseteq \mathcal{T}$ and $\mathcal{D} \setminus \mathcal{E} \neq \emptyset$, respectively. Namely, $\mathcal{D} \setminus \mathcal{E} \neq \emptyset$ if and only if $A^c \in \mathcal{D}$ for some $A \in \mathcal{D}$.

For the various connectedness properties, also the real line $\mathbb{R}$ is the main source of intuition. (Recall that its usual topology can be derived from both an order and a metric.) However, to make nice pictures, one can rather use the plane $\mathbb{C} = \mathbb{R}^2$.

2. A few basic facts on relations

In the sequel, the reader will actually be assumed to be acquainted only with the most basic notions and notations concerning the elements of a fixed *ground set* X and its *power set* $\mathcal{P}(X) = \{A : \quad A \subseteq X\}$.

For any $a, b \in X$, the sets $\{a\} = \{x \in X : \ x = a\}$, $\{a, b\} = \{a\} \cup \{b\}$ and $(a, b) = \big\{ \{a\}, \{a, b\} \big\}$ are called the *singlelon, doubleton* and *ordered pair* formed from the elements a and b, respectively.

For any two sets X and Y, the set $X \times Y = \{ (x, y) : \ x \in X, \ y \in Y \}$ is called the *Cartesian product* of the sets X and Y. And, any subset F of the product set $X \times Y$ is called a *relation on X to Y*.

In particular, a relation on X to itself is called a *relation on X*. And, for instance, the sets $\Delta_X = \{ (x, x) : \ x \in X \}$ and $X^2 = X \times X$ are called the *identity and universal relations on X*, respectively.

If F is a relation on X to Y, then by the above definitions we can also state that F is a relation on $X \cup Y$. However, for several purposes, the latter view of the relation F would be quite unnatural.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{ y \in Y : \ (x, y) \in F \}$ and $F[A] = \bigcup \{ F(x) : \ x \in A \}$ are called the *images of x and A under F,* respectively.

If $(x, y) \in F$, then instead of $y \in F(x)$, we may also write $x \, F \, y$. However, instead of $F[A]$, we cannot write $F(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

Now, the sets $D_F = \{ x \in X : \ F(x) \neq \emptyset \}$ and $R_F = F[X]$ may be called the *domain* and *range* of F, respectively. If in particular $D_F = X$, then we may say that F is a *relation of X to Y*, or that F is a *non-partial relation on X to Y*.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation on X*. While, a function $*$ of X^2 to X is called a *binary operation on X*. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

If x is a function of a set I to X, then by using the values $x_i = x(i)$, with $i \in I$, we also define $(x_i)_{i \in I} = x$ and $\{x_i\}_{i \in I} = x[I]$. Thus, the function x may also be considered as an *indexed family* of elements of X.

Now, for an indexed family $\big(A_i\big)_{i \in I}$ of subsets of X, the sets $\bigcap_{i \in I} A_i = \{ x \in X : \ \forall \ i \in I : \ x \in A_i \}$ and $\bigcup_{i \in I} A_i = \{ x \in X : \ \exists \ i \in I : \ x \in A_i \big\}$ may be called the *intersection and union* of the sets A_i, respectively.

Moreover, by denoting by X^I the family of all functions of I of to X, the set $\prod_{i \in I} A_i = \big\{ x \in X^I : \ \forall \ i \in I : \ x_i \in A_i \big\}$ may be called the *Cartesian product* of the sets A_i. Thus, in particular, we also have $X^I = \prod_{i \in I} X$.

If F is a relation on X to Y, then we can easily see that $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement* F^c and the *inverse* F^{-1} can be defined such that $F^c(x) = Y \setminus F(x)$ for all $x \in X$ and $F^{-1}(y) = \{ x \in X : \ y \in F(x) \}$ for all $y \in Y$. Thus, we also have $F^c = X \times Y \setminus F$ and $F^{-1} = \{ (y, x) \in Y \times X : \ (x, y) \in F \}$.

Moreover, if in addition G is a relation on Y to Z, then the *composition* $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $G \circ F = \{ (x, z) \in X \times Z : \ \ \exists \ y \in Y : \ (x, y) \in F, \ \ (y, z) \in G \}$.

While, if G is a relation on Z to W, then the *box product* $F \boxtimes G$ can be naturally defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Note that the box product can be defined for any family of relations.

If F is a relation on X to Y, then a relation Φ of D_F to Y is called a *selection relation of* F if $\Phi \subseteq F$, i.e., $\Phi(x) \subseteq F(x)$ for all $x \in D_F$. By using the Axiom of Choice, it can be seen that every relation is the union of its selection functions.

For a relation F on X to Y, we may naturally define two *set-valued functions* φ_F of X to $\mathcal{P}(Y)$ and Φ_F of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_F(x) = F(x)$ for all $x \in X$ and $\Phi_F(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more general objects than relations on X to Y. They were briefly called *corelations* on X to Y in [194, 204, 205].

Now, a relation R on X may be briefly defined to be *reflexive* on X if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For $A \subseteq X$, *Pervin's relation* $R_A = A^2 \cup A^c \times X$, with $A^c = X \setminus A$, is an important preorder on X. While, for a *pseudometric* d on X, *Weil's surrounding* $B_r = \{(x, y) \in X^2 : \ d(x, y) < r\}$, with $r > 0$, is an important tolerance on X.

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap \left(A^c\right)^2$ is already an equivalence on X. And, more generally if $\mathcal{A}$ is a *cover (partition)* of X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

Now, for any relation R on X, we may also naturally define $R^0 = \Delta_X$ and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may naturally define $R^\infty = \bigcup_{n=0}^\infty R^n$. Thus, R^∞ is the smallest preorder relation on X containing R [65].

3. A few basic facts on ordered sets

If $\leq$ is a relation on X, then motivated by Birkhoff [13, p. 1] the ordered pair $X(\leq) = (X, \leq)$ is called a *goset* (generalized ordered set) [184]. In particular, it is called a *proset* (preordered set) if the relation $\leq$ is a preorder on X.

Quite similarly, a goset $X(\leq)$ is called a *poset* (partially ordered set) if the relation $\leq$ is a partial order on X. The importance of posets lies mainly in the fact that any family of sets forms a poset with set inclusion.

A function f of one goset $X(\leq)$ to another $Y(\leq)$ is called *increasing* if $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in X$. The function f can now be briefly called *decreasing* if it is increasing as a function of $X(\leq)$ to the dual $Y(\geq)$.

Moreover, a unary operation φ on a goset $X = X(\leq)$ is called *extensive, intensive, involutive and idempotent* if, under our former notations, $\varphi^0 = \Delta_X$ and $\varphi^2 = \varphi \circ \varphi$, we have $\varphi^0 \leq \varphi$, $\varphi \leq \varphi^0$, $\varphi^2 = \varphi^0$ and $\varphi^2 = \varphi$, respectively.

In particular, an increasing extensive (intensive) operation is called a *preclosure (preinterior) operation*. And, an idempotent preclosure (preinterior) operation is called a *closure (interior) operation*.

Moreover, an extensive (intensive) idempotent operation is called a *semiclosure (semiinterior) operation*. And, an increasing involutive (idempotent) operation is called an *involution (projection) operation*.

If f is a function of one goset X to another Y and g is a function of Y to X such that, for any $x \in X$ and $y \in Y$, we have $f(x) \leq y$ if and only if $x \leq g(y)$, then we say that f and g form a *Galos connection* between X and Y [33].

While, if f is a function of one goset X to another Y and φ is an unary operation on X such that, for any $u, v \in X$, we have $f(u) \leq f(v)$ if and only if $u \leq \varphi(v)$, then we say that f and φ form a *Pataki connection* between X and Y [184].

If f and g form a Galois connection between X and Y, then we also say that f is a *g–normal function* of X to Y. While, if f and φ form a Pataki connection between X and Y, then we also say that f is a *φ–regular function* of X to Y.

Thus, if f is a g–normal function of X to Y and $\varphi = g \circ f$, then we can at once see that $f(u) \leq f(v) \iff u \leq g\big(f(v)\big) \iff u \leq (g \circ f)(v) \iff u \leq \varphi(v)$ for all $u, v \in X$. Therefore, f is φ-regular.

Conversely, if f is a φ–regular function of X onto Y and g is a function of Y to X such that $\varphi = g \circ f$, then we can quite similarly see that f is g–normal. Thus, regular functions are somewhat less general than the normal ones.

However, if f is a φ–regular function of one proset X to another Y, then we can already prove that f is increasing, φ is a closure operation on X and $f = f \circ \varphi$. Therefore, Pataki connections have to be investigated before the Galois ones.

In practical situations, we usually have an increasing function f of one poset X to another Y, and try to find a function g of Y to X (or at least an unary operation φ on X) such that f could be g–normal (φ–regular).

Galois and Pataki connections occur in almost every branches of mathematics. They allow of transposing notions and statements from one world of our imagination to another one. (For their theories and applications, see [14, 64, 59, 33, 36].)

Some examples and generalizations of Galois and Pataki connections can also be found in [179, 183, 20, 194, 198, 202] and [190, 206, 209, 199, 203]. However, it is frequently enough to consider such connections only for corelations.

For any corelation F on X to Y, we can easily define a corelation G on Y to X such that $G(B) = \big\{x \in X : \ F\big(\{x\}\big) \subseteq B\big\}$. And, we can try to find conditions on the corelation F in order that F could be G–normal.

However, in the sequel, we shall only investigate the operation Φ defined by $\Phi(A) = \big(G \circ F\big)(A) = \big\{x \in X : \ F\big(\{x\}\big) \subseteq F(A)\big\}$ for all $A \subseteq X$. Namely, if F is union-preserving, then it will be a compatible closure operation on X.

4. A few basic facts on relators

Instead of open sets, Hausdorff [71], Kuratowski [85], Weil [215], Tukey [214], Efremovič and Švarc [46, 47], Kowalsky [83], Császár [27], Doičinov [37], Herrlich [72] and others [156, 76, 22, 125] offered some more powerful tools.

For instance, from the works of Davis [34], Pervin [139] and Hunsaker and Lindgren [74], it should have been completely clear that topologies, closures and proximities should not be studied without generalized uniformities.

Considering several papers and some books on generalized uniformities and their induced structures, the second author in [162, 178, 180] offered *relators* (families of relations) as the most suitable basic term on which analysis should be based on.

Thus, if $\mathcal{R}$ is a family of relations on X to Y $\big($i. e., $\mathcal{R} \subseteq \mathcal{P}(X \times Y)\big)$, then $\mathcal{R}$ is called a *relator* on X to Y, and the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space*.

If in particular $\mathcal{R}$ is a relator on X to itself, then $\mathcal{R}$ is simply called a relator on X. Moreover, by identifying singletons with their elements, we write $X(\mathcal{R})$ instead of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \big\{\{X\}, \{X, X\}\big\} = \big\{\{X\}\big\}$.

A relator $\mathcal{R}$ on X to Y, or a relator space $(X, Y)(\mathcal{R})$, is called *simple* if $\mathcal{R} = \{R\}$ for some relation R. Simple relator spaces $X(R)$ and $(X, Y)(R)$ were called *gosets* and *formal context* in [197] and [59], respectively.

Moreover, a relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive on X. Thus, we may also naturally speak of *preorder, tolerance*, and *equivalence relators.*

For instance, for a family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{ R_A : \ A \in \mathcal{A}\}$, where $R_A = A^2 \cup A^c \times X$, is an important preorder relator on X. Such relators were first used by Davis [34], Pervin [139] and Levine [99].

While, for a family $\mathcal{D}$ of *pseudo-metrics* on X, the family $\mathcal{R}_{\mathcal{D}} = \{B_r^d : \ r > 0, d \in \mathcal{D}\}$, where $B_r^d = \{(x, y) : \ d(x, y) < r\}$, is an important tolerance relator on X. Such relators were first considered by Weil [215].

Moreover, if $\mathfrak{S}$ is a family of *partitions* of X, then the family $\mathcal{R}_{\mathfrak{S}} = \{ S_{\mathcal{A}} : \mathcal{A} \in \mathfrak{S}\}$, where $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$, is an equivalence relator on X. Such practically important relators were first investigated by Levine [98].

If $\star$ is a unary operation for relations on X to Y, then for any relator $\mathcal{R}$ on X to Y we may naturally define $\mathcal{R}^{\star} = \big\{ R^{\star} : \ R \in \mathcal{R}\big\}$. However, this plausible notation may cause confusions whenever, for instance, $\star = c$.

In particular, for any relator $\mathcal{R}$ on X, we may naturally define $\mathcal{R}^{\infty} = \big\{ R^{\infty} : R \in \mathcal{R}\big\}$. Moreover, we may also naturally define $\mathcal{R}^{\partial} = \big\{ S \subseteq X^2 : \ S^{\infty} \in \mathcal{R}\big\}$. Namely, thus the operations ∞ and ∂ form a Galois connection.

Quite similarly, if $*$ is a binary operation for relations, then for any two relators $\mathcal{R}$ and $\mathcal{S}$ we may naturally define $\mathcal{R} * \mathcal{S} = \big\{ R * S : \ R \in \mathcal{R}, \ S \in \mathcal{S}\big\}$. However, this plausible notation may again cause confusions whenever, for instance, $* = \cup$.

Therefore, in the sequel we shall rather write $\mathcal{R} \vee \mathcal{S} = \big\{R \cup S : \ R \in \mathcal{R}, \ S \in \mathcal{S}\big\}$. Moreover, for instance, we shall also write $\mathcal{R} \triangledown \mathcal{R}^{-1} = \big\{ R \cup R^{-1} : \ R \in \mathcal{R}\big\}$. Note that thus $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is a symmetric relator such that $\mathcal{R} \triangledown \mathcal{R}^{-1} \subseteq \mathcal{R} \vee \mathcal{R}^{-1}$.

A function $\square$ of the family of all relators on X to Y is called a *direct (indirect) unary operation for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^{\square} = \square(\mathcal{R})$ is a relator on X to Y (on Y to X).

For instance, c and -1 are involution operations for relators. While, ∞ and ∂ are projection operations for relators. Moreover, the operation $\square = c$, ∞ or ∂ is inversion compatible in the sense that $\mathcal{R}^{\square -1} = \mathcal{R}^{-1 \square}$.

More generally, a function $\mathfrak{F}$ of the family of all relators on X to Y is called a *structure for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ is in a power set depending only on X and Y.

For instance, if $\operatorname{cl}_{\mathcal{R}}(B) = \bigcap \{R^{-1}[B] : \ R \in \mathcal{R}\}$ for every relator $\mathcal{R}$ on X to Y and $B \subseteq Y$, then the function $\mathfrak{F}$, defined by $\mathfrak{F}_{\mathcal{R}} = \operatorname{cl}_{\mathcal{R}}$, is a structure for relators such that $\mathfrak{F}_{\mathcal{R}} \subseteq \mathcal{P}(Y) \times X$, and thus $\mathfrak{F}_{\mathcal{R}} \in \mathcal{P}\big(\mathcal{P}(Y) \times X\big)$.

A structure $\mathfrak{F}$ for relators is called *increasing* if $\mathcal{R} \subseteq \mathcal{S}$ implies $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and S on X to Y. And, $\mathfrak{F}$ is called *quasi-increasing* if $R \in \mathcal{R}$ implies $\mathfrak{F}_R = \mathfrak{F}_{\{R\}} \subseteq \mathfrak{F}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X to Y.

Moreover, the structure $\mathfrak{F}$ is called *union-preserving* if $\mathfrak{F}_{\bigcup_{i \in I} \mathcal{R}_i} = \bigcup_{i \in I} \mathfrak{F}_{\mathcal{R}_i}$ for any family $(\mathcal{R}_i)_{i \in I}$ of relators on X to Y. It can be shown that $\mathfrak{F}$ is union-preserving if and only if $\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_R$ for any relator $\mathcal{R}$ on X to Y [194].

By using Pataki connections, important closure operations for relators can be derived from union-preserving structures. However, more generally, one can also find first the Galois adjoint $\mathfrak{G}$ of such a structure $\mathfrak{F}$, and then take $\square_{\mathfrak{F}} = \mathfrak{G} \circ \mathfrak{F}$.

5. The induced proximal closure and interior

Notation 5.1. In this and the next two sections, we shall assume that $\mathcal{R}$ is a relator on X to Y.

Definition 5.2. For any $A \subseteq X$ and $B \subseteq Y$, we write:

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subseteq B$ for some $R \in \mathcal{R}$;

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$.

Thus, $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ are relations on $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ which are called the *proximal closure and interior relations* generated by $\mathcal{R}$, respectively.

Remark 5.3. The origins of these relations go back to Efremović's proximity δ [46] and Smirnov's strong inclusion $\Subset$ [157], respectively.

While, the convenient notations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$, instead of the more usual ones $\delta_{\mathcal{R}}$ and $\Subset_{\mathcal{R}}$, were first used by the second author in [162, 169, 167, 170].

For an easy illustration of the relation $\operatorname{Cl}_{\mathcal{R}}$, we can at once state

Example 5.4. If d is a function of $X \times Y$ to $[0, +\infty]$, and

$$\mathcal{R}_d = \{ B_r^d : \ r > 0 \} \qquad \text{with} \qquad B_r^d = \{ (x, y) \in X \times Y : \ d(x, y) < r \},$$

then for any $A \subseteq X$ and $B \subseteq Y$ we have

$$A \in \operatorname{Cl}_{\mathcal{R}_d}(B) \iff d(A, B) = 0,$$

with $d(A, B) = \inf \{ d(x, y) : \ x \in A, \ y \in B \}$. That is, A is near to B.

The forthcoming simple, but important theorems have been proved in several former papers on relators written by the second author and his former PhD students.

Theorem 5.5. *For any $B \subseteq Y$, we have*

(1) $\operatorname{Cl}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \operatorname{Int}_{\mathcal{R}}(Y \setminus B)$; *(2)* $\operatorname{Int}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \operatorname{Cl}_{\mathcal{R}}(Y \setminus B)$.

Remark 5.6. By using appropriate complementations, assertion (1) can be written in the more concise form that $\operatorname{Cl}_{\mathcal{R}} = \left(\operatorname{Int}_{\mathcal{R}} \circ \mathcal{C}_Y \right)^c = \left(\operatorname{Int}_{\mathcal{R}} \right)^c \circ \mathcal{C}_Y$.

Theorem 5.7. *We have*

(1) $\operatorname{Cl}_{\mathcal{R}^{-1}} = \operatorname{Cl}_{\mathcal{R}}^{-1}$; *(2)* $\operatorname{Int}_{\mathcal{R}^{-1}} = \mathcal{C}_Y \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}_X$.

Theorem 5.8. *We have*

(1) $\operatorname{Cl}_{\mathcal{R}}(\emptyset) = \emptyset$ *and* $\operatorname{Cl}_{\mathcal{R}}^{-1}(\emptyset) = \emptyset$ *if* $\mathcal{R} \neq \emptyset$;

(2) $\operatorname{Cl}_{\mathcal{R}}(B_1) \subseteq \operatorname{Cl}_{\mathcal{R}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$ *and* $\operatorname{Cl}_{\mathcal{R}}^{-1}(A_1) \subseteq \operatorname{Cl}_{\mathcal{R}}^{-1}(A_2)$ *if* $A_1 \subseteq A_2 \subseteq X$.

Theorem 5.9. *We have*

(1) $\operatorname{Int}_{\mathcal{R}}(X) = \mathcal{P}(X)$ *and* $\operatorname{Int}_{\mathcal{R}}^{-1}(\emptyset) = \mathcal{P}(Y)$ *if* $\mathcal{R} \neq \emptyset$;

(2) $\operatorname{Int}_{\mathcal{R}}(B_1) \subseteq \operatorname{Int}_{\mathcal{R}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$ *and* $\operatorname{Int}_{\mathcal{R}}^{-1}(A_2) \subseteq \operatorname{Int}_{\mathcal{R}}^{-1}(A_1)$ *if* $A_1 \subseteq A_2 \subseteq X$.

Remark 5.10. Conversely, it can be shown that, for any such relation Int on $\mathcal{P}(Y)$ to $\mathcal{P}(X)$, there exists a nonvoid relator $\mathcal{R}$ on X to Y such that $\operatorname{Int} = \operatorname{Int}_{\mathcal{R}}$. (See [169].) Thus, generalized proximity relations should not be studied without generalized uniformities.

Theorem 5.11. *We have*

(1) $\operatorname{Cl}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \operatorname{Cl}_R$; (2) $\operatorname{Int}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \operatorname{Int}_R$.

Corollary 5.12. *The mapping*

(1) $\mathcal{R} \mapsto \operatorname{Cl}_{\mathcal{R}}$ *is intersection-preserving;* (2) $\mathcal{R} \mapsto \operatorname{Int}_{\mathcal{R}}$ *is union-preserving.*

6. The induced topological closure and interior

Definition 6.1. In particular, for any $x \in X$ and $B \subseteq Y$, we write:

(1) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$; (2) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$.

Thus, $\operatorname{cl}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ are relations on $\mathcal{P}(Y)$ to X which are called the *topological closure and interior relations* generated by the relator $\mathcal{R}$, respectively.

Now, by specializing Definition 5.1 and our former observations, we can easily establish the following facts.

Example 6.2. Under the notations of Example 5.4, for any $x \in X$ and $B \subseteq Y$, we have

$$x \in \operatorname{cl}_{\mathcal{R}}(B) \iff d(x, B) = 0,$$

with $d(x, B) = d\big(\{x\}, B\big)$. That is, x is near to B.

Theorem 6.3. *For any $x \in X$ and $B \subseteq Y$, we have*

(1) $x \in \operatorname{int}_{\mathcal{R}}(B)$ *if and only if* $R(x) \subseteq B$ *for some* $R \in \mathcal{R}$;

(2) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ *if and only if* $R(x) \cap B \neq \emptyset$ *for all* $R \in \mathcal{R}$.

Corollary 6.4. *For any $A \subseteq X$ and $B \subseteq Y$,*

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ *implies that* $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$;

(2) $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *implies that* $A \in \operatorname{Cl}_{\mathcal{R}}(B)$.

Remark 6.5. Clearly, the converse implications need not be true. Thus, the relations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ are, in general, more powerful tools than $\operatorname{cl}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$.

Theorem 6.6. *For any $B \subseteq Y$, we have*

(1) $\operatorname{cl}_{\mathcal{R}}(B) = X \setminus \operatorname{int}_{\mathcal{R}}(Y \setminus B)$; (2) $\operatorname{int}_{\mathcal{R}}(B) = X \setminus \operatorname{cl}_{\mathcal{R}}(Y \setminus B)$.

Remark 6.7. By using appropriate complementations, assertion (1) can be written in the more concise form that $\operatorname{cl}_{\mathcal{R}} = \big(\operatorname{int}_{\mathcal{R}} \circ \mathcal{C}_Y\big)^c = \big(\operatorname{int}_{\mathcal{R}}\big)^c \circ \mathcal{C}_Y$.

Theorem 6.8. *We have*

(1) $\operatorname{cl}_{\mathcal{R}}(\emptyset) = \emptyset$ *if* $\mathcal{R} \neq \emptyset$;

(2) $\operatorname{cl}_{\mathcal{R}}(B_1) \subseteq \operatorname{cl}_{\mathcal{R}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$.

Theorem 6.9. *We have*

(1) $\operatorname{int}_{\mathcal{R}}(X) = X$ *if* $\mathcal{R} \neq \emptyset$;

(2) $\operatorname{int}_{\mathcal{R}}(B_1) \subseteq \operatorname{int}_{\mathcal{R}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$.

Remark 6.10. Conversely, it can be shown that, for any such relation int on $\mathcal{P}(Y)$ to X, there exists a nonvoid relator $\mathcal{R}$ on X to Y such that $\operatorname{int} = \operatorname{int}_{\mathcal{R}}$. (See again [169].) Thus, generalized closure operations should not also be studied without generalized uniformities.

Theorem 6.11. *We have*

(1) $\operatorname{cl}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \operatorname{cl}_R$; (2) $\operatorname{int}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \operatorname{int}_R$.

Corollary 6.12. *The mapping*

(1) $\mathcal{R} \mapsto \operatorname{cl}_{\mathcal{R}}$ *is intersection-preserving;* (2) $\mathcal{R} \mapsto \operatorname{int}_{\mathcal{R}}$ *is union-preserving.*

Concerning the relations $\operatorname{cl}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$, we can also prove the following

Theorem 6.13. *For any* $B \subseteq Y$, *we have*

(1) $\operatorname{cl}_{\mathcal{R}}(B) = \bigcap_{R \in \mathcal{R}} R^{-1}[B]$; (2) $\operatorname{int}_{\mathcal{R}}(B) = \bigcup_{R \in \mathcal{R}} R^{-1}[B^c]^c$.

Corollary 6.14. *For any* $R \in \mathcal{R}$, $A \subseteq X$ *and* $B \subseteq Y$, *we have*

$$A \subseteq \operatorname{int}_R(B) \iff \operatorname{cl}_{R^{-1}}(A) \subseteq B.$$

Remark 6.15. This corollary shows that the mappings

$$A \mapsto \operatorname{cl}_{R^{-1}}(A) \qquad \text{and} \qquad B \mapsto \operatorname{int}_R(B),$$

where $A \subseteq X$ and $B \subseteq Y$, form a Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

This important closure-interior Galois connection, introduced first in [196], and used in [202], is not independent from the more familiar upper and lower bound Galois connection mentioned in [187].

7. The induced fat and dense sets

Definition 7.1. For any $B \subseteq Y$, we write:

(1) $B \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$; (2) $B \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(B) = X$.

Thus, $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are families of subsets of Y whose members are called the *fat and dense sets* generated by the relator $\mathcal{R}$, respectively.

Remark 7.2. The importance of the dense sets is well–established in topology. However, the fat sets have formerly been explicitly used only by the second author in [166, 169, 178, 182, 201].

At a Topological Symposium [166], the second author tried to persuade the audience, without any success, that the fat and dense sets are, in general, much better tools than the topologically open and closed ones.

Now, by using the corresponding properties of the relations $\operatorname{cl}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$, we can easily establish the following theorems which, together with the results of the next section, will show the advantages of the fat and dense sets.

Theorem 7.3. *For any* $B \subseteq Y$, *we have*
(1) $B \in \mathcal{E}_{\mathcal{R}}$ *if and only if* $R(x) \subseteq B$ *for some* $x \in X$ *and* $R \in \mathcal{R}$;
(2) $B \in \mathcal{D}_{\mathcal{R}}$ *if and only if* $R(x) \cap B \neq \emptyset$ *for all* $x \in X$ *and* $R \in \mathcal{R}$.

Remark 7.4. Thus, in particular, we have $R(x) \in \mathcal{E}_{\mathcal{R}}$ and $R(x)^c \notin \mathcal{D}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$.

Moreover, by using the notation $\mathcal{U}_{\mathcal{R}}(x) = \operatorname{int}_{\mathcal{R}}^{-1}(x) = \{ B \subseteq Y : x \in \operatorname{int}_{\mathcal{R}}(B) \}$, we can see that $\mathcal{E}_{\mathcal{R}} = \bigcup_{x \in X} \mathcal{U}_{\mathcal{R}}(x)$.

Theorem 7.5. *For any* $B \subseteq Y$, *we have*
(1) $B \in \mathcal{D}_{\mathcal{R}}$ *if and only if* $X = R^{-1}[B]$ *for all* $R \in \mathcal{R}$;
(2) $B \in \mathcal{E}_{\mathcal{R}}$ *if and only if* $X \neq R^{-1}[B^c]$ *for some* $R \in \mathcal{R}$.

Hint. Recall that, by Theorem 6.13, we have $\operatorname{cl}_{\mathcal{R}}(B) = \bigcap_{R \in \mathcal{R}} R^{-1}[B]$ for all $B \subseteq Y$. Therefore, $\operatorname{cl}_{\mathcal{R}}(B) = X$ if and only if $R^{-1}[B] = X$ for all $R \in \mathcal{R}$.

Theorem 7.6. *For any* $B \subseteq Y$, *we have*
(1) $B \in \mathcal{D}_{\mathcal{R}} \iff B^c \notin \mathcal{E}_{\mathcal{R}}$; *(2)* $B \in \mathcal{E}_{\mathcal{R}} \iff B^c \notin \mathcal{D}_{\mathcal{R}}$.

Theorem 7.7. *For any* $B \subseteq Y$, *we have*
(1) $B \in \mathcal{D}_{\mathcal{R}}$ *if and only if* $B \cap E \neq \emptyset$ *for all* $E \in \mathcal{E}_{\mathcal{R}}$;
(2) $B \in \mathcal{E}_{\mathcal{R}}$ *if and only if* $B \cap D \neq \emptyset$ *for all* $D \in \mathcal{D}_{\mathcal{R}}$.

Hint. In principle this theorem can be derived from Theorem 7.6. However, it can be more easily proved with the help of Theorem 7.3.

Theorem 7.8. *We have*

(1) $\emptyset \notin \mathcal{D}_{\mathcal{R}}$ *if* $X \neq \emptyset$ *and* $\mathcal{R} \neq \emptyset$;

(2) $B \in \mathcal{D}_{\mathcal{R}}$ *and* $B \subseteq C \subseteq Y$ *imply* $C \in \mathcal{D}_{\mathcal{R}}$.

Theorem 7.9. *We have*

(1) $Y \in \mathcal{E}_{\mathcal{R}}$ *if* $X \neq \emptyset$ *and* $\mathcal{R} \neq \emptyset$;

(2) $B \in \mathcal{E}_{\mathcal{R}}$ *and* $B \subseteq C \subseteq Y$ *imply* $C \in \mathcal{E}_{\mathcal{R}}$.

Remark 7.10. Conversely, it can be shown that if $\mathcal{A}$ is a nonvoid, ascending family of subsets of a nonvoid set X, then there exists a nonvoid, preorder relator on X such that $\mathcal{A} = \mathcal{E}_{\mathcal{R}}$. (See [185].) Thus, stacks should not also be studied without generalized uniformities.

Theorem 7.11. *We have*

(1) $\mathcal{E}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathcal{E}_R$; *(2)* $\mathcal{D}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathcal{D}_R$.

Corollary 7.12. *The mapping*

(1) $\mathcal{R} \mapsto \mathcal{E}_{\mathcal{R}}$ *is union-preserving;* *(2)* $\mathcal{R} \mapsto \mathcal{D}_{\mathcal{R}}$ *is intersection-preserving.*

Concerning the families $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$, we can also easily prove the following more particular theorems.

Theorem 7.13. *The following assertions are equivalent:*

(1) $\emptyset \notin \mathcal{D}_{\mathcal{R}}$; *(2)* $\mathcal{E}_{\mathcal{R}} \neq \emptyset$;

(3) $Y \in \mathcal{E}_{\mathcal{R}}$; *(4)* $\mathcal{D}_{\mathcal{R}} \neq \mathcal{P}(Y)$; *(5)* $X \neq \emptyset$ *and* $\mathcal{R} \neq \emptyset$.

Theorem 7.14. *The following assertions are equivalent:*

(1) $\emptyset \notin \mathcal{E}_{\mathcal{R}}$; (2) $\mathcal{D}_{\mathcal{R}} \neq \emptyset$;

(3) $Y \in \mathcal{D}_{\mathcal{R}}$; (4) $\mathcal{E}_{\mathcal{R}} \neq \mathcal{P}(Y)$; (5) $X = R^{-1}[\,Y\,]$ *if* $R \in \mathcal{R}$.

Hint. Note that assertion (5), in a detailed form, means only that for any $x \in X$ and $R \in \mathcal{R}$ we have $x \in R^{-1}[\,Y\,]$. That is, there exists $y \in Y$ such that $x \in R^{-1}(y)$, i. e., $y \in R(x)$. Consequently, $R(x) \neq \emptyset$ for all $x \in X$ and $R \in \mathcal{R}$. That is, X is the domain of each member of $\mathcal{R}$.

Remark 7.15. If the assertions (5) of Theorems 7.13 and 7.14 hold, then the relator $\mathcal{R}$ on X to Y, or the relator space $(X, Y)(\mathcal{R})$, may be naturally called *non-degerated* and *non-partial*, respectively.

In addition to Theorem 7.13 and 7.14, it is also worth mentioning that if in particular $\mathcal{R}$ is $\mathcal{E}$–simple in the sense that $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_S$ for some relation S on X to Y, then the stack $\mathcal{E}_{\mathcal{R}}$ has a base $\mathcal{B}$ with $\operatorname{card}(\mathcal{B}) \leq \operatorname{card}(X)$. (See Pataki [134] .)

8. The induced open and closed sets

Notation 8.1. In this section, we shall already assume that $\mathcal{R}$ is a relator on X.

Definition 8.2. For any $A \subseteq X$, we write:

(1) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$; (2) $A \in \tau\!\!\!-_{\mathcal{R}}$ if $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$.

The members of the families $\tau_{\mathcal{R}}$ and $\tau\!\!\!-_{\mathcal{R}}$ are called the *proximally open and closed sets* generated by $\mathcal{R}$, respectively.

Remark 8.3. The families $\tau_{\mathcal{R}}$ and $\tau\!\!\!-_{\mathcal{R}}$ were first used by the second author in [167, 169] .

In particular, the practical notation $\tau\!\!\!-_{\mathcal{R}}$ has been suggested to the second author by János Kurdics.

By using the results of Section 5, we can easily prove the following theorems which, together with some forthcoming theorems, will show that the proximally open and closed sets are also better tools than the topologically open and closed ones.

Theorem 8.4. *For any* $A \subseteq X$, *we have*

(1) $A \in \tau_{\mathcal{R}}$ *if and only if* $R[\,A\,] \subseteq A$ *for some* $R \in \mathcal{R}$;

(2) $A \in \tau\!\!\!-_{\mathcal{R}}$ *if and only if* $A \cap R[\,A^c\,] = \emptyset$ *for some* $R \in \mathcal{R}$.

Theorem 8.5. *For any* $A \subseteq X$, *we have*

(1) $A \in \tau\!\!\!-_{\mathcal{R}} \iff A^c \in \tau_{\mathcal{R}}$; (2) $A \in \tau_{\mathcal{R}} \iff A^c \in \tau\!\!\!-_{\mathcal{R}}$.

Theorem 8.6. *We have*

(1) $\tau\!\!\!-_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$; (2) $\tau_{\mathcal{R}} = \tau\!\!\!-_{\mathcal{R}^{-1}}$.

Theorem 8.7. *If* $\mathcal{R} \neq \emptyset$, *then*

(1) $\{\emptyset, X\} \subseteq \tau_{\mathcal{R}}$; (2) $\{\emptyset, X\} \subseteq \tau\!\!\!-_{\mathcal{R}}$.

Remark 8.8. Conversely, it can be shown that if $\mathcal{A}$ is a family of subsets of X containing $\emptyset$ and X, then there exists a nonvoid, preorder relator $\mathcal{R}$ on X such that $\mathcal{A} = \tau_{\mathcal{R}}$. (See again [185].) Thus, minimal structures should not also be studied without generalized uniformities.

Theorem 8.9. *We have*

(1) $\tau_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \tau_R$; (2) $\tau\!\!\!-_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \tau\!\!\!-_R$.

Corollary 8.10. *The mappings $\mathcal{R} \mapsto \tau_{\mathcal{R}}$ and $\mathcal{R} \mapsto \tau\!\!\!-_{\mathcal{R}}$ are union-preserving.*

Definition 8.11. For any $A \subseteq X$, we write:

(1) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$; (2) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$.

The members of the families $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{F}_{\mathcal{R}}$ are called the *topologically open and closed sets* generated by $\mathcal{R}$, respectively.

By using the results of Section 6, we can easily prove the following theorems which will already indicate some disadvantages of the topologically open and closed sets.

Theorem 8.12. *For any $A \subseteq X$, we have*

(1) $A \in \mathcal{T}_{\mathcal{R}}$ if and only if for each $x \in A$ there exists $R \in \mathcal{R}$ such that $R(x) \subseteq A$;

(2) $A \in \mathcal{F}_{\mathcal{R}}$ if and only if for each $x \in A^c$ there exists $R \in \mathcal{R}$ such that $A \cap R(x) = \emptyset$.

Theorem 8.13. *For any $A \subseteq X$, we have*

(1) $A \in \mathcal{F}_{\mathcal{R}} \iff A^c \in \mathcal{T}_{\mathcal{R}}$; *(2)* $A \in \mathcal{T}_{\mathcal{R}} \iff A^c \in \mathcal{F}_{\mathcal{R}}$.

Theorem 8.14. *We have*

(1) $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$; *(2)* $\tau\!\!\!-_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Remark 8.15. In particular, for any $R \in \mathcal{R}$, we have

(1) $\tau_R = \mathcal{T}_R$; (2) $\tau\!\!\!-_R = \mathcal{F}_R$.

Theorem 8.16. *We have*

(1) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$; *(2)* $\mathcal{D}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{X\}$.

Remark 8.17. Hence, by using global complementations, we can easily infer that $\mathcal{F}_{\mathcal{R}} \subseteq \big(\mathcal{D}_{\mathcal{R}}\big)^c \cup \{X\}$ and $\mathcal{D}_{\mathcal{R}} \subseteq \big(\mathcal{F}_{\mathcal{R}}\big)^c \cup \{X\}$.

Theorem 8.18. *For any $A \subseteq X$ we have*

(1) $A \in \mathcal{E}_{\mathcal{R}}$ if $V \subseteq A$ for some $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$;

(2) $A \in \mathcal{D}_{\mathcal{R}}$ only if $A \setminus W \neq \emptyset$ for all $W \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$.

Hint. To prove (2), note that if $W \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$, then $W \in \mathcal{F}_{\mathcal{R}}$ and $W \neq X$. Therefore, $W^c \in \mathcal{T}_{\mathcal{R}}$ and $W^c \neq \emptyset$. Hence, by using Theorem 8.16, we can infer that $W^c \in \mathcal{E}_{\mathcal{R}}$. Therefore, if $A \in \mathcal{D}_{\mathcal{R}}$, then by Theorem 7.7 we necessarily have $A \setminus W = A \cap W^c \neq \emptyset$.

Theorem 8.19. *We have*

(1) $\emptyset \in \mathcal{F}_{\mathcal{R}}$ *if* $\mathcal{R} \neq \emptyset$; (2) $\mathcal{A} \subseteq \mathcal{F}_{\mathcal{R}}$ *implies* $\bigcap \mathcal{A} \in \mathcal{F}_{\mathcal{R}}$.

Theorem 8.20. *We have*

(1) $X \in \mathcal{T}_{\mathcal{R}}$ *if* $\mathcal{R} \neq \emptyset$; (2) $\mathcal{A} \subseteq \mathcal{T}_{\mathcal{R}}$ *implies* $\bigcup \mathcal{A} \in \mathcal{T}_{\mathcal{R}}$.

Remark 8.21. Conversely, it can be shown that if $\mathcal{A}$ is a family of subsets of X such that $X \in \mathcal{A}$ and $\mathcal{A}$ is closed under arbitrary unions, then there exists a nonvoid, preorder relator $\mathcal{R}$ on X such that $\mathcal{A} = \mathcal{T}_{\mathcal{R}}$. (See again [185] .) Thus, generalized topologies should not also be studied without generalized uniformities.

Unfortunately, in contrast to Theorems 5.11, 6.11, 7.11 and 8.9, we can only prove the following

Theorem 8.22. *The mappings* $\mathcal{R} \mapsto \mathcal{T}_{\mathcal{R}}$ *and* $\mathcal{R} \mapsto \mathcal{F}_{\mathcal{R}}$ *are increasing.*

Remark 8.23. Thus, in particular

(1) $\bigcup_{R \in \mathcal{R}} \mathcal{T}_R \subseteq \mathcal{T}_{\mathcal{R}}$; (2) $\bigcup_{R \in \mathcal{R}} \mathcal{F}_R \subseteq \mathcal{F}_{\mathcal{R}}$.

9. Further structures derived from relators

Notation 9.1. In this section, we shall assume that $\mathcal{R}$ is a relator on X to Y.

Now, according to a former paper of the second author [180] , we may also have

Definition 9.2. For for any $A \subseteq X$, $B \subseteq Y$, $x \in X$, and $y \in Y$ we write:

(1) $B \in \mathrm{Ub}_{\mathcal{R}}(A)$ and $A \in \mathrm{Lb}_{\mathcal{R}}(B)$ if $A \times B \subseteq R$ for some $R \in \mathcal{R}$;

(2) $y \in \mathrm{ub}_{\mathcal{R}}(A)$ if $\{y\} \in \mathrm{Ub}_{\mathcal{R}}(B)$; (3) $x \in \mathrm{lb}_{\mathcal{R}}(B)$ if $\{x\} \in \mathrm{Lb}_{\mathcal{R}}(A)$.

Remark 9.3. Thus, for instance, we evidently have $\mathrm{Lb}_{\mathcal{R}} = \mathrm{Ub}_{\mathcal{R}^{-1}} = \mathrm{Ub}_{\mathcal{R}}^{-1}$.

In particular, we may also naturally have the following

Definition 9.4. If $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we write:

(1) $\max_{\mathcal{R}}(A) = A \cap \mathrm{ub}_{\mathcal{R}}(A)$; (2) $\min_{\mathcal{R}}(A) = A \cap \mathrm{lb}_{\mathcal{R}}(A)$;

(3) $\sup_{\mathcal{R}}(A) = \min_{\mathcal{R}}\big(\mathrm{ub}_{\mathcal{R}}(A)\big)$; (4) $\inf_{\mathcal{R}}(A) = \max_{\mathcal{R}}\big(\mathrm{lb}_{\mathcal{R}}(A)\big)$.

(5) $\mathrm{Max}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \mathrm{Ub}_{\mathcal{R}}(A)$; (6) $\mathrm{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \mathrm{Lb}_{\mathcal{R}}(A)$,

(7) $\mathrm{Sup}_{\mathcal{R}}(A) = \mathrm{Min}_{\mathcal{R}}\big(\mathrm{Lb}_{\mathcal{R}}(A)\big)$; (8) $\mathrm{Inf}_{\mathcal{R}}(A) = \mathrm{Max}_{\mathcal{R}}\big(\mathrm{Ub}_{\mathcal{R}}(A)\big)$.

Remark 9.5. Thus, for instance, it can be shown that

$$A \in \mathrm{Ub}_{\mathcal{R}}(A) \iff A \in \mathrm{Lb}_{\mathcal{R}}(A) \iff A \in \mathrm{Min}_{\mathcal{R}}(A) \iff A \in \mathrm{Inf}_{\mathcal{R}}(A) .$$

The following theorem, proved first in [180] , shows that the present algebraic structures are not independent of the former topological ones.

Theorem 9.6. *We have*

(1) $\mathrm{lb}_{\mathcal{R}} = \mathrm{int}_{\mathcal{R}^c} \circ \mathcal{C}_Y$; (2) $\mathrm{int}_{\mathcal{R}} = \mathrm{lb}_{\mathcal{R}^c} \circ \mathcal{C}_Y$;

(3) $\mathrm{Lb}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y$; (4) $\mathrm{Int}_{\mathcal{R}} = \mathrm{Lb}_{\mathcal{R}^c} \circ \mathcal{C}_Y$.

Proof. For any $R \in \mathcal{R}$, $A \subseteq X$ and $B \subseteq Y$ we have

$$A \times B \subseteq R \iff \forall\, a \in A : \ B \subseteq R(a) \iff \forall\, a \in A : \ R(a)^c \subseteq B^c$$
$$\iff \forall\, a \in A : \ R^c(a) \subseteq B^c \iff R^c[\,A\,] \subseteq B^c.$$

Hence, by the corresponding definitions, it is clear we also have

$$A \in \mathrm{Lb}_{\mathcal{R}}(B) \iff A \in \mathrm{Int}_{\mathcal{R}^c}(\,B^c\,) \iff A \in \big(\mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y\,\big)(B)\,.$$

Therefore, assertion (3), and thus in particular (1) is also true. □

Remark 9.7. By our former results, it is clear that the relations $\mathrm{Cl}_{\mathcal{R}}$, $\mathrm{Int}_{\mathcal{R}}$, $\mathrm{Ub}_{\mathcal{R}}$ and $\mathrm{Lb}_{\mathcal{R}}$ are equivalent tools in the relator space $(X, Y)(\mathcal{R})$.

In this respect, it is worth mentioning that, by using nets instead of sets, we can define some much stronger tools in the relator space $(X, Y)(\mathcal{R})$.

Definition 9.8. A function x of a preordered set $\Gamma = \Gamma(\leq)$ to the set X will be called a Γ–*net* in X. And, for any $A \subseteq X$, we shall say that:

(1) x is *fatly* in A if $x^{-1}[\,A\,]$ is a fat subset of Γ;

(2) x is *densely* in A if $x^{-1}[\,A\,]$ is a dense subset of Γ.

Remark 9.9. Note that, by definition, $x^{-1}[\,A\,]$ is a fat subset of Γ if and only if $x^{-1}[\,A\,] \in \mathcal{E}_{\leq}$. That is, by Theorem 7.3, there exists $\alpha \in \Gamma$ such $\leq(\alpha) \subseteq x^{-1}[\,A\,]$, i.e., $[\,\alpha, +\infty[\, \subseteq x^{-1}[\,A\,]$. That is, for each $\beta \geq \alpha$, we have $\beta \in x^{-1}[\,A\,]$, i.e., $x_\beta = x(\beta) \in A$.

And quite similarly, $x^{-1}[\,A\,]$ is a dense subset of Γ if and only if for each $\alpha \in \Gamma$ there exists $\beta \geq \alpha$ such that $x_\beta \in A$. Therefore, instead of the terms "fatly" and "densely", we could also use the generally accepted terms "eventually" and "frequently".

Now, extending the ideas of Efremović and Švarc [47] and the second author [161, 162], we may also naturally have the following

Definition 9.10. For any two Γ-nets x in X and y in Y, and $a \in X$, we write:

(1) $x \in \mathrm{Lim}_{\mathcal{R}}(y)$ if the net (x, y) is fatly in each $R \in \mathcal{R}$;

(2) $x \in \mathrm{Adh}_{\mathcal{R}}(y)$ if the net (x, y) is densely in each $R \in \mathcal{R}$;

(3) $a \in \mathrm{lim}_{\mathcal{R}}(y)$ if $a_\Gamma \in \mathrm{Lim}_{\mathcal{R}}(y)$; (4) $a \in \mathrm{adh}_{\mathcal{R}}(y)$ if $a_\Gamma \in \mathrm{Adh}_{\mathcal{R}}(y)$;

where a_Γ means now the constant net $(a)_{\alpha \in \Gamma} = \Gamma \times \{a\}$.

Remark 9.11. Thus, by Remark 9.8 and the equalities

$$(x, y)_\alpha = (x, y)(\alpha) = \big(\,x(\alpha),\, y(\alpha)\big) = (x_\alpha, y_\alpha)\,,$$

we have $x \in \mathrm{Adh}_{\mathcal{R}}(y)$ if and only if, for each $R \in \mathcal{R}$ and $\alpha \in \Gamma$, there exists $\beta \geq \alpha$ such that $(\,x_\beta, y_\beta) \in R$.

Moreover, for an easy illustration of the relation $\mathrm{Adh}_{\mathcal{R}}$, we can also state

Example 9.12. If d is a function of $X \times Y$ to $[0, +\infty\,]$, and

$$\mathcal{R}_d = \big\{\, B_r^d : \ \ r > 0 \,\big\} \qquad \text{with} \qquad B_r^d = \{\,(x, y) \in X \times Y : \ \ d\,(x, y) < r\,\}\,,$$

then for any two nets x and y in X and Y, respectively, we have

$$x \in \operatorname{Adh}_{\mathcal{R}_d}(y) \iff \underline{\lim}_{\alpha \to +\infty} d\,(x_\alpha, y_\alpha) = 0\,.$$

Remark 9.13. Definitions 9.7 and 9.9 can be extended to the more general case when $\Gamma = \Gamma(\mathcal{U})$ is an arbitrary relator space and x and y are relations on Γ to X and Y, respectively, in two natural ways.

Namely, in the latter case, beside the set $x^{-1}[A] = \{\alpha \in \Gamma : \; x(\alpha) \cap A \neq \emptyset\}$, we may also naturally consider the set $x^{-1}[A^c]^c = \{\alpha \in \Gamma : \; x(\alpha) \subseteq A\}$.

However, to express the relation $\operatorname{Cl}_{\mathcal{R}}$ in term of the relation $\operatorname{Lim}_{\mathcal{R}}$, preordered nets are sufficient. Namely, we can prove the following

Theorem 9.14. *For any $A \subseteq X$ and $B \subseteq Y$, we have $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if and only if there exist nets x in A and y in B such that $x \in \operatorname{Lim}_{\mathcal{R}}(y)$ $\big(x \in \operatorname{Adh}_{\mathcal{R}}(y)\big)$.*

Corollary 9.15. *For any $a \in X$ and $B \subseteq Y$, we have $a \in \operatorname{cl}_{\mathcal{R}}(B)$ if and only if there exist a net y in B such that $a \in \lim_{\mathcal{R}}(y)$ $\big(a \in \operatorname{adh}_{\mathcal{R}}(y)\big)$.*

Moreover, it is also worth noticing that we also have the following

Theorem 9.16. *For any Γ–net y in Y, we have:*

(1) $\lim_{\mathcal{R}}(y) = \bigcap_{R \in \mathcal{R}} \underline{\lim}_{\alpha \to \infty} R^{-1}(y_\alpha)$; *(2)* $\operatorname{adh}_{\mathcal{R}}(y) = \bigcap_{R \in \mathcal{R}} \overline{\lim}_{\alpha \to \infty} R^{-1}(y_\alpha)$.

Remark 9.17. By Definition 9.10, it is clear that

$$\operatorname{Lim}_{\mathcal{R}}(y) = \bigcap_{R \in \mathcal{R}} \operatorname{Lim}_R(y) \qquad \text{and} \qquad \lim_{\mathcal{R}}(y) = \bigcap_{R \in \mathcal{R}} \lim_R(y)\,.$$

Thus, in particular the net y may, for instance, be naturally called *convergence Cauchy* if $\lim_R(y) \neq \emptyset$ for all $R \in \mathcal{R}$. Note that in this case y need not be convergent in the sense that $\lim_{\mathcal{R}}(y) \neq \emptyset$.

10. Regular structures for relators

Notation 10.1. In this and the next section, we shall assume that $\mathfrak{F}$ is a structure and $\square$ is a unary operation for relators on X to Y.

In accordance with our former terminology, we shall use the following

Definition 10.2. We say that:

(1) $\mathfrak{F}$ is *upper $\square$–semiregular* if $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$ implies $\mathcal{R} \subseteq \mathcal{S}^{\square}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y;

(2) $\mathfrak{F}$ is *lower $\square$–semiregular* if $\mathcal{R} \subseteq \mathcal{S}^{\square}$ implies $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y.

Remark 10.3. Now, the structure $\mathfrak{F}$ may be naturally called *$\square$–regular* if it is both upper and lower $\square$–semiregular.

In this case, because of the fundamental work of Pataki [135], we may also say that $\mathfrak{F}$ and $\square$ form a Pataki connection.

Recall that Pataki connections should actually be derived from the corresponding Galois ones. However, in the sequel, we shall not need such Galois connections.

Definition 10.4. For any relator $\mathcal{R}$ on X to Y, we define

$$\mathcal{R}^{\Box_{\mathfrak{F}}} = \left\{ S \subseteq X \times Y : \quad \mathfrak{F}_S \subseteq \mathfrak{F}_{\mathcal{R}} \right\}.$$

Thus, $\Box_{\mathfrak{F}}$ is a direct unary operation for relators which will be called the *Pataki operation* generated by the structure $\mathfrak{F}$.

Remark 10.5. Actually, this definition could only be naturally applied to increasing or quasi-increasing structures for relators.

However, by using Definition 10.4, we can easily prove the following

Theorem 10.6. *If $\mathfrak{F}$ is $\Box$–regular, then $\Box = \Box_{\mathfrak{F}}$.*

Proof. By the corresponding definitions, for any relator $\mathcal{R}$ and relation S on X to Y, we have

$$S \in \mathcal{R}^{\Box} \iff \{S\} \subseteq \mathcal{R}^{\Box} \iff \mathfrak{F}_{\{S\}} \subseteq \mathfrak{F}_{\mathcal{R}} \iff \mathfrak{F}_S \subseteq \mathfrak{F}_{\mathcal{R}} \iff S \in \mathcal{R}^{\Box_{\mathfrak{F}}}.$$

Therefore, $\mathcal{R}^{\Box} = \mathcal{R}^{\Box_{\mathfrak{F}}}$, and thus the required equality is also true. $\Box$

Remark 10.7. Note that if, for instance, $\mathfrak{F}$ is only lower $\Box$–semiregular, then we can only prove that $\mathcal{R}^{\Box} \subseteq \mathcal{R}^{\Box_{\mathfrak{F}}}$ for any relator $\mathcal{R}$ on X to Y.

From Theorem 10.6, we can immediately derive

Corollary 10.8. *There exists at most one unary operation $\Box$ for relators such that $\mathfrak{F}$ is $\Box$–regular.*

In addition to Definition 10.2, we may also naturally use the following

Definition 10.9. The structure $\mathfrak{F}$ will be called *regular* if it is $\Box$–regular for some unary operation $\Box$ for relators.

Namely, thus as an immediate consequence of Theorem 10.6, we can also state

Theorem 10.10. *The following assertions are equivalent:*

(1) $\mathfrak{F}$ *is regular;* *(2)* $\mathfrak{F}$ *is* $\Box_{\mathfrak{F}}$*–regular.*

The appropriateness of our present definitions is also apparent from the following

Theorem 10.11. *The following assertions are equivalent:*

(1) $\Box_{\mathfrak{F}}$ *is extensive;*

(2) $\mathfrak{F}$ *is quasi-increasing;* *(3)* $\mathfrak{F}$ *is upper* $\Box_{\mathfrak{F}}$*–semiregular.*

Proof. If (2) holds and $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$, then for any $R \in \mathcal{R}$ we have not only $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathcal{R}}$, but also $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathcal{S}}$. Hence, by Definition 10.4, it follows that $R \in \mathcal{S}^{\Box_{\mathfrak{F}}}$. Therefore, $\mathcal{R} \subseteq \mathcal{S}^{\Box_{\mathfrak{F}}}$, and thus (3) also holds.

On the other hand, if (3) holds and $\mathcal{R}$ is a relator on X to Y, then from the trivial inclusion $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{R}}$, we can already infer that $\mathcal{R} \subseteq \mathcal{R}^{\Box_{\mathfrak{F}}}$. Therefore, (1) also holds.

Finally, if (1) holds, then for any relator $\mathcal{R}$ on X to Y, we have $\mathcal{R} \subseteq \mathcal{R}^{\Box_{\mathfrak{F}}}$. Therefore, for any $R \in \mathcal{R}$ we also have $R \in \mathcal{R}^{\Box_{\mathfrak{F}}}$. Hence, by Definition 10.4, it follows that $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathcal{R}}$. Therefore, (2) also holds. $\Box$

Now, as an immediate consequence of the latter two theorems, we can also state

Corollary 10.12. *The following assertions are equivalent:*

(1) $\mathfrak{F}$ *is regular;* *(2)* $\mathfrak{F}$ *is quasi-increasing and lower* $\Box_{\mathfrak{F}}$*–semiregular.*

Moreover, by using Theorem 10.11 and Definition 10.4, we can also easily prove

Theorem 10.13. *If* $\mathfrak{F}$ *is increasing, then*

(1) $\Box_{\mathfrak{F}}$ *is a preclosure;* *(2)* $\mathfrak{F}$ *is upper* $\Box_{\mathfrak{F}}$*–semiregular.*

Proof. From Theorem 10.11, we can see that now $\Box_{\mathfrak{F}}$ is extensive and (2) holds. Moreover, if $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{R} \subseteq \mathcal{S}$, then because of the increasingness of $\mathfrak{F}$ we also have $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$. Hence, by Definition 10.4, it is clear that $\mathcal{R}^{\Box_{\mathfrak{F}}} \subseteq \mathcal{S}^{\Box_{\mathfrak{F}}}$. Therefore, $\Box_{\mathfrak{F}}$ is also increasing, and thus (1) also holds. □

Now, in addition to this theorem, we can also easily prove the following

Theorem 10.14. *If* $\mathfrak{F}$ *is union-preserving, then* $\mathfrak{F}$ *is regular.*

Proof. Note that a union–preserving structure is increasing. Thus, by Corollary 10.12, we need only show that $\mathfrak{F}$ is lower $\Box_{\mathfrak{F}}$–semiregular.

For this, suppose that $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{R} \subseteq \mathcal{S}^{\Box_{\mathfrak{F}}}$ and $\Omega \in \mathfrak{F}_{\mathcal{R}}$. Then, since $\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_R$, there exists $R \in \mathcal{R}$ such that $\Omega \in \mathfrak{F}_R$. Thus, since $\mathcal{R} \subseteq \mathcal{S}^{\Box_{\mathfrak{F}}}$, we also have $R \in \mathcal{S}^{\Box_{\mathfrak{F}}}$. Hence, by Definition 10.4, it follows that $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathcal{S}}$. Therefore, we also have $\Omega \in \mathfrak{F}_{\mathcal{S}}$. Consequently, $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$. This proves that $\mathfrak{F}$ is lower $\Box_{\mathfrak{F}}$–semiregular. □

11. Further theorems on regular structures

The importance of regular structures lies mainly in the following

Theorem 11.1. *If* $\mathfrak{F}$ *is regular, then*

(1) $\Box_{\mathfrak{F}}$ *is a closure;* *(2)* $\mathfrak{F}$ *is increasing;*

(3) $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}$ *for any relator* $\mathcal{R}$ *on* X *to* Y.

Proof. From Theorem 10.10, we know that $\mathfrak{F}$ is $\Box_{\mathfrak{F}}$–regular. Hence, by Theorem 10.11, we can see that $\Box_{\mathfrak{F}}$ is extensive. Therefore, $\mathcal{R} \subseteq \mathcal{R}^{\Box_{\mathfrak{F}}}$ for any relator $\mathcal{R}$ on X to Y.

Thus, if $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{R} \subseteq \mathcal{S}$, then by using that $\mathcal{S} \subseteq \mathcal{S}^{\Box_{\mathfrak{F}}}$, we can see that $\mathcal{R} \subseteq \mathcal{S}^{\Box}$ also holds. Hence, by using the lower $\Box_{\mathfrak{F}}$–semiregularity of $\mathfrak{F}$, we can infer that $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$. Therefore, (2) is true.

Now, from the inclusion $\mathcal{R} \subseteq \mathcal{R}^{\Box_{\mathfrak{F}}}$, by using (2), we can infer that $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}$. Moreover, from the inclusion $\mathcal{R}^{\Box_{\mathfrak{F}}} \subseteq \mathcal{R}^{\Box_{\mathfrak{F}}}$, by using the lower $\Box_{\mathfrak{F}}$–semiregularity of $\mathfrak{F}$, we can infer that $\mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}} \subseteq \mathfrak{F}_{\mathcal{R}}$. Therefore, (3) is also true.

On the other hand, from Theorem 10.12, we can see that $\Box_{\mathfrak{F}}$ is a preclosure operation. Therefore, to prove (1), we need only show that $\Box_{\mathfrak{F}}$ is idempotent. For this, note that, by the extensivity of $\Box_{\mathfrak{F}}$, we have $\mathcal{R}^{\Box_{\mathfrak{F}}} \subseteq \mathcal{R}^{\Box_{\mathfrak{F}}\Box_{\mathfrak{F}}}$. Moreover, by (3), we have $\mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}\Box_{\mathfrak{F}}}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}} = \mathfrak{F}_{\mathcal{R}}$. Hence, by using the upper $\Box_{\mathfrak{F}}$–semiregularity of $\mathfrak{F}$, we can infer that $\mathcal{R}^{\Box_{\mathfrak{F}}\Box_{\mathfrak{F}}} \subseteq \mathcal{R}^{\Box_{\mathfrak{F}}}$. Therefore, the corresponding equality is also true. □

Now, in particular, we can also easily prove the following

Theorem 11.2. *The following assertions are equivalent:*

(1) $\square$ *is a closure;* *(2)* $\square$ *is* $\square$*–regular;*

(3) there exists a $\square$*–regular structure* $\mathfrak{F}$ *for relators.*

Proof. If (1) holds and $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{R}^{\square} \subseteq \mathcal{S}^{\square}$, then by the extensivity of $\square$, we also have $\mathcal{R} \subseteq \mathcal{S}^{\square}$. Therefore, $\square$ is upper $\square$–semiregular. While, if $\mathcal{R} \subseteq \mathcal{S}^{\square}$, then by the increasingness and the idempotency of $\square$, we also have $\mathcal{R}^{\square} \subseteq \mathcal{S}^{\square\square} = \mathcal{S}^{\square}$. Therefore, $\square$ is lower $\square$–semiregular. Consequently, (2) also holds.

Now, since (2) trivially implies (3), we need only show that (3) also implies (1). For this note, that if (3) holds, then by Theorem 10.6 we necessarily have $\square = \square_{\mathfrak{F}}$. Moreover, by Theorem 11.1, $\square_{\mathfrak{F}}$ is a closure operation. $\square$

From this theorem, by Theorem 10.6, it is clear that in particular we also have

Corollary 11.3. *If* $\Diamond$ *is a closure operation for relators, then* $\Diamond = \square_{\Diamond}$.

Moreover, from Theorem 11.2, by using Theorem 11.2, we can immediately derive

Theorem 11.4. *The following assertions are equivalent:*

(1) $\mathfrak{F}$ *is regular;*

(2) $\square_{\mathfrak{F}}$ *is a closure, and for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y *we have*

$$\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}} \quad \Longleftrightarrow \quad \mathcal{R}^{\square_{\mathfrak{F}}} \subseteq \mathcal{S}^{\square_{\mathfrak{F}}}.$$

However, it is now more important to note that we also have the following

Theorem 11.5. *The following assertions are equivalent:*

(1) $\mathfrak{F}$ *is regular,*

(2) $\mathfrak{F}$ *is increasing, and for every relator* $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\square_{\mathfrak{F}}}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$.

Proof. If (1) holds, then by Theorem 11.1 the structure $\mathfrak{F}$ is increasing, and for any relator $\mathcal{R}$ on X to Y we have $\mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} = \mathfrak{F}_{\mathcal{R}}$. Moreover, if $\mathcal{S}$ is a relator on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$, then by using the upper $\square_{\mathfrak{F}}$–regularity of $\mathfrak{F}$ we can see that $\mathcal{S} \subseteq \mathcal{R}^{\square_{\mathfrak{F}}}$. Thus, in particular, (2) also holds.

On the other hand, if (2) holds, and $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$, then from the assumed maximality property of $\mathcal{R}^{\square_{\mathfrak{F}}}$ we can see that $\mathcal{S} \subseteq \mathcal{R}^{\square_{\mathfrak{F}}}$. Therefore, $\mathfrak{F}$ is upper $\square_{\mathfrak{F}}$–semiregular.

Conversely, if $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y such that $\mathcal{S} \subseteq \mathcal{R}^{\square_{\mathfrak{F}}}$, then by using the assumed increasingness of $\mathfrak{F}$ we can see that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$. Hence, by the assumed inclusion $\mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} \subseteq \mathfrak{F}_{\mathcal{R}}$, it follows that $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}}$. Therefore, $\mathfrak{F}$ is also lower $\square_{\mathfrak{F}}$–regular, and thus (1) also holds. $\square$

From this theorem, by Theorem 11.1, it is clear that in particular we also have

Corollary 11.6. *If* $\mathfrak{F}$ *is regular, then for any relator* $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\square_{\mathfrak{F}}}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}}$.

Finally, we note that, by [209, Theorem 32] and [135, Theorem 1.5], the following two theorems are also true.

Theorem 11.7. *The following assertions are equivalent:*

(1) $\square$ *is an involution,*

(2) for any two relators $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y, *we have*

$$\mathcal{R}^{\square} \subseteq \mathcal{S} \quad \iff \quad \mathcal{R} \subseteq \mathcal{S}^{\square}.$$

Theorem 11.8. *The following assertions are equivalent:*

(1) $\square$ *is a semiclosure,*

(2) for every relator $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\square}$ *is the largest relator on* X *to* Y *such that* $\mathcal{R}^{\square} = \mathcal{S}^{\square}$,

(3) there exists a structure $\mathfrak{F}$ *for relators such that, for every relator* $\mathcal{R}$ *on* X *to* Y, $\mathcal{S} = \mathcal{R}^{\square}$ *is the largest relator on* X *to* Y *such that* $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.

Remark 11.9. Two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y may be naturally called $\mathfrak{F}$–*equivalent* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.

Moreover, the relator $\mathcal{R}$ may be naturally called $\mathfrak{F}$-*simple* if it is $\mathfrak{F}$–equivalent to a singleton relator.

Thus, the relator $\mathcal{R}$ is to be called *properly simple*, instead of simple, if it is equal to a singleton relator.

12. Important closure operations for relators

Notation 12.1. In this and the next section, we shall assume that $\mathcal{R}$ is a relator on X to Y.

Definition 12.2. The relators

$$\mathcal{R}^{*} = \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathcal{R} : \quad R \subseteq S \big\},$$
$$\mathcal{R}^{\#} = \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subseteq S[A] \big\},$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subseteq S(x) \big\}$$

and

$$\mathcal{R}^{\triangle} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ u \in X : \quad \exists\ R \in \mathcal{R} : \quad R(u) \subseteq S(x) \big\}$$

are called the *uniform, proximal, topological and paratopological closures (refinements)* of the relator $\mathcal{R}$, respectively.

Thus, we can we easily establish the following two theorems.

Theorem 12.3. *We have*

$$\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge} \subseteq \mathcal{R}^{\triangle}.$$

Theorem 12.4. *We have*

$$\mathcal{R}^{\triangle} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \big\};$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad x \in \operatorname{int}_{\mathcal{R}}\big(S(x) \big) \big\};$$
$$\mathcal{R}^{\#} = \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad A \in \operatorname{Int}_{\mathcal{R}}\big(S[A] \big) \big\}.$$

Now, by using this theorem and Definition 10.4, we can also easily prove

Theorem 12.5. *We have*

(1) $\mathcal{R}^{\#}=\mathcal{R}^{\square_{\mathrm{Int}}}$,   (2) $\mathcal{R}^{\wedge}=\mathcal{R}^{\square_{\mathrm{int}}}$,   (3) $\mathcal{R}^{\triangle}=\mathcal{R}^{\square_{\mathcal{E}}}$,

Proof. We shall only prove that $\mathcal{R}^{\square_{\mathrm{Int}}}\subseteq\mathcal{R}^{\#}$. The proof of the converse inclusion, and those of (2) and (3), will be left to the reader.

For this, we can note that if $S\in\mathcal{R}^{\square_{\mathrm{Int}}}$, then by Definition 10.4 S is a relation on X to Y such that $\mathrm{Int}_S\subseteq\mathrm{Int}_{\mathcal{R}}$, and so $\mathrm{Int}_S(B)\subseteq\mathrm{Int}_{\mathcal{R}}(B)$ for all $B\subseteq Y$.

Thus, in particular, for any $A\subseteq X$, we have $\mathrm{Int}_S\big(S[A]\big)\subseteq\mathrm{Int}_{\mathcal{R}}\big(S[A]\big)$. Hence, by using that $A\in\mathrm{Int}_S\big(S[A]\big)$, we can already infer that $A\in\mathrm{Int}_{\mathcal{R}}\big(S[A]\big)$. Therefore, by Theorem 12.3, $S\in\mathcal{R}^{\#}$ also holds. □

From this theorem, by using our former results, we can immediately derive

Theorem 12.6. $\#$, $\wedge$ *and* $\triangle$ *are closure operations for relators on* X *to* Y.

Proof. From Theorems 5.11, 6.11 and 3.11, we know that the structures Int , int , and $\mathcal{E}$ are union-preserving. Thus, by Theorems 10.14 and 11.1, the operations $\square_{\mathrm{Int}}$, $\square_{\mathrm{int}}$ and $\square_{\mathcal{E}}$ are closures. Therefore, by Theorem 12.5, the required assertions are also true. □

Remark 12.7. By using the definition of the operation $*$, we can easily see that $*$ is also a closure operation for relators.

It can actually be derived, by a similar procedure, from the structure Lim . While, the structure lim can lead only to the operation $\wedge$.

Now, by using Theorems 12.3 and 12.6, we can also easily prove the following

Theorem 12.8. *We have*

(1) $\mathcal{R}^{\#}=\big(\mathcal{R}^{*}\big)^{\#}=\big(\mathcal{R}^{\#}\big)^{*}$,

(2) $\mathcal{R}^{\wedge}=\big(\mathcal{R}^{\Diamond}\big)^{\wedge}=\big(\mathcal{R}^{\wedge}\big)^{\Diamond}$ *with* $\Diamond=*$ *and* $\#$,

(3) $\mathcal{R}^{\triangle}=\big(\mathcal{R}^{\Diamond}\big)^{\triangle}=\big(\mathcal{R}^{\triangle}\big)^{\Diamond}$ *with* $\Diamond=*$, $\#$ *and* $\wedge$.

Proof. To prove (1), note that, by Theorems 12.3 and 12.6, we have

$$\mathcal{R}^{\#}\subseteq\big(\mathcal{R}^{\#}\big)^{*}\subseteq\mathcal{R}^{\#\#}=\mathcal{R}^{\#}\qquad\text{and}\qquad\mathcal{R}^{\#}\subseteq\mathcal{R}^{*\#}\subseteq\mathcal{R}^{\#\#}=\mathcal{R}^{\#}\,.$$

Therefore, the corresponding equalities are also true. □

Now, since the structures Int , int , and $\mathcal{E}$ are union-preserving, by Theorems 12.5 and the corresponding results of Sections 10 and 11, we can also state the following two theorems.

Theorem 12.9. *For any relator* $\mathcal{S}$ *on* X *to* Y, *we have:*

(1) $\mathcal{S}\subseteq\mathcal{R}^{\triangle}\iff\mathcal{S}^{\triangle}\subseteq\mathcal{R}^{\triangle}\iff\mathcal{E}_{\mathcal{S}}\subseteq\mathcal{E}_{\mathcal{R}}\iff\mathcal{D}_{\mathcal{R}}\subseteq\mathcal{D}_{\mathcal{S}}$;

(2) $\mathcal{S}\subseteq\mathcal{R}^{\wedge}\iff\mathcal{S}^{\wedge}\subseteq\mathcal{R}^{\wedge}\iff\mathrm{int}_{\mathcal{S}}\subseteq\mathrm{int}_{\mathcal{R}}\iff\mathrm{cl}_{\mathcal{R}}\subseteq\mathrm{cl}_{\mathcal{S}}$;

(3) $\mathcal{S}\subseteq\mathcal{R}^{\#}\iff\mathcal{S}^{\#}\subseteq\mathcal{R}^{\#}\iff\mathrm{Int}_{\mathcal{S}}\subseteq\mathrm{Int}_{\mathcal{R}}\iff\mathrm{Cl}_{\mathcal{R}}\subseteq\mathrm{Cl}_{\mathcal{S}}$.

Remark 12.10. From (3), by using that $\mathrm{Lb}_{\mathcal{R}}=\mathrm{Int}_{\mathcal{R}^c}\circ\mathcal{C}$, we can easily see that

$$\mathrm{Lb}_{\mathcal{R}}\subset\mathrm{Lb}_{\mathcal{S}}\iff\mathrm{Int}_{\mathcal{R}^c}\circ\mathcal{C}\subset\mathrm{Int}_{\mathcal{S}^c}\circ\mathcal{C}\iff$$
$$\mathrm{Int}_{\mathcal{R}^c}\subset\mathrm{Int}_{\mathcal{S}^c}\iff\mathcal{R}^c\subset\mathcal{S}^{c\#}\iff\mathcal{R}\subset\mathcal{S}^{c\#c}.$$

Therefore, under the notation $\circledast = c\#c$, the structure Lb is $\circledast$–regular. Thus, by Theorem 10.6, $\square_{\mathrm{Lb}} = \circledast$. Moreover, by Theorem 11.1, $\circledast$ is a closure operation for relators.

The letter fact can also be easily proved directly by using that c is an involution and $\#$ is a closure operation.

Theorem 12.11. *The following assertions are true:*

(1) $\mathcal{S} = \mathcal{R}^{\triangle}$ *is the largest relator on* X *to* Y *such that* $\mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}}$ $\big(\mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{R}} \big)$, *or equivalently* $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$ $\big(\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{S}} \big)$;

(2) $\mathcal{S} = \mathcal{R}^{\wedge}$ *is the largest relator on* X *to* Y *such that* $\operatorname{int}_{\mathcal{S}} \subseteq \operatorname{int}_{\mathcal{R}}$ $\big(\operatorname{int}_{\mathcal{S}} = \operatorname{int}_{\mathcal{R}} \big)$, *or equivalently* $\operatorname{cl}_{\mathcal{R}} \subseteq \operatorname{cl}_{\mathcal{S}}$ $\big(\operatorname{cl}_{\mathcal{R}} = \operatorname{cl}_{\mathcal{S}} \big)$;

(3) $\mathcal{S} = \mathcal{R}^{\#}$ *is the largest relator on* X *to* Y *such that* $\operatorname{Int}_{\mathcal{S}} \subseteq \operatorname{Int}_{\mathcal{R}}$ $\big(\operatorname{Int}_{\mathcal{S}} = \operatorname{Int}_{\mathcal{R}} \big)$, *or equivalently* $\operatorname{Cl}_{\mathcal{R}} \subseteq \operatorname{Cl}_{\mathcal{S}}$ $\big(\operatorname{Cl}_{\mathcal{R}} = \operatorname{Cl}_{\mathcal{S}} \big)$.

Remark 12.12. To prove similar results for the operation $*$, the structures Lim and Adh have to be used.

13. Further results on the operations $\wedge$ and $\triangle$

A preliminary form of the following theorem was already proved in [162].

Theorem 13.1. *If* $\mathcal{R}$ *is nonvoid relator on* X *to* Y, *then for any* $B \subseteq Y$ *we have:*

(1) $\operatorname{Int}_{\mathcal{R}^{\wedge}}(B) = \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$; *(2)* $\operatorname{Cl}_{\mathcal{R}^{\wedge}}(B) = \mathcal{P}\big(\operatorname{cl}_{\mathcal{R}}(B)^c\big)^c$.

Proof. If $A \in \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$, then $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$. Therefore, by Theorem 6.2, for each $x \in A$ there exists $R_x \in \mathcal{R}$ such that $R_x(x) \subseteq B$. Now, by defining

$$S(x) = R_x(x) \quad \text{for all} \quad x \in A \qquad \text{and} \qquad S(x) = Y \quad \text{for all} \quad x \in A^c,$$

we can see that $S \in \mathcal{R}^{\wedge}$ such that $S[A] \subseteq B$. Therefore, by Definition 5.2, we also have $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$. Consequently, $\mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big) \subseteq \operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$.

The converse inclusion follows immediately from Corollary 6.4 and Theorem 12.11. Moreover, (2) can, in principle, be immediately derived from (1) by using Theorems 5.5 and 6.6. $\square$

Hence, by using Definitions 8.2 and 8.11 and Theorem 8.9, we can easily infer

Corollary 13.2. *If* $\mathcal{R}$ *is a nonvoid relator on* X, *then*

(1) $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}^{\wedge}} \mathcal{T}_R$, *(2)* $\tau\!\!\!-_{\mathcal{R}^{\wedge}} = \mathcal{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}^{\wedge}} \mathcal{F}_R$.

Hence, by using Definition 12.2 and Theorem 12.8, we can immediately derive

Corollary 13.3. *If* $\mathcal{R}$ *is a nonvoid relator on* X, *then*

(1) $\tau_{\mathcal{R}^{\triangle}} = \mathcal{T}_{\mathcal{R}^{\triangle}}$; *(2)* $\tau\!\!\!-_{\mathcal{R}^{\triangle}} = \mathcal{F}_{\mathcal{R}^{\triangle}}$.

Proof. By Definition 12.2, it is clear that $X^2 \in \mathcal{R}^{\triangle}$, and thus $\mathcal{R}^{\triangle} \neq \emptyset$. Hence, by using Corollary 13.2 and Theorem 12.8, we can see that $\tau_{\mathcal{R}^{\triangle}} = \tau_{\mathcal{R}^{\triangle\wedge}} = \mathcal{T}_{\mathcal{R}^{\triangle}}$. $\square$

Remark 13.4. Note that if $\mathcal{R} = \emptyset$, but $X \neq \emptyset$, then by Definition 12.2 we have $\mathcal{R}^{\triangle} = \emptyset$. Hence, by Theorems 8.4 and 8.12, we can see that $\tau_{\mathcal{R}^{\triangle}} = \emptyset$, but

$\mathcal{T}_{\mathcal{R}^{\triangle}} = \{\emptyset\}$. Therefore, in this case, the equalities stated in Corollary 13.3, and thus also those stated in Theorem 13.1 and Corollary 13.2 do not hold.

In addition to Theorem 13.1, we can also easily prove the following

Theorem 13.5. *If $\mathcal{R}$ is a nonvoid relator on X to Y, then for any $B \subseteq Y$ we have :*

(1) $\operatorname{Int}_{\mathcal{R}^{\triangle}}(B) = \{\emptyset\}$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{Int}_{\mathcal{R}^{\triangle}}(B) = \mathcal{P}(X)$ *if* $B \in \mathcal{E}_{\mathcal{R}}$*;*

(2) $\operatorname{Cl}_{\mathcal{R}^{\triangle}}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^{\triangle}}(B) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $B \in \mathcal{D}_{\mathcal{R}}$*.*

Proof. If $A \in \operatorname{Int}_{\mathcal{R}^{\triangle}}(B)$, then there exists $S \in \mathcal{R}^{\triangle}$ such that $S[A] \subseteq B$. Therefore, if $A \neq \emptyset$, then there exists $x \in X$ such that $S(x) \subseteq B$. Hence, since $S(x) \in \mathcal{E}_{\mathcal{R}}$, it follows that $B \in \mathcal{E}_{\mathcal{R}}$. Therefore, the first part of (1) is true.

To prove the second part of (1), it is enough to note only that if $B \in \mathcal{E}_{\mathcal{R}}$, then $R = X \times B \in \mathcal{R}^{\triangle}$ such that $R[A] \subseteq B$, and thus $A \in \operatorname{Int}_{\mathcal{R}^{\triangle}}(B)$ for all $A \subseteq X$.

Now, to complete the proof, it remains only to note that (2) can, in principle, be immediately derived from (1) by using Theorem 5.5. □

From this theorem, by using Definition 6.1, we can immediately derive

Corollary 13.6. *If $\mathcal{R}$ is nonvoid relator on X to Y , then for any $B \subseteq Y$, we have :*

(1) $\operatorname{cl}_{\mathcal{R}^{\triangle}}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{cl}_{\mathcal{R}^{\triangle}}(B) = X$ *if* $B \in \mathcal{D}_{\mathcal{R}}$ *;*

(2) $\operatorname{int}_{\mathcal{R}^{\triangle}}(B) = \emptyset$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{int}_{\mathcal{R}^{\triangle}}(B) = X$ *if* $B \in \mathcal{E}_{\mathcal{R}}$*.*

Now, by using this corollary, we can also easily prove the following

Corollary 13.7. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}^{\triangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$ *;* *(2)* $\mathcal{F}_{\mathcal{R}^{\triangle}} = \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}$ *.*

Proof. If $A \in \mathcal{T}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\}$, then $\emptyset \neq A \subseteq \operatorname{int}_{\mathcal{R}}(A)$. Hence, if $\mathcal{R} \neq \emptyset$, then by using Corollary 13.6 we can infer that $A \in \mathcal{E}_{\mathcal{R}}$. Therefore, $\mathcal{T}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, and thus $\mathcal{T}_{\mathcal{R}^{\triangle}} \subseteq \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$.

Conversely, if $\mathcal{R} \neq \emptyset$ and $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 13.6 we have $A \subseteq X = \operatorname{int}_{\mathcal{R}^{\triangle}}(B)$, and thus $A \in \mathcal{T}_{\mathcal{R}^{\triangle}}$. Therefore, $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}^{\triangle}}$. Hence, since $\emptyset \in \mathcal{T}_{\mathcal{R}^{\triangle}}$ is always true, we can infer that $\mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}^{\triangle}}$, and thus (1) also holds.

On the other hand, if $\mathcal{R} = \emptyset$, then by Definition 12.2 we can see that

$$\mathcal{R}^{\triangle} = \emptyset \quad \text{if} \quad X \neq \emptyset \qquad \text{and} \qquad \mathcal{R}^{\triangle} = \{\emptyset\} \quad \text{if} \quad X = \emptyset .$$

Therefore, by Theorem 8.12, we have $\mathcal{T}_{\mathcal{R}^{\triangle}} = \{\emptyset\}$. Moreover, by Theorem 7.3, we can see that $\mathcal{E}_{\mathcal{R}} = \emptyset$, and thus (1) also holds. □

Now, by using this corollary, we can also easily prove the following

Corollary 13.8. *If $\mathcal{R}$ is a non-partial relator on X, then*

(1) $\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\}$ *,* *(2)* $\mathcal{D}_{\mathcal{R}} = \big(\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}^{\triangle}}\big) \cup \{X\}$ *.*

Proof. Recall that by Remark 7.15 and Theorem 7.14, we now have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$. Therefore, assertion (1) of this corollary follows from that of Corollary 13.7.

Moreover, by (1) and Theorems 7.6 and 8.13, for any $A \subseteq X$ we have

$$A \in \mathcal{D}_{\mathcal{R}} \iff A^c \notin \mathcal{E}_{\mathcal{R}} \iff A^c \notin \mathcal{T}_{\mathcal{R}^\vartriangle} \setminus \{\emptyset\} \iff A^c \notin \mathcal{T}_{\mathcal{R}^\vartriangle} \text{ or } A^c = \emptyset$$
$$\iff A \notin \mathcal{F}_{\mathcal{R}^\vartriangle} \text{ or } A = X \iff A \in \big(\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}^\vartriangle}\big) \cup \{X\}.$$

□

14. The importance of the operations ∞ and ∂

Notation 14.1. In this and the next three sections, we shall already assume that $\mathcal{R}$ is a relator on X.

Concerning the operation ∞, we shall first prove the following

Theorem 14.2. *The following assertions hold:*

(1) ∞ *is a closure operation for relations on* X*;*

(2) for any two relations R *and* S *on* X*, we have*

$$S \subseteq R^\infty \iff S^\infty \subseteq R^\infty \iff \tau_R \subseteq \tau_S \iff \tau\!\!\!-_R \subseteq \tau\!\!\!-_S\,;$$

(3) for any relation R *on* X*,* $S = R^\infty$ *is the largest relation on* X *such that* $\tau_R \subseteq \tau_S$ $\big(\tau_R = \tau_S\big)$*, or equivalently* $\tau\!\!\!-_R \subseteq \tau\!\!\!-_S$ $\big(\tau\!\!\!-_R = \tau\!\!\!-_S\big)$.

Proof. We shall only prove that, for any two relations R and S on X,

(a) $\tau_R \subseteq \tau_S$ implies $S \subseteq R^\infty$; (b) $S \subseteq R^\infty$ implies $\tau_R \subseteq \tau_S$.

Therefore, the function $\mathfrak{F}$, defined by $\mathfrak{F}_R = \mathcal{P}(X) \setminus \tau_R$ for all relation R on X, is an ∞–regular structure for relations. Thus, analogously to the results of Section 12, the remaining assertions of the theorem can also be proved.

To prove (a), note that if $x \in X$, then because of the inclusion $R \subseteq R^\infty$ and the transitivity of R^∞ we have

$$R[\,R^\infty(x)\,] \subseteq R^\infty[\,R^\infty(x)\,] = \big(R^\infty \circ R^\infty\big)(x) \subseteq R^\infty(x)\,.$$

Therefore, by Theorem 8.4, $R^\infty(x) \in \tau_{\mathcal{R}}$. Now, if $\tau_R \subseteq \tau_S$ holds, then we can see that $R^\infty(x) \in \tau_S$, and thus $S[\,R^\infty(x)\,] \subseteq R^\infty(x)$. Hence, by using the reflexivity of R^∞, we can already infer that $S(x) \subseteq R^\infty(x)$. Therefore, $S \subseteq R^\infty$ also holds.

While, to prove (b), note that if $A \in \tau_R$, then by again Theorem 8.4 we have $R[\,A\,] \subseteq A$. Hence, by induction, we can see that $R^n[\,A\,] \subseteq A$ for all $n \in \mathbb{N}$. Now, since $R^0[\,A\,] = \Delta_X[\,A\,] = A$ also holds, we can already state that

$$R^\infty[\,A\,] = \left(\bigcup_{n=0}^{\infty} R^n\right)[\,A\,] = \bigcup_{n=0}^{\infty} R^n[\,A\,] \subseteq \bigcup_{n=0}^{\infty} A = A\,.$$

Therefore, if $S \subseteq R^\infty$ holds, then we have $S[A] \subseteq R^\infty[\,A\,] \subseteq A$, and thus $A \in \tau_S$ also holds. □

Remark 14.3. A preliminary form of this theorem and the fact that

$$R^\infty(x) = \bigcap\ \{A \in \tau_R: \quad x \in A\}$$

for all $x \in X$, and thus $R^\infty = \bigcap\{R_A: \ A \in \tau_R\}$, were first proved by Mala [107].

Now, as an immediate consequence of Theorems 8.9 and 14.2, we can also state

Theorem 14.4. *We have*

(1) $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\infty}} = \tau_{\mathcal{R}^{\#}}$; (2) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\infty}} = \mathcal{T}_{\mathcal{R}^{\#}}$;

(3) $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\infty\#}} = \tau_{\mathcal{R}^{\#\infty}}$; (4) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\infty\#}} = \mathcal{T}_{\mathcal{R}^{\#\infty}}$.

Proof. To prove (1), recall that $\mathcal{R}^{\infty} = \{R^{\infty} : \ R \in \mathcal{R}\}$. Thus, by Theorems 8.9 and 14.2, we have $\tau_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \tau_R = \bigcup_{R \in \mathcal{R}} \tau_{R^{\infty}} = \tau_{\mathcal{R}^{\infty}}$. Moreover, by Theorem 12.11, we have $\operatorname{Int}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^{\#}}$, and thus also $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\#}}$. □

Remark 14.5. Concerning the operation ∞, one can also prove that

(1) $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*\infty} \subseteq \mathcal{R}^{\infty *} \subseteq \mathcal{R}^{*}$; (2) $\mathcal{R}^{*\infty} = \mathcal{R}^{\infty * \infty}$ and $\mathcal{R}^{\infty *} = \mathcal{R}^{* \infty *}$.

However, it is more important to note that now we can also prove the following

Theorem 14.6. *We have*

$$\mathcal{R}^{\square_{\tau}} = \mathcal{R}^{\#\partial} .$$

Proof. If $S \in \mathcal{R}^{\#\partial}$, then by the definition of ∂ we have $S^{\infty} \in \mathcal{R}^{\#}$. Hence, by using Theorems 14.2 and 14.4, we can see that $\tau_S = \tau_{S^{\infty}} \subseteq \tau_{\mathcal{R}^{\#}} = \tau_{\mathcal{R}}$. Therefore, by Definition 10.4, $S \in \mathcal{R}^{\square_{\tau}}$ also holds.

Conversely, if $S \in \mathcal{R}^{\square_{\tau}}$, then Definition 10.4 S is a relation on X to Y such that $\tau_S \subseteq \tau_{\mathcal{R}}$. Moreover, if $A \subseteq X$, then by using that $S \subseteq S^{\infty}$ and S^{∞} is transitive, we can note that

$$S\left[S^{\infty}[A]\right] \subseteq S^{\infty}\left[S^{\infty}[A]\right] = \left(S^{\infty} \circ S^{\infty}\right)[A] \subseteq S^{\infty}[A],$$

and thus $S^{\infty}[A] \in \tau_S$. Therefore, by the inclusion $\tau_S \subseteq \tau_{\mathcal{R}}$, for any $A \subseteq X$ we also have $S^{\infty}[A] \in \tau_{\mathcal{R}}$, and thus $S^{\infty}[A] \in \operatorname{Int}_{\mathcal{R}}\left(S^{\infty}[A]\right)$. Hence, by using that $A \subseteq \mathcal{S}^{\infty}[A]$, we can infer that $A \in \operatorname{Int}_{\mathcal{R}}\left(S^{\infty}[A]\right)$ also holds. Therefore, by Theorem 12.4, $S^{\infty} \in \mathcal{R}^{\#}$, and thus $S \in \mathcal{R}^{\#\partial}$ also holds. □

Now, analogously to the results of Section 12, we can also easily prove

Theorem 14.7. *The following assertions hold:*

(1) $\#\partial$ *is a closure operation for relators on* X*;*

(2) for any relator $\mathcal{S}$ *on* X*, we have*

$$\mathcal{S} \subseteq \mathcal{R}^{\#\partial} \iff \mathcal{S}^{\#\partial} \subseteq \mathcal{R}^{\#\partial} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} ;$$

(3) $\mathcal{S} = \mathcal{R}^{\#\partial}$ *is the largest relator on* X *such that* $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $\left(\tau_{\mathcal{S}} = \tau_{\mathcal{R}}\right)$*, or equivalently* $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}}$ $\left(\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}\right)$.

By using the definition of ∂, and our former results, this theorem can be reformulated in the following more convenient form.

Theorem 14.8. *The following assertions hold:*

(1) $\#\infty$ *is a projection operation for relators on* X*;*

(2) for any $\mathcal{S}$ *on* X*, we have*

$$\mathcal{S}^{\infty} \subseteq \mathcal{R}^{\#} \iff \mathcal{S}^{\#\infty} \subseteq \mathcal{R}^{\#\infty} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} .$$

(3) $\mathcal{S} = \mathcal{R}^{\#\infty}$ *is the largest preorder relator on* X *such that* $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $\left(\tau_{\mathcal{S}} = \tau_{\mathcal{R}}\right)$*, or equivalently* $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}}$ $\left(\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}\right)$.

Remark 14.9. It can be shown that the following assertions are also equivalent:

(1) $S^{\infty} \subseteq \mathcal{R}^{\#}$; (2) $\mathcal{S}^{\# \infty} \subseteq \mathcal{R}^{\#}$; (3) $\mathcal{S}^{\infty \#} \subseteq \mathcal{R}^{\infty \#}$.

The advantage of the projection operations $\# \infty$ and $\infty \#$ over the closure operation $\# \partial$ lies mainly in the fact that, in contrast to $\# \partial$, they are *stable* in the sense that they leave the relator $\{X^2\}$ fixed.

15. Further theorems on the operations ∞ and ∂

In addition to Theorem 14.6, we can also easily prove the following

Theorem 15.1. *We have*

$$\mathcal{R}^{\square \mathcal{T}} = \mathcal{R}^{\wedge \partial}.$$

Proof. If $\mathcal{R} \neq \emptyset$, then by Definition 10.4, Remark 8.15, Corollary 13.2 and Theorems 14.7 and 12.8, it is clear that, for any relation S on X, we have

$$S \in \mathcal{R}^{\square \mathcal{T}} \iff \mathcal{T}_S \subseteq \mathcal{T}_{\mathcal{R}} \iff \tau_S \subseteq \tau_{\mathcal{R}^{\wedge}} \iff S \in \mathcal{R}^{\wedge \# \partial} \iff S \in \mathcal{R}^{\wedge \partial}.$$

While, if $\mathcal{R} = \emptyset$, then by using Theorem 8.12 we can see that $\mathcal{T}_{\mathcal{R}} = \{\emptyset\}$. Thus, by Definition 10.4, we have

$$\mathcal{R}^{\square \mathcal{T}} = \big\{ S \subseteq X^2 : \quad \mathcal{T}_S \subseteq \mathcal{T}_{\mathcal{R}} \big\} = \big\{ S \subseteq X^2 : \quad \mathcal{T}_S \subseteq \{\emptyset\} \big\}.$$

Hence, since $X \in \mathcal{T}_S$ for any relation S on X, it is clear that

$$\mathcal{R}^{\square \mathcal{T}} = \emptyset \quad \text{if} \quad X \neq \emptyset \qquad \text{and} \qquad \mathcal{R}^{\square \mathcal{T}} = \{\emptyset\} \quad \text{if} \quad X = \emptyset.$$

Moreover, if $\mathcal{R} = \emptyset$, then by using Definition 12.2, we can see that

$$\mathcal{R}^{\wedge} = \emptyset \quad \text{if} \quad X \neq \emptyset \qquad \text{and} \qquad \mathcal{R}^{\wedge} = \{\emptyset\} \quad \text{if} \quad X = \emptyset.$$

Hence, since $\mathcal{R}^{\wedge \partial} = \{ S \subseteq X^2 : \quad S^{\infty} \in \mathcal{R}^{\wedge} \}$, it is clear that

$$\mathcal{R}^{\wedge \partial} = \emptyset \quad \text{if} \quad X \neq \emptyset \qquad \text{and} \qquad \mathcal{R}^{\wedge \partial} = \{\emptyset\} \quad \text{if} \quad X = \emptyset.$$

Therefore, the required equality is also true if $\mathcal{R} = \emptyset$. □

Unfortunately, the structure $\mathcal{T}$ is not union-preserving. Namely, we have

Example 15.2. If $x_1 \in X$ and $x_2 \in X \setminus \{x_1\}$, and

$$R_i = \{x_i\}^2 \cup \big(X \setminus \{x_i\} \big)^2$$

for all $i = 1, 2$, then it is clear that $\mathcal{R} = \{R_1, R_2\}$ is an equivalence relator on X. Moreover, by using Theorem 8.12, we can easily see that

$$\{x_1, x_2\} \in \mathcal{T}_{\mathcal{R}} \setminus \big(\mathcal{T}_{R_1} \cup \mathcal{T}_{R_2} \big), \qquad \text{and thus} \qquad \mathcal{T}_{\mathcal{R}} \not\subseteq \mathcal{T}_{R_1} \cup \mathcal{T}_{R_2}.$$

Therefore, in contrast to Theorem 14.7, we can only prove the following

Theorem 15.3. *The following assertions are true:*

(1) $\wedge \partial$ *is a preclosure operation for relators on* X*;*

(2) for any two relators $\mathcal{R}$ *and* $\mathcal{S}$ *on* X*, we have*

$$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}} \implies \mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge \partial} \implies \mathcal{S}^{\wedge \partial} \subseteq \mathcal{R}^{\wedge \partial}.$$

Proof. From Theorem 8.22, we know that the structure $\mathcal{T}$ is increasing. Thus, by Theorem 10.13, $\Box_{\mathcal{T}}$ is a preclosure operation for relators and $\mathcal{T}$ is upper $\Box_{\mathcal{T}}$–semiregular. Thus, in particular, $\mathcal{T}_{\mathcal{S}^{\wedge}} \subseteq \mathcal{T}_{\mathcal{R}}$ implies $\mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\Box_{\mathcal{T}}}$. Hence, since $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{S}^{\wedge}}$ and $\mathcal{R}^{\Box_{\mathcal{T}}} = \mathcal{R}^{\wedge\partial}$, we can already see that $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}}$ implies $\mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge\partial}$. Moreover, since ∂ is a projection (modification) operation for relators, we can also note that $\mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge\partial} \implies \mathcal{S}^{\wedge\partial} \subseteq \mathcal{R}^{\wedge\partial\partial} \implies \mathcal{S}^{\wedge\partial} \subseteq \mathcal{R}^{\wedge\partial}$. □

Remark 15.4. If $\operatorname{card}(X) > 2$, then by using the equivalence relator $\mathcal{R} = \{X^2\}$ Mala [107, Example 5.3] proved that there does not exist a largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$.

Moreover, Pataki [135, Example 7.2] proved that $\mathcal{T}_{\mathcal{R}^{\wedge\partial}} \not\subseteq \mathcal{T}_{\mathcal{R}}$ and $\wedge\partial$ is not idempotent. (Actually, it can be proved that $\mathcal{R}^{\wedge\partial\wedge} \not\subseteq \mathcal{R}^{\wedge\partial}$ also holds [183, Example 10.11].)

Thus, by Theorem 11.1, the increasing structure $\mathcal{T}$ is not regular. Moreover, by Theorems 11.2, there does not exist a structure $\mathfrak{F}$ for relators such that $\mathfrak{F}$ is $\wedge\partial$–regular. And, by Theorem 11.8, there does not exist a structure $\mathfrak{F}$ for relators such that, for every relator $\mathcal{R}$ on X, $\mathcal{S} = \mathcal{R}^{\wedge\partial}$ is the largest relator on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$.

However, from Theorem 14.8, by using Corollary 13.2, we can easily derive

Theorem 15.5. *The following assertions are true:*

(1) $\wedge\infty$ *is a modification operation for relators on* X*;*

(2) for any two nonvoid relators $\mathcal{R}$ *and* $\mathcal{S}$ *on* X*, we have*
$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}} \iff \mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge\infty}$*;*

(3) for any nonvoid relator $\mathcal{R}$ *on* X*,* $\mathcal{S} = \mathcal{R}^{\wedge\infty}$ *is the largest preorder relator on* X *such that* $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}}$ $(\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}})$*, or equivalently* $\mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}}$ $(\mathcal{F}_{\mathcal{S}} = \mathcal{F}_{\mathcal{R}})$*.*

Proof. To prove (2), note that by Corollary 13.2 and Theorem 14.8, we have

$$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \tau_{\mathcal{R}^{\wedge}} \subseteq \tau_{\mathcal{S}^{\wedge}} \iff \mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge\#} \iff \mathcal{S}^{\wedge\#\infty} \subseteq \mathcal{R}^{\wedge\#\infty}.$$

Moreover, by Theorem 12.8, we have $\mathcal{R}^{\wedge\#} = \mathcal{R}^{\wedge}$ and also $\mathcal{S}^{\wedge\#} = \mathcal{S}^{\wedge}$. □

Remark 15.6. In the light of the several disadvantages of the structure $\mathcal{T}$, it is rather curious that most of the works in topology and analysis have been based on open sets suggested by Tietze [213] and Alexandroff [4], and standardized by Bourbaki [18], Kelley [80] and Engelking [52].

Moreover, it also a very striking fact that, despite the results of Pervin [139], Fletcher and Lindgren [55], and the second author [185], minimal structures, generalized topologies and stacks are still intensively investigated by a great number of mathematicians without using generalized uniformities.

16. Reflexive and topological relators

The subsequent definitions and theorems on a relator $\mathcal{R}$ on X have been mainly taken from [167, 209].

Definition 16.1. The relator $\mathcal{R}$ is called *reflexive* if each member R of $\mathcal{R}$ is a reflexive relation on X.

Remark 16.2. Thus, the following assertions are equivalent:

(1) $\mathcal{R}$ is reflexive;

(2) $x \in R(x)$ for all $x \in X$ and $R \in \mathcal{R}$;

(3) $A \subseteq R[A]$ for all $A \subseteq X$ and $R \in \mathcal{R}$.

The importance of reflexive relators is also apparent from the following two obvious theorems.

Theorem 16.3. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is reflexive;*

(2) $A \subseteq \mathrm{cl}_{\mathcal{R}}(A)$ *for all* $A \subseteq X$*;* *(3)* $\mathrm{int}_{\mathcal{R}}(A) \subseteq A$ *for all* $A \subseteq X$.

Proof. To prove the implication (3) $\Longrightarrow$ (1), note that, for any $x \in X$ and $R \in \mathcal{R}$, we have $R(x) \subseteq R(x)$, and thus $x \in \mathrm{int}_{\mathcal{R}}\big(R(x)\big)$. Therefore, if (3) holds, then $x \in R(x)$, and thus (1) also holds. □

Remark 16.4. In addition to this theorem, it is also worth mentioning that the relator $\mathcal{R}$ is reflexive if and only if the relation $\delta_{\mathcal{R}} = \bigcap \mathcal{R}$ is reflexive. Namely, by using Theorem 6.13, we can show that $\mathrm{cl}_{\mathcal{R}}(x) = \delta_{\mathcal{R}}^{-1}(x)$ for all $x \in X$.

Theorem 16.5. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is reflexive;*

(2) $A \in \mathrm{Int}_{\mathcal{R}}(B)$ *implies* $A \subseteq B$ *for all* $A, B \subseteq X$*;*

(3) $A \cap B \neq \emptyset$ *implies* $A \in \mathrm{Cl}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq X$.

Remark 16.6. In addition to the above two theorems , it is also worth mentioning that if $\mathcal{R}$ is reflexive, then

(1) $\mathrm{Int}_{\mathcal{R}}$ is a transitive relation;

(2) $B \in \mathrm{Cl}_{\mathcal{R}}(A)$ implies $\mathcal{P}(X) = \mathrm{Cl}_{\mathcal{R}}(A)^c \cup \mathrm{Cl}_{\mathcal{R}}^{-1}(B)$;

(3) $\mathrm{int}_{\mathcal{R}}\big(A \setminus \mathrm{int}_{\mathcal{R}}(A)\big) = \emptyset$ and $\mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A) \setminus A\big) = \emptyset$ for all $A \subseteq X$.

Thus, for instance, for any $A \subseteq X$ we have $A \in \mathcal{F}_{\mathcal{R}}$ if and only if $\mathrm{cl}_{\mathcal{R}}(A) \setminus A \in \mathcal{T}_{\mathcal{R}}$.

Definition 16.7. We say that:

(1) $\mathcal{R}$ is *quasi-topological* if $x \in \mathrm{int}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}\big(R(x)\big)\big)$ for all $x \in X$ and $R \in \mathcal{R}$;

(2) $\mathcal{R}$ is *topological* if for any $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$.

The appropriateness of these definitions is already quite obvious from the following four theorems.

Theorem 16.8. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topological;*

(2) $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}$ *for all* $A \subseteq X$*;*

(3) $\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$ *for all* $A \subseteq X$ *;*
(4) $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{T}_{\mathcal{R}}$ *for all* $x \in X$ *and* $R \in \mathcal{R}$ *.*

Theorem 16.9. *The following assertions are equivalent:*
(1) $\mathcal{R}$ *is topological;* *(2)* $\mathcal{R}$ *is reflexive and quasi-topological.*

Remark 16.10. By Theorem 16.8, the relator $\mathcal{R}$ may be called *weakly (strongly) quasi-topological* if $\operatorname{cl}_{\mathcal{R}}(x) \in \mathcal{F}_{\mathcal{R}}$ $\big(R(x) \in \mathcal{T}_{\mathcal{R}}\big)$ for all $x \in X$ and $R \in \mathcal{R}$.

Moreover, by Theorem 16.9, the relator $\mathcal{R}$ may be called *weakly (strongly) toplogical* if it is reflexive and weakly (strongly) quasi-topological.

Theorem 16.11. *The following assertions are equivalent:*
(1) $\mathcal{R}$ *is topological;*
(3) $\operatorname{int}_{\mathcal{R}}(A) = \bigcup \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$ *for all* $A \subseteq X$ *;*
(3) $\operatorname{cl}_{\mathcal{R}}(A) = \bigcap \mathcal{F}_{\mathcal{R}} \cap \mathcal{P}^{-1}(A)$ *for all* $A \subseteq X$ *.*

Now, as an immediate consequence of this theorem, we can also state

Corollary 16.12. *If* $\mathcal{R}$ *is topological, then for any* $A \subset X$, *we have*
(1) $A \in \mathcal{E}_{\mathcal{R}}$ *if and only if there exists* $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ *such that* $V \subseteq A$ *;*
(2) $A \in \mathcal{D}_{\mathcal{R}}$ *if and only if for all* $W \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$ *we have* $A \setminus W \neq \emptyset$ *.*

However, it is now more important to note that we can also prove the following

Theorem 16.13. *The following assertions are equivalent:*
(1) $\mathcal{R}$ *is topological;*
(2) $\mathcal{R}$ *is topologically equivalent to* $\mathcal{R}^{\wedge\infty}$ *;*
(3) $\mathcal{R}$ *is topologically equivalent to a preorder relator on* X *.*

Proof. To prove the implication (1) $\Longrightarrow$ (3), note that if (1) holds, then by Definition 16.7, for any $x \in X$ and $R \in \mathcal{R}$, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$. Hence, by considering the Pervin relator

$$\mathcal{S} = \mathcal{R}_{\mathcal{T}_{\mathcal{R}}} = \big\{ R_V : \quad V \in \mathcal{T}_{\mathcal{R}} \big\}, \qquad \text{where} \qquad R_V = V^2 \cup V^c \times X,$$

we can note that $\mathcal{R} \subseteq \mathcal{S}^{\wedge}$, and thus $\mathcal{R}^{\wedge} \subseteq \mathcal{S}^{\wedge\wedge} = \mathcal{S}^{\wedge}$. Moreover, since

$$R_V(x) = V \quad \text{if} \quad x \in V \qquad \text{and} \qquad R_V(x) = X \quad \text{if} \quad x \in V^c,$$

we can also note that $\mathcal{S} \subseteq \mathcal{R}^{\wedge}$, and thus $\mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge\wedge} = \mathcal{R}^{\wedge}$. Therefore, we actually have $\mathcal{R}^{\wedge} = \mathcal{S}^{\wedge}$, and thus $\mathcal{R}$ is topologically equivalent to $\mathcal{S}$. Hence, since $\mathcal{S}$ is a preorder relator on X, we can already see that (3) also holds. □

17. Proximal relators

For a relator $\mathcal{R}$ on X, in addition to Definition 16.7 and Remark 16.10, we may also have the following

Definition 17.1. We say that:

(1) $\mathcal{R}$ is *semi-proximal* if $A \in \operatorname{Int}_{\mathcal{R}}\big[\operatorname{Int}_{\mathcal{R}}\big(R[A]\big)\big]$ for all $A \subseteq X$ and $R \in \mathcal{R}$;

(2) $\mathcal{R}$ is *quasi-proximal* if $A \in \operatorname{Int}_{\mathcal{R}}\big[\tau_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{R}}\big(R[A]\big)\big]$ for all $A \subseteq X$ and $R \in \mathcal{R}$;

(3) $\mathcal{R}$ is *proximal* if for any $A \subseteq X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[A]$;

(4) $\mathcal{R}$ is *weakly proximal* if for any $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$.

Remark 17.2. Hence, it is clear that "quasi-proximal" implies "semi-proximal", and "proximal" implies "weakly proximal". Moreover, since $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can also note that "weakly proximal" implies "topological".

Furthermore, by using the corresponding definitions, we can also easily see that the relator $\mathcal{R}$ is quasi-proximal if and only if, for any $A \subseteq X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $A \in \operatorname{Int}_{\mathcal{R}}(V)$ and $V \in \operatorname{Int}_{\mathcal{R}}\big(R[A]\big)$.

The appropriateness of definitions (2) and (3) is also quite obvious from the following analogues of Theorems 16.9, 16.10 and 16.13.

Theorem 17.3. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximal;* *(2)* $\mathcal{R}$ *is reflexive and quasi-proximal.*

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then for any $A \subseteq X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[A]$. Hence, since A may be $\{x\}$ for any $x \in X$, and $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can see that $\mathcal{R}$ is not only reflexive, but also topological.

Moreover, since $V \in \tau_{\mathcal{R}}$, we can also note $V \in \operatorname{Int}_{\mathcal{R}}(V)$. Hence, by using that $A \subseteq V$ and $V \subseteq R[A]$, we can already infer that the inclusions $A \in \operatorname{Int}_{\mathcal{R}}(V)$ and $V \in \operatorname{Int}_{\mathcal{R}}\big(R[A]\big)$ are also true. Therefore, by Remark 17.2, $\mathcal{R}$ is quasi-proximal, and thus (2) also holds. $\square$

Theorem 17.4. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximal;*

(2) $Int_{\mathcal{R}}(A) = \bigcap \big\{ \mathcal{P}(V): \ A \supseteq V \in \tau_{\mathcal{R}} \big\}$ *for all* $A \subseteq X$;

(3) $\operatorname{Cl}_{\mathcal{R}}(A) = \bigcap \big\{ \mathcal{P}(X) \setminus \mathcal{P}(W^c): \ A \subseteq W \in \tau_{\mathcal{R}} \big\}$ *for all* $A \subseteq X$.

Proof. To check the equivalence of (1) and (2), note that, for any $A, B \subseteq X$, we have $B \in \bigcap \big\{ \mathcal{P}(V): \ A \supseteq V \in \tau_{\mathcal{R}} \big\}$ if and only if there exists $V \in \tau_{\mathcal{R}}$ such that $V \subseteq A$ and $B \subseteq V$. Thus, in particular, $\bigcap \big\{ \mathcal{P}(V): \ A \supseteq V \in \tau_{\mathcal{R}} \big\} \subseteq \operatorname{Int}_{\mathcal{R}}(A)$ is always true.

Moreover, if $A \subseteq X$ and $R \in \mathcal{R}$, then because of $R[A] \subseteq R[A]$, we always have $A \in \operatorname{Int}_{\mathcal{R}}\big(R[A]\big)$. Therefore, if the essential part of (2) holds, then there exists $V \in \tau_{\mathcal{R}}$ such that $V \subseteq R[A]$ and $A \subseteq V$, and thus (1) also holds. $\square$

Thus, since $\mathcal{P}(A) = \{B: \ B \subseteq A\}$ and $\mathcal{P}(A) = \operatorname{Int}_{\Delta_X}(A)$, we can also state

Corollary 17.5. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximal;*

(2) $\operatorname{Int}_{\mathcal{R}}(A) = \mathcal{P}\big[\tau_{\mathcal{R}} \cap \mathcal{P}(A)\big]$ *for all* $A \subseteq X$;

(3) $\operatorname{Int}_{\mathcal{R}}(A) = \operatorname{Int}_{\Delta_X}\big[\tau_{\mathcal{R}} \cap \operatorname{Int}_{\Delta_X}(A)\big]$ *for all* $A \subseteq X$.

However, it is now more important to note that we also have the following

Theorem 17.6. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximal;*

(2) $\mathcal{R}$ *is proximally equivalent to* $\mathcal{R}^{\infty}$ *or* $\mathcal{R}^{\#\infty}$;

(3) $\mathcal{R}$ *is proximally equivalent to a preorder relator on* X.

In principle, each theorem on topological and quasi-topological relators can be immediately derived from a corresponding theorem on proximal and quasi-proximal relators by using the following two theorems.

Theorem 17.7. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topological;*

(2) $\mathcal{R}^{\wedge}$ *is semi-proximal;* *(3)* $\mathcal{R}^{\wedge}$ *is quasi-proximal;*

(4) $\{x\} \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big[\operatorname{Int}_{\mathcal{R}^{\wedge}}\big(R(x)\big)\big]$ *for all* $x \in X$ *and* $R \in \mathcal{R}$.

Proof. If (4) holds, then for any $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(R(x)\big)$ such that $\{x\} \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Hence, if $\mathcal{R} \neq \emptyset$, then by using Theorem 13.1 we can infer that $x \in \operatorname{int}_{\mathcal{R}}(V)$ and $V \subseteq \operatorname{int}_{\mathcal{R}}\big(R(x)\big)$. Therefore, $x \in \operatorname{int}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}\big(R(x)\big)\big)$, and thus (1) also holds.

Conversely, assume now that (1) holds and $A \subseteq X$ and $S \in \mathcal{R}^{\wedge}$. Define $V = \operatorname{int}_{\mathcal{R}}\big(S[A]\big)$. Then, if $\mathcal{R} \neq \emptyset$, by Theorem 16.8 and Corollary 13.2, we have $V \in \mathcal{T}_{\mathcal{R}} = \tau_{\mathcal{R}^{\wedge}}$. Moreover, since $V \subseteq \operatorname{int}_{\mathcal{R}}\big(S[A]\big)$, by Theorem 13.1 we also have $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)$. Therefore, $V \in \tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)$.

On the other hand, since $S \in \mathcal{R}^{\wedge}$ and $S[A] \subseteq S[A]$, we can also note that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)$. Hence, by using Theorem 13.1, we can infer that $A \subseteq \operatorname{int}_{\mathcal{R}}\big(S[A]\big) = V$. Moreover, since $V \in \tau_{\mathcal{R}^{\wedge}}$, we can also note that $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Hence, since $A \subseteq V$, we can infer that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Therefore, since $V \in \tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)$, we also have $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big[\tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)\big]$. This shows that (3) also holds. Moreover, it is clear that (3) $\Longrightarrow$ (2) $\Longrightarrow$ (4).

Thus, to complete the proof it remains only to note that if $\mathcal{R} = \emptyset$, then $\mathcal{R}$ is topological. Moreover, $\mathcal{R}^{\wedge} = \emptyset$ if $X \neq \emptyset$ and $\mathcal{R}^{\wedge} = \{\emptyset\}$ if $X = \emptyset$. Thus, $\mathcal{R}^{\wedge}$ is proximal. □

Now, as an immediate consequence of Theorems 17.3 and 17.7, we can also state

Theorem 17.8. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological,* *(2)* $\mathcal{R}^{\wedge}$ *is proximal.*

Remark 17.9. From Definition 12.2, it is clear that the relator $\mathcal{R}^{\wedge}$ is reflexive if and only if $\mathcal{R}$ is reflexive.

However, if $\mathcal{R} \not\subseteq \{X^2\}$, then there exists $R \in \mathcal{R}$ such that $R \neq X^2$. Therefore, there exist $x, y \in X$ such that $x \notin R(x)$. Thus, $S = \{x\} \times R(y) \cup \{x\}^c \times X$ is a non-reflexive relation on X such that $S \in \mathcal{R}^{\vartriangle}$. Therefore, $\mathcal{R}^{\vartriangle}$ cannot be reflexive.

18. Some basic facts on the elementwise unions of relators

Definition 18.1. For any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y, the relator

$$\mathcal{R} \vee \mathcal{S} = \big\{ R \cup S : \quad R \in \mathcal{R}, \quad S \in \mathcal{S} \big\}$$

is called the *elementwise union* of the relators $\mathcal{R}$ and $\mathcal{S}$.

Remark 18.2. If somewhat more generally $\mathcal{R} = \big(R_i\big)_{i \in I}$ and $\mathcal{S} = \big(S_i\big)_{i \in I}$, where R_i and S_i are relations on X to Y, then we may also naturally define

$$\mathcal{R} \triangledown \mathcal{S} = \big(R_i \cup S_i\big)_{i \in I}.$$

Thus, in particular for any relator $\mathcal{R}$ on X, we may also naturally write
$$\mathcal{R} \triangledown \mathcal{R}^{-1} = \{ R \cup R^{-1} : \ R \in \mathcal{R} \} \quad \text{and} \quad \mathcal{R} \vee \mathcal{R}^{-1} = \{ R \cup S^{-1} : \ R, S \in \mathcal{R} \}.$$

The importance of the relator $\mathcal{R} \vee \mathcal{S}$ is already apparent from the following

Theorem 18.3. *For any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y, we have*

(1) $\operatorname{Int}_{\mathcal{R} \vee \mathcal{S}} = \operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}}$; *(2)* $\operatorname{Cl}_{\mathcal{R} \vee \mathcal{S}} = \operatorname{Cl}_{\mathcal{R}} \cup \operatorname{Cl}_{\mathcal{S}}$.

Proof. If $B \subseteq Y$ and $A \in \operatorname{Int}_{\mathcal{R} \vee \mathcal{S}}(B)$, then there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that
$$(R \cup S) [A] \subseteq B .$$
Hence, by using that $(R \cup S) [A] = R [A] \cup S [A]$, we can already infer that $R [A] \cup S [A] \subseteq B$, and thus
$$R [A] \subseteq B \qquad \text{and} \qquad S [A] \subseteq B .$$
Therefore, $A \in \operatorname{Int}_{\mathcal{R}}(B)$ and $A \in \operatorname{Int}_{\mathcal{S}}(B)$, and thus
$$A \in \operatorname{Int}_{\mathcal{R}}(B) \cap \operatorname{Int}_{\mathcal{S}}(B) = \big(\operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}} \big)(B) .$$
This shows that
$$\operatorname{Int}_{\mathcal{R} \vee \mathcal{S}}(B) \subseteq \big(\operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}} \big)(B)$$
for all $B \subseteq Y$, and thus $\operatorname{Int}_{\mathcal{R} \vee \mathcal{S}} \subseteq \operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}}$ also holds.

The converse inclusion can be proved quite similarly. Moreover, assertion (2) can be derived from (1) by using Theorem 5.5. □

Now, as an immediate consequence of this theorem we can also state

Corollary 18.4. *For any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have*

(1) $\tau_{\mathcal{R} \vee \mathcal{S}} = \tau_{\mathcal{R}} \cap \tau_{\mathcal{S}}$; *(2)* $\bar{\tau}_{\mathcal{R} \vee \mathcal{S}} = \bar{\tau}_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{S}}$.

Proof. To prove (1), note that for any $A \subseteq X$, we have
$$A \in \tau_{\mathcal{R} \vee \mathcal{S}} \iff A \in \operatorname{Int}_{\mathcal{R} \vee \mathcal{S}}(A) \iff A \in \big(\operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}} \big)(A)$$
$$\iff A \in \operatorname{Int}_{\mathcal{R}}(A) \cap \operatorname{Int}_{\mathcal{S}}(A) \iff A \in \operatorname{Int}_{\mathcal{R}}(A) , \quad A \in \operatorname{Int}_{\mathcal{S}}(A)$$
$$\iff A \in \tau_{\mathcal{R}} , \quad A \in \tau_{\mathcal{S}} \iff A \in \tau_{\mathcal{R}} \cap \tau_{\mathcal{S}} .$$
□

Hence, by Theorem 8.6, it is clear that in particular we also have

Corollary 18.5. *For any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have*

(1) $\tau_{\mathcal{R} \vee \mathcal{S}^{-1}} = \tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{S}}$; *(2)* $\bar{\tau}_{\mathcal{R} \vee \mathcal{S}^{-1}} = \bar{\tau}_{\mathcal{R}} \cap \tau_{\mathcal{S}}$;

From Theorem 18.3, we can also immediately derive

Theorem 18.6. *For any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y, we have*

(1) $\operatorname{int}_{\mathcal{R} \vee \mathcal{S}} = \operatorname{int}_{\mathcal{R}} \cap \operatorname{int}_{\mathcal{S}}$; *(2)* $\operatorname{cl}_{\mathcal{R} \vee \mathcal{S}} = \operatorname{cl}_{\mathcal{R}} \cup \operatorname{cl}_{\mathcal{S}}$.

Now, as an immediate consequence of this theorem, we can also state

Corollary 18.7. *For any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have*

(1) $\mathcal{T}_{\mathcal{R} \vee \mathcal{S}} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{S}}$; *(2)* $\mathcal{F}_{\mathcal{R} \vee \mathcal{S}} = \mathcal{F}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{S}}$.

However, an analogue of Corollary 18.5 cannot be stated. Moreover, by using Theorem 18.6, we can only prove

Corollary 18.8. *For any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y, *we have*

(1) $\mathcal{E}_{\mathcal{R}\vee\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \cap \mathcal{E}_{\mathcal{S}}$; *(2)* $\mathcal{D}_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{S}} \subseteq \mathcal{D}_{\mathcal{R}\vee\mathcal{S}}$.

Remark 18.9. If $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y, then analogously to Definition 18.1 we may also naturally consider the *elementwise intersection*

$$\mathcal{R} \wedge \mathcal{S} = \big\{ R \cap S : \quad R \in \mathcal{R}\,, \quad S \in \mathcal{S} \big\}\,.$$

Namely, thus the relator $\mathcal{R}$ may, for instance, be naturally called *uniformly filtered* if $\mathcal{R} \wedge \mathcal{R} \subseteq \mathcal{R}^*$. That is, for any $R,\, S \in \mathcal{R}$ there exists $U \in \mathcal{R}$ such that $U \subseteq R \cap S$.

Thus, it can be shown that $\mathcal{R}$ is uniformly filtered if and only if $\mathcal{R}$ and $\mathcal{R} \wedge \mathcal{R}$ are *uniformly equivalent* in the sense that $\mathcal{R}^* = \big(\mathcal{R} \wedge \mathcal{R}\big)^*$. Or equivalently, $\mathcal{R}^*$ is *properly filtered* in the sense that $\mathcal{R}^* \wedge \mathcal{R}^* \subseteq \mathcal{R}^*$, or equivalently $\mathcal{R}^* \wedge \mathcal{R}^* = \mathcal{R}^*$.

Now, by using the above definition, we can also easily prove the following

Theorem 18.10. *If* $\mathcal{R}$ *is a uniformly filtered relator on* X *to* Y, *then for any* $\square \in \{ *,\, \#,\, \wedge,\, \vartriangle \}$, *we have*

$$\big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^{\square} = \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^{\square}.$$

Proof. By the corresponding definitions, we have $\mathcal{R} \triangledown \mathcal{R}^{-1} \subseteq \mathcal{R} \vee \mathcal{R}^{-1}$, and thus

$$\big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^{*} \subseteq \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^{*}.$$

On the other hand, if $V \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^{*}$, then there exist $R,\, S \in \mathcal{R}$ such that $R \cup S^{-1} \subseteq V$. Moreover, since $\mathcal{R}$ is uniformly filtered, there exists $U \in \mathcal{R}$ such that $U \subseteq R \cap S$. Hence, we can already see that $U \cup U^{-1} \subseteq R \cup S^{-1} \subseteq V$, and thus $V \in \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^{*}$. Therefore,

$$\big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^{*} \subseteq \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^{*},$$

and thus the corresponding equality is also true. Hence, since $*\,\square = \square$, it is clear that the required equality is also true. □

Thus, for instance, we can also state the following

Corollary 18.11. *If* $\mathcal{R}$ *is a uniformly filtered relator on* X, *then*

(1) $\tau_{\mathcal{R}\triangledown\mathcal{R}^{-1}} = \tau_{\mathcal{R}\vee\mathcal{R}^{-1}}$; *(2)* $\bar{\tau}_{\mathcal{R}\triangledown\mathcal{R}^{-1}} = \bar{\tau}_{\mathcal{R}\vee\mathcal{R}^{-1}}$;

(3) $\mathcal{T}_{\mathcal{R}\triangledown\mathcal{R}^{-1}} = \mathcal{T}_{\mathcal{R}\vee\mathcal{R}^{-1}}$; *(4)* $\mathcal{F}_{\mathcal{R}\triangledown\mathcal{R}^{-1}} = \mathcal{F}_{\mathcal{R}\vee\mathcal{R}^{-1}}$.

Remark 18.12. Analogously to Remark 18.9, a relator $\mathcal{R}$ on X to Y may be naturally called *topologically filtered* if the relator $\mathcal{R}^{\wedge}$ is properly filtered. However, since in general $R[A] \cap S[A] \not\subseteq (R \cap S)[A]$, to define "proximally filtered" we have two natural possibilities [167].

Moreover, for instance, a relator $\mathcal{R}$ on X may be naturally called *quasi-topologically filtered* if the relator $\mathcal{R}^{\wedge\infty}$ is properly filtered. Namely, thus it can be shown that $\mathcal{R}$ is quasi-topologically filtered if and only if the family $\mathcal{T}_{\mathcal{R}}$ is closed under binary intersections.

19. Further results on the elementwise unions of relators

Concerning the relator $\mathcal{R} \vee \mathcal{S}$, we can also easily prove the following

Theorem 19.1. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y and $\square \in \{ *, \#, \wedge \}$, then*

$$\big(\mathcal{R} \vee \mathcal{S} \big)^{\square} = \mathcal{R}^{\square} \cap \mathcal{S}^{\square}.$$

Proof. We shall only prove the $\square = \#$ particular case of the above equality. For this, note that if $V \in (\mathcal{R} \vee \mathcal{S})^{\#}$, then for each $A \subseteq X$ there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that

$$(R \cup S)[A] \subseteq V[A].$$

Hence, by using that $(R \cup S)[A] = R[A] \cup S[A]$, we can already infer that $R[A] \cup S[A] \subseteq V[A]$, and thus

$$R[A] \subseteq V[A] \qquad \text{and} \qquad S[A] \subseteq V[A].$$

Therefore, $V \in \mathcal{R}^{\#}$ and $V \in \mathcal{S}^{\#}$, and thus $V \in \mathcal{R}^{\#} \cap \mathcal{S}^{\#}$.

On the other hand, if $V \in \mathcal{R}^{\#} \cap \mathcal{S}^{\#}$, then $V \in \mathcal{R}^{\#}$ and $V \in \mathcal{S}^{\#}$. Therefore, for each $A \subseteq X$, there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that

$$R[A] \subseteq V[A] \qquad \text{and} \qquad S[A] \subseteq V[A].$$

Hence, it follows that

$$(R \cup S)[A] = R[A] \cup S[A] \subseteq V[A],$$

and thus $V \in (\mathcal{R} \vee \mathcal{S})^{\#}$. □

Remark 19.2. By using a similar argument, concerning the operation $\vartriangle$, we can only prove that

$$\big(\mathcal{R} \vee \mathcal{S} \big)^{\vartriangle} \subseteq \mathcal{R}^{\vartriangle} \cap \mathcal{S}^{\vartriangle}.$$

From Theorem 19.1, we can easily derive the following

Corollary 19.3. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y and $\square \in \{ *, \#, \wedge \}$, then*

(1) $\big(\mathcal{R} \vee \mathcal{S} \big)^{\square} = \big(\mathcal{R}^{\square} \vee \mathcal{S}^{\square} \big)^{\square}$; *(2)* $\mathcal{R}^{\square} \cap \mathcal{S}^{\square} = \big(\mathcal{R}^{\square} \cap \mathcal{S}^{\square} \big)^{\square}$.

Proof. By Theorem 19.1 and the idempotency of $\square$, it is clear that

$$(\mathcal{R} \vee \mathcal{S})^{\square} = \mathcal{R}^{\square} \cap \mathcal{S}^{\square} = \mathcal{R}^{\square\square} \cap \mathcal{S}^{\square\square} = \big(\mathcal{R}^{\square} \vee \mathcal{S}^{\square} \big)^{\square}.$$

□

Remark 19.4. From assertion (1), it is clear that

$$(\mathcal{R} \vee \mathcal{S})^{\wedge} \subseteq \big(\mathcal{R}^{\#} \vee \mathcal{S}^{\#} \big)^{\wedge} \subseteq \big(\mathcal{R}^{\wedge} \vee \mathcal{S}^{\wedge} \big)^{\wedge} = (\mathcal{R} \vee \mathcal{S})^{\wedge},$$

and thus in particular $(\mathcal{R} \vee \mathcal{S})^{\wedge} = \big(\mathcal{R}^{\#} \vee \mathcal{S}^{\#} \big)^{\wedge}$ is also true.

While, from assertion (2), we can at once see that the relator $\mathcal{R}^{\square} \cap \mathcal{S}^{\square}$ is always $\square$–*fine*. Moreover, if $\mathcal{R}$ and $\mathcal{S}$ are $\square$–fine, then $\mathcal{R} \cap \mathcal{S}$ is also $\square$–fine.

In addition to Theorem 19.1, we can also easily prove the following

Theorem 19.5. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X to Y and $\square \in \{ *, \#, \wedge \}$, then the following assertions are equivalent:*

(1) $\mathcal{R} \vee \mathcal{S} \subseteq \big(\mathcal{R} \cap \mathcal{S} \big)^{\square}$;

(2) $\big(\mathcal{R} \vee \mathcal{S} \big)^{\square} \subseteq \big(\mathcal{R} \cap \mathcal{S} \big)^{\square}$; *(3)* $\big(\mathcal{R} \vee \mathcal{S} \big)^{\square} = \big(\mathcal{R} \cap \mathcal{S} \big)^{\square}$;

(4) $\mathcal{R}^{\square}\cap\mathcal{S}^{\square}\subseteq\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$; *(5)* $\mathcal{R}^{\square}\cap\mathcal{S}^{\square}=\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$.

Proof. Since $\square$ is a closure operation for relators, it is clear that assertions (1) and (2) are equivalent.

Moreover, we can note that $\mathcal{R}\cap\mathcal{S}\subseteq\mathcal{R}\vee\mathcal{S}$, and thus $(\mathcal{R}\cap\mathcal{S})^{\square}\subseteq(\mathcal{R}\vee\mathcal{S})^{\square}$. Therefore, assertions (2) and (3) are equivalent.

On the other hand, by Theorem 19.1, it is clear that the equivalences $(2)\Longleftrightarrow(4)$ and $(3)\Longleftrightarrow(5)$ are also true. $\square$

Now, combining Theorems 18.3 and 19.5, we can also easily establish

Theorem 19.6. *For any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}\vee\mathcal{S}\subseteq\big(\mathcal{R}\cap\mathcal{S}\big)^{\#}$;

(2) $\operatorname{Int}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Int}_{\mathcal{R}}\cap\operatorname{Int}_{\mathcal{S}}$; *(3)* $\operatorname{Cl}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Cl}_{\mathcal{R}}\cup\operatorname{Cl}_{\mathcal{S}}$.

Proof. If assertion (1) holds, then by Theorem 19.5 we also have $(\mathcal{R}\cap\mathcal{S})^{\#}=(\mathcal{R}\vee\mathcal{S})^{\#}$. Hence, by Theorem 12.9, it follows that $\operatorname{Int}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Int}_{\mathcal{R}\vee\mathcal{S}}$. Therefore, by Theorem 18.3, assertion (2) also holds.

On the other hand, if assertion (2) holds, then by Theorem 18.3 we also have $\operatorname{Int}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Int}_{\mathcal{R}\vee\mathcal{S}}$. Hence, again by Theorem 12.9, it follows that $(\mathcal{R}\cap\mathcal{S})^{\#}=(\mathcal{R}\vee\mathcal{S})^{\#}$. Therefore, in particular, assertion (1) also holds.

Finally, to complete the proof, we note that the equivalence of assertions (2) and (3) can be easily proved with the help of Theorem 5.5. $\square$

Analogously to this theorem, we can also prove the following

Theorem 19.7. *For any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}\vee\mathcal{S}\subseteq\big(\mathcal{R}\cap\mathcal{S}\big)^{\wedge}$;

(2) $\operatorname{int}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{int}_{\mathcal{R}}\cap\operatorname{int}_{\mathcal{S}}$; *(3)* $\operatorname{cl}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{cl}_{\mathcal{R}}\cup\operatorname{cl}_{\mathcal{S}}$.

20. Quasi-proximally and quasi-topologically minimal relators

Analogously to the definition of a minimal topology, we may naturally introduce

Definition 20.1. A relator $\mathcal{R}$ on X will be called

(1) *quasi-proximally minimal* if $\tau_{\mathcal{R}}\subseteq\{\emptyset,\,X\}$;

(2) *quasi-topologically minimal* if $\mathcal{T}_{\mathcal{R}}\subseteq\{\emptyset,\,X\}$.

Remark 20.2. If in particular $\mathcal{R}\neq\emptyset$, then by Theorems 8.7 and 8.14 we have $\{\emptyset,\,X\}\subseteq\tau_{\mathcal{R}}\subseteq\mathcal{T}_{\mathcal{R}}$. Therefore, in this case, we may write equalities instead of inclusions in the above definition.

The use of the term quasi-proximally and quasi-topologically instead of proximally and topologically is only motivated by the fact that the families $\tau_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ are, in general, much weaker tools than the relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$.

Now, as an immediate consequence of Definition 20.1, we can state

Theorem 20.3. *If $\mathcal{R}$ is a quasi-topologically minimal relator on X, then $\mathcal{R}$ is quasi-proximally minimal.*

Proof. By Theorem 8.14, we have $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X. Moreover, if $\mathcal{R}$ is quasi-topologically minimal, then we also have $\mathcal{T}_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Therefore, in this case $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$ also holds. Thus, $\mathcal{R}$ is quasi-proximally minimal. □

Moreover, by using Definition 20.1, we can also easily prove the following

Theorem 20.4. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically minimal;

(2) $\mathcal{R}^{\wedge}$ is quasi-proximally minimal.

Proof. Note that if $\mathcal{R} \neq \emptyset$, then by Corollary 13.2 we have $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$. Therefore, $\tau_{\mathcal{R}^{\wedge}} \subseteq \{\emptyset, X\}$ if and only if $\mathcal{T}_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Thus, assertions (1) and (2) are equivalent.

While, if $\mathcal{R} = \emptyset$, then it is clear that $\mathcal{T}_{\mathcal{R}} = \{\emptyset\}$. Therefore, assertion (1) holds. Moreover, we can also note that

(a) if $X \neq \emptyset$, then $\mathcal{R}^{\wedge} = \emptyset$, and thus $\tau_{\mathcal{R}^{\wedge}} = \emptyset$;

(b) if $X = \emptyset$, then $\mathcal{R}^{\wedge} = \{\emptyset\}$, and thus $\tau_{\mathcal{R}^{\wedge}} = \{\emptyset\}$.

Therefore, assertion (2) also holds.

Consequently, if $\mathcal{R} = \emptyset$, then assertions (1) and (2) trivially hold. Thus, in particular, they are equivalent. □

Remark 20.5. Note that $\mathcal{R} \subseteq \mathcal{R}^{\wedge}$, and thus $\tau_{\mathcal{R}} \subseteq \tau_{\mathcal{R}^{\wedge}}$. Therefore, the quasi-proximal minimality of $\mathcal{R}^{\wedge}$ implies that of $\mathcal{R}$. Thus, Theorem 20.3 can be derived from Theorem 20.4.

Now, as an immediate consequence of Theorem 20.4, we can also state

Corollary 20.6. *If $\mathcal{R}$ is a topologically fine relator on X, then $\mathcal{R}$ is quasi-proximally minimal if and only if $\mathcal{R}$ is quasi-topologically minimal.*

Proof. In this case, we have $\mathcal{R}^{\wedge} = \mathcal{R}$. Therefore, by Theorem 20.4, the required assertion is true. □

In addition to this corollary, it is also worth proving the following

Theorem 20.7. *If $\mathcal{R}$ is a proximally simple relator on X, then $\mathcal{R}$ is quasi-proximally minimal if and only if $\mathcal{R}$ is quasi-topologically minimal.*

Proof. Now, there exists a relation S on X that $\mathcal{R}^{\#} = \{S\}^{\#}$, and thus also $\mathcal{R}^{\wedge} = \{S\}^{\wedge}$. Hence, by using Theorem 12.11 and Remark 8.15, we can see that

$$\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\#}} = \tau_{\{S\}^{\#}} = \tau_{\{S\}} = \mathcal{T}_{\{S\}} = \mathcal{T}_{\{S\}^{\wedge}} = \mathcal{T}_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}.$$

Therefore, by Definition 20.1, the required assertion is true. □

Concerning quasi-minimal relators, we can also easily prove the following two theorems.

Theorem 20.8. *A relator $\mathcal{R}$ on X is quasi-proximally minimal if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$ and $\mathcal{R}^{-1}$ is quasi-proximally minimal.*

Proof. Recall that $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\square}}$ for all $\square \in \{\infty, *, \#\}$, and moreover

$$\tau_{\mathcal{R}^{-1}} = \tau\!\!\!-_{\mathcal{R}} = \{ A^c : \quad A \in \tau_{\mathcal{R}} \}.$$

Therefore, by Definition 20.1, the required assertion is true. □

Remark 20.9. From this theorem, for instance, we can see that the relator $\mathcal{R}$ is quasi-proximally minimal if and only if any one of the relators $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally minimal.

Theorem 20.10. *A relator $\mathcal{R}$ on X is quasi-topologically minimal if and only if any one of the relators $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$ and $\mathcal{R}^{\wedge\infty}$ is quasi-topologically minimal.*

Proof. Recall that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$ for all $\square \in \{*, \#, \wedge, \wedge\infty\}$. Therefore, by Definition 20.1, the required assertion is true. □

Remark 20.11. Note that $\mathcal{R}^\infty \subseteq \mathcal{R}^\wedge$, and thus $\mathcal{T}_{\mathcal{R}^\infty} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, if $\mathcal{R}$ is quasi-topologically minimal, then $\mathcal{R}^\infty$ is also quasi-topologically minimal.

21. The main characterizations of quasi-minimal relators

From Theorem 20.4, we can see that the properties of the quasi-topologically minimal relators can, in priciple, be immediately derived from those of the quasi-proximally minimal ones.

Therefore, it is of major importance to prove the following basic characterization theorem of quasi-proximally minimal relators.

Theorem 21.1. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally minimal;*

(2) $\mathcal{R} \subseteq \{X^2\}^\partial$; *(3)* $\mathcal{R}^\infty \subseteq \{X^2\}$;

(4) $\mathcal{R}^\# \subseteq \{X^2\}^\partial$; *(5)* $\mathcal{R}^{\#\infty} \subseteq \{X^2\}$.

Proof. By taking $\mathcal{S} = \{X^2\}$, we can note that

$$\tau_{\mathcal{S}} = \{\emptyset, X\} \qquad \text{and} \qquad \mathcal{S} = \mathcal{S}^\#.$$

Moreover, by using Theorem 14.7 and the Galois property of the operations ∞ and ∂, we can easily see that

$$\tau_{\mathcal{R}} \subseteq \{\emptyset, X\} \iff \tau_{\mathcal{R}} \subseteq \tau_{\mathcal{S}} \iff \mathcal{R} \subseteq \mathcal{S}^{\#\partial}$$
$$\iff \mathcal{R}^\infty \subseteq \mathcal{S}^\# \iff \mathcal{R}^\infty \subseteq \mathcal{S} \iff \mathcal{R} \subseteq \mathcal{S}^\partial.$$

Therefore, assertions (1), (2) and (3) are equivalent.

Now, by using Theorem 20.8 and the above equivalences, we can see that assertions (1), (4) and (5) are also equivalent. □

Remark 21.2. Note that, by Theorem 20.8, instead of $\#$ we may also write ∞, $*$ or -1 in the assertions (4) and (5) of the above theorem.

Detailed reformulations of assertion (3) of Theorem 21.1 give the following

Corollary 21.3. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally minimal;*

(2) for each $R \in \mathcal{R}$ *we have* $R^\infty = X^2$;

(3) *for each* $R \in \mathcal{R}$ *and* $a, b \in X$ *there exists* $n \in \{0\} \cup \mathbb{N}$ *such that* $(a, b) \in R^n$ *;*

(4) *for each* $R \in \mathcal{R}$ *and* $a, b \in X$ *there exist* $n \in \{0\} \cup \mathbb{N}$ *and a family* $(x_i)_{i=0}^n$ *in* X *such that* $x_0 = a$*,* $x_n = b$ *and* $(x_{i-1}, x_i) \in R$ *for all* $i = 1, 2, \ldots, n$*.*

Proof. To derive this from Theorem 21.1, recall that

$$\mathcal{R}^\infty = \{ R^\infty : \ R \in \mathcal{R} \} \qquad \text{with} \qquad R^\infty = \bigcup_{n=0}^{\infty} R^n ,$$

where $R^n = \Delta_X$ if $n = 0$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. □

Remark 21.4. From the equivalence of assertions (1) and (4) in this corollary, we can see that, for Euclidean and metric spaces, our quasi-proximal minimalness coincides with the well-chainedness (chain-connectedness) studied by G. Cantor in 1883. (See Thron [212, p. 29], and also Wilder [216].)

While, from the equivalence of assertions (1) and (3) in Theorem 21.1, we can see that, for uniformities and nonvoid relators, our quasi-proximal minimalness coincides with the *well-chainedness* and *proper well-chainedness* studied mainly by Levine [100] and Kurdics, Pataki and Száz [86, 90, 91, 137].

Now, as an immediate consequence of Theorems 20.4 and 21.1, we can also state

Theorem 21.5. *For a relator* $\mathcal{R}$ *on* X*, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topologically minimal;*

(2) $\mathcal{R}^\wedge \subseteq \{X^2\}^\partial$*;* *(3)* $\mathcal{R}^{\wedge\infty} \subseteq \{X^2\}$*.*

Remark 21.6. By Theorems 21.1 and 21.5, a relator $\mathcal{R}$ on X to Y may be naturally called $\square$*–minimal*, for some unary operation $\square$ for relators on X to Y, if $\mathcal{R}^\square \subseteq \{X \times Y\}$.

Moreover, in particular a relator $\mathcal{R}$ on X may be naturally called *quasi–*$\square$*–minimal*, for some unary operation $\square$ for relators on X, if it is $\square\infty$–minimal. That is, $\mathcal{R}^{\square\infty} \subseteq \{X^2\}$.

22. Further characterizations of quasi-minimal relators

A simple reformulation of Definition 20.1 gives the following

Theorem 22.1. *For a relator* $\mathcal{R}$ *on* X*, the following assertions hold:*

(1) $\mathcal{R}$ *is quasi-proximally minimal if and only if* $\bar{\tau}_{\mathcal{R}} \subseteq \{\emptyset, X\}$*;*

(2) $\mathcal{R}$ *is quasi-topologically minimal if and only if* $\mathcal{F}_{\mathcal{R}} \subseteq \{\emptyset, X\}$*.*

Proof. By Theorem 8.5, for any $A \subseteq X$, we have $A \in \bar{\tau}_{\mathcal{R}}$ if and only if $A^c \in \tau_{\mathcal{R}}$. Hence, it is clear that $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$ if and if $\bar{\tau}_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Therefore, assertion (1) is true. Assertion (2) can be proved quite similarly by using Theorem 8.13. □

Concerning quasi-proximally minimal relators, we can also easily prove

Theorem 22.2. *For a relator* $\mathcal{R}$ *on* X *with* $\emptyset \notin \mathcal{R}$*, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally minimal;*

(2) $X = A \cup B$ *implies* $A \in \mathrm{Cl}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq X$ *with* $A, B \neq \emptyset$*;*

(3) $A \in \mathrm{Int}_{\mathcal{R}}(B)$ *implies* $B \not\subseteq A$ *for all* $A, B \subseteq X$ *with* $A \neq \emptyset$ *and* $B \neq X$*.*

Proof. If (1) does not hold, then $\tau_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$. Therefore, there exists $A \in \tau_{\mathcal{R}}$ such that $A \neq \emptyset$ and $A \neq X$. Hence, since $A \subseteq A$ and $A \in \mathrm{Int}_{\mathcal{R}}(A)$, it is clear that (3) does not also hold. Therefore, (3) implies (1).

Conversely, if (3) does not hold, then there exist $A, B \subseteq X$ such that

$$A \neq \emptyset, \qquad B \neq X, \qquad B \subseteq A \qquad \text{and} \qquad A \in \mathrm{Int}_{\mathcal{R}}(B).$$

Hence, by the definition of the relation $\mathrm{Int}_{\mathcal{R}}$, it is clear that we also have

$$A \in \mathrm{Int}_{\mathcal{R}}(A) \qquad \text{and} \qquad B \in \mathrm{Int}_{\mathcal{R}}(B),$$

and thus $A, B \in \tau_{\mathcal{R}}$. Now, we can already see that $\tau_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$, and thus (1) does not also hold. Therefore, (1) also implies (3).

Namely, if $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$, then because of $A, B \in \tau_{\mathcal{R}}$ we also have $A, B \in \{\emptyset, X\}$. Hence, since $A \neq \emptyset$ and $B \neq X$, we can infer that $A = X$ and $B = \emptyset$. Therefore, because of $A \in \mathrm{Int}_{\mathcal{R}}(B)$, we actually have $X \in \mathrm{Int}_{\mathcal{R}}(\emptyset)$. Thus, there exists $R \in \mathcal{R}$ such that $R[X] \subseteq \emptyset$, and thus $R[X] = \emptyset$. This implies that $R = \emptyset$, and thus $\emptyset \in \mathcal{R}$. And, this is a contradiction.

Now, to complete the proof, it remains only to show that (2) and (3) are also equivalent. For this, note that if for instance (2) does not hold, then there exist $A, B \subseteq X$ such that

$$A, B \neq \emptyset, \qquad X = A \cup B \qquad \text{and} \qquad A \notin \mathrm{Cl}_{\mathcal{R}}(B).$$

Hence, by using that $\mathrm{Cl}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \mathrm{Int}_{\mathcal{R}}(B^c)$, we can infer that $A \in \mathrm{Int}_{\mathcal{R}}(B^c)$. Moreover, since $B \neq \emptyset$ and $X = A \cup B$, we can also note that $B^c \neq X$ and $B^c \subseteq A$. Therefore, assertion (3) does not also hold. This shows that (3) implies (2). □

Remark 22.3. Note that the implications (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (1) do not required the extra condition on $\mathcal{R}$ that $\emptyset \notin \mathcal{R}$.

Moreover, if $\mathcal{R}$ is a quasi-proximally minimal relator on X such that $\emptyset \in \mathcal{R}$, then by the definitions of ∞ and Theorem 21.1 we necessarily have

$$\Delta_X = \emptyset^\infty \in \mathcal{R}^\infty \subseteq \{X^2\},$$

and thus $\Delta_X = X^2$. Therefore, X is either the empty set or a singleton. Consequently, in Theorem 22.2, instead of $\emptyset \notin \mathcal{R}$ we may also naturally assume that $\mathrm{card}(X) > 1$.

Theorem 22.4. *If* $\mathrm{card}(X) > 1$*, then for a relator* $\mathcal{R}$ *on* X *the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topologically minimal;*

(2) $X = A \cup B$ *implies* $A \cap \mathrm{cl}_{\mathcal{R}}(B) \neq \emptyset$ *for all* $A, B \subseteq X$ *with* $A, B \neq \emptyset$*;*

(3) $A \subseteq \mathrm{int}_{\mathcal{R}}(B)$ *implies* $B \not\subseteq A$ *for all* $A, B \subseteq X$ *with* $A \neq \emptyset$ *and* $B \neq X$*.*

Proof. If $\mathcal{R} \neq \emptyset$, then by Theorems 20.4, 22.2 and 13.1 it is clear that the following assertions are equivalent:

(a) $\mathcal{R}$ is quasi-topologically minimal; (b) $\mathcal{R}^{\wedge}$ is quasi-proximally minimal;

(c) $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$ implies $B \not\subseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$;

(d) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ implies $B \not\subseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$.

Therefore, in this case, assertions (1) and (3) are equivalent.

While, if $\mathcal{R} = \emptyset$, then it is clear that $\mathcal{T}_{\mathcal{R}} = \{\emptyset\}$, and thus (1) trivially holds. Moreover, in this case, we can note that $\operatorname{int}_{\mathcal{R}}(B) = \emptyset$. Therefore, if $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$, then $A = \emptyset$. Thus, (3) also trivially holds.

Now, it remains only to show that (2) and (3) are also equivalent. For this, one can recall that $\operatorname{cl}_{\mathcal{R}}(B) = X \setminus \operatorname{int}_{\mathcal{R}}\big(B^c\big)$ for all $B \subseteq X$. Therefore, a similar argument as in the proof of Theorem 22.2 can be applied. □

Remark 22.5. Note that in this theorem, instead of $\operatorname{card}(X) > 1$ we may also assume that $\emptyset \notin \mathcal{R}^{\wedge}$. That is, there exists $x \in X$ such that for any $R \in \mathcal{R}$ we have $R(x) \not\subseteq \emptyset$, and thus $R(x) \neq \emptyset$.

23. Paratopologically minimal relators

Analogously to the definition of a minimal topology, a stack (ascending family) $\mathcal{A}$ of subsets of a set X may be naturally called minimal if $\mathcal{A} \subseteq \{X\}$.

Therefore, having in mind the family $\mathcal{E}_{\mathcal{R}}$ of all fat sets generated by a relator $\mathcal{R}$, we may also naturally introduce the following

Definition 23.1. A relator $\mathcal{R}$ on X to Y will be called *paratopologically minimal* if

$$\mathcal{E}_{\mathcal{R}} \subseteq \{Y\}.$$

Remark 23.2. Note that if a relator $\mathcal{R}$ on X to Y is non-degenerated in the sense that both X and $\mathcal{R}$ are nonvoid, then by Theorem 7.13 we have $Y \in \mathcal{E}_{\mathcal{R}}$. Therefore, in this case, we may write equality instead of inclusion in the above definition.

The following theorems will show that paratopological minimalness is a much stronger property than quasi-topological minimalness.

Theorem 23.3. *If $\mathcal{R}$ is a paratopologically minimal relator on X, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically minimal.*

Proof. By Theorem 8.16, we have $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X. Moreover, if $\mathcal{R}$ is paratopologically minimal, then we also have $\mathcal{E}_{\mathcal{R}} \subseteq \{X\}$. Therefore, in this case $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \{X\}$ also holds. Hence, we can infer that $\mathcal{T}_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Therefore, $\mathcal{R}$ is quasi-topologically minimal. Now, by Theorem 20.3, we can see that $\mathcal{R}$ is quasi-proximally minimal too. □

From this theorem, we can easily derive the following stronger statement.

Corollary 23.4. *If $\mathcal{R}$ is a paratopologically minimal relator on X, then the relator $\mathcal{R}^{\triangle}$ is also both quasi-proximally and quasi-topologically minimal.*

Proof. By Theorem 12.11, we have $\mathcal{E}_{\mathcal{R}^{\triangle}} = \mathcal{E}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X to Y. Therefore, if in particlar $\mathcal{R}$ is a paratopologically minimal relator on X, then the relator $\mathcal{R}^{\triangle}$ is also paratopologically minimal. Thus, by Theorem 23.3, it has the required quasi-minimalness properties. □

Now, in addition to this corollary, we can also easily prove the following

Theorem 23.5. *For a non-partial relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically minimal;*

(2) $\mathcal{R}^{\triangle}$ *is quasi-proximally minimal;*

(3) $\mathcal{R}^{\triangle}$ *is quasi-topologically minimal.*

Proof. From Corollary 23.4, we know that (1) implies (2). Moreover, from Theorem 12.8, we know that $\mathcal{R}^{\triangle\wedge} = \mathcal{R}^{\triangle}$. Therefore, by Corollary 20.6, assertions (2) and (3) are equivalent.

Thus, we need only prove that (3) also implies (1). For this, note that if (3) holds, then by Corollary 13.8 and Definition 20.1 we have

$$\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\} \subseteq \{\,\emptyset\,,\ X\,\} \setminus \{\emptyset\} = \{X\}\,.$$

Therefore, by Definition 23.1, $\mathcal{R}$ is paratopologically minimal. □

Now, combining Theorems 21.1 and 23.5, we can also state

Theorem 23.6. *For a non-partial relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically minimal;*

(2) $\mathcal{R}^{\triangle} \subseteq \{\,X^2\,\}^{\partial}$; *(3)* $\mathcal{R}^{\triangle\infty} \subseteq \{\,X^2\,\}$.

Remark 23.7. Note that the implications $(1) \Longrightarrow (2) \Longleftrightarrow (3)$ in the above two theorems do not require the relator $\mathcal{R}$ to be non-partial.

Moreover, by Theorem 23.6, a non-partial relator $\mathcal{R}$ on X is paratopologically minimal if and only if it is quasi–$\triangle$–minimal in the sense of Remark 21.6.

24. Further characterizations of paratopologically minimal relators

By using Definition 23.1, we can also easily prove the following

Theorem 24.1. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically minimal;* *(2)* $\mathcal{R} \subseteq \{\,X \times Y\,\}$.

Proof. If $R \in \mathcal{R}$, then by Theorem 7.3 we have $R(x) \in \mathcal{E}_{\mathcal{R}}$ for all $x \in X$. Hence, if (1) holds, i.e., $\mathcal{E}_{\mathcal{R}} \subseteq \{\,Y\,\}$, we can infer that $R(x) = Y$ for all $x \in X$, and thus $R = X \times Y$. This shows that either $\mathcal{R} = \emptyset$ or $\mathcal{R} = \{\,X \times Y\,\}$. Therefore, (2) also holds.

Conversely, if (2) holds, then either $\mathcal{R} = \emptyset$ or $\mathcal{R} = \{\,X \times Y\,\}$. Now, if $\mathcal{R} = \emptyset$, then by Theorem 7.3 we can see that $\mathcal{E}_{\mathcal{R}} = \emptyset$. While, if $\mathcal{R} = \{\,X \times Y\,\}$, then we can note that $\mathcal{E}_{\mathcal{R}} = \emptyset$ if $X = \emptyset$ and $\mathcal{E}_{\mathcal{R}} = \{Y\}$ if $X \neq \emptyset$. Therefore, in both cases, $\mathcal{E}_{\mathcal{R}} \subseteq \{\,Y\,\}$, and thus (1) also holds. □

Remark 24.2. By this theorem and Remark 21.6, a relator $\mathcal{R}$ on X to Y is paratopologically minimal if and only if it is $\Box$–minimal with $\Box$ being the identity operation for relators.

Now, analogously to Theorems 20.8 and 20.10, we can also easily prove

Theorem 24.3. *A relator $\mathcal{R}$ on X to Y is paratopologically minimal if and only if any one of the relators $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$, $\mathcal{R}^\vartriangle$ and $\mathcal{R}^{-1}$ is paratopologically minimal.*

Proof. By Theorems 12.8 and 12.11, we have $\mathcal{E}_\mathcal{R} = \mathcal{E}_{\mathcal{R}^\Box}$ for all $\Box \in \{*, \#, \wedge, \vartriangle\}$. Therefore, by Definition 23.1, the paratopological minimalness of $\mathcal{R}$ is equivalent to that of $\mathcal{R}^\Box$.

Moreover, we evidently have $\mathcal{R} \subseteq \{X \times Y\}$ if and only if $\mathcal{R}^{-1} \subseteq \{Y \times X\}$. Therefore, by Theorem 24.1, the paratopological minimalness of $\mathcal{R}$ is also equivalent to that of $\mathcal{R}^{-1}$. $\Box$

Remark 24.4. Note that $\mathcal{R}^\infty \subseteq \mathcal{R}^\vartriangle$, and thus $\mathcal{E}_{\mathcal{R}^\infty} \subseteq \mathcal{E}_\mathcal{R}$. Therefore, if $\mathcal{R}$ is paratopologically minimal, then $\mathcal{R}^\infty$ is also paratopologically minimal.

Moreover, as some useful reformulation of Definition 23.1, we can also easily prove the following two theorems.

Theorem 24.5. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically minimal;* *(2)* $\mathcal{P}(Y) \setminus \{\emptyset\} \subseteq \mathcal{D}_\mathcal{R}$.

Proof. If (1) holds, then $\mathcal{E}_\mathcal{R} \subseteq \{Y\}$. Hence, by using Theorem 7.6, we can see that

$$B \neq \emptyset \implies B^c \neq Y \implies B^c \notin \mathcal{E}_\mathcal{R} \implies B \in \mathcal{D}_\mathcal{R}$$

for all $B \subseteq Y$. Therefore, (2) also holds.

Conversely, if (2) holds, then by using Theorem 7.6 we can similarly see that

$$B \in \mathcal{E}_\mathcal{R} \implies B^c \notin \mathcal{D}_\mathcal{R} \implies B^c = \emptyset \implies B = Y.$$

Therefore, $\mathcal{E}_\mathcal{R} \subseteq \{Y\}$, and thus (1) also holds. $\Box$

Theorem 24.6. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically minimal;*

(2) $\operatorname{Int}_\mathcal{R}(B) \subseteq \{\emptyset\}$ *for all* $B \subseteq Y$ *with* $B \neq Y$;

(3) $\mathcal{P}(X) \setminus \{\emptyset\} \subseteq \operatorname{Cl}_\mathcal{R}(B)$ *for all* $B \subseteq Y$ *with* $B \neq \emptyset$.

Proof. If $S = X \times Y$, then we can note that $\{S\}^\# = \{\mathcal{S}\}$. Moreover, by using Theorems 24.1 and 12.9, we can see that

$$(1) \iff \mathcal{R} \subseteq \{S\} \iff \mathcal{R} \subseteq \{S\}^\# \iff \operatorname{Int}_\mathcal{R} \subseteq \operatorname{Int}_S.$$

On the other hand, we can also note that

$$S[A] = \emptyset \quad \text{if} \quad A = \emptyset \qquad \text{and} \qquad S[A] = Y \quad \text{if} \quad A \neq \emptyset$$

for all $A \subseteq X$. Hence, by using Definition 5.2, we can see that

$$\operatorname{Int}_S(B) = \{\emptyset\} \quad \text{if} \quad B \neq Y \qquad \text{and} \qquad \operatorname{Int}_S(B) = \mathcal{P}(X) \quad \text{if} \quad B = Y$$

for all $B \subseteq Y$. Therefore,

$$(1) \iff \operatorname{Int}_{\mathcal{R}} \subseteq \operatorname{Int}_{S} \iff \forall\ B \subseteq Y : \quad \operatorname{Int}_{\mathcal{R}}(B) \subseteq \operatorname{Int}_{S}(B)$$
$$\iff \forall\ B \in \mathcal{P}(Y) \setminus \{Y\} : \quad \operatorname{Int}_{\mathcal{R}}(B) \subseteq \{\emptyset\} \iff (2).$$

The equivalence of assertions (2) and (3) can be easily proved with the help of Theorem 5.5. □

Now, as an immediate consequence of this theorem, we can also state

Theorem 24.7. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically minimal;

(2) $\operatorname{int}_{\mathcal{R}}(B) = \emptyset$ for all $B \subseteq Y$ with $B \neq Y$;

(3) $X = \operatorname{cl}_{\mathcal{R}}(B)$ for all $B \subseteq Y$ with $B \neq \emptyset$.

Proof. To prove the equivalence of assertions (2) of Theorems 24.6 and 24.7, note that for any $B \subseteq Y$, we have

$$\operatorname{int}_{\mathcal{R}}(B) = \emptyset \iff \operatorname{Int}_{\mathcal{R}}(B) \subseteq \{\emptyset\}.$$

□

From this theorem, it is clear that in particular we also have

Corollary 24.8. *If in particular $\mathcal{R}$ is a paratopologically minimal relator on X, then $\mathcal{T}_{\mathcal{R}} \setminus \{X\} \subseteq \{\emptyset\}$, and thus also $\tau_{\mathcal{R}} \setminus \{X\} \subseteq \{\emptyset\}$.*

Remark 24.9. Analogously to the various minimal relators, the corresponding maximal relators can also be naturally introduced.

However, these are certainly less important than the corresponding minimal ones which are generalizations of well-chained (chain-connected) uniformities.

25. Quasi-proximally and quasi-topologically connected relators

Analogously to the definition of a connected topology, we may naturally introduce the following

Definition 25.1. A relator $\mathcal{R}$ on X will be called

(1) *quasi-proximally connected* if $\tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}} \subseteq \{\emptyset, X\}$;

(2) *quasi-topologically connected* if $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{\emptyset, X\}$.

Remark 25.2. If in particular $\mathcal{R} \neq \emptyset$, then by Theorems 8.7 and 8.14 we have $\{\emptyset, X\} \subseteq \tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$. Therefore, in this case, we may write equalities instead of inclusions in the above definition.

By using Definitions 20.1 and 25.1, we can easily establish the following

Theorem 25.3. *If $\mathcal{R}$ is a quasi-proximally (quasi-topologically) minimal relator on X, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) connected.*

Proof. If $\mathcal{R}$ is a quasi-proximally minimal relator on X, then by Definition 20.1 $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Thus, in particular we also have $\tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Therefore, by Definition 25.1, $\mathcal{R}$ is quasi-proximally connected.

This proves the first statement of the theorem. The second statement can be proved quite similarly. □

Now, as an immediate consequence of Theorems 23.3 and 25.3, we can also state

Corollary 25.4. *If $\mathcal{R}$ is a paratopologically minimal relator on X, then $\mathcal{R}$ is both quasi-proximally quasi-topologically connected.*

Moreover, analogously to the corresponding results of Section 20, we can also prove the following statements.

Theorem 25.5. *If $\mathcal{R}$ is a quasi-topologically connected relator on X, then $\mathcal{R}$ is quasi-proximally connected.*

Proof. By Theorem 8.14 and Definition 25.1, we have $\tau_{\mathcal{R}} \cap \digamma_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Therefore, by Definition 25.1, $\mathcal{R}$ is quasi-proximally connected. $\square$

Theorem 25.6. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically connected;

(2) $\mathcal{R}^{\wedge}$ is quasi-proximally connected.

Proof. If $\mathcal{R} \neq \emptyset$, then by Corollary 13.2 we have $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$ and $\digamma_{\mathcal{R}^{\wedge}} = \mathcal{F}_{\mathcal{R}}$, and thus also $\tau_{\mathcal{R}^{\wedge}} \cap \digamma_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$. Therefore, by Definition 25.1, assertions (1) and (2) are equivalent.

While, if $\mathcal{R} = \emptyset$, then from the proof of Theorem 20.4 we know that $\mathcal{R}$ is quasi-topologically minimal and $\mathcal{R}^{\wedge}$ is quasi-proximally minimal. Hence, by using Theorem 25.3, we can infer that $\mathcal{R}$ is quasi-topologically connected and $\mathcal{R}^{\wedge}$ is quasi-proximally connected. $\square$

Corollary 25.7. *If $\mathcal{R}$ is a topologically fine relator on X, then $\mathcal{R}$ is quasi-proximally connected if and only if $\mathcal{R}$ is quasi-topologically connected.*

Theorem 25.8. *If $\mathcal{R}$ is a proximally simple relator on X, then $\mathcal{R}$ is quasi-proximally connected if and only if $\mathcal{R}$ is quasi-topologically connected.*

Proof. From the proof of Theorem 20.7, we know that $\tau_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}$. Hence, it is clear that $\digamma_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}}$, and thus also $\tau_{\mathcal{R}} \cap \digamma_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$. Therefore, by Definition 25.1, the required assertion is also true. $\square$

Theorem 25.9. *A relator $\mathcal{R}$ on X is quasi-proximally connected if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$ and $\mathcal{R}^{-1}$ is quasi-proximally connected.*

Proof. Recall that, for any $\square \in \{\infty, *, \#\}$, we have $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\square}}$. Hence, it is clear that $\digamma_{\mathcal{R}} = \digamma_{\mathcal{R}^{\square}}$, and thus also $\tau_{\mathcal{R}} \cap \digamma_{\mathcal{R}} = \tau_{\mathcal{R}^{\square}} \cap \digamma_{\mathcal{R}^{\square}}$.

Moreover, we also have $\tau_{\mathcal{R}^{-1}} = \digamma_{\mathcal{R}}$, and thus also $\tau_{\mathcal{R}^{-1}} \cap \digamma_{\mathcal{R}^{-1}} = \digamma_{\mathcal{R}} \cap \tau_{\mathcal{R}} = \tau_{\mathcal{R}} \cap \digamma_{\mathcal{R}}$. Hence, by Definition 25.1, it is clear that the required assertion is true. $\square$

Remark 25.10. From this theorem, for instance, we can see that the relator $\mathcal{R}$ is quasi-proximally connected if and only if any one of the relators $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally connected.

Theorem 25.11. *A relator $\mathcal{R}$ on X is quasi-topologically connected if and only if any one of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$ and $\mathcal{R}^{\wedge}$ is quasi-topologically connected.*

Proof. Recall that, for any $\square \in \{*, \#, \wedge\}$, we have $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$. Hence, it is clear that $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\square}}$, and thus also $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}} \cap \mathcal{F}_{\mathcal{R}^{\square}}$. Therefore, by Definition 25.1, the required assertion is true. $\square$

Remark 25.12. From Remark 20.11, we know that $\mathcal{T}_{\mathcal{R}^\infty} \subseteq \mathcal{T}_{\mathcal{R}}$. Hence, it follows that $\mathcal{F}_{\mathcal{R}^\infty} \subseteq \mathcal{F}_{\mathcal{R}}$, and thus also $\mathcal{T}_{\mathcal{R}^\infty} \cap \mathcal{F}_{\mathcal{R}^\infty} \subseteq \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$. Therefore, if $\mathcal{R}$ is quasi-topologically connected, then $\mathcal{R}^\infty$ is also quasi-topologically connected.

26. The main characterizations of quasi-connected relators

From Theorem 25.6, we can see that the properties of quasi-topologically connected relators can, in priciple, be immediately derived from those of the quasi-proximally connected ones.

Therefore, it is of major importance to note that, by using the relator

$$\mathcal{R} \vee \mathcal{R}^{-1} = \left\{ R \cup S^{-1} : \quad R, S \in \mathcal{R} \right\},$$

we can also prove the following

Theorem 26.1. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally connected;*

(2) $\mathcal{R} \vee \mathcal{R}^{-1}$ *is quasi-proximally minimal.*

Proof. By Corollary 18.5, we have

$$\tau_{\mathcal{R} \vee \mathcal{R}^{-1}} = \tau_{\mathcal{R}} \cap \tau_{\mathcal{R}}\,.$$

Thus, by Definitions 20.1 and 25.1, assertions (1) and (2) are equivalent. □

Now, as an immediate consequence of Theorems 25.6 and 26.1, we can also state

Theorem 26.2. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topologically connected;*

(2) $\mathcal{R}^\wedge \vee \mathcal{R}^\vee$ *is quasi-proximally minimal,*

Proof. From Theorems 25.6 and 26.1, we can see that

$$\begin{aligned} \mathcal{R} \text{ is quasi-topologically connected} \quad &\iff \quad \mathcal{R}^\wedge \text{ quasi-proximally connected} \\ &\iff \quad \mathcal{R}^\wedge \vee \left(\mathcal{R}^\wedge\right)^{-1} \text{ is quasi-proximally minimal.} \end{aligned}$$

Thus, since $\mathcal{R}^\vee$ is defined by $\left(\mathcal{R}^\wedge\right)^{-1}$, assertions (1) and (2) are also equivalent. □

Remark 26.3. The latter two theorems show that the properties of the quasi-proximally and quasi-topologically connected relators can, in principle, be also immediately derived from those of the quasi-proximally minimal ones.

The fact that minimalness is a more important notion than connectedness was first established by Kurdics, Pataki and Száz [86, 91, 137] by using well-chainedness instead of minimalness and the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ instead of $\mathcal{R} \vee \mathcal{R}^{-1}$.

Now, from Theorem 26.1, by using Theorem 21.1, we can easily derive

Theorem 26.4. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally connected;*

(2) $\mathcal{R} \vee \mathcal{R}^{-1} \subseteq \{ X^2 \}^\partial$ *;* *(3)* $\left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^\infty \subseteq \{ X^2 \}$ *;*

(4) $\mathcal{R}^{\#} \cap \left(\mathcal{R}^{\#}\right)^{-1} \subseteq \{ X^2 \}^{\partial}$ *;* *(5)* $\left(\mathcal{R}^{\#} \cap \left(\mathcal{R}^{\#}\right)^{-1}\right)^{\infty} \subseteq \{ X^2 \}$ *.*

Proof. To obtain assertions (4) and (5), instead of the equalities

$$\left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^{\#} = \mathcal{R}^{\#} \cap \left(\mathcal{R}^{-1}\right)^{\#} = \mathcal{R}^{\#} \cap \left(\mathcal{R}^{\#}\right)^{-1},$$

it is better to use Theorem 25.9 and the equivalence of assertions (1), (2) and (3). □

Moreover, from Theorem 26.2, by using Theorem 21.1, we can similarly derive

Theorem 26.5. *For a relator* $\mathcal{R}$ *on* X*, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topologically connected;*

(2) $\mathcal{R}^{\wedge} \vee \mathcal{R}^{\vee} \subseteq \{ X^2 \}^{\partial}$ *;* *(3)* $\left(\mathcal{R}^{\wedge} \vee \mathcal{R}^{\vee}\right)^{\infty} \subseteq \{ X^2 \}$ *.*

Remark 26.6. By Theorems 26.1, 20.8 and 26.2, a relator $\mathcal{R}$ on X may be naturally called quasi-$\square$–connected, for some unary operation $\square$ for relators on X, if the relator $\mathcal{R}^{\square} \vee \left(\mathcal{R}^{\square}\right)^{-1}$ is quasi-proximally minimal.

27. Further characterizations of quasi-connected relators

Now, in addition to Theorems 26.1 and 26.4, we can also prove the following

Theorem 27.1. *For a relator* $\mathcal{R}$ *on* X *with* $\emptyset \notin \mathcal{R}$*, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally connected;*

(2) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ *and* $B^c \in \operatorname{Int}_{\mathcal{R}}(A^c)$ *imply* $B \not\subseteq A$ *for all* $A, B \subseteq X$ *with* $A \neq \emptyset$ *and* $B \neq X$*;*

(3) $X = A \cup B$ *implies that either* $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ *or* $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ *for all* $A, B \subseteq X$ *with* $A, B \neq \emptyset$ *.*

Proof. Clearly, $\emptyset \notin \mathcal{R}$ implies $\emptyset \notin \mathcal{R} \cup \mathcal{R}^{-1}$. Thus, by using Theorems 26.1, 22.2, 18.3 and 5.7, we can see that the following assertions are equivalent:

(a) $\mathcal{R}$ is quasi-proximally connected;

(b) $\mathcal{R} \vee \mathcal{R}^{-1}$ is quasi-proximally minimal,

(c) $X = A \cup B$ implies $A \in \operatorname{Cl}_{\mathcal{R} \vee \mathcal{R}^{-1}}(B)$ for all $A, B \subseteq X$ with $A, B \neq \emptyset$;

(d) $X = A \cup B$ implies that either $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ or $A \in \operatorname{Cl}_{\mathcal{R}^{-1}}(B)$ for all $A, B \subseteq X$ with $A, B \neq \emptyset$;

(d) $X = A \cup B$ implies that either $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ or $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ for all $A, B \subseteq X$ with $A, B \neq \emptyset$.

Therefore, assertions (1) and (3) are equivalent.

Now, it remains only to show that (2) and (3) are also equivalent. For this, note that if for instance (2) does not hold, then there exist $A, B \subseteq X$ such that

$$A \neq \emptyset, \qquad B \neq X, \quad B \subseteq A, \qquad A \in \operatorname{Int}_{\mathcal{R}}(B) \qquad \text{and} \qquad B^c \in \operatorname{Int}_{\mathcal{R}}(A^c).$$

Hence, by using that $\operatorname{Int}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \operatorname{Cl}_{\mathcal{R}}(B^c)$, we can infer that

$$A \notin \operatorname{Cl}_{\mathcal{R}}(B^c) \qquad \text{and} \qquad B^c \notin \operatorname{Cl}_{\mathcal{R}}(A).$$

Moreover, we can also note $B^c \neq \emptyset$ and $X = A \cup B^c$. Therefore, (3) does not also hold. This shows that (3) implies (2). □

Remark 27.2. By Remark 22.3, the implications (3) $\iff$ (2) $\implies$ (1) do not require the extra condition that $\emptyset \notin \mathcal{R}$.

Moreover, analogously to Theorem 22.4, we can also prove the following

Theorem 27.3. *If* $\operatorname{card}(X) > 1$, *then for a relator* $\mathcal{R}$ *on* X, *the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topologically connected;*

(2) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ *and* $B^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$ *imply* $B \not\subseteq A$ *for all* $A, B \subseteq X$ *with* $A \neq \emptyset$ *and* $B \neq X$;

(3) $X = A \cup B$ *implies that either* $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *or* $\operatorname{cl}_{\mathcal{R}}(A) \cap B \neq \emptyset$ *for all* $A, B \subseteq X$ *with* $A, B \neq \emptyset$.

Proof. If $\mathcal{R} \neq \emptyset$, then by Theorems 25.6, 27.1 and 13.1 it is clear that the following assertions are equivalent:

(a) $\mathcal{R}$ is quasi-topologically connected;

(b) $\mathcal{R}^{\wedge}$ is quasi-proximally connected;

(c) $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$ and $B^c \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(A^c)$ imply $B \not\subseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$;

(d) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ and $B^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$ imply $B \not\subseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$.

Therefore, in this case, assertions (1) and (2) are equivalent.

While, if $\mathcal{R} = \emptyset$, then it is clear that $\mathcal{T}_{\mathcal{R}} = \{\emptyset\}$, and thus (1) trivially holds. Moreover, in this case, we can note that $\operatorname{int}_{\mathcal{R}}(B) = \emptyset$ for all $B \subseteq Y$. Therefore, if $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ and $B^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$, then $A = \emptyset$ and $B^c = \emptyset$, i.e., $B = X$. Thus, (2) also trivially holds.

Now, it remains only to show that (2) and (3) are also equivalent. For this, one can note that $\operatorname{cl}_{\mathcal{R}}(B) = X \setminus \operatorname{int}_{\mathcal{R}}(B^c)$ for all $B \subseteq X$. Therefore, a similar argument as in the proof of Theorem 22.1 can be applied. □

28. Relationships between quasi-connectedness and mild continuity

Concerning quasi-proximally connected relators, we can also prove the following

Theorem 28.1. *For a relator* $\mathcal{R}$ *on* X, *the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally connected;*

(2) $f^{-1} \circ f \notin \mathcal{R}^{\#}$ *for any function* f *of* X *onto* $\{0, 1\}$;

(3) $f^{-1} \circ f \notin \mathcal{R}^{\# \infty}$ *for any function* f *of* X *onto* $\{0, 1\}$.

Proof. If (2) does not hold, then there exists a function f of X onto $\{0, 1\}$ such that $f^{-1} \circ f \in \mathcal{R}^{\#}$. Define

$$A = f^{-1}(0) \qquad \text{and} \qquad E = f^{-1} \circ f \, .$$

Then, since $A = \{ x \in X : \ \ f(x) = 0 \}$, it is clear that A is a proper, nonvoid subset of X such that $A^c = \big\{ x \in X : \ \ f(x) = 1 \big\} = f^{-1}(1)$. Moreover, we can note that E is a relation on X such that

$$E[A] = \big(f^{-1} \circ f \big) [f^{-1}(0)] = f^{-1} \left[f \left[f^{-1}(0) \right] \right] = f^{-1}(0) = A \, ,$$

and quite similarly

$$E[A^c] = \big(f^{-1} \circ f \big) [f^{-1}(1)] = f^{-1} \left[f \left[f^{-1}(1) \right] \right] = f^{-1}(1) = A^c \, .$$

On the other hand, since $f^{-1} \circ f \in \mathcal{R}^{\#}$, we can also state that there exist $R, S \in \mathcal{R}$ such that

$$R[A] \subseteq E[A] \qquad \text{and} \qquad S[A^c] \subseteq E[A^c] \, .$$

Hence, since $E[A] = A$ and $E[A^c] = A^c$, we can see that $R[A] \subseteq A$ and $S[A^c] \subseteq A^c$. Therefore, $A, \, A^c \in \tau_{\mathcal{R}}$, and thus $A \in \tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}}$. Hence, it is clear that $\tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}} \not\subseteq \{ \emptyset, \, X \}$, and thus (1) does not also hold. Consequently, (1) implies (2).

Conversely, if (1) does not hold, then there exists a proper, nonvoid subset A of X such that $A \in \tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}}$, and thus $A, \, A^c \in \tau_{\mathcal{R}}$. Therefore, there exist $R, \, S \in \mathcal{R}$ such that

$$R[A] \subseteq A \qquad \text{and} \qquad S[A^c] \subseteq A^c \, .$$

Now, by defining

$$f(x) = 0 \quad \text{if} \quad x \in A \qquad \text{and} \qquad f(x) = 1 \quad \text{if} \quad x \in A^c \, ,$$

we can see that f is a function of X onto $\{0, 1\}$. Moreover, we can show that $f^{-1} \circ f \in \mathcal{R}^{\#}$, and thus (2) does not also hold. Consequently, (2) also implies (1).

Namely, if $E = f^{-1} \circ f$, then for any $x, \, y \in X$ we have

$$\begin{aligned} (x, \, y) \in E \iff y \in E(x) \iff & \ y \in \big(f^{-1} \circ f \big)(x) \\ \iff & \ y \in f^{-1}\big(f(x) \big) \iff f(y) = f(x) \, . \end{aligned}$$

Therefore, if $V \subseteq X$, then for any $y \in X$ we have

$$y \in E[V] \iff \exists \ x \in V : \ \ y \in E(x) \iff \exists \ x \in V : \ \ f(x) = f(y) \, ,$$

and thus

$$E[V] = \begin{cases} \emptyset & \text{if} \quad V = \emptyset \, , \\ A & \text{if} \quad \emptyset \neq V \subseteq A \, , \\ A^c & \text{if} \quad \emptyset \neq V \subseteq A^c \, , \\ X & \text{if} \quad A \cap V \neq \emptyset , \quad A^c \cap V \neq \emptyset . \end{cases}$$

Hence, we can see that $R[V] = \emptyset = E[V]$ if $V = \emptyset$,

$R[A] \subseteq A = E[V]$ if $\emptyset \neq V \subseteq A$ and $S[A^c] \subseteq A^c = E[V]$ if $\emptyset \neq V \subseteq A^c$,

and $S[V] \subseteq X = E[V]$ if $A \cap V \neq \emptyset$ and $A^c \cap V \neq \emptyset$. Therefore, $E \in \mathcal{R}^{\#}$, and thus $f^{-1} \circ f \in \mathcal{R}^{\#}$.

Now, to complete the proof, it remains to prove only that assertions (2) and (3) are also equivalent. For this, note that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*}$, and thus in particular $\mathcal{R}^{\# \infty} \subseteq \mathcal{R}^{\# *} = \mathcal{R}^{\#}$. Therefore, $E \notin \mathcal{R}^{\#}$ implies $E \notin \mathcal{R}^{\# \infty}$, and thus (2) implies (3).

Moreover, for any $x, y \in X$, we have $(x, y) \in E$ if and only if $f(x) = f(y)$. Therefore, E is an equivalence relation on X, and thus in particular $E^{\infty} = E$. Hence, it is clear that $E \in \mathcal{R}^{\#}$ implies $E = E^{\infty} \in \mathcal{R}^{\# \infty}$. Therefore, $E \notin \mathcal{R}^{\# \infty}$ implies $E \notin \mathcal{R}^{\#}$, and thus (3) also implies (2). $\square$

From this theorem, by using Theorem 25.6, we can immediately derive

Theorem 28.2. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically connected;

(2) $f^{-1} \circ f \notin \mathcal{R}^{\wedge}$ for any function f of X onto $\{0, 1\}$;

(3) $f^{-1} \circ f \notin \mathcal{R}^{\wedge \infty}$ for any function f of X onto $\{0, 1\}$.

Proof. By Theorem 25.6, $\mathcal{R}$ is quasi-topologically connected if and only if $\mathcal{R}^{\wedge}$ is quasi-proximally connected. That is, by Theorem 28.1,

$$f^{-1} \circ f \notin \mathcal{R}^{\wedge \#}, \qquad \text{resp.} \qquad f^{-1} \circ f \notin \mathcal{R}^{\wedge \# \infty}$$

for any function f of X onto $\{0, 1\}$. Hence, by using that $\mathcal{R}^{\wedge \#} = \mathcal{R}^{\wedge}$, we can already see that assertions (1), (2) and (3) are also equivalent. $\square$

Remark 28.3. Because of Theorems 28.1 and 28.2, a relator $\mathcal{R}$ on X may be naturally called $\square$–*connected*, for some unary operation $\square$ for relators on X, if $f^{-1} \circ f \notin \mathcal{R}^{\square}$ for any function f of X onto $\{0, 1\}$. Moreover, in particular the relator $\mathcal{R}$ may be naturally called quasi–$\square$–connected if it is $\square\infty$–connected.

Hence, by noticing that $f^{-1} \circ f = f^{-1} \circ \Delta_{\{0, 1\}} \circ f$, we can see that the relator $\mathcal{R}$ is $\square$–connected (quasi–$\square$–connected) if only if only the constant functions of X to $\{0, 1\}$ can be mildly $\square$–continuous (quasi–$\square$–continuous) with respect to the relators $\mathcal{R}$ and $\{\Delta_{\{0, 1\}}\}$. (Concerning continuity properties, see [199].)

29. Quasi-hyperconnected relators

Analogously to the definition of a hyperconnected topology, we may also naturally introduce the following

Definition 29.1. A relator $\mathcal{R}$ on X will be called

(1) *quasi-proximally hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$

(2) *quasi-topologically hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.

Remark 29.2. Thus, the relator $\mathcal{R}$ is quasi-proximally (quasi-topologically) hyperconnected if and only if the family $\tau_{\mathcal{R}} \setminus \{\emptyset\}$ $\big(\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}\big)$ has a certain pairwise intersection property.

Theorem 29.3. *If $\mathcal{R}$ is a quasi-proximally (quasi-topologically) minimal relator on X, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) hyperconnected.*

Proof. If $\mathcal{R}$ is a quasi-proximally minimal relator on X, then $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Hence, we can infer that

$$\tau_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \{\emptyset, X\} \setminus \{\emptyset\} = \begin{cases} \emptyset & \text{if} \quad X = \emptyset, \\ \{X\} & \text{if} \quad X \neq \emptyset. \end{cases}$$

Therefore, if $A, B \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$, then we necessarily have $X \neq \emptyset$, and moreover $A \cap B = X \cap X = X$. Thus, $\mathcal{R}$ is quasi-proximally hyperconnected.

This proves the first statement of the theorem. The second statement can be proved quite similarly. □

From this theorem, by using Theorem 23.3, we can immediately derive

Corollary 29.4. *If $\mathcal{R}$ is a paratopologically minimal relator on X, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically hyperconnected.*

Concerning quasi-hyperconnected relators, we can also easily prove the following

Theorem 29.5. *If $\mathcal{R}$ is a quasi-proximally (quasi-topologically) hyperconnected relator on X, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) connected.*

Proof. If $\mathcal{R}$ is not quasi-proximally connected, then $\tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$. Thus, there exists $A \subseteq X$ such that $A \in \tau_{\mathcal{R}}$ and $A \in \bar{\tau}_{\mathcal{R}}$, but $A \neq \emptyset$ and $A \neq X$. Hence, by using Theorem 8.5, we can infer that $A^c \in \tau_{\mathcal{R}}$ and $A^c \neq \emptyset$. Therefore, $A, A^c \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$ such that $A \cap A^c = \emptyset$. Thus, $\mathcal{R}$ cannot be quasi-proximally hyperconnected.

This proves the first statement of the theorem. The second statement can be proved quite similarly. □

Remark 29.6. This theorem shows that Theorem 25.3 and Corollary 25.4 are actually consequences of Theorem 29.1 and Corollary 29.2.

Now, analogously to Theorems 25.5 and 25.6, we can also easily prove the following two theorems.

Theorem 29.7. *If $\mathcal{R}$ is quasi-topologically hyperconnected relator on X, then $\mathcal{R}$ is quasi-proximally hyperconnected.*

Proof. By Theorem 8.14, we have $\tau_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ has the binary intersection property, then $\tau_{\mathcal{R}} \setminus \{\emptyset\}$ also has this property. Thus, by Definition 29.1, the required assertion is true. □

Theorem 29.8. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically hyperconnected;

(2) $\mathcal{R}^{\wedge}$ is quasi-proximally hyperconnected.

Proof. If $\mathcal{R} \neq \emptyset$, then by Corollary 13.2 we have $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$, and thus also $\tau_{\mathcal{R}^{\wedge}} \setminus \{\emptyset\} = \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, by Definition 29.1, assertions (1) and (2) are equivalent.

While, if $\mathcal{R} = \emptyset$, then from the proof of Theorem 20.4 we know that $\mathcal{R}$ is quasi-topologically minimal and $\mathcal{R}^{\wedge}$ is quasi-proximally minimal. Hence, by Theorem 29.3, we can see that $\mathcal{R}$ is quasi-topologically hyperconnected and $\mathcal{R}^{\wedge}$ is quasi-proximally hyperconnected. □

Moreover, analogously to Theorems 25.9 and 25.11, we can also prove the following two theorems.

Theorem 29.9. *A relator $\mathcal{R}$ on X is quasi-proximally hyperconnected if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$ and $\mathcal{R}^{\#}$ is quasi-proximally hyperconnected.*

Remark 29.10. From this theorem, for instance, we can see that the relator $\mathcal{R}$ is quasi-proximally hyperconnected if and only if any one of the relators $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally hyperconnected.

Theorem 29.11. *A relator $\mathcal{R}$ on X is quasi-topologically hyerconnected if and only if any one of the relators $\mathcal{R}^*$, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\wedge\infty}$ is quasi-topologically hyperconnected.*

Remark 29.12. From Remark 20.11, we know that $\mathcal{T}_{\mathcal{R}^\infty} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus $\mathcal{T}_{\mathcal{R}^\infty} \setminus \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if $\mathcal{R}$ is quasi-topologically hyperconnected, then $\mathcal{R}^\infty$ is also quasi-topologically hyperconnected.

From Definition 29.1, by using Theorems 8.5 and 8.13, we can also easily derive the following two theorems.

Theorem 29.13. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-proximally hyperconnected;*

(2) $A \cup B \neq X$ *for all* $A, B \in \overline{\tau}_{\mathcal{R}} \setminus \{X\}$;

(3) $A \setminus B \neq \emptyset$ *for all* $A \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$ *and* $B \in \overline{\tau}_{\mathcal{R}} \setminus \{X\}$.

Theorem 29.14. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topologically hyperconnected;*

(2) $A \cup B \neq X$ *for all* $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$;

(3) $A \setminus B \neq \emptyset$ *for all* $A \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ *and* $B \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$.

Proof. For instance, if $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$, then by Theorem 8.13 we evidently have $A^c, B^c \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if (1) holds, then $A^c \cap B^c \neq \emptyset$ also holds. Hence, since $(A \cup B)^c = A^c \cap B^c$, we can infer that $(A \cup B)^c \neq \emptyset$, and thus $A \cup B \neq X$. Therefore, (1) implies (2). □

30. Quasi-ultraconnected relators

Analogously to the definition of an ultraconnected topology, we may also naturally introduce the following

Definition 30.1. A relator $\mathcal{R}$ on X will be called

(1) *quasi-proximally ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \overline{\tau}_{\mathcal{R}} \setminus \{\emptyset\}$

(2) *quasi-topologically ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$.

Remark 30.2. Thus, the relator $\mathcal{R}$ is quasi-proximally (quasi-topologically) hyperconnected if and only if the family $\overline{\tau}_{\mathcal{R}} \setminus \{\emptyset\}$ $\big(\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}\big)$ has a certain pairwise intersection property.

Now, analogously to our former statements on hyperconnected relators, we can also easily prove the following assertions.

Theorem 30.3. *If $\mathcal{R}$ is a quasi-proximally (quasi-topologically) minimal relator on X, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) ultraconnected.*

Corollary 30.4. *If $\mathcal{R}$ is a paratopologically minimal relator on X, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically ultraconected.*

Theorem 30.5. *If $\mathcal{R}$ is a quasi-proximally (quasi-topologically) ultraconnected relator on X, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) connected.*

Proof. If $\mathcal{R}$ is not quasi-proximally connected, then $\tau_{\mathcal{R}} \cap \tau\!\!\!\!-_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$. Thus, there exists $A \subseteq X$ such that $A \in \tau_{\mathcal{R}}$ and $A \in \tau\!\!\!\!-_{\mathcal{R}}$, but $A \neq \emptyset$ and $A \neq X$. Hence, by using Theorem 8.5, we can infer that $A^c \in \tau\!\!\!\!-_{\mathcal{R}}$ and $A^c \neq \emptyset$. Therefore, $A, A^c \in \tau\!\!\!\!-_{\mathcal{R}} \setminus \{\emptyset\}$ such that $A \cap A^c = \emptyset$. Thus, $\mathcal{R}$ cannot be quasi-proximally ultraconnected.

This proves the first statement of the theorem. The second statement can be proved quite similarly. □

Theorem 30.6. *If $\mathcal{R}$ is quasi-topologically ultraconnected relator on X, then $\mathcal{R}$ is quasi-proximally ultraconnected.*

Theorem 30.7. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically ultraconnected;

(2) $\mathcal{R}^{\wedge}$ is quasi-proximally ultraconnected.

Theorem 30.8. *A relator $\mathcal{R}$ on X is quasi-proximally ultraconnected if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$ and $\mathcal{R}^{\#}$ is quasi-proximally ultraconnected.*

Remark 30.9. From this theorem, we can see that the relator $\mathcal{R}$ is quasi-proximally connected if, for instance, any one of the relators and $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally ultraconnected.

Theorem 30.10. *A relator $\mathcal{R}$ on X is quasi-topologically ultraconnected if and only if any one of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\wedge\infty}$ is quasi-topologically ultraconnected.*

Remark 30.11. From Remark 20.11, we know that $\mathcal{T}_{\mathcal{R}^{\infty}} \subseteq \mathcal{T}_{\mathcal{R}}$. Hence, we can infer that $\mathcal{F}_{\mathcal{R}^{\infty}} \subseteq \mathcal{F}_{\mathcal{R}}$, and thus $\mathcal{F}_{\mathcal{R}^{\infty}} \setminus \{\emptyset\} \subseteq \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if $\mathcal{R}$ is quasi-topologically ultraconnected, then $\mathcal{R}^{\infty}$ is also quasi-topologically ultraconnected.

Theorem 30.12. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-proximally ultraconnected;

(2) $A \cup B \neq X$ for all $A, B \in \tau_{\mathcal{R}} \setminus \{X\}$;

(3) $A \setminus B \neq \emptyset$ for all $A \in \tau\!\!\!\!-_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \tau_{\mathcal{R}} \setminus \{X\}$.

Theorem 30.13. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically ultraconnected;

(2) $A \cup B \neq X$ for all $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{X\}$;

(3) $A \setminus B \neq \emptyset$ for all $A \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \mathcal{T}_{\mathcal{R}} \setminus \{X\}$.

Proof. For instance, if (1) holds and $A \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \mathcal{T}_{\mathcal{R}} \setminus \{X\}$, then because of $B^c \in \mathcal{F} \setminus \{\emptyset\}$, we have $A \setminus B = A \cap B^c \neq \emptyset$. Therefore, (1) implies (3). □

Remark 30.14. This theorem shows that our quasi-topologically ultraconnectedness also extends the *strong connectedness* of Levine [95] studied also by Leuschen and Sims [94].

Namely, it can be easily seen that assertion (2) of Theorem 30.13 can be reformulated in the form that $X = A \cup B$, together with $A, B \in \mathcal{T}_{\mathcal{R}}$, implies that either $A = X$ or $B = X$.

Now, in addition to the above theorems, we can also easily prove the following

Theorem 30.15. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-proximally ultraconnected;

(2) $\mathcal{R}^{-1}$ quasi-proximally hyperconnected.

Proof. By Theorem 8.6, we have $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$, and thus also $\tau_{\mathcal{R}} \setminus \{\emptyset\} = \tau_{\mathcal{R}^{-1}} \setminus \{\emptyset\}$ for any relator $\mathcal{R}$ on X. Therefore, $\tau_{\mathcal{R}} \setminus \{\emptyset\}$ has the binary intersection property if and only if $\tau_{\mathcal{R}^{-1}} \setminus \{\emptyset\}$ has this property. Thus, by Definition 30.1, assertions (1) and (2) are equivalent. □

Remark 30.16. This theorem shows that, in contrast to the independence of quasi-topological ulraconnectedness and quasi-topological hyperconnedtedness [158, p. 29], the quasi-proximal ultraconnectedness is not completely independent of the quasi-proximal hyperconnectedness.

31. Hyperconnected relators

Because of a reformulation of the definition of a hyperconnected topology mentioned in Section 1, we may also naturally introduce the following

Definition 31.1. A relator $\mathcal{R}$ on X to Y will be called *hyperconnected* if

$$\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}\,.$$

Remark 31.2. This property can be expressed in a more instructive form that the identity function Δ_Y of Y is *fatness reversing.*

Therefore, some of the forthcoming results can be greatly generalized according to the ideas of a former paper [201] of the second author.

Theorem 31.3. *If $\mathcal{R}$ is a hyperconnected relator on X, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically hyperconnected.*

Proof. By Theorem 8.16 and Definition 31.1, we have $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$. Therefore, if $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$, then we have both $A \in \mathcal{E}_{\mathcal{R}}$ and $A \neq \emptyset$. Hence, by using Theorem 7.7, we can infer that $A \cap B \neq \emptyset$. Thus, by Definition 29.1, $\mathcal{R}$ is quasi-topologically hyperconnected. Now, by Theorem 29.7, we can state that $\mathcal{R}$ is also quasi-proximally hyperconnected. □

From this theorem, by using Theorem 29.5, we can immediately derive

Corollary 31.4. *If $\mathcal{R}$ is a hyperconnected relator on X, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically connected.*

However, as a certain converse to the above results, we can only prove

Theorem 31.5. *If $\mathcal{R}$ is a paratopologically minimal relator on an arbitrary set X to a nonvoid set Y, then $\mathcal{R}$ is hyperconnected.*

Proof. By Theorem 24.5, we have $\mathcal{P}(Y) \setminus \{\emptyset\} \subseteq \mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{P}(Y)$, it is clear that $\mathcal{E}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{D}_{\mathcal{R}}$ also holds. Moreover, since $Y \neq \emptyset$, we can note that $\mathcal{P}(Y) \setminus \{\emptyset\} \neq \emptyset$, and thus also $\mathcal{D}_{\mathcal{R}} \neq \emptyset$. Hence, by Theorem 7.14, we can see that $\emptyset \notin \mathcal{E}_{\mathcal{R}}$, and thus $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, we actually have $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$, and thus $\mathcal{R}$ is hyperconnected. □

Remark 31.6. Note that if in particular $\mathcal{R}$ is a relator on X to $\emptyset$, then because of $\mathcal{R} \subseteq \mathcal{P}(X \times Y) = \mathcal{P}(\emptyset) = \{\emptyset\}$ we have either $\mathcal{R} = \emptyset$ or $\mathcal{R} = \{\emptyset\}$.

Hence, by using Theorem 7.3 and 7.6, we can see that either $\mathcal{E}_{\mathcal{R}} = \emptyset$ and $\mathcal{D}_{\mathcal{R}} = \{\emptyset\}$, or $\mathcal{E}_{\mathcal{R}} = \{\emptyset\}$ and $\mathcal{D}_{\mathcal{R}} = \emptyset$. Therefore, in the latter case $\mathcal{R}$ is not hyperconnected despite that in both cases it is paratopologically minimal.

By using the corresponding definitions, we can also easily prove the following

Theorem 31.7. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is hyperconnected;*

(2) $R(x) \in \mathcal{D}_{\mathcal{R}}$ *for all* $x \in X$ *and* $R \in \mathcal{R}$*;*

(3) $R(x) \cap S(y) \neq \emptyset$ *for all* $x, y \in X$ *and* $R, S \in \mathcal{R}$.

Proof. Since by Remark 7.4 we have $R(x) \in \mathcal{E}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$, it is clear that assertion (1), i.e., the inclusion $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$, implies (2).

On the other hand, if $A \in \mathcal{E}_{\mathcal{R}}$, then there exists $x \in X$ and $R \in \mathcal{R}$, such that $R(x) \subseteq A$. Moreover, if (2) holds, then we have $R(x) \in \mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{D}_{\mathcal{R}}$ is ascending, we can already infer that $A \in \mathcal{D}_{\mathcal{R}}$ also holds. This shows that $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$, and thus (1) also holds. Therefore, (2) also implies (1).

The equivalence of (2) and (3) can be proved most directly by noticing that, for any $x \in X$ and $R \in \mathcal{R}$, we have

$$R(x) \in \mathcal{D}_{\mathcal{R}} \iff X \subseteq \mathrm{cl}_{\mathcal{R}}\big(R(x)\big) \iff \forall\, y \in X:\ \forall\, S \in \mathcal{R}:\ S(y) \cap R(x) \neq \emptyset.$$

□

Remark 31.8. According to [167], a relator $\mathcal{R}$ on X to Y may be called *semi-directed* if (3) holds. Thus, a relator is hyperconnected if and only if it is semi-directed.

Moreover, the relator $\mathcal{R}$ may be called *quasi-directed* if $R(x) \cap S(y) \in \mathcal{E}_{\mathcal{R}}$ holds for all $x, y \in X$ and $R, S \in \mathcal{R}$. Thus, a non-partial, quasi-directed relator is semi-directed.

From Theorem 31.7, we can also immediately derive

Corollary 31.9. *If $\mathcal{R}$ is a hyperconnected relator on X to Y, then $\mathcal{R}$ is non-partial.*

Proof. Namely, by Theorem 31.7, we have $R(x) = R(x) \cap R(x) \neq \emptyset$ for all $x \in X$ and $R \in \mathcal{R}$. □

Remark 31.10. Moreover, if for instance $R = \big\{ (x, y) \in \mathbb{R}^2 : \quad x \leq y \big\}$ and $S = \big\{ (x, y) \in \mathbb{R}^2 : \quad |x - y| < r \big\}$ for some $r > 0$, then by using Theorem 31.7 we can also at once see that $\mathcal{R} = \{R\}$ is hyperconnected, but $\mathcal{S} = \{S\}$ is not hyperconnected.

However, it is now more important to note that, by using Theorem 31.7 and the plausible notation $\mathcal{R}^{-1} \circ \mathcal{R} = \big\{ S^{-1} \circ R : \quad R, S \in \mathcal{R} \big\}$, we can also easily prove some more instructive characterizations of hyperconnected relators.

Theorem 31.11. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is hyperconnected;*

(2) $X^2 = S^{-1} \circ R$ *for all* $R, S \in \mathcal{R}$*;*

(3) $\mathcal{R}^{-1} \circ \mathcal{R} \subseteq \{X^2\}$. *(4)* $X^2 = \bigcap \mathcal{R}^{-1} \circ \mathcal{R}$.

Proof. Note that, for any $x, y \in X$ and $R, S \in \mathcal{R}$, we have

$$R(x) \cap S(y) \neq \emptyset \iff y \in S^{-1}[R(x)] \iff y \in \big(S^{-1} \circ R\big)(x) \iff (x, y) \in S^{-1} \circ R.$$

Therefore, by Theorem 31.7, assertions (1) and (2) are equivalent.

Thus, to complete the proof, it remains only to note that (3) and (4) are only concise reformulations of (2). □

Remark 31.12. By using the equality $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1}$, assertion (4) can be written in the shorter form that $X^2 = \rho_{\mathcal{R}^{-1} \circ \mathcal{R}}$.

Moreover, by using the cross product of relations [193], assertion (4) can also reformulated in the shorter form that $\Delta_Y \in \mathcal{E}_{\mathcal{R} \boxtimes \mathcal{R}}$.

32. Further characterizations of hyperconnected relators

Now, analogously to Theorems 29.11 and 10.10, we can also easily prove

Theorem 32.1. *A relator* $\mathcal{R}$ *on* X *to* Y *is hyperconnected if and only if any one of the relators* $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$ *and* $\mathcal{R}^\triangle$ *is hyperconnected.*

Proof. By Theorems 12.8 and 12.11, we have $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^\square}$ and $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^\square}$ for all $\square \in \{*, \#, \wedge, \triangle\}$. Therefore, by Definition 31.1, the required assertion is also true. □

However, it is now more important to note that by using Corollary 13.7, we can also prove the following

Theorem 32.2. *For a non-partial relator* $\mathcal{R}$ *on* X*, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is hyperconnected;*

(2) $\mathcal{R}^\triangle$ *is quasi-proximally connected;*

(3) $\mathcal{R}^\triangle$ *is quasi-topologically connected.*

Proof. By Definition 25.1, assertion (3) is equivalent to the inclusion

(a) $\mathcal{T}_{\mathcal{R}^\triangle} \cap \mathcal{F}_{\mathcal{R}^\triangle} \subseteq \{\emptyset, X\}$.

Moreover, by using Corollary 13.7, we can see that inclusion (a) is equivalent to the inclusion

(b) $\big(\mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}\big) \cap \big(\big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}\big) \subseteq \{\emptyset, X\}$.

However, this inclusion can easily be seen to be equivalent to the simplified inclusions

(c) $\mathcal{E}_{\mathcal{R}} \setminus \mathcal{D}_{\mathcal{R}} \subseteq \{\emptyset, X\}$, (d) $\mathcal{E}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{D}_{\mathcal{R}} \cup \{X\}$.

Namely, because of $\mathcal{E}_{\mathcal{R}} \setminus \mathcal{D}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}} \cap \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big)$, assertion (b) trivially implies (c). Moreover, if (b) does not hold, then there exists $A \subseteq X$ such that

$$A \in \big(\mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}\big) \cap \big(\big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}\big), \qquad \text{but} \qquad A \notin \{\emptyset, X\}.$$

This implies that

$$A \in \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \quad \text{and} \quad A \in \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}, \qquad \text{but} \qquad A \neq \emptyset \quad \text{and} \quad A \neq X,$$

Hence, we can already infer that

$$A \in \mathcal{E}_{\mathcal{R}} \setminus \mathcal{D}_{\mathcal{R}}, \qquad \text{but} \qquad A \notin \{\emptyset, X\}.$$

Therefore, (c) does not also hold. This shows that (c) also implies (b). Therefore, assertions (b) and (c) are equivalent.

The equivalence of assertions (c) and (d) can be proved even more easily. Namely, if (d) does not hold, then there exists $A \subseteq X$ such that

$$A \in \mathcal{E}_{\mathcal{R}} \setminus \{\emptyset\}, \qquad \text{but} \qquad A \notin \mathcal{D}_{\mathcal{R}} \cup \{X\}.$$

This, implies that

$$A \in \mathcal{E}_{\mathcal{R}} \quad \text{and} \quad A \neq \emptyset, \qquad \text{but} \qquad A \notin \mathcal{D}_{\mathcal{R}} \quad \text{and} \quad A \neq X.$$

Hence, we can infer that

$$A \in \mathcal{E}_{\mathcal{R}} \setminus \mathcal{D}_{\mathcal{R}}, \qquad \text{but} \qquad A \notin \{\emptyset, X\}.$$

Therefore, (c) does not also hold. This shows that (c) implies (d). The converse implication can be proved quite similarly.

Now, to complete the proof, it is enough to note only that, since $\mathcal{R}$ is non-partial, we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ and $X \in \mathcal{D}_{\mathcal{R}}$. Therefore, inclusion (d) is equivalent to the more simple inclusion $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$. Thus, assertion (3) is equivalent to (1).

Moreover, by Corollary 25.7, assertions (2) and (3) are also equivalent. Namely, the relator $\mathcal{R}^{\vartriangle}$ is topologically fine in the sense that $\mathcal{R}^{\vartriangle\wedge} = \mathcal{R}^{\vartriangle}$. □

Remark 32.3. This theorem shows that the properties of non-partial hyperconnected relators can, in principle, be immediately derived from those of the quasi-proximally connected ones.

For instance, from our former Theorems 26.1 and 26.4, by using Theorem 32.2, we can immediately derive the following

Theorem 32.4. *For a non-partial relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is hyperconnected;*

(2) $\mathcal{R}^{\vartriangle} \vee \mathcal{R}^{\triangledown}$ *is quasi-proximally minimal;*

(3) $\mathcal{R}^{\vartriangle} \vee \mathcal{R}^{\triangledown} \subseteq \{X^2\}^{\partial}$*;* *(4)* $\big(\mathcal{R}^{\vartriangle} \vee \mathcal{R}^{\triangledown}\big)^{\infty} \subseteq \{X^2\}$.

By using Theorem 31.7, and some basic properties of the families $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$, we can also easily prove the following two theorems.

Theorem 32.5. *For a relator $\mathcal{R}$ on Y to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is hyperconnected;*

(2) $A^c \notin \mathcal{E}_{\mathcal{R}}$ *for all* $A \in \mathcal{E}_{\mathcal{R}}$*;*

(3) $A \cap B \neq \emptyset$ *for all* $A, B \in \mathcal{E}_{\mathcal{R}}$.

Theorem 32.6. *For a relator $\mathcal{R}$ on Y to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is hyperconnected;*

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *or* $A^c \in \mathcal{D}_{\mathcal{R}}$ *for all* $A \subseteq Y$*;*

(3) $A \in \mathcal{D}_{\mathcal{R}}$ *or* $B \in \mathcal{D}_{\mathcal{R}}$ *whenever* $Y = A \cup B$.

Proof. For instance if (3) does not hold, then there exist $A, B \subseteq X$ such that $Y = A \cup B$, but $A \notin \mathcal{D}_{\mathcal{R}}$ and $B \notin \mathcal{D}_{\mathcal{R}}$. Hence, by using Theorem 7.6, we can infer that $A^c \in \mathcal{E}_{\mathcal{R}}$ and $B^c \in \mathcal{E}_{\mathcal{R}}$. Moreover, we can also note that

$$A^c \cap B^c = (A \cup B)^c = Y^c = \emptyset.$$

Therefore, by Theorem 32.5, assertions (1) does not also holds. This shows that (1) implies (3). □

33. Some particular theorems on minimal and connected relators

In addition to Theorem 23.3, Corollary 20.6 and Theorem 20.7, we can also prove

Theorem 33.1. *For a weakly proximal relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically minimal;*

(2) $\mathcal{R}$ *is quasi-proximally minimal;* *(3)* $\mathcal{R}$ *is quasi-topologically minimal.*

Proof. From Theorems 23.3 and 20.3, we know that $(1) \Longrightarrow (3) \Longrightarrow (2)$ even if $\mathcal{R}$ is not supposed to be weakly proximal. Therefore, we need only prove that now (2) also implies (1).

For this, note that if (1) does not hold, then by Theorem 24.1 we have $\mathcal{R} \not\subseteq \{X^2\}$. Therefore, there exists $R \in \mathcal{R}$ such that $R \neq X^2$. Thus, there exist $x, y \in X$ such that $(x, y) \notin R$. Hence, we can infer that $y \notin R(x)$, and thus $R(x) \neq X$. Moreover, since $\mathcal{R}$ is weakly proximal, there exists $A \in \tau_{\mathcal{R}}$ such that $x \in A \subseteq R(x)$, and thus $\emptyset \neq A \neq X$. This shows that $\tau_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$, and thus (2) does not also hold. Therefore, (2) implies (1). □

Quite similarly, we can also prove the following theorem which will now be rather proved as a consequence of the above theorem.

Theorem 33.2. *For a topological relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically minimal;* *(2)* $\mathcal{R}$ *is quasi-topologically minimal.*

Proof. If $\mathcal{R}$ is a topological relator on X, then from Theorem 17.8 we can see that $\mathcal{R}^{\wedge}$ is a proximal relator on X. Thus, by Theorem 33.1, the following assertions are equivalent:

(a) $\mathcal{R}^{\wedge}$ is paratopologically minimal; (b) $\mathcal{R}^{\wedge}$ is quasi-topologically minimal.

Moreover, from Theorems 24.3 and 20.10 we can see that (a) is equivalent to (1), and (b) is equivalent to (2). Therefore, (1) and (2) are also equivalent. □

Remark 33.3. Now, for an easy illustration of Theorems 33.2 and 24.7, one can note that if in particular $\mathcal{T}$ is a topology on X, then under the notations

$$\operatorname{int}(A) = \bigcup \mathcal{T} \cap \mathcal{P}(A) \qquad \text{and} \qquad \mathcal{E} = \big\{ A \subseteq X : \quad \operatorname{int}(A) \neq \emptyset \big\},$$

the following assertions are equivalent:

(1) $\mathcal{E} = \{X\}$; (2) $\mathcal{T} = \{\emptyset, X\}$; (3) $\operatorname{int}(A) = \emptyset$ for $A \in \mathcal{P}(X) \setminus \{X\}$.

However, it is now more important to note that, in addition to Theorem 33.2, we can also prove the following

Theorem 33.4. *For a topological relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{D}_{\mathcal{R}}$ *;*

(2) $\mathcal{R}$ *is hyperconnected;* *(3)* $\mathcal{R}$ *is quasi-topologically hyperconnected.*

Proof. From Theorem 31.3, we know that (2) always implies (3). Moreover, if (2) holds, then by Definition 31.1 we have $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$. Hence, by using that $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, we can see that (1) also holds even if $\mathcal{R}$ is not supposed to be topological.

On the other hand, if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 16.12 we can state that there exists $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subseteq A$. Hence, if (1) holds we can infer that $V \in \mathcal{D}_{\mathcal{R}}$. Now, since $\mathcal{D}_{\mathcal{R}}$ is ascending, we can also state that $A \in \mathcal{D}_{\mathcal{R}}$. Therefore, $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$, and thus (2) also holds.

Quite similarly, if $A, B \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 16.12 we can state that there exist $V, W \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such $V \subseteq A$ and $W \subseteq B$. Therefore, if (3) holds, then $V \cap W \neq \emptyset$, and thus $A \cap B \neq \emptyset$ is also true. Now, by Theorem 32.5, we can see that (2) also holds. □

The following two theorems show that quasi-ultraconnected relators are less important than the quasi-hyperconnected ones.

Theorem 33.5. *If $\mathcal{R}$ is a T_1–separating relator on X and* $\operatorname{card}(X) > 1$, *then $\mathcal{R}$ is not quasi-topologically ultraconnected.*

Proof. By the assumption, for any $x, y \in X$, with $x \neq y$, there exists $R \in \mathcal{R}$ such that $x \notin R(y)$, and thus $R(y) \cap \{x\} = \emptyset$. Hence, by Theorem 6.3, we can see that $y \notin \operatorname{cl}_{\mathcal{R}}(\{x\})$, and thus $\operatorname{cl}_{\mathcal{R}}(\{x\}) \subseteq \{x\}$. Therefore, $\{x\} \in \mathcal{F}_{\mathcal{R}}$, and thus also $\{x\} \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ for all $x \in X$. Thus, if $\mathcal{R}$ is quasi-topologically ultraconnected, i.e., the family $\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ has the binary intersection property, then the family $\{\{x\}\}_{x \in X}$ also has the binary intersection property. Therefore, $\{x\} \cap \{y\} \neq \emptyset$ for all $x, y \in X$. Hence, we can infer that X is either the empty set or a singleton, and thus $\operatorname{card}(X) \leq 1$. This contradiction shows that $\mathcal{R}$ cannot be quasi-topologically ultraconnected, □

Theorem 33.6. *For a weakly topological relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topologically ultraconnected;*

(2) $\operatorname{cl}_{\mathcal{R}}(x) \cap \operatorname{cl}_{\mathcal{R}}(y) \neq \emptyset$ *for all* $x, y \in X$*;*

(3) $\operatorname{cl}_{\mathcal{R}}(A) \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *for all* $\emptyset \neq A, B \subseteq X$.

Proof. By Remark 16.10 and Theorem 16.3, for any $x \in X$ we have

$$\operatorname{cl}_{\mathcal{R}}(x) = \operatorname{cl}_{\mathcal{R}}\big(\{x\}\big) \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}\,.$$

Moreover, if (1) holds, then the family $\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ has the binary intersection property. Thus, in this case, the family $\{\operatorname{cl}_{\mathcal{R}}(x)\}_{x \in X}$ also has the binary intersection property. Therefore, (2) also holds.

On the other hand, if (2) holds, then by using that $\operatorname{cl}_{\mathcal{R}}(x) = \operatorname{cl}_{\mathcal{R}}\big(\{x\}\big) \subseteq \operatorname{cl}_{\mathcal{R}}\big(A\big)$ for all $x \in A \subseteq X$, we can at once see that (3) also holds. While, if (3) holds and $x, y \in X$, then by taking $A = \{x\}$ and $B = \{y\}$, we can at once see that (2) also holds.

Therefore, it remains to show only that (2) also implies (1). For this, note that if $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$, then by taking $x \in A$ and $y \in B$, we have

$$\operatorname{cl}_{\mathcal{R}}(x) = \operatorname{cl}_{\mathcal{R}}\big(\{x\}\big) \subseteq \operatorname{cl}_{\mathcal{R}}\big(A\big) \subseteq A \quad \text{and} \quad \operatorname{cl}_{\mathcal{R}}(y) = \operatorname{cl}_{\mathcal{R}}\big(\{y\}\big) \subseteq \operatorname{cl}_{\mathcal{R}}\big(B\big) \subseteq B\,.$$

Moreover, if (2) holds, then $\operatorname{cl}_{\mathcal{R}}(x) \cap \operatorname{cl}_{\mathcal{R}}(y) \neq \emptyset$, and thus $A \cap B \neq \emptyset$. Therefore, (1) also holds. □

Remark 33.7. Note that the implications $(3) \Longleftrightarrow (2) \Longrightarrow (1)$ do not require any extra assumptions on the relator $\mathcal{R}$.

Moreover, instead of the weak-topologicalness of $\mathcal{R}$, it is enough to assume only that $\mathcal{R}$ is weakly quasi-topological and $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1}$ is non-partial.

34. Quasi-door, quasi-superset and quasi-submaximal relators

Analogously to the definition of a door topology, we may naturally introduce the following

Definition 34.1. A relator $\mathcal{R}$ on X will be called

(1) a *quasi-proximally door relator* if $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \bar{\tau}_{\mathcal{R}}$;

(2) a *quasi-topologically door relator* if $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$.

Now, by using this definition, we can easily establish the following two theorems.

Theorem 34.2. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is a quasi-proximally door relator;

(2) $\mathcal{P}(X) \setminus \tau_{\mathcal{R}} \subseteq \bar{\tau}_{\mathcal{R}}$; *(3) $\mathcal{P}(X) \setminus \bar{\tau}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$.*

Theorem 34.3. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is a quasi-topologically door relator;

(2) $\mathcal{P}(X) \setminus \mathcal{T}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$; *(3) $\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.*

Proof. To prove the implication $(2) \Longrightarrow (1)$, note that if (2) holds, then we have $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \big(\mathcal{P}(X) \setminus \mathcal{T}_{\mathcal{R}}\big) \subseteq \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$. Therefore, $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$, and thus (1) also holds. □

Remark 34.4. Now, for instance, we can also easily see that $\mathcal{R}$ is a quasi-topologically door relator on X if and only if, for any $A \subseteq X$, we have either $A \in \mathcal{T}_{\mathcal{R}}$ or $A^c \in \mathcal{T}_{\mathcal{R}}$.

Namely, if for instance $\mathcal{R}$ is a quasi-topologically door relator on X, then by Theorem 34.3 we have $\mathcal{P}(X) \setminus \mathcal{T}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$. Therefore, if $A \subseteq X$ such that $A \notin \mathcal{T}_{\mathcal{R}}$, then we necessarily we have $A \in \mathcal{F}_{\mathcal{R}}$. Hence, by Theorem 8.13, it follows that $A^c \in \mathcal{T}_{\mathcal{R}}$.

Because of a reformulation of the definition of a superset topology mentioned in Section 1, we may also naturally introduce the following

Definition 34.5. A relator $\mathcal{R}$ on X will be called

(1) *quasi-proximally superset relator* if $\mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$;

(2) *quasi-topologically superset relator* if $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.

Thus, we can easily prove the following two theorems.

Theorem 34.6. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is a quasi-proximally superset relator;*

(2) $\mathcal{E}_{\mathcal{R}} \setminus \tau_{\mathcal{R}} = \emptyset$; *(3)* $\mathcal{P}(X) = \bar{\tau}_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{R}}$;

(4) $\mathcal{P}(X) \setminus \bar{\tau}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$; *(5)* $\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \subseteq \bar{\tau}_{\mathcal{R}}$.

Theorem 34.7. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is a quasi-topologically superset relator;*

(2) $\mathcal{E}_{\mathcal{R}} \setminus \mathcal{T}_{\mathcal{R}} = \emptyset$; *(3)* $\mathcal{P}(X) = \mathcal{F}_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{R}}$;

(4) $\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$; *(5)* $\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Proof. It is clear that the inclusion $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$ is equivalent to the property $\mathcal{E}_{\mathcal{R}} \setminus \mathcal{T}_{\mathcal{R}} = \emptyset$. Therefore, assertions (1) and (2) are equivalent.

Moreover, if (3) does not hold, then there exists $A \subseteq X$ such that $A \notin \mathcal{F}_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{R}}$, and thus $A \notin \mathcal{F}_{\mathcal{R}}$ and $A \notin \mathcal{D}_{\mathcal{R}}$. Hence, by using the equality $A = (A^c)^c$ and Theorems 7.6 and 8.13, we can infer that $A^c \in \mathcal{E}_{\mathcal{R}}$ and $A^c \notin \mathcal{T}_{\mathcal{R}}$. Therefore, $\mathcal{E}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}(1)$, thus (1) does not also hold. Consequently, (1) implies (3).

The converse implication $(3) \Longrightarrow (1)$ can be proved quite similarly. Therefore, assertions (1) and (3) are also equivalent. Moreover, analogously to Theorem 34.3, it is clear that assertions (3), (4) and (5) are also equivalent. □

Concerning superset relators, we can also easily prove the following

Theorem 34.8. *For a non-partial relator $\mathcal{R}$ on X, the following assertions hold:*

(1) $\mathcal{R}$ *is a quasi-proximally superset relator if and only if* $\mathcal{E}_{\mathcal{R}} = \tau_{\mathcal{R}} \setminus \{\emptyset\}$;

(2) $\mathcal{R}$ *is a quasi-topologically superset relator if and only if* $\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.

Proof. By Theorems 8.14 and 8.16, we have $\tau_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X. Moreover, if $\mathcal{R}$ is non-partial, then by Theorem 7.14 we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$. Therefore, in this case, $\mathcal{R}$ is a a quasi-proximally (quasi-topologically) superset relator if and only if $\mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}} \setminus \{\emptyset\}$ $\big(\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}\big)$. □

Analogously to the definition of a submaximal topology, we may also naturally introduce the following

Definition 34.9. A relator $\mathcal{R}$ on X will be called

(1) *quasi-proximally submaximal* if $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$;

(2) *quasi-topologically submaximal* if $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.

Thus, analogously to Theorems 34.6 and 34.7, we can also easily prove the following two theorems.

Theorem 34.10. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-proximally submaximal;

(2) $\mathcal{D}_{\mathcal{R}} \setminus \tau_{\mathcal{R}} = \emptyset$; *(3) $\mathcal{P}(X) = \overline{\tau}_{\mathcal{R}} \cup \mathcal{E}_{\mathcal{R}}$;*

(4) $\mathcal{P}(X) \setminus \overline{\tau}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$; *(5) $\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}} \subseteq \overline{\tau}_{\mathcal{R}}$.*

Theorem 34.11. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically submaximal;

(2) $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{T}_{\mathcal{R}} = \emptyset$; *(3) $\mathcal{P}(X) = \mathcal{F}_{\mathcal{R}} \cup \mathcal{E}_{\mathcal{R}}$;*

(4) $\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$; *(5) $\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.*

35. Relationships among door, superset and submaximality properties

Now, in contrast to Theorems 20.3, 25.5, 29.7 and 30.6, we have the following

Theorem 35.1. *If $\mathcal{R}$ is a quasi-proximally door, superset, resp. submaximal relator on X, then $\mathcal{R}$ is a quasi-topologically door, superset, resp. submaximal relator on X.*

Proof. By Theorem 8.14, we have $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$ and $\tau_{\mathcal{R}} \cup \overline{\tau}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X. Hence, by the corresponding definitions, it is clear that the required implications are true.

For instance, if $\mathcal{R}$ is a quasi-proximallly door relator on X, then by Definition 34.1 and the above observation, we have $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \overline{\tau}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$, and thus also $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$. Therefore, by Definition 34.1, $\mathcal{R}$ is a quasi-topologically door relator on X. □

Theorem 35.2. *If $\mathcal{R}$ is a nonvoid, quasi-proximally (quasi-topologically) door relator on X, then $\mathcal{R}$ is a quasi-proximally (quasi-topologically) submaximal relator on X.*

Proof. Suppose first that $\mathcal{R}$ is a quasi-topologically door relator on X and $A \in \mathcal{D}_{\mathcal{R}}$. Then, by the corresponding definitions, we have $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$ and $X = \mathrm{cl}_{\mathcal{R}}(A)$.

Now, if $A \notin \mathcal{T}_{\mathcal{R}}$, then because of $X \in \mathcal{T}_{\mathcal{R}}$ we can state that $A \neq X$. Moreover, because of $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$, we can state that $A \in \mathcal{F}_{\mathcal{R}}$, and thus $\mathrm{cl}_{\mathcal{R}}(A) \subseteq A$. Hence, since $X = \mathrm{cl}_{\mathcal{R}}(A)$, we can infer that $X \subseteq A$, and thus $A = X$. This contradiction proves that $A \in \mathcal{T}_{\mathcal{R}}$. Therefore, $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus $\mathcal{R}$ is quasi-topologically submaximal.

Next, suppose that $\mathcal{R}$ is a quasi-proximally door relator on X and $A \in \mathcal{D}_{\mathcal{R}}$. Then, by Definition 34.1 and Theorem 7.5, we have $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \overline{\tau}_{\mathcal{R}}$ and $X = R^{-1}[\,A\,]$ for all $R \in \mathcal{R}$.

Now, if $A \notin \tau_{\mathcal{R}}$, then because of $X \in \tau_{\mathcal{R}}$ we can state that $A \neq X$. Moreover, because of $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$, we can state that $A \in \tau\!\!\!-_{\mathcal{R}}$. Hence, by using Theorem 8.6, we can infer that $A \in \tau_{\mathcal{R}^{-1}}$. Therefore, by Theorem 8.4, there exists $R \in \mathcal{R}$ such that $R^{-1}[A] \subseteq A$. Hence, since $X = R^{-1}[A]$, we can infer that $X \subseteq A$, and thus $A = X$. This contradiction proves that $A \in \tau_{\mathcal{R}}$. Therefore, $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$, and thus $\mathcal{R}$ is quasi-proximally submaximal. □

Remark 35.3. Note that if $\mathcal{R}$ is a quasi-proximally door relator on X, then because of $\tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}} = \mathcal{P}(X)$ and $\tau_{\emptyset} \cup \tau\!\!\!-_{\emptyset} = \emptyset$ we necessarily have $\mathcal{R} \neq \emptyset$.

While, if $\mathcal{R}$ is a quasi-topologically door relator on X, then by using that $\mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}} = \mathcal{P}(X)$ and $\mathcal{T}_{\emptyset} \cup \mathcal{F}_{\emptyset} = \{\emptyset, X\}$ we can only prove $\mathcal{R} \neq \emptyset$ if $\operatorname{card}(X) > 1$.

Theorem 35.4. *If $\mathcal{R}$ is a hyperconnected, quasi-proximally (quasi-topologically) submaximal relator on X, then*

(1) $\mathcal{R}$ is a quasi-proximally (quasi-topologically) door relator on X;

(2) $\mathcal{R}$ is a quasi-proximally (quasi-topologically) superset relator on X.

Proof. Now, by Definitions 31.1 and 34.9, we have $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$ $(\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}})$. Therefore, $\mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$ $(\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}})$, and thus assertion (2) is true. Therefore, actually we need only prove assertion (1).

For this, suppose that $\mathcal{R}$ is a hyperconnected, quasi-proximally submaximal relator on X. Then, by Definitions 31.1 and 34.9, we have $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$. Now, if $A \in \mathcal{D}_{\mathcal{R}}$, then because of $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$, we also have $A \in \tau_{\mathcal{R}}$. While, if $A \in \mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}$, then by Theorem 7.6 we have $A^c \in \mathcal{E}_{\mathcal{R}}$. Hence, by using that $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$, we can infer that $A^c \in \mathcal{D}_{\mathcal{R}}$. Thus, again by $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$, we also have $A^c \in \tau_{\mathcal{R}}$. Hence, by Theorem 8.13, we can infer that $A \in \tau\!\!\!-_{\mathcal{R}}$. Therefore, in both cases, we have $A \in \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$. This proves that $\mathcal{P}(X) \subseteq \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$, and thus $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$. Therefore, $\mathcal{R}$ is a quasi-proximally door relator on X.

Thus, we have proved the first statement of (1). The second statement of (1) can be proved quite similarly. □

Now, as an immediate consequence of Theorems 35.2 and 35.4, we can also state

Corollary 35.5. *For a nonvoid, hyperconnected relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is a quasi-proximally (quasi-topologically) door relator on X;

(2) $\mathcal{R}$ is a quasi-proximally (quasi-topologically) submaximal relator on X.

Remark 35.6. Note that the implication $(2) \Longrightarrow (1)$ does not require the extra condition that $\mathcal{R} \neq \emptyset$.

However, $\mathcal{D}_{\emptyset} = \mathcal{P}(X)$, but $\mathcal{T}_{\emptyset} = \{\emptyset\}$. Therefore, $\emptyset$ is a topologically submaximal relator on X if and only if $X = \emptyset$.

Concerning quasi-topologically superset relators, we can also easily prove the following two theorems.

Theorem 35.7. *If $\mathcal{R}$ is a quasi-topologically superset relator on X, then $\mathcal{R}$ is a strongly quasi-topological relator on X.*

Proof. Now, for any $x \in X$ and $R \in \mathcal{R}$, we have $R(x) \in \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, by Remark 16.10, the required assertion is true. □

Theorem 35.8. *If $\mathcal{R}$ is a quasi-topologically filtered, quasi-topologically superset relator on X such that $\mathcal{R}$ is not quasi-topologically maximal, then $\mathcal{R}$ is quasi-topologically hyperconnected.*

Proof. Assume on the contrary that $\mathcal{R}$ is not quasi-topologically hyperconnected. Then by Definition 29.1, there exist $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $A \cap B = \emptyset$. Thus, for any $x \in X$, we have

$$\{x\} = \big(A \cup \{x\} \big) \cap \big(B \cup \{x\} \big).$$

Moreover, because of $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, we also have $A, B \in \mathcal{E}_{\mathcal{R}}$. Thus, since $\mathcal{E}_{\mathcal{R}}$ is ascending, we can also state that $A \cup \{x\}$, $B \cup \{x\} \in \mathcal{E}_{\mathcal{R}}$. Hence, by using that $\mathcal{R}$ is a quasi-topologically superset relator, and thus $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can infer that

$$A \cup \{x\}, \ B \cup \{x\} \in \mathcal{T}_{\mathcal{R}}.$$

Now, since $\mathcal{R}$ is quasi-topologically filtered, and thus $\mathcal{T}_{\mathcal{R}}$ is closed under binary intersection, we can also state that

$$\big(A \cup \{x\} \big) \cap \big(B \cup \{x\} \big) \in \mathcal{T}_{\mathcal{R}}.$$

Therefore, $\{x\} \in \mathcal{T}_{\mathcal{R}}$, and thus $\{x\} \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.

Hence, by using that $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, we can infer that $\{x\} \in \mathcal{E}_{\mathcal{R}}$. Now, since $x \in X$ was arbitrary and $\mathcal{E}_{\mathcal{R}}$ is ascending, it is clear that $\mathcal{P}(X) \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$. Hence, by using again that $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can infer that $\mathcal{P}(X) \setminus \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, we actually have $\mathcal{T}_{\mathcal{R}} = \mathcal{P}(X)$, and thus $\mathcal{R}$ is quasi-topologically maximal. This contradiction proves that $\mathcal{R}$ is quasi-topologically hyperconnected. □

36. Resolvable and irresolvable relators

Because of a reformulation of the definition of a resolvable topology, mentioned in Section 1, we may also naturally introduce the following

Definition 36.1. A relator $\mathcal{R}$ on X to Y will be called *resolvable* if

$$\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} \neq \emptyset.$$

The importance of this definition can easily be clarified by the following

Example 36.2. If $X = \mathbb{R}$ and

$$R_n = \big\{ (x, y) \in X^2 : \quad |x - y| < n^{-1} \big\}$$

for all $n \in \mathbb{N}$, then $\mathcal{R} = \{ R_n \}_{n=1}^{\infty}$ is a resolvable tolerance relator on X.

To prove the resolvability of $\mathcal{R}$, note that

$$R_n(x) = \,]\, x - n^{-1}, \ x + n^{-1} \,[$$

for all $x \in X$ and $n \in \mathbb{N}$. Moreover, recall that every nonvoid, open interval in $\mathbb{R}$ contains both rational and irrational numbers. Therefore, $\mathbb{Q} \in \mathcal{D}_{\mathcal{R}}$ and $\mathbb{Q} \notin \mathcal{E}_{\mathcal{R}}$, and thus $\mathbb{Q} \in \mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$.

By using Theorem 7.6, Definition 36.1 can be reformulated in the following

Theorem 36.3. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ is resolvable; *(2) $\mathcal{D}_{\mathcal{R}} \not\subseteq \mathcal{E}_{\mathcal{R}}$;*

(3) there exists $A \in \mathcal{D}_{\mathcal{R}}$ such that $A^c \in \mathcal{D}_{\mathcal{R}}$.

Now, by calling the relator $\mathcal{R}$ to be *irresolvable* if it is not resolvable, we can also easily establish the following

Theorem 36.4. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ is irresolvable; *(2) $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} = \emptyset$;* *(3) $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$.*

Hence, by Definition 31.1, it is clear that in particular we also have

Corollary 36.5. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{D}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}}$; *(2) $\mathcal{R}$ is irresolvable and hyperconnected.*

Moreover, by using Theorem 36.4 and Definitions 34.5 and 34.9, we can also easily establish the following

Theorem 36.6. *If $\mathcal{R}$ is an irresolvable, quasi-proximally (quasi-topologically) superset relator on X, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) submaximal.*

Proof. Now, by Theorem 36.4 and Definition 34.5, we have $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$ $(\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}})$. Therefore, $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$ $(\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}})$, and thus by Definition 34.9 the required assertion is true. □

Now, by using Definition 36.1 and Theorem 36.3, we can also easily prove following counterpart of Theorem 32.6.

Theorem 36.7. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ is irresolvable;

(2) $A^c \notin \mathcal{D}_{\mathcal{R}}$ for all $A \in \mathcal{D}_{\mathcal{R}}$;

(3) $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{D}_{\mathcal{R}}$.

Proof. If (2) does not hold, then there exists $A \in \mathcal{D}_{\mathcal{R}}$ such that $A^c \in \mathcal{D}_{\mathcal{R}}$. Therefore, by Theorem 36.3, $\mathcal{R}$ is resolvable, and thus (1) does not also hold. Consequently, (1) implies (2).

While, if (3) does not hold, then there exists $A, B \in \mathcal{D}_{\mathcal{R}}$ such that $A \cap B = \emptyset$. Hence, we can infer that $B \subseteq A^c$, and thus $A^c \in \mathcal{D}_{\mathcal{R}}$. Therefore, (2) does not also hold. Consequently, (2) implies (3).

Finally, if (1) does not hold, then by Theorem 36.3 there exists $A \in \mathcal{D}_{\mathcal{R}}$ such that $A^c \in \mathcal{D}_{\mathcal{R}}$. Thus, since $A \cap A^c = \emptyset$, assertion (3) does not also hold. Consequently, (3) also implies (1). □

Moreover, analogously to Theorems 32.6 and 32.1, we can also easily prove the following two theorems.

Theorem 36.8. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ is irresolvable;

(2) $A \in \mathcal{E}_{\mathcal{R}}$ or $A^c \in \mathcal{E}_{\mathcal{R}}$ for all $A \subseteq Y$;

(3) $A \in \mathcal{E}_{\mathcal{R}}$ or $B \in \mathcal{E}_{\mathcal{R}}$ whenever $Y = A \cup B$.

Theorem 36.9. *A relator $\mathcal{R}$ on X to Y is resolvable (irresolvable) if and only if any one of the relators $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$ and $\mathcal{R}^\vartriangle$ is resolvable (irresolvable).*

37. An illustrating example

The following example, given by Pataki [137], will show that even a very particular quasi-proximally minimal relator need not be topologically minimal. Thus, the converse of Theorem 20.3 is not true.

Example 37.1. If $X = \{1, 2, 3\}$ and $R_1, R_2 \subseteq X^2$ such that

$R_1(1) = X$, $\quad R_1(2) = \{1, 2\}$, $\quad R_1(3) = \{1, 3\}$,

$R_2(1) = \{1, 2\}$, $\quad R_2(2) = X$, $\quad R_2(3) = \{2, 3\}$,

then $\mathcal{R} = \{R_1, R_2\}$ is a tolerance relator on X such that:

(1) $\mathcal{R}$ is quasi-proximally minimal;

(2) $\mathcal{R}$ is both irresolvable and hyperconnected;

(3) $\mathcal{R}$ is neither paratopologically nor quasi-topologically minimal;

(4) $\mathcal{R}$ is neither quasi-proximally nor quasi-topologically door, superset and submaximal;

(5) $\mathcal{R}$ is both quasi-proximally and quasi-topologically connected, hyperconnected and ultraconnected.

It can be easily seen that R_1 and R_2 are reflexive and symmetric relations on X. Therefore, $\mathcal{R}$ is a tolerance relator on X. Moreover, by using Theorem 31.7, we can easily see that $\mathcal{R}$ is hyperconnected. Thus, by Corollary 31.4 and Theorem 31.3, $\mathcal{R}$ is both quasi-proximally and quasi-topologically connected and hyperconnected.

On the other hand, by using Theorem 8.12, we can easily see that

$$\mathcal{T}_{\mathcal{R}} = \{\emptyset, \{1, 2\}, X\}.$$

Therefore, $\mathcal{T}_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$, and thus $\mathcal{R}$ is not quasi-topologically minimal. Hence, by Theorem 23.3, it follows that $\mathcal{R}$ is also not paratopologically minimal. (This is also immediate from the fact that $\{1, 2\} = R_1(2) \in \mathcal{E}_{\mathcal{R}}$.)

Now, by using Theorem 8.13, we can also note that

$$\mathcal{F}_{\mathcal{R}} = \{\emptyset, \{3\}, X\}.$$

Therefore, $\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ also has the binary intersection property, and thus $\mathcal{R}$ is quasi-topologically ultraconnected. Hence, by Theorem 30.6, it follows that $\mathcal{R}$ is also quasi-proximally ultraconnected.

On the other hand, concerning the set $A = \{1, 2\}$, we can also easily see that

$$R_i[A] = R_i(1) \cup R_i(2) = X \not\subseteq A$$

for all $i = 1, 2$, and thus $A \notin \tau_{\mathcal{R}}$. Hence, by using that $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can already infer that $\tau_{\mathcal{R}} = \{\emptyset, X\}$, and thus $\mathcal{R}$ is quasi-proximally minimal.

Now, we can also note that

$$\mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}} = \{\emptyset, \{3\}, \{1, 2\}, X\} \neq \mathcal{P}(X).$$

Therefore, $\mathcal{R}$ is not a quasi-topologically door relator. Moreover, by using Theorems 7.3 and 7.6, we can also easily see that

$$\mathcal{E}_{\mathcal{R}} = \big\{ \{1, 2\}, \ \{1, 3\}, \ \{2, 3\}, \ X \big\} = \mathcal{D}_{\mathcal{R}} .$$

Therefore, $\mathcal{E}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}$, and thus $\mathcal{R}$ is not also a quasi-topologically superset and submaximal relator. Hence, by Theorem 35.1, we can see that $\mathcal{R}$ is also not a quasi-proximally door, superset and submaximal relator. Moreover, since $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} = \emptyset$, we can also state that $\mathcal{R}$ is not resolvable.

Remark 37.2. In connection with the above relator $\mathcal{R}$, it is also noteworthy that

$$\big(R_i \circ R_j\big)(x) = R_i\,[\,R_j(x)\,] = \bigcup_{y \in R_j(x)} R_i(y) = X$$

for all $x \in X$ and $i, j = 1, 2$. Therefore, $R_i \circ R_j = X^2$ for all $i = 1, 2$, and thus

$$\mathcal{R} \circ \mathcal{R} = \big\{ R \circ S : \quad R, S \in \mathcal{R} \big\} = \big\{ X^2 \big\} .$$

Hence, in particular we can see that $\mathcal{R}^2 = \big\{ R^2 : \ \ R \in \mathcal{R} \big\} = \big\{ X^2 \big\}$, and thus $\mathcal{R}$ is 2–well-chained in a natural sense.

Moreover, if $\mathcal{R}$ is as in Example 37.1, then by Theorem 20.7 and Corollary 20.6, it is clear that $\mathcal{R}$ cannot be proximally simple and topologically fine. However, by using direct arguments, we can prove some much better assertions.

Example 37.3. If $\mathcal{R}$ is as in Example 37.1, then

(1) $\mathcal{R}$ is not uniformly, proximally and topologically simple ;

(2) $\mathcal{R}$ is quasi-proximally, quasi-topologically and paratopologically simple.

Now, by using the preorder relations $U = X^2$ and $V = A^2 \cup A^c \times X$ with $A = \{1, 2\}$, we can easily see that

$$\tau_{\mathcal{R}} = \big\{ \emptyset, \ X \big\} = \tau_{\{U\}} \qquad \text{and} \qquad \mathcal{T}_{\mathcal{R}} = \big\{ \emptyset, \ A, \ X \big\} = \mathcal{T}_{\{V\}} .$$

Hence, by using Theorem 12.9, we can already infer that

$$\mathcal{R}^{\#\infty} = \big\{ U \big\}^{\#\infty} \qquad \text{and} \qquad \mathcal{R}^{\wedge\infty} = \big\{ V \big\}^{\wedge\infty} .$$

Therefore, $\mathcal{R}$ is both quasi-proximally and quasi-topologically simple.

Moreover, if W is a relation on X such that

$$W(1) = \{\, 1, 2 \,\}, \qquad W(2) = \{\, 2, 3 \,\}, \qquad W(3) = \{\, 1, 3 \,\},$$

then by using Theorem 7.3 we can easily see that $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\{W\}}$. Hence, by using Theorem 12.9, we can infer that $\mathcal{R}^{\vartriangle} = \{W\}^{\vartriangle}$. Therefore, $\mathcal{R}$ is also paratopologically simple.

Next, we show that $\mathcal{R}$ is not topologically simple. For this, assume on the contrary that $\mathcal{R}$ is topologically simple. Then, there exists a relation S on X such that $\mathcal{R}^{\wedge} = \{\, S \,\}^{\wedge}$. Hence, by using that $\wedge$ is extensive, we can infer that $R_1, R_2 \in \{\, S \,\}^{\wedge}$ and $S \in \mathcal{R}^{\wedge}$. Thus, in particular by the definition of $\wedge$, we have both $S(3) \subseteq R_1(3)$ and $S(3) \subseteq R_2(3)$, and moreover either $R_1(3) \subseteq S(3)$ or $R_2(3) \subseteq S(3)$. Hence, by using that $R_1(3) = \{1, 3\}$ and $R_2(3) = \{2, 3\}$, we can infer that either $\{1, 3\} \subseteq \{3\}$ or $\{2, 3\} \subseteq \{3\}$. This contradiction shows that $\mathcal{R}$ cannot be topologically simple.

Hence, it is clear that $\mathcal{R}$ cannot also be $\Box$–simple for any operation $\Box$ for relators with $\Box \wedge = \wedge$. Thus, in particular, $\mathcal{R}$ cannot also be uniformly and proximally simple.

Remark 37.4. Concerning the relator $\mathcal{R}$, considered in Example 37.1, we can also note that $X^2 \notin \mathcal{R}$, and thus $\mathcal{R}$ cannot be $\Box$–fine for any operation $\Box$ for relators with $X^2 \in \mathcal{R}^{\Box}$.

38. Another illustrating example

Recall that the relator considered in Example 37.1 is quasi-topologically connected. Therefore, to see that the converse of Theorem 25.5 is also not true, we have to consider another example.

The following somewhat more difficult example, given also by Pataki [137], will show that even a very particular quasi-proximally connected relator need not be quasi-topologically connected.

Example 38.1. If $X = \{ i \}_{i=1}^4$ and $R_i \subseteq X^2$ for all $i \in X$ such that

$$R_1(1) = \{1, 2\}, \qquad R_1(2) = X, \qquad R_1(3) = R_1(4) = \{2, 3\ 4\},$$
$$R_2(1) = X, \qquad R_2(2) = \{1, 2\}, \qquad R_2(3) = R_2(4) = \{1, 3, 4\},$$
$$R_3(1) = R_3(2) = \{1, 2, 4\}, \qquad R_3(3) = \{3, 4\}, \qquad R_3(4) = X,$$
$$R_4(1) = R_4(2) = \{1, 2, 3\}, \qquad R_4(3) = X, \qquad R_4(4) = \{3, 4\},$$

then $\mathcal{R} = \{ R_i \}_{i=1}^4$ is a tolerance relator on X such that:

(1) $\mathcal{R}$ is not resolvable, hyperconnected and paratopologically minimal;

(2) $\mathcal{R}$ is quasi-proximally minimal, connected, hyperconnected and ultraconnected;

(3) $\mathcal{R}$ is neither quasi-proximally nor quasi-topologically door, superset and submaximal;

(4) $\mathcal{R}$ is not quasi-topologically minimal, connected, hyperconnected and ultraconnected.

It can again be easily seen that each R_i is a reflexive and symmetric relation on X. Therefore, $\mathcal{R}$ is a tolerance relator on X. Moreover, we can at once see that $R_1(1) \cap R_3(3) = \emptyset$. Therefore, by Theorem 31.7, we can state that $\mathcal{R}$ is not hyperconnected. Hence, by using Theorem 31.5, we can infer that $\mathcal{R}$ is not paratopologically minimal. (This statement is also immediate from the fact that $\{1, 2\} = R_1(1) \in \mathcal{E}_{\mathcal{R}}$.)

On the other hand, by using Theorems 8.12 and 8.13, we can see that

$$\mathcal{T}_{\mathcal{R}} = \{ \emptyset, \{1, 2\}, \{3, 4\}, X \} = \mathcal{F}_{\mathcal{R}}.$$

Therefore, $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$, and thus $\mathcal{R}$ is not quasi-topologically connected. Hence, by using Theorems 25.3, 29.5 and 30.5, we can infer that $\mathcal{R}$ is also not quasi-topologically minimal, hyperconnected and ultraconnected. (The latter statements are now also quite obvious by the corresponding definitions.)

On the other hand, concerning the sets $A = \{1, 2\}$ and $B = \{3, 4\}$ we can also easily see that

$$R_i[A] = R_i(1) \cup R_i(2) \not\subseteq A \qquad \text{and} \qquad R_i[B] = R_i(3) \cup R_i(4) \not\subseteq B$$

for all $i \in X$, and thus $A, B \notin \tau_{\mathcal{R}}$. Hence, by using that $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can infer that $\tau_{\mathcal{R}} = \{ \emptyset, X \}$, and thus $\mathcal{R}$ is quasi-proximally minimal. Hence, by using Theorems 25.3, 29.3 and 30.3, we can infer that $\mathcal{R}$ is also quasi-proximally connected, hyperconnected and ultraconnected. (The latter statements are now also quite obvious by the corresponding definitions.)

Now, we can also note that $\mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}} \neq \mathcal{P}(X)$. Therefore, $\mathcal{R}$ is not a quasi-topologically door relator. Moreover, by using Theorems 7.3 and 7.6, we can also easily see that

$$\mathcal{E}_{\mathcal{R}} = \big\{ \{1,\, 2\},\ \{3,\, 4\},\ \{1,\, 2,\, 3\},\ \{1,\, 2,\, 4\},\ \{1,\, 3,\, 4\},\ \{2,\, 3,\, 4\},\ X \big\}$$

and $\mathcal{D}_{\mathcal{R}} = \{\{1,\, 2,\, 3\}, \{1,\, 2,\, 4\}, \{1,\, 3,\, 4\}, \{2,\, 3,\, 4\}, X\}$. Therefore, $\mathcal{E}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}$, and thus $\mathcal{R}$ is not also a quasi-topologically superset and submaximal relator. Hence, by Theorem 35.1, we can see that $\mathcal{R}$ is also not a quasi-proximally door, superset and submaximal relator. Moreover, since $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} = \emptyset$, we can also state that $\mathcal{R}$ is not resolvable.

Remark 38.2. In connection with the above relator $\mathcal{R}$, it is also noteworthy that

$$\mathcal{R}^2 = \big\{ R^2 :\ R \in \mathcal{R} \big\} = \big\{ X^2 \big\}, \qquad \text{but} \qquad \mathcal{R} \circ \mathcal{R} = \big\{ R \circ S :\ \ R,\, S \in \mathcal{R} \big\} \not\subseteq \big\{ X^2 \big\}.$$

Namely, for instance, we have $R_1 \,[\, R_3(3)\,] = R_1 \,[\, \{3,\, 4\}\,] = \{2,\, 3,\, 4\} \neq X$.

Moreover, if $\mathcal{R}$ is as in Example 38.1, then again by Theorem 20.7 and Corollary 20.6, it is clear that $\mathcal{R}$ cannot be proximally simple and topologically fine. However, by using direct arguments, we can again prove some better assertions.

Example 38.3. If $\mathcal{R}$ is as in Example 38.1, then

(1) $\mathcal{R}$ is not uniformly and proximally simple;

(2) $\mathcal{R}$ is quasi-proximally, quasi-topologically, topologically and paratopologically simple.

By taking $U = X^2$, we can note that U is an equivalence relation on X such that $\tau_{\mathcal{R}} = \big\{ \emptyset,\, X \big\} = \tau_{\{U\}}$. Hence, by using Theorem 12.9, we can infer that $\mathcal{R}^{\#\infty} = \{U\}^{\#\infty}$, and thus $\mathcal{R}$ is quasi-proximally simple.

Moreover, by taking $V = A^2 \cup B^2$, with $A = \{1,\, 2\}$ and $B = \{3,\, 4\}$, we can note that V is an equivalence relation on X such that

$$V(1) = V(2) = \{1,\, 2\} \qquad\qquad \text{and} \qquad\qquad V(3) = V(4) = \{3,\, 4\}.$$

Hence, it is clear that in addition $\mathcal{R} \subseteq \{V\}^{\wedge}$, we also have $V \in \mathcal{R}^{\wedge}$. Therefore, $\mathcal{R}^{\wedge} = \{V\}^{\wedge}$, and thus $\mathcal{R}$ is topologically simple. Hence, it is clear that $\mathcal{R}$ is also quasi-topologically simple. Moreover, since $\wedge\ \triangle {=} \triangle$, we can also state that $\mathcal{R}$ is also paratopologically simple.

Next, we show directly that $\mathcal{R}$ is not proximally simple. For this, assume on the contrary that $\mathcal{R}$ is proximally simple. Then, there exists a relation S on X such that $\mathcal{R}^{\#} = \{S\}^{\#}$. Then, by using that $\#$ is extensive, we can infer that $\mathcal{R} \subseteq \{S\}^{\#}$ and $S \in \mathcal{R}^{\#}$. Thus, in particular we have $S(3) \subseteq R_1(3)$ and $S(3) \subseteq R_2(3)$, and thus $S(3) \subseteq R_1(3) \cap R_2(3) = \{3,\, 4\}$. Moreover, quite similarly we can also see that $S(4) \subseteq \{3,\, 4\}$. Therefore, for the set $A = \{3,\, 4\}$, we have $S\,[\,A\,] \subseteq A$. On the other hand, since $S \in \mathcal{R}^{\#}$, we have $R_i\,[\,A\,] \subseteq S\,[\,A\,]$, and thus $R_i\,[\,A\,] \subseteq A$ for some $i \in X$. However, this is a contradiction since $\operatorname{card}(A) = 2$, while $\operatorname{card}\big(R_i\,[\,A\,]\big) \geq 3$ for all $i \in X$. Therefore, $\mathcal{R}$ is not proximally simple. Hence, since $*\# = \#$, it is clear that $\mathcal{R}$ cannot also be uniformly simple.

Remark 38.4. Concerning the relator $\mathcal{R}$, considered in Example 38.1, we can also note that $X^2 \notin \mathcal{R}$, and thus $\mathcal{R}$ cannot be $\square$–fine for any operation $\square$ for relators with $X^2 \in \mathcal{R}^{\square}$.

Remark 38.5. Simple and quasi-simple relators have formerly been intensively investigated by Száz and Mala [167, 105, 110, 111, 108].

However, the characterization of paratopologically simple relators and the existence of non-paratopologically simple relators were serious problems.

They were first established by J. Deák and G. Pataki. (See [134].) In particular, Pataki has constructed a non-paratopologically simple equivalence relator.

This justified an old conjecture of the second author that, in addition to preordered nets, multi-preordered nets have also to be intensively investigated.

39. Two further illustrating examples

The following example, suggested probably also by Pataki [135], will show that even some very particular quasi-topologically minimal relators need not be paratopologically minimal. Thus, in particular, the converse of Theorem 23.3 is not true.

Example 39.1. If $X = \mathbb{R}$ and R is a relation on X such that

$$R(x) = \{x-1\} \cup [x, +\infty[$$

for all $x \in X$, then $\mathcal{R} = \{R\}$ is a reflexive relator on X such that:

(1) $\mathcal{R}$ is not paratopologically minimal;

(2) $\mathcal{R}$ is both resolvable and hyperconnected;

(3) $\mathcal{R}$ is neither quasi-proximally nor quasi-topologically door, superset and submaximal;

(4) $\mathcal{R}$ is both quasi-proximally and quasi-topologically minimal, connected, hyperconnected and ultraconnected.

It is clear that R is a reflexive relation on X, and thus $\mathcal{R}$ is a reflexive relator on X. Moreover, we can at once see that $R(x) \cap R(y) \neq \emptyset$ for all $x, y \in X$. Thus, by Theorem 31.7, $\mathcal{R}$ is hyperconnected.

On the other hand, we can at once see that $R \neq X^2$, and thus $\mathcal{R} \not\subseteq \{X^2\}$. Therefore, by Theorems 24.1, $\mathcal{R}$ is not paratopologically minimal. Moreover, we can also note that $\mathbb{N} \cap R(x) \neq \emptyset$ and $R(x) \not\subseteq \mathbb{N}$ for all $x \in X$. Therefore, by Theorem 7.3, $\mathbb{N} \in \mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$, and thus $\mathcal{R}$ is resolvable.

Now, actually it remains only to show that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$, and thus $\mathcal{R}$ is quasi-topologically minimal. Namely, in this case, by Theorems 20.3, 25.3, 29.3 and 30.3, the remaining parts of assertion (4) are also true. Moreover, by Definitions 34.1, 34.5 and 31.9, assertion (3) is also true.

For the proof of $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$, note that if $A \in \mathcal{T}_{\mathcal{R}}$, then by Theorem 8.12, for any $a \in A$, we have $R(a) \subseteq A$, and thus $\{a-1\} \cup [a, +\infty[\subseteq A$. Therefore, if $x \in A$, then $\{x-1\} \cup [x, +\infty[\subseteq A$, and thus in particular $x-1 \in A$. Therefore, $\{x-2\} \cup [x-1, +\infty[\subseteq A$, and thus in particular $x-2 \in A$. Hence, is clear that we can only have either $A = \emptyset$ or $A = X$. Therefore, $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$.

Remark 39.2. If $\mathcal{R}$ is as in Example 39.1, then it is also worth noticing that

$$R(x) = \{ x-1 \} \cup [\, x \,,\, +\infty \,[\; \in \mathcal{E}_{\mathcal{R}} \,,$$

but

$$\mathrm{cl}_{\mathcal{R}}(x) = R^{-1}(x) =]-\infty \,,\, x\,] \cup \{ x+1 \} \notin \{ \emptyset ,\, X \} = \mathcal{F}_{\mathcal{R}}$$

for all $x \in X$.

Therefore, despite of $\mathcal{T}_{\mathcal{R}} = \{ \emptyset ,\, X \}$, $\mathcal{E}_{\mathcal{R}}$ is quite a large subfamily of $\mathcal{P}(X)$. Moreover, the relator $\mathcal{R}$ is very far from being even weakly quasi-topological.

The following example will show that, despite of the close resemblance of Definitions 29.1 and 30.1, quasi-proximal and quasi-topological ultraconnectedness properties are quite independent from the corresponding hyperconnectedness ones.

Example 39.3. If $X = \{1,\, 2,\, 3\}$ and $R_1,\, R_2 \subseteq X^2$ such that

$$R_1(1) = \{1\}\,, \qquad R_1(2) = X\,, \qquad R_1(3) = X\,,$$
$$R_2(1) = X\,, \qquad R_2(2) = \{2\}\,, \qquad R_2(3) = X\,,$$

then $\mathcal{R} = \{ R_1,\, R_2 \}$ is a preorder relator on X such that:

(1) $\mathcal{R}$ is both quasi-proximally and quasi-topologically ultraconnected;

(2) $\mathcal{R}$ is neither quasi-proximally nor quasi-topologically hyperconnected.

For this, note that $R_1 = \{1\}^2 \cup \{1\}^c \times X$ and $R_2 = \{2\}^2 \cup \{2\}^c \times X$. Therefore, by a basic property of Pervin relations, R_1 and R_2 are preorder (reflexive and transitive) relations on X. Thus, $\mathcal{R}$ is a preorder relator on X.

Moreover, by using some further basic properties of Pervin relations, we can see that

$$\tau_{\mathcal{R}} = \big\{ \emptyset ,\, \{1\} ,\{2\} ,\, X \big\} \qquad \text{and} \qquad \mathcal{T}_{\mathcal{R}} = \big\{ \emptyset ,\, \{1\} ,\{2\} ,\, \{1,\, 2\} ,\, X \big\} ,$$

and thus

$$\tau\!\!\!-_{\mathcal{R}} = \big\{ \emptyset ,\, \{1,\, 3\} ,\, \{2,\, 3\} ,\, X \big\} \qquad \text{and} \qquad \mathcal{F}_{\mathcal{R}} = \big\{ \emptyset ,\, \{3\} ,\, \{1,\, 3\} ,\, \{2,\, 3\} ,\, X \big\} .$$

Therefore, the families $\tau\!\!\!-_{\mathcal{R}} \setminus \{\emptyset\}$ and $\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ have the binary intersection property, but the families $\tau_{\mathcal{R}} \setminus \{\emptyset\}$ and $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ do not have the binary intersection property.

Remark 39.4. By using the equality $R_A^{-1} = R_{A^c}$, we can quite easily see that

$$R_1^{-1}(1) = X\,, \qquad R_1^{-1}(2) = \{2,\, 3\}\,, \qquad R_1^{-1}(3) = \{2,\, 3\}\,,$$
$$R_2^{-1}(1) = \{1,\, 3\}\,, \qquad R_2^{-1}(2) = X\,, \qquad R_2^{-1}(3) = \{1,\, 3\}\,.$$

Hence, by some another basic properties of Pervin relations, it is clear that

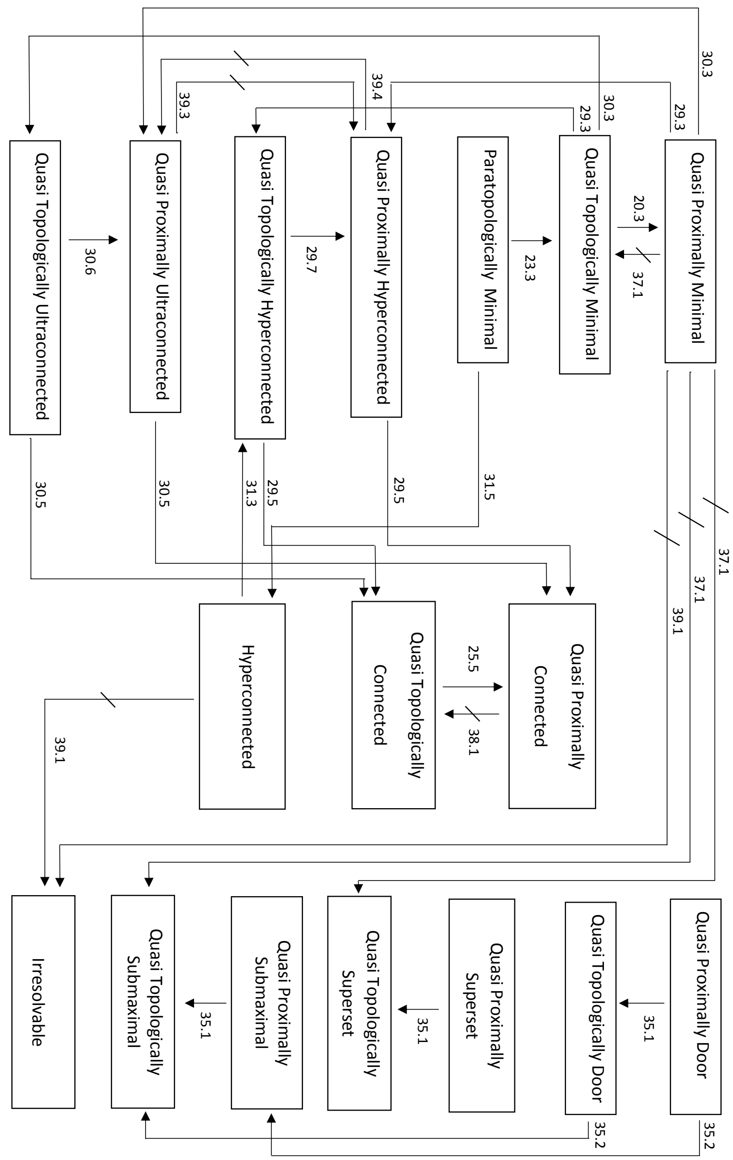
$$\tau_{\mathcal{R}^{-1}} = \big\{ \emptyset ,\, \{1,\, 3\} ,\, \{2,\, 3\} ,\, X \big\} \qquad \text{and} \qquad \mathcal{T}_{\mathcal{R}^{-1}} = \big\{ \emptyset ,\, \{1,\, 3\} ,\, \{2,\, 3\} ,\, X \big\} ,$$

and thus

$$\tau\!\!\!-_{\mathcal{R}^{-1}} = \big\{ \emptyset ,\, \{1\} ,\{2\} ,\, X \big\} \qquad \text{and} \qquad \mathcal{F}_{\mathcal{R}^{-1}} = \big\{ \emptyset ,\, \{1\} ,\{2\} ,\, X \big\} .$$

Therefore, we can also state that $\mathcal{R}^{-1}$ is a both quasi-proximally and quasi-topologically hyperconnected preorder relator on X such that $\mathcal{R}^{-1}$ is neither quasi-proximally nor quasi-topologically ultraconnected.

For a nonvoid relator $\mathcal{R}$ on a nonvoid set X, the following implications are true:



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Muwafaq Salih, Institute of Mathematics, University of Debrecen, H-4032 Debrecen, Egyetem tér 1, Hungary

Email address: muwafaq.salih@science.unideb.hu

Árpád Száz, Institute of Mathematics, University of Debrecen, H-4032 Debrecen, Egyetem tér 1, Hungary

Email address: szaz@science.unideb.hu