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THE DIGITAL SMASH PRODUCT

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ABSTRACT. In this paper, we construct the smash product from the digital viewpoint and prove some its properties such as associativity, distributivity, and commutativity. Moreover, we present the digital suspension and the digital cone for an arbitrary digital image and give some examples of these new concepts.

1. INTRODUCTION

Digital topology with interesting applications has been a popular topic in computer science and mathematics for several decades. Many researchers such as Rosenfeld [21, 22], Kong [18, 17], Kopperman [19], Boxer, Herman [14], Kovalevsky [20], Bertrand and Malgouyres would like to obtain some information about digital objects using topology and algebraic topology.

The first study in this area was done by Rosenfeld [21] at the end of 1970s. He introduced the concept of continuity of a function from a digital image to another digital image. Later Boxer [1] presents a continuous function, a retraction, and a homotopy from the digital viewpoint. Boxer et al. [7] calculate the simplicial homology groups of some special digital surfaces and compute their Euler characteristics.

Ege and Karaca [9] introduce the universal coefficient theorem and the Eilenberg-Steenrod axioms for digital simplicial homology groups. They also obtain some results on the Künneth formula and the Hurewicz theorem in digital images. Ege and Karaca [10] investigate the digital simplicial cohomology groups and especially define the cup product. For other significant studies, see [13, 12, 16].

Karaca and Cinar [15] construct the digital singular cohomology groups of the digital images equipped with Khalimsky topology. Then they examine the Eilenberg-Steenrod axioms, the universal coefficient theorem, and the Künneth formula for a cohomology theory. They also introduce a cup product and give general properties of this new operation. Cinar and Karaca [8] calculate the digital homology groups of various digital surfaces and give some results related to Euler characteristics for some digital connected surfaces.

This paper is organized as follows: First, some information about the digital topology is given in the section of preliminaries. In the next section, we define the smash product for digital images. Then, we show that this product has some properties such as associativity, distributivity, and commutativity. Finally, we investigate a suspension and a cone for any digital image and give some examples.

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2. Preliminaries

Let $\mathbb{Z}^n$ be the set of lattice points in the n-dimensional Euclidean space. We call that (X, κ) is a digital image where X is a finite subset of $\mathbb{Z}^n$ and κ is an adjacency relation for the members of X. Adjacency relations on $\mathbb{Z}^n$ are defined as follows: Two points $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ in $\mathbb{Z}^n$ are called c_l-adjacent [2] for $1 \leq l \leq n$ if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices i such that $|p_i - q_i| \neq 1$, $p_i = q_i$. It is easy to see that $c_1 = 2$ (see Figure 1) in $\mathbb{Z}$,

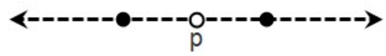

Figure 1. 2-adjacency in $\mathbb{Z}$

$c_1 = 4$ and $c_2 = 8$ (see Figure 2) in $\mathbb{Z}^2$,

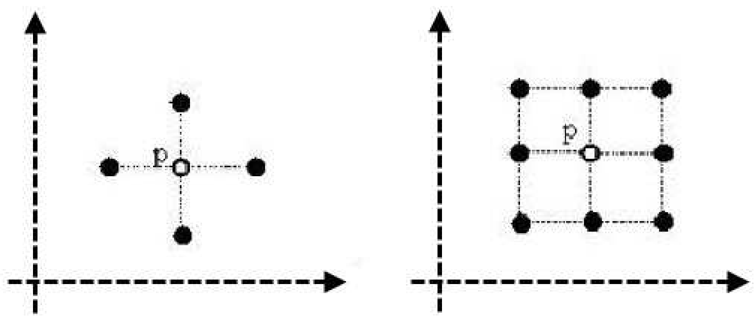

Figure 2. 4 and 8 adjacencies in $\mathbb{Z}^2$

and $c_1 = 6$, $c_2 = 18$ and $c_3 = 26$ (see Figure 3) in $\mathbb{Z}^3$.

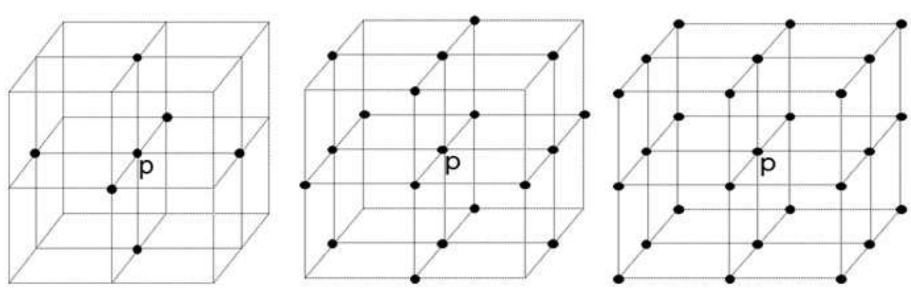

Figure 3. 6, 18 and 26 adjacencies in $\mathbb{Z}^3$

A κ-*neighbor* of p in $\mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ which is κ-adjacent to p. A digital image X is κ-*connected* [14] if and only if for each distinct points $x, y \in X$, there exists a set $\{a_0, a_1, \ldots, a_r\}$ of points of X such that $x = a_0$, $y = a_r$, and a_i and a_{i+1} are κ-adjacent where $i \in \{0, 1, \ldots, r-1\}$. A κ-component of a digital image X is a maximal κ-connected subset of X. Let $a, b \in \mathbb{Z}$ with $a < b$. A digital interval [1] is defined as follows:

$$[x, y]_{\mathbb{Z}} = \{a \in \mathbb{Z} \mid x \leq a \leq y, x, y \in \mathbb{Z}\},$$

where 2-adjacency relation is assumed.

In a digital image (X, κ), *a digital κ-path* [3] from x to y is a $(2, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \to X$ such that $f(0) = x$ and $f(m) = y$ where $x, y \in X$. Let $f : (X, \kappa) \to (Y, \lambda)$ be a function. If the image under f of every κ-connected subset of X is κ-connected, then f is called (κ, λ)-continuous [2].

A function $f : (X, \kappa) \to (Y, \lambda)$ is (κ, λ)-continuous [22, 2] if and only if for any κ-adjacent points $a, b \in X$, the points $f(a)$ and $f(b)$ are equal or λ-adjacent. A function $f : (X, \kappa) \to (Y, \lambda)$ is an *isomorphism* [4] if f is a (κ, λ)-continuous bijection and f^{-1} is (λ, κ)-continuous.

Definition 2.1. [2] Suppose that $f,\ g : (X, \kappa) \to (Y, \lambda)$ are (κ, λ)-continuous maps. If there exist a positive integer m and a function

$$F : X \times [0, m]_{\mathbb{Z}} \to Y$$

with the following conditions, then F is called *a digital* (κ, λ)*-homotopy* between f and g, and we say that f and g are digitally (κ, λ)-homotopic in Y, denoted by $f \simeq_{(\kappa,\lambda)} g$.

(i) For all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$.

(ii) For all $x \in X$, $F_x : [0, m] \to Y$ defined by $F_x(t) = F(x, t)$ is $(2, \lambda)$-continuous.

(iii) For all $t \in [0, m]_{\mathbb{Z}}$, $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ is (κ, λ)-continuous.

A digital image (X, κ) is *κ-contractible* [1] if the identity map on X is (κ, κ)-homotopic to a constant map on X.

A (κ, λ)-continuous map $f : X \to Y$ is (κ, λ)-homotopy equivalence [3] if there exists a (λ, κ)-continuous map $g : Y \to X$ such that

$$g \circ f \simeq_{(\kappa,\kappa)} 1_X \quad \text{and} \quad f \circ g \simeq_{(\lambda,\lambda)} 1_Y$$

where 1_X and 1_Y are the identity maps on X and Y, respectively. Moreover, we say that X and Y have the same (κ, λ)*-homotopy type.*

For the cartesian product of two digital images X_1 and X_2, the adjacency relation [6] is defined as follows: Two points $x_i, y_i \in (X_i, \kappa_i)$, (x_0, y_0) and (x_1, y_1) are $k_*(\kappa_1, \kappa_2)$-adjacent in $X_1 \times X_2$ if and only if one of the following is satisfied:

- $x_0 = x_1$ and $y_0 = y_1$; or
- $x_0 = x_1$ and y_0 and y_1 are κ_1-adjacent; or
- x_0 and x_1 are κ_0-adjacent and $y_0 = y_1$; or
- x_0 and x_1 are κ_0-adjacent and y_0 and y_1 are κ_1-adjacent.

Definition 2.2. [3] A (κ, λ)-continuous surjection $f : X \to Y$ is (κ, λ)-shy if

- for each $y \in Y$, $f^{-1}(\{y\})$ is κ-connected, and
- for each $y_0, y_1 \in Y$, if y_0 and y_1 are λ-adjacent, then $f^{-1}(\{y_0, y_1\})$ is κ-connected.

Theorem 2.3. [5] *For a continuous surjection $f(X, \kappa) \to (Y, \lambda)$, if f is an isomorphism, then f is shy. On the other hand, if f is shy and injective, then f is an isomorphism.*

The *wedge* of two digital images (X, κ) and (Y, λ), denoted by $X \vee Y$, is the union of the digital images (X', μ) and (Y', μ), where [4]

- X' and Y' have a single point p;
- If $x \in X'$ and $y \in Y'$ are μ-adjacent, then either $x = p$ or $y = p$;
- (X', μ) and (X, κ) are isomorphic; and
- (Y', μ) and (Y, λ) are isomorphic.

Theorem 2.4. [5] *Two continuous surjections*

$$f : (A, \alpha) \to (C, \gamma) \quad \textit{and} \quad g : (B, \beta) \to (D, \delta)$$

are shy maps if and only if $f \times g : (A \times B, k_*(\alpha, \beta)) \to (C \times D, k_*(\gamma, \delta))$ *is a shy map.*

Sphere-like digital images is defined as follows [4]:

$$S_n = [-1, 1]_{\mathbb{Z}}^{n+1} \backslash \{0_{n+1}\} \subset \mathbb{Z}^{n+1},$$

where 0_n is the origin point of $\mathbb{Z}^n$. For $n = 0$ and $n = 1$, the sphere-like digital images are shown in Figure 4.

$$S_0 = \{c_0 = (1, 0), c_1 = (-1, 0)\},$$

$$S_1 = \{c_0 = (1, 0), c_1 = (1, 1), c_2 = (0, 1), c_3 = (-1, 1), c_4 = (-1, 0), c_5 = (-1, -1), \\ c_6 = (0, -1), c_7 = (1, -1)\}.$$

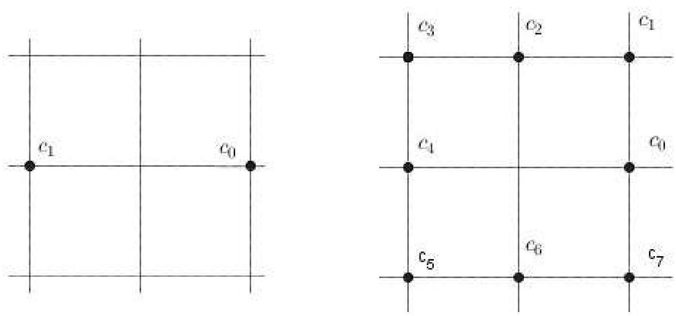

FIGURE 4. Digital 0−sphere S_0 and digital 1-sphere S_1

3. THE DIGITAL SMASH PRODUCT

In this section, we define the digital smash product which has some important relations with a digital homotopy theory.

Definition 3.1. Let (X, κ) and (Y, λ) be two digital images. The *digital smash product* $X \wedge Y$ is defined to be the quotient digital image $(X \times Y)/(X \vee Y)$ with the adjacency relation $k_*(\kappa, \lambda)$, where $X \vee Y$ is regarded as a subset of $X \times Y$.

Before giving some properties of the digital smash product, we prove some theorems which will be used later.

Theorem 3.2. *Let* X_a *and* Y_a *be digital images for each element* a *of an index set* A. *For each* $a \in A$, *if* $f_a \simeq_{(\kappa,\lambda)} g_a : X_a \to Y_a$ *then*

$$\prod_{a \in A} f_a \simeq_{(\kappa^n,\lambda^n)} \prod_{a \in A} g_a,$$

where n *is the cardinality of the set* A.

Proof. Let $F_a : X_a \times [0,m]_{\mathbb{Z}} \to Y_a$ be a digital (κ,λ)-homotopy between f_a and g_a, where $[0,m]_{\mathbb{Z}}$ is a digital interval. Then

$$F : (\prod_{a\in A} X_a) \times [0,m]_{\mathbb{Z}} \to \prod_{a\in A} Y_a$$

defined by

$$F((x_a),t) = (F_a(x_a,t))$$

is a digital continuous function, where t is an element of $[0,m]_{\mathbb{Z}}$ since the functions F_a are digital continuous for each element $a \in A$. Therefore F is a digital (κ^n,λ^n)-homotopy between $\prod_{a\in A} f_a$ and $\prod_{a\in A} g_a$. □

Theorem 3.3. *If each $f_a : X_a \to Y_a$ is a digital (κ,λ)-homotopy equivalence for all $a \in A$, then $\prod_{a\in A} f_a$ is a digital (κ^n,λ^n)-homotopy equivalence, where n is the cardinality of the set A.*

Proof. Let $g_a : Y_a \to X_a$ be a (λ,κ)-homotopy inverse to f_a, for each $a \in A$. Then we obtain the following relations:

$$(\prod_{a\in A} g_a)(\prod_{a\in A} f_a) = \prod_{a\in A}(g_a \times f_a) \simeq_{(\lambda^n,\kappa^n)} \prod_{a\in A}(1_{X_a}) = 1_{\prod_{a\in A} X_a},$$

$$(\prod_{a\in A} f_a)(\prod_{a\in A} g_a) = \prod_{a\in A}(f_a \times g_a) \simeq_{(\kappa^n,\lambda^n)} \prod_{a\in A}(1_{Y_a}) = 1_{\prod_{a\in A} Y_a}.$$

So we conclude that $\prod_{a\in A} f_a$ is a digital (κ^n,λ^n)-homotopy equivalence. □

Theorem 3.4. *Let (X,κ), (Y,λ) and (Z,σ) be digital images. If $p : (X,\kappa) \to (Y,\lambda)$ is a $(\kappa,\lambda)-$shy map and (Z,σ) is a σ-connected digital image, then*

$$p \times 1 : (X \times Z, k_*(\kappa\times\sigma)) \to (Y \times Z, k_*(\lambda\times\sigma))$$

is a $(\kappa\times\sigma,\lambda\times\sigma)$-shy map, where $1_Z : (Z,\sigma) \to (Z,\sigma)$ is an identity function.

Proof. Since (Z,σ) is a σ-connected digital image, then for $y \in Y$ and $z \in Z$, we have

$$\begin{aligned}(p\times 1_Z)^{-1}(y,z) &= (p^{-1}(y), 1_Z^{-1}(z))\\ &= (p^{-1}(y), z).\end{aligned}$$

Thus, for each $y \in Y$ and $z \in Z$, $(p \times 1_Z)^{-1}(y,z)$ is κ-connected by the definition of the adjacency of the cartesian product of digital images. Moreover, the map 1_Z preserves the connectivity, that is, for every $z_0, z_1 \in Z$ such that z_0 and z_1 are σ-adjacent, $1_Z(\{z_0,z_1\}) = \{z_0,z_1\}$ is σ-connected. It is easy to see that

$$\begin{aligned}(p\times 1_Z)^{-1}(\{y_0,y_1\},\{z_0,z_1\}) &= (p^{-1}(\{y_0,y_1\}), 1_Z^{-1}(\{z_0,z_1\}))\\ &= (p^{-1}(\{y_0,y_1\}), (\{z_0,z_1\})).\end{aligned}$$

Hence for each $y_0, y_1 \in Y$ and $z_0, z_1 \in Z$, $(p\times 1_Z)^{-1}(\{y_0,y_1\},\{z_0,z_1\})$ is a $k_*(\kappa,\sigma)$-connected using the definition of the adjacency of the Cartesian product of digital images. □

Theorem 3.5. *Let A and B be digital subsets of (X,κ) and (Y,λ), respectively. If $f,g : (X,A) \to (Y,B)$ are (κ,λ)-continuous functions such that $f \simeq_{(\kappa,\lambda)} g$, then the induced maps $\bar{f},\bar{g} : (X/A,\kappa) \to (Y/B,\lambda)$ are digitally (κ,λ)-homotopic.*

Proof. Let $F : (X \times I, A \times I) \to (Y, B)$ be a digital (κ, λ)-homotopy between f and g where $I = [0, m]_{\mathbb{Z}}$. It is clear that F induces a digital function $\bar{F} : (X/A) \times I \to Y/B$ such that the following square diagram is commutative, where p and q are shy maps:

$$\begin{array}{ccc} X \times I & \xrightarrow{\ F\ } & Y \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle q} \\ (X/A) \times 1 & \xrightarrow[\bar{F}]{} & Y/B. \end{array}$$

Since $q \circ F$ is digitally continuous, $p \times 1$ is a shy map and $\bar{F}(p \times 1) = q \circ F$, $\bar{F}$ is a digital continuous map. Hence $\bar{F}$ is a digital (κ, λ)-homotopy map between $\bar{f}$ and $\bar{g}$. □

We are ready to present some properties of the digital smash product. The following theorem gives a relation between the digital smash product and the digital homotopy.

Theorem 3.6. *Given digital images (X, κ), (Y, λ), (A, σ), (B, α) and two digital functions $f : X \to A$ and $g : Y \to B$, there exists a function $f \wedge g : X \wedge Y \to A \wedge B$ with the following properties:*
(i) If $h : A \to C$, $k : B \to D$ are digital functions, then

$$(h \wedge k) \circ (f \wedge g) = (h \circ f) \wedge (k \circ g).$$

(ii) If $f \simeq_{(\kappa,\sigma)} f^{'} : X \to A$ and $g \simeq_{(\lambda,\alpha)} g^{'} : Y \to B$, then

$$f \wedge g \simeq_{(k_*(\kappa,\lambda),k_*(\sigma,\alpha))} f^{'} \wedge g^{'}.$$

Proof. The digital function $f \times g : X \times Y \to A \times B$ has the property that

$$(f \times g)(X \vee Y) \subset A \times B.$$

Hence $f \times g$ induces a digital function $f \wedge g : X \wedge Y \to A \wedge B$ and property (i) is obvious. As for (ii), the digital homotopy F between $f \times g$ and $f^{'} \times g^{'}$ can be constructed as follows: We know that

$$f \simeq_{(\kappa,\sigma)} f^{'} \quad \text{and} \quad g \simeq_{(\lambda,\alpha)} g^{'}.$$

By Theorem 3.2, we have

$$f \times g \simeq_{(k_*(\kappa,\lambda),k_*(\sigma,\alpha))} f^{'} \times g^{'}.$$

F is a digital homotopy of functions of pairs from $(X \times Y, X \vee Y)$ to $(A \times B, A \vee B)$. Consequently a digital homotopy between $f \wedge g$ and $f^{'} \wedge g^{'}$ is induced by Theorem 3.5. □

Theorem 3.7. *If f and g are digital homotopy equivalences, then $f \wedge g$ is a digital homotopy equivalence.*

Proof. Let $f : (X, \kappa) \to (Y, \lambda)$ be a (κ, λ)-homotopy equivalence. Then there exists a (λ, κ)-continuous function $f^{'} : (Y, \lambda) \to (X, \kappa)$ such that

$$f \circ f^{'} \simeq_{(\lambda,\lambda)} 1_Y \text{ and } f^{'} \circ f \simeq_{(\kappa,\kappa)} 1_X.$$

Moreover, let $g : (A, \sigma) \to (B, \alpha)$ be a (σ, α)-homotopy equivalence. Then there is a (α, σ)-continuous function $g^{'} : (B, \alpha) \to (A, \sigma)$ such that

$$g \circ g^{'} \simeq_{(\alpha,\alpha)} 1_B \text{ and } g^{'} \circ g \simeq_{(\sigma,\sigma)} 1_A.$$

By Theorem 3.6, there exist digital functions

$$f \wedge g : X \wedge A \to Y \wedge B \quad \text{and} \quad f^{'} \wedge g^{'} : Y \wedge B \to X \wedge A$$

such that

$$(f \wedge g) \circ (f^{'} \wedge g^{'}) = 1_{Y \wedge B},$$
$$(f \circ f^{'}) \wedge (g \circ g^{'}) = 1_{Y \wedge B},$$

and

$$(f^{'} \wedge g^{'}) \circ (f \wedge g) = 1_{X \wedge A},$$
$$(f^{'} \circ f) \wedge (g^{'} \circ g) = 1_{X \wedge A}.$$

So $f \wedge g$ is a digital homotopy equivalence. □

The following theorem shows that the digital smash product is associative.

Theorem 3.8. *Let (X, κ), (Y, λ) and (Z, σ) be digital images. $(X \wedge Y) \wedge Z$ is digitally isomorphic to $X \wedge (Y \wedge Z)$.*

Proof. Consider the following diagram:

$$\begin{array}{ccc} X \times Y \times Z & \xrightarrow{1} & X \times Y \times Z \\ \downarrow {\scriptstyle p \times 1} & & \downarrow {\scriptstyle 1 \times p} \\ (X \wedge Y) \times Z & & X \times (Y \wedge Z) \\ \downarrow {\scriptstyle p} & & \downarrow {\scriptstyle p} \\ (X \wedge Y) \wedge Z & \xrightarrow[f]{} & X \wedge (Y \wedge Z) \end{array}$$

where p represents for the digital shy maps of the form $X \times Y \to X \wedge Y$. By Theorem 3.4, $p \times 1$ and $1 \times p$ are digital shy maps. $1 : X \times Y \times Z \to X \times Y \times Z$ induces functions

$$f : (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z) \quad \text{and} \quad g : X \wedge (Y \wedge Z) \to (X \wedge Y) \wedge Z.$$

These functions are clearly injections. By Theorem 2.3, f is a digital isomorphism. □

The next theorem gives the distributivity property for the digital smash product.

Theorem 3.9. *Let (X, κ), (Y, λ) and (Z, σ) be digital images. $(X \vee Y) \wedge Z$ is digitally isomorphic to $(X \wedge Z) \vee (Y \wedge Z)$.*

Proof. Suppose that p represents for the digital shy maps of the form $X \times Y \to X \wedge Y$ and q stands for the digital shy maps of the form $X \times Y \to X \vee Y$. We may obtain the following diagram:

$$\begin{array}{ccc} (X \times Y) \times Z & \xrightarrow{m} & (X \times Z) \times (Y \times Z) \\ {\scriptstyle p} \downarrow & & \downarrow {\scriptstyle p \times p} \\ (X \times Y) \wedge Z & & (X \wedge Z) \times (Y \wedge Z) \\ {\scriptstyle q \wedge 1} \downarrow & & \downarrow {\scriptstyle q} \\ (X \vee Y) \wedge Z & \xrightarrow[f]{} & (X \wedge Z) \vee (Y \wedge Z). \end{array}$$

From Theorem 2.4, $p \times p$ is a digital shy map and by Theorem 3.4, $q \wedge 1$ is also a digital shy map. The function $m : (X \times Y) \times Z \to (X \times Z) \times (Y \times Z)$ induces a digital function

$$f : (X \wedge Z) \times (Y \wedge Z) \to (X \times Z) \times (Y \times Z).$$

Obviously f is a one-to-one function. By Theorem 2.3, f is a digital isomorphism. □

Theorem 3.10. *Let (X, κ) and (Y, λ) be digital images. $X \wedge Y$ is digitally isomorphic to $Y \wedge X$.*

Proof. If we suppose that g stands for the digital shy maps $Y \times X \to Y \wedge X$ and p represents for the digital shy maps of the form $X \times Y \to X \wedge Y$, we get the following diagram:

$$\begin{array}{ccc} X \times Y & \xrightarrow{u} & Y \times X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X \wedge Y & \xrightarrow[f]{} & Y \wedge X. \end{array}$$

The switching map $u : X \times Y \to Y \times X$ induces a digital shy map $f : X \wedge Y \to Y \wedge X$. Additionally, f is a one-to-one. Hence, f is a digital isomorphism from Theorem 2.3. □

Definition 3.11. The *digital suspension* of a digital image X, denoted by sX, is defined to be $X \wedge S_1$.

Example 1. Choose a digital image $X = S_0$. Then we get the following digital images in Figure 5.

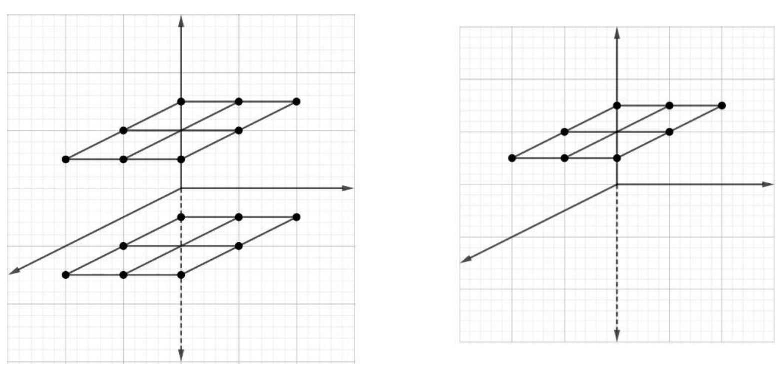

FIGURE 5. $S_1 \times S_0$ and $S_1 \wedge S_0$

Theorem 3.12. *Let x_0 be the base point of a digital image X. Then sX is digitally isomorphic to the quotient digital image*

$$(X \times [a, b]_{\mathbb{Z}})/(X \times \{a\} \cup \{x_0\} \times [a, b] \cup X \times \{b\}),$$

where the cardinality of $[a, b]_{\mathbb{Z}}$ is equal to 8.

Proof. The function

$$[a,b]_{\mathbb{Z}} \xrightarrow{\theta} S_1$$

is a digital shy map defined by $\theta(t_i) = c_i \bmod 8$, where $c_i \in S_1$ and $i \in \{0, 1, \ldots, 7\}$. Hence if $p : X \times S_1 \to X \wedge S_1$ is a digital shy map, then the digital function

$$X \times [a,b]_{\mathbb{Z}} \xrightarrow{1\times\theta} X \times S_1 \xrightarrow{p} X \wedge S_1$$

is also a digital shy map, and its effect is to identify together points of

$$X \times \{a\} \cup \{x_0\} \times [a,b]_{\mathbb{Z}} \cup X \times \{b\}.$$

The digital composite function $p \circ (1 \times \theta)$ induces a digital isomorphism

$$(X \times [a,b]_{\mathbb{Z}})/(X \times \{a\} \cup \{x_0\} \times [a,b]_{\mathbb{Z}} \cup X \times \{b\}) \to X \wedge S_1 = sX.$$

□

Definition 3.13. The *digital cone* of a digital image X, denoted by cX, is defined to be $X \wedge I$, where $I = [0,1]_{\mathbb{Z}}$.

Example 2. Take a digital image $X = S_0$. Then we have the following digital images in Figure 6.

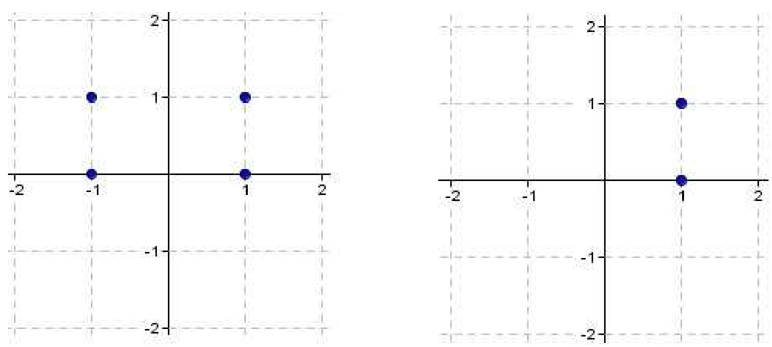

FIGURE 6. $S_0 \times I$ and $S_0 \wedge I$

Theorem 3.14. *For any digital image (X, κ), the digital cone cX is a contractible digital image.*

Proof. Since $I = [0,1]_{\mathbb{Z}}$ is digitally contractible to the point $\{0\}$,

$$cX = X \wedge I \simeq_{(2,2)} X \wedge \{0\}$$

is obviously a single point. □

Corollary 1. *For $m \in \mathbb{N}$, $S_m \wedge I$ is equal to $S_m \wedge S_0$, where $I = [0,1]_{\mathbb{Z}}$ is the digital interval and S_0 is a digital 0-sphere.*

Proof. Since S_0 and I consist of two points, we get the required result. □

4. THE OPEN PROBLEM

For each $m, n \geq 0$, can we prove that digital $(m+n)$-sphere S_{m+n} is isomorphic to $S_m \wedge S_n$?

5. Conclusion and future works

This paper introduces some notions such as the smash product, the suspension, and the cone for digital images. Since they are significant topics related to homotopy, homology, and cohomology groups in algebraic topology, we believe that the results in the paper can be useful for future studies in digital topology.

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