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NEW GENERAL DECAY RESULT FOR A FOURTH-ORDER MOORE-GIBSON-THOMPSON EQUATION WITH MEMORY

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Abstract. In this paper, we consider the fourth-order Moore-Gibson-Thompson equation with memory recently introduced by (Milan J. Math. 2017, 85: 215-234) that proposed the fourth-order model. We discuss the well-posedness of the solution by using Faedo-Galerkin method. On the other hand, for a class of relaxation functions satisfying $g'(t) \leq -\xi(t)M(g(t))$ for M to be increasing and convex function near the origin and $\xi(t)$ to be a non-increasing function, we establish the explicit and general energy decay result, from which we can improve the earlier related results.

1. Introduction

The Moore-Gibson-Thompson (MGT) equation is one of the equations of nonlinear acoustics describing acoustic wave propagation in gases and liquids [13, 15, 30] and arising from modeling high frequency ultrasound waves [9, 18] accounting for viscosity and heat conductivity as well as effect of the radiation of heat on the propagation of sound. This research field is highly active due to a wide range of applications such as the medical and industrial use of high intensity ultrasound in lithotripsy, thermotherapy, ultraound cleaning, etc. The classical nonlinear acoustics models include the Kuznetson's equation, the Westervelt equation and the Kokhlov-Zabolotskaya-Kuznetsov equation.

In order to gain a better understanding of the nonlinear MGT equation, we shall begin with the linearized model. In [15], Kaltenbacher, Lasiecka and Marchand investigated the following linearized MGT equation

$$\tau u_{ttt} + \alpha u_{tt} + c^2 \mathcal{A}u + b\mathcal{A}u_t = 0. \tag{1.1}$$

For equation (1.1), they disclosed a critical parameter $\gamma = \alpha - \frac{c^2\tau}{b}$ and showed that when $\gamma > 0$, namely in the subcritical case, the problem is well-posed and its solution is exponentially stable; while $\gamma = 0$, the energy is conserved. Since its appearance, an increasing interest has been developed to study the MGT equation, see [4, 5, 8, 10, 14]. Caixeta, Lasiecka and Cavalcanti [4] considered the following

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nonlinear equation

$$\tau u_{ttt} + \alpha u_{tt} + c^2 \mathcal{A}u + b\mathcal{A}u_t = f(u, u_t, u_{tt}). \tag{1.2}$$

They proved that the underlying PDE generates a well-posed dynamical system which admits a global and finite dimensional attractor. They also overcomed the difficulty of lacking the Lyapunov function and the lack of compactness of the trajectory.

Now, we concentrate on the stabilization of MGT equation with memory which has received a considerable attention recently. For instance, Lasiecka and Wang [17] studied the following equation:

$$\tau u_{ttt} + \alpha u_{tt} + b\mathcal{A}u_t + c^2 \mathcal{A}u - \int_0^t g(t-s)\mathcal{A}w(s)ds = 0, \tag{1.3}$$

where $\alpha - \frac{c^2\tau}{b} \geq 0$ and the form of w classifies the memory into three types. By imposing the assumption on the relaxation function g, for a positive constant c_0, as

$$g'(t) \leq -c_0 g(t), \tag{1.4}$$

they discussed the effect of memory described by three types on decay rates of the energy when $\alpha - \frac{c^2\tau}{b} > 0$. Moreover, in the critical case $\alpha - \frac{c^2\tau}{b} = 0$, they proved an exponential rate of decay for the solution of (1.3) under "the right mixture" of memory. Lasiecka and Wang [18] showed the general decay result of the equation (1.3) when $w = u$, and established their result under weaker condition on g. In [9], Filippo et al. investigated the critical case of equation (1.3) (that is $\alpha b - c^2\tau = 0$) for $w = u$ and g satisfies (1.4), and obtained an exponential decay result if and only if $\mathcal{A}$ is a bounded operator. When $\int_0^t$ is replaced by $\int_0^\infty$, (1.3) turns to

$$\tau u_{ttt} + \alpha u_{tt} + b\mathcal{A}u_t + c^2 \mathcal{A}u - \int_0^\infty g(s)\mathcal{A}w(t-s)ds = 0. \tag{1.5}$$

Alves et al. [1] investigated the uniform stability of equation (1.5) encompassing three different types of memory in a history space set by the linear semigroup theory. Moreover, we refer the reader to [3, 6, 7, 12, 24, 25, 26, 28] for other works of the equation(s) with memory.

More recently, Filippo and Vittorino [11] considered the fourth-order MGT equation

$$u_{tttt} + \alpha u_{ttt} + \beta u_{tt} + \gamma \mathcal{A}u_{tt} + \delta \mathcal{A}u_t + \varrho \mathcal{A}u = 0. \tag{1.6}$$

They investigated the stability properties of the related solution semigroup. And, according to the values of certain stability numbers depending on the strictly positive parameters α, β, γ, δ, ϱ, they established the necessary and sufficient condition for exponential stability. For other related results on the higher-order equations, please see [20, 27, 34, 35, 36, 37] and the references therein.

Motivated by the above results, we intend to study the following abstract version of the fourth-order Moore-Gibson-Thompson (MGT) equation with a memory term

$$u_{tttt} + \alpha u_{ttt} + \beta u_{tt} + \gamma \mathcal{A}u_{tt} + \delta \mathcal{A}u_t + \varrho \mathcal{A}u - \int_0^t g(t-s)\mathcal{A}u(s)ds = 0, \tag{1.7}$$

where α, β, γ, δ, ϱ are strictly positive constants, $\mathcal{A}$ is a strictly positive self-adjoint linear operator defined in a real Hilbert space H where the (dense) embedding $\mathcal{D}(\mathcal{A}) \subset H$ need not to be compact. And we consider the following initial conditions

$$u(0) = u_0, \qquad u_t(0) = u_1, \qquad u_{tt}(0) = u_2, \qquad u_{ttt}(0) = u_3. \tag{1.8}$$

A natural question that arised in dealing with the general decay of fourth-order MGT equation with memory:

- Can we get a general decay result for a class of relaxation functions satisfying $g'(t) \leq -\xi(t)M(g(t))$ for M to be increasing and convex function near the origin and $\xi(t)$ to be a nonincreasing function?

Mustafa answered this question for viscoelastic wave equations in [31, 32]. Messaoudi and Hassan [29] considered the similar question for memory-type Timoshenko system in the cases of equal and non-equal speeds of wave propagation. Moreover, they extended the range of polynomial decay rate optimality from $p \in \left[1, \frac{3}{2}\right)$ to $p \in [1, 2)$ when g satisfies $g'(t) \leq -\xi(t)g^p(t)$. We refer to [19] for the non-equal wave speeds case. And, Liu et al. [22, 23] also concerned with the similar question for third-order MGT equations with memory term.

The aim of this paper is to establish the well-posedness and answer the above mention question for fourth-order MGT equation with memory (1.7). We first use the Faedo-Galerkin method to prove the well-posedness result. We then use the idea developed by Mustafa in [31, 32], taking into consideration the nature of fourth-order MGT equation, to prove new general decay results for the case $\gamma - \frac{\delta}{\alpha} > 0$ and $\beta - \frac{\alpha\varrho}{\delta} > 0$, based on the perturbed energy method and on some properties of convex functions. Our result substantially improves and generalizes the earlier related results in previous literature.

The rest of our paper is organized as follows. In Section 2, we give some assumptions and state our main results. In Section 3, we give the proof of well-posedness. In Section 4, we state and prove some technical lemmas that are relevant in the entire work. In Section 5, we prove the general decay result.

2. Preliminaries and main results

In this section, we consider the following assumptions and state our main results. We use $c > 0$ to denote a positive constant which does not depend on the initial data.

First, we consider the following assumptions as in [11] for $(A1)$, in [18] for $(A3)$, $(A5)$ and in [31] for $(A2)$, $(A4)$:

(A1) $\gamma - \frac{\delta}{\alpha} > 0$ and $\beta - \frac{\alpha\varrho}{\delta} > 0$.

(A2) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing differentiable function such that

$$0 < g(0) < \frac{2\alpha\varrho}{\delta}(\alpha\gamma - \delta), \qquad \varrho - \int_0^{+\infty} g(s)ds = l > 0.$$

(A3) $g''(t) \geq 0$ almost everywhere.

(A4) There exists a non-increasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ and a C^1 function $M : [0, \infty) \to [0, \infty)$ which is either linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r \leq g(0)$, with $M(0) = M'(0) = 0$, such that

$$g'(t) \leq -\xi(t)M(g(t)), \quad \forall\ t \geq 0. \tag{2.1}$$

(A5) There exists $\lambda_0 > 0$ such that $\mathcal{A}$ satisfies $\|u\|^2 \leq \lambda_0 \left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2$ for all $u \in H$.

Remark 1. ([31, Remark 2.8]) (1) From assumption $(A2)$, we deduce that

$$g(t) \to 0 \qquad as \qquad t \to +\infty \qquad and \qquad g(t) \leq \frac{\varrho - l}{t}, \qquad \forall\ t > 0.$$

Furthermore, from the assumption $(A4)$, we obtain that there exists $t_0 \geq 0$ large enough such that

$$g(t_0) = r \qquad and \qquad g(t) \leq r, \qquad \forall\ t \geq t_0.$$

The non-increasing property of $g(t)$ and $\xi(t)$ gives

$$0 < g(t_0) \leq g(t) \leq g(0) \qquad and \qquad 0 < \xi(t_0) \leq \xi(t) \leq \xi(0), \qquad \forall\ t \in [0, t_0].$$

A combination of these with the continuity of H, for two constants $a, d > 0$, yields

$$a \leq \xi(t) M(g(t)) \leq d, \qquad \forall\ t \in [0, t_0].$$

Consequently, for any $t \in [0, t_0]$, we get

$$g'(t) \leq -\xi(t) M(g(t)) \leq -a = -\frac{a}{g(0)} g(0) \leq -\frac{a}{g(0)} g(t)$$

and, hence,

$$g(t) \leq -\frac{g(0)}{a} g'(t), \qquad \forall\ t \in [0, t_0]. \tag{2.2}$$

(2) If M is a strictly increasing and strictly convex C^2 function on $(0, r]$, with $M(0) = M'(0) = 0$, then it has an extension $\overline{M}$, which is strictly increasing and strictly convex C^2 function on $(0, \infty)$. For example, if we set $M(r) = A$, $M'(r) = B$, $M''(r) = C$, we can define $\overline{M}$, for any $t > r$, by

$$\overline{M} = \frac{C}{2} t^2 + (B - Cr) t + \left(A + \frac{C}{2} r^2 - Br \right).$$

Then, inspired by the notations in [11], we define the Hilbert spaces

$$H_r := \mathcal{D}\left(\mathcal{A}^{\frac{r}{2}} \right), \qquad r \in \mathbb{R}.$$

In order to simplify the notation, we denote the usual space H_0 by H. The phase space of our problem is

$$\mathcal{H} = \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}} \right) \times \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}} \right) \times \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}} \right) \times H.$$

Moreover, we will denote the inner product of H by $(\cdot, \cdot)$ and its norm by $\| \cdot \|$.

After that, we introduce the following energy functional

$$\begin{aligned}
&E(t)\\
=&\frac{1}{2}\left[\left\| u_{ttt} + \alpha u_{tt} + \frac{\alpha\varrho}{\delta} u_t \right\|^2 + \frac{\delta}{\alpha} \left(\frac{\varrho - G(t)}{\varrho} \right) \left\| \mathcal{A}^{\frac{1}{2}} u_{tt} + \alpha \mathcal{A}^{\frac{1}{2}} u_t + \frac{\alpha\varrho}{\delta} \mathcal{A}^{\frac{1}{2}} u \right\|^2 \right.\\
&+ \frac{\delta}{\alpha\varrho} G(t) \left\| \mathcal{A}^{\frac{1}{2}} u_{tt} + \alpha \mathcal{A}^{\frac{1}{2}} u_t \right\|^2 + \left(\gamma - \frac{\delta}{\alpha} \right) \left\| \mathcal{A}^{\frac{1}{2}} u_{tt} \right\|^2 + \left(\gamma - \frac{\delta}{\alpha} \right) \frac{\alpha\varrho}{\delta} \left\| \mathcal{A}^{\frac{1}{2}} u_t \right\|^2\\
&+ 2 \int_0^t g(t-s) \left(\mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(s), \mathcal{A}^{\frac{1}{2}} u_{tt} + \alpha \mathcal{A}^{\frac{1}{2}} u_t \right) ds + \frac{\alpha\varrho}{\delta} \left(g \circ \mathcal{A}^{\frac{1}{2}} u \right)(t)\\
&\left. - \alpha \left(g' \circ \mathcal{A}^{\frac{1}{2}} u \right)(t) + \alpha g(t) \left\| \mathcal{A}^{\frac{1}{2}} u \right\|^2 + \left(\beta - \frac{\alpha\varrho}{\delta} \right) \| u_{tt} \|^2 + \frac{\alpha\varrho}{\delta} \left(\beta - \frac{\alpha\varrho}{\delta} \right) \| u_t \|^2 \right],
\end{aligned}$$

where $G(t) = \int_0^t g(s) ds$ and for any $v \in L^2_{loc}\left(\mathbb{R}^+; L^2(\Omega) \right)$,

$$(g \circ v)(t) := \int_\Omega \int_0^t g(t-s) \left(v(t) - v(s) \right)^2 ds dx.$$

As in [31], we set, for any $0 < \nu < 1$,

$$C_\nu = \int_0^\infty \frac{g^2(s)}{\nu g(s) - g'(s)} ds \qquad and \qquad h(t) = \nu g(t) - g'(t).$$

The following lemmas play an important role in the proof of our main results.

Lemma 2.1. *([31]) Assume that condition (A2) holds. Then for any $u \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$, we have*

$$\int_\Omega \left(\int_0^t g(t-s) \left(\mathcal{A}^{\frac{1}{2}} u(s) - \mathcal{A}^{\frac{1}{2}} u(t) \right) ds \right)^2 dx \leq C_\nu \left(h \circ \mathcal{A}^{\frac{1}{2}} u \right)(t), \qquad \forall\ t \geq 0.$$

Lemma 2.2. *(Jensen's inequality) Let P: $[b, c] \to \mathbb{R}$ be a convex function. Assume that functions $f : \Omega \to [b, c]$ and $h : \Omega \to \mathbb{R}$ are integrable such that $h(x) \geq 0$, for any $x \in \Omega$ and $\int_\Omega h(x)dx = k > 0$. Then*

$$P\left(\frac{1}{k} \int_\Omega f(x)h(x)dx \right) \leq \frac{1}{k} \int_\Omega P(f(x))h(x)dx.$$

Lemma 2.3. *([2])(The generalized Young inequality) If f is a convex function defined on a real vector space X and ite convex conjugate is denoted by f^*, then*

$$AB \leq f^*(A) + f(B), \tag{2.3}$$

where

$$f^*(s) = s\left(f'\right)^{-1}(s) - f\left[\left(f'\right)^{-1}(s)\right]. \tag{2.4}$$

We are now in a position to state the well-posedness and the general decay result for problem (1.7)-(1.8).

Theorem 2.4. *(Well-posedness) Assume that (A1) − (A5) hold. Then, for given $(u_0, u_1, u_2, u_3) \in \mathcal{H}$ and $T > 0$, there exists a unique weak solution u of problem (1.7)-(1.8) such that*

$$u \in C\left([0, T]; \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right) \cap C^1\left([0, T]; H\right).$$

Theorem 2.5. *(General decay) Let $(u_0, u_1, u_2, u_3) \in \mathcal{H}$. Assume that (A1)-(A5) hold. Then there exist positive constants k_1 and k_2 such that, along the solution of problem (1.7)-(1.8), the energy functional satisfies*

$$E(t) \leq k_2 M_1^{-1}\left(k_1 \int_{g^{-1}(r)}^t \xi(s)ds \right), \qquad \forall\ t \geq g^{-1}(t), \tag{2.5}$$

where $M_1(t) = \int_t^r \frac{1}{sM'(s)} ds$ and M_1 is strictly decreasing and convex on $(0, r]$, with $\lim_{t\to 0} M_1(t) = +\infty$.

Remark 2. Assume that $M(s) = s^p$, $1 \leq p < 2$ in (A4), then by simple calculations, we see that the decay rate of $E(t)$ is given by, for constants $\overline{k}$, $\widetilde{k}$ and C,

$$E(t) \leq \begin{cases} C \exp\left(-\widetilde{k} \int_0^t \xi(s)ds \right), & if \quad p = 1, \\ \overline{k}\left(1 + \int_0^t \xi(s)ds \right)^{-\frac{1}{p-1}}, & if \quad 1 < p < 2. \end{cases} \tag{2.6}$$

3. Proof of the well-posedness

In this section, we will prove the global existence and uniqueness of the solution of problem (1.7)-(1.8). Firstly, we give the following lemmas.

Lemma 3.1. *If* $0 < g(0) < \frac{2\alpha\varrho}{\delta}(\alpha\gamma - \delta)$, *then there is* $\sigma > 0$ *such that* $\alpha\left(\gamma - \frac{\delta}{\alpha}\right) - \frac{\delta g(0)}{2(\alpha-\sigma)\varrho} > 0$.

Proof. Since $g(0) < \frac{2\alpha\varrho}{\delta}(\alpha\gamma - \delta)$, all we need is to show

$$\frac{2\alpha\varrho}{\delta}(\alpha - \sigma)\left(\gamma - \frac{\delta}{\alpha}\right) \to \frac{2\alpha\varrho}{\delta}(\alpha\gamma - \delta) \qquad as \quad \sigma \to 0,$$

which is trivially true. □

Lemma 3.2. *Assume that* (A1)-(A5) *hold. Then, the energy functional* $E(t)$ *satisfies, for all* $t \geq 0$,

$$\begin{aligned}
&\frac{1}{2}\left[\left\|u_{ttt} + \alpha u_{tt} + \frac{\alpha\varrho}{\delta}u_t\right\|^2 + \frac{\delta}{\alpha}\left(\frac{\varrho - G(t)}{\varrho}\right)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt} + \alpha\mathcal{A}^{\frac{1}{2}}u_t + \frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u\right\|^2\right.\\
&+\left(\gamma - \frac{\delta}{\alpha}\right)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2 + \left(\gamma - \frac{\delta}{\alpha}\right)\frac{\alpha\varrho}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2 - \alpha\left(g' \circ \mathcal{A}^{\frac{1}{2}}u\right)(t)\\
&\left.+\alpha g(t)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2 + \left(\beta - \frac{\alpha\varrho}{\delta}\right)\|u_{tt}\|^2 + \frac{\alpha\varrho}{\delta}\left(\beta - \frac{\alpha\varrho}{\delta}\right)\|u_t\|^2\right]\\
\leq& E(t)\\
\leq&\frac{1}{2}\left[\left\|u_{ttt} + \alpha u_{tt} + \frac{\alpha\varrho}{\delta}u_t\right\|^2 + \frac{\delta}{\alpha}\left(\frac{\varrho - G(t)}{\varrho}\right)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt} + \alpha\mathcal{A}^{\frac{1}{2}}u_t + \frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u\right\|^2\right.\\
&+\left(\gamma - \frac{\delta}{\alpha}\right)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2 + \left(\gamma - \frac{\delta}{\alpha}\right)\frac{\alpha\varrho}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2 - \alpha\left(g' \circ \mathcal{A}^{\frac{1}{2}}u\right)(t)\\
&+\frac{2\alpha\varrho}{\delta}\left(g \circ \mathcal{A}^{\frac{1}{2}}u\right)(t) + \alpha g(t)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2 + \left(\beta - \frac{\alpha\varrho}{\delta}\right)\|u_{tt}\|^2\\
&\left.+\frac{\alpha\varrho}{\delta}\left(\beta - \frac{\alpha\varrho}{\delta}\right)\|u_t\|^2 + \frac{2\delta}{\alpha\varrho}G(t)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt} + \alpha\mathcal{A}^{\frac{1}{2}}u_t\right\|^2\right].
\end{aligned}$$

Proof. From the definition of $E(t)$, we have

$$\begin{aligned}
&E(t)\\
=&\frac{1}{2}\left[\left\|u_{ttt} + \alpha u_{tt} + \frac{\alpha\varrho}{\delta}u_t\right\|^2 + \frac{\delta}{\alpha}\left(\frac{\varrho - G(t)}{\varrho}\right)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt} + \alpha\mathcal{A}^{\frac{1}{2}}u_t + \frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u\right\|^2\right.\\
&+\frac{\delta}{\alpha\varrho}G(t)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt} + \alpha\mathcal{A}^{\frac{1}{2}}u_t\right\|^2 + \left(\gamma - \frac{\delta}{\alpha}\right)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2 + \left(\gamma - \frac{\delta}{\alpha}\right)\frac{\alpha\varrho}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2\\
&+2\int_0^t g(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(t) - \mathcal{A}^{\frac{1}{2}}u(s), \mathcal{A}^{\frac{1}{2}}u_{tt} + \alpha\mathcal{A}^{\frac{1}{2}}u_t\right)ds + \frac{\alpha\varrho}{\delta}\left(g \circ \mathcal{A}^{\frac{1}{2}}u\right)(t)\\
&\left.-\alpha\left(g' \circ \mathcal{A}^{\frac{1}{2}}u\right)(t) + \alpha g(t)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2 + \left(\beta - \frac{\alpha\varrho}{\delta}\right)\|u_{tt}\|^2 + \frac{\alpha\varrho}{\delta}\left(\beta - \frac{\alpha\varrho}{\delta}\right)\|u_t\|^2\right].
\end{aligned}$$

Then, we estimate the sixth term of the above equality

$$\begin{aligned}
&2\left|\int_0^t g(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(t) - \mathcal{A}^{\frac{1}{2}}u(s), \mathcal{A}^{\frac{1}{2}}u_{tt} + \alpha\mathcal{A}^{\frac{1}{2}}u_t\right)ds\right|\\
\leq&\int_0^t g(t-s)\left[\frac{\alpha\varrho}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u(t) - \mathcal{A}^{\frac{1}{2}}u(s)\right\|^2 + \frac{\delta}{\alpha\varrho}\left\|\mathcal{A}^{\frac{1}{2}}u_{tt} + \alpha\mathcal{A}^{\frac{1}{2}}u_t\right\|^2\right]ds
\end{aligned}$$

$$=\frac{\alpha\varrho}{\delta}\left(g\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)+\frac{\delta}{\alpha\varrho}G(t)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}+\alpha\mathcal{A}^{\frac{1}{2}}u_t\right\|^2.$$

A combination of the above results, we complete the proof of lemma. □

Now, we prove the well-posedness result of problem (1.7)-(1.8).

Proof of Theorem 2.1. The proof is given by Faedo-Galerkin method and combines arguments from [16, 39, 38]. We present only the main steps.

Step 1. Approximate problem

We construct approximations of the solution u by the Faedo-Galerkin method as follows. For every $m\geq 1$, let $W_m=\text{span}\{w_1,\cdots,w_m\}$ be a Hilbertian basis of the space $H_0^1(\Omega)$. We choose four sequences (u_0^m), (u_1^m), (u_2^m) and (u_3^m) in W_m such that $u_0^m\to u_0$ strongly in $\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)$, $u_1^m\to u_1$ strongly in $\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)$, $u_2^m\to u_2$ strongly in $\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)$ and $u_3^m\to u_3$ strongly in H. We define now the approximations:

$$u^m(t)=\sum_{j=1}^{m}a_j^m(t)w_j(x), \tag{3.1}$$

where $u^m(t)$ are solutions to the finite dimensional Cauchy problem (written in normal form):

$$\begin{aligned}&\int_\Omega u_{tttt}^m(t)w_j dx+\alpha\int_\Omega u_{ttt}^m(t)w_j dx+\beta\int_\Omega u_{tt}^m(t)w_j dx\\&+\gamma\int_\Omega\mathcal{A}^{\frac{1}{2}}u_{tt}^m(t)\mathcal{A}^{\frac{1}{2}}w_j dx+\delta\int_\Omega\mathcal{A}^{\frac{1}{2}}u_t^m(t)\mathcal{A}^{\frac{1}{2}}w_j dx\\&+\varrho\int_\Omega\mathcal{A}^{\frac{1}{2}}u^m(t)\mathcal{A}^{\frac{1}{2}}w_j dx-\int_0^t g(t-s)\int_\Omega\mathcal{A}^{\frac{1}{2}}u^m(t)\mathcal{A}^{\frac{1}{2}}w_j dxds=0\end{aligned} \tag{3.2}$$

with initial conditions

$$(u^m(0),u_t^m(0),u_{tt}^m(0),u_{ttt}^m(0))=(u_0^m,u_1^m,u_2^m,u_3^m). \tag{3.3}$$

According to the standard theory of ordinary differential equation, the finite dimensional problem (3.2)-(3.3) has a local solution $(u^m(t),u_t^m(t),u_{tt}^m(t),u_{ttt}^m(t))$ in some interval $[0,T_m)$ with $0<T_m\leq T$, for every $m\in\mathbb{N}$. Next, we present some estimates that allow us to extend the local solutions to the interval $[0,T]$, for any given $T>0$.

Step 2. Weak solutions

Multiplying equation (3.2) by $a_{jttt}^m+\alpha a_{jtt}^m+\frac{\alpha\varrho}{\delta}a_{jt}^m$ and integrating over Ω, we have

$$\begin{aligned}&\frac{d}{dt}E^m(t)+\alpha\left(\beta-\frac{\alpha\varrho}{\delta}\right)\|u_{tt}^m\|^2-\frac{\alpha g'(t)}{2}\left\|\mathcal{A}^{\frac{1}{2}}u^m\right\|^2\\&+\frac{\delta}{2\alpha\varrho}g(t)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u^m\right\|^2+\frac{\alpha}{2}\left(g''\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)\\=&-\left[\alpha\left(\gamma-\frac{\delta}{\alpha}\right)-\frac{\delta g(t)}{2\alpha\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right\|^2\\&+\int_0^t g'(t-s)\left(\mathcal{A}^{\frac{1}{2}}u^m(t)-\mathcal{A}^{\frac{1}{2}}u^m(s),\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right)ds+\frac{\alpha\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t),\end{aligned}$$

where

$$
\begin{aligned}
&E^m(t)\\
=&\frac{1}{2}\left[\left\|u_{ttt}^m+\alpha u_{tt}^m+\frac{\alpha\varrho}{\delta}u_t^m\right\|^2+\frac{\delta}{\alpha}\left(\frac{\varrho-G(t)}{\varrho}\right)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m+\alpha\mathcal{A}^{\frac{1}{2}}u_t^m+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u^m\right\|^2\right.\\
&+\frac{\delta}{\alpha\varrho}G(t)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m+\alpha\mathcal{A}^{\frac{1}{2}}u_t^m\right\|^2+\left(\gamma-\frac{\delta}{\alpha}\right)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right\|^2+\left(\gamma-\frac{\delta}{\alpha}\right)\frac{\alpha\varrho}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u_t^m\right\|^2\\
&+2\int_0^t g(t-s)\left(\mathcal{A}^{\frac{1}{2}}u^m(t)-\mathcal{A}^{\frac{1}{2}}u^m(s),\mathcal{A}^{\frac{1}{2}}u_{tt}^m+\alpha\mathcal{A}^{\frac{1}{2}}u_t^m\right)ds\\
&+\frac{\alpha\varrho}{\delta}\left(g\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)-\alpha\left(g'\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)+\alpha g(t)\left\|\mathcal{A}^{\frac{1}{2}}u^m\right\|^2\\
&\left.+\left(\beta-\frac{\alpha\varrho}{\delta}\right)\|u_{tt}^m\|^2+\frac{\alpha\varrho}{\delta}\left(\beta-\frac{\alpha\varrho}{\delta}\right)\|u_t^m\|^2\right].
\end{aligned}
\tag{3.4}
$$

From assumptions $(A1)-(A3)$ and Lemma 3.1, we get, for $\varepsilon\in(0,\alpha)$,

$$
\begin{aligned}
&\int_0^t g'(t-s)\left(\mathcal{A}^{\frac{1}{2}}u^m(t)-\mathcal{A}^{\frac{1}{2}}u^m(s),\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right)ds\\
\leq&-\frac{(\alpha-\varepsilon)\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)-\frac{\delta}{2(\alpha-\varepsilon)\varrho}\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right\|^2\int_0^t g'(t-s)ds\\
=&-\frac{(\alpha-\varepsilon)\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)+\frac{\delta(g(0)-g(t))}{2(\alpha-\varepsilon)\varrho}\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right\|^2
\end{aligned}
$$

and so

$$
\begin{aligned}
&-\left[\alpha\left(\gamma-\frac{\delta}{\alpha}\right)-\frac{\delta g(t)}{2\alpha\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right\|^2\\
&+\int_0^t g'(t-s)\left(\mathcal{A}^{\frac{1}{2}}u^m(t)-\mathcal{A}^{\frac{1}{2}}u^m(s),\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right)ds+\frac{\alpha\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)\\
\leq&-\left[\alpha\left(\gamma-\frac{\delta}{\alpha}\right)-\frac{\delta g(t)}{2\alpha\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right\|^2-\frac{(\alpha-\varepsilon)\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)\\
&+\frac{\delta(g(0)-g(t))}{2(\alpha-\varepsilon)\varrho}\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right\|^2+\frac{\alpha\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)\\
=&-\left[\alpha\left(\gamma-\frac{\delta}{\alpha}\right)-\frac{\delta g(0)}{2(\alpha-\varepsilon)\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right\|^2+\frac{\varepsilon\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)\\
&-\left[\frac{\delta g(t)}{2(\alpha-\varepsilon)\varrho}-\frac{\delta g(t)}{2\alpha\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m\right\|^2\\
\leq&0.
\end{aligned}
\tag{3.5}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{d}{dt}E^m(t)+\alpha\left(\beta-\frac{\alpha\varrho}{\delta}\right)\|u_{tt}^m\|^2-\frac{\alpha g'(t)}{2}\left\|\mathcal{A}^{\frac{1}{2}}u^m\right\|^2\\
&+\frac{\delta}{2\alpha\varrho}g(t)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^m+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u^m\right\|^2+\frac{\alpha}{2}\left(g''\circ\mathcal{A}^{\frac{1}{2}}u^m\right)(t)\leq 0.
\end{aligned}
\tag{3.6}
$$

Integrating (3.6) from 0 to $t\leq T_m$, one has

$$
E^m(t)+\int_0^t\left[\alpha\left(\beta-\frac{\alpha\varrho}{\delta}\right)\|u_{tt}^m\|^2-\frac{\alpha g'(\tau)}{2}\left\|\mathcal{A}^{\frac{1}{2}}u^m\right\|^2\right.
$$

$$+\frac{\delta}{2\alpha\varrho}g(\tau)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}^{m}+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u^{m}\right\|^{2}+\frac{\alpha}{2}\left(g''\circ\mathcal{A}^{\frac{1}{2}}u^{m}\right)(\tau)]d\tau$$

$$\leq E^{m}(0). \tag{3.7}$$

Now, since the sequences $(u_0^m)_{m\in N}$, $(u_1^m)_{m\in N}$, $(u_2^m)_{m\in N}$ and $(u_3^m)_{m\in N}$ converge and using $(A1)-(A3)$, we can find a positive constant C independent of m such that

$$E^{m}(t)\leq C. \tag{3.8}$$

Therefore, using the fact $\varrho-\int_0^t g(s)ds\geq l$, the last estimate (3.8) together with (3.4) give us, for all $m\in N, T_m=T$, we deduce that

$$\begin{aligned}
&(u^{m})_{m\in N} \quad is \quad bounded \quad in \quad L^{\infty}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)\\
&(u_{t}^{m})_{m\in N} \quad is \quad bounded \quad in \quad L^{\infty}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)\\
&(u_{tt}^{m})_{m\in N} \quad is \quad bounded \quad in \quad L^{\infty}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)\\
&(u_{ttt}^{m})_{m\in N} \quad is \quad bounded \quad in \quad L^{\infty}\left(0,T;H\right).
\end{aligned} \tag{3.9}$$

Consequently, we may conclude that

$$\begin{aligned}
&u^{m}\rightharpoonup u \quad weak^{*} \quad in \quad L^{\infty}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)\\
&u_{t}^{m}\rightharpoonup u_{t} \quad weak^{*} \quad in \quad L^{\infty}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)\\
&u_{tt}^{m}\rightharpoonup u_{tt} \quad weak^{*} \quad in \quad L^{\infty}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)\\
&u_{ttt}^{m}\rightharpoonup u_{ttt} \quad weak^{*} \quad in \quad L^{\infty}\left(0,T;H\right).
\end{aligned}$$

From (3.9), we get that $(u^m)_{m\in N}$ is bounded in $L^{\infty}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)$. Then, $(u^m)_{m\in N}$ is bounded in $L^{2}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)$. Since $(u_t^m)_{m\in N}$ is bounded in $L^{\infty}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)$, $(u_t^m)_{m\in N}$ is bounded in $L^{2}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)$. Consequently, $(u_{tt}^m)_{m\in N}$ is bounded in $L^{2}\left(0,T;\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right)$ and $(u_{ttt}^m)_{m\in N}$ is bounded in $L^{2}\left(0,T;H\right)$. Moreover, $(u^m)_{m\in N}$ is bounded in $H^3(0,T;H^1(\Omega))$.

Since the embedding $H^3(0,T;H^1(\Omega))\hookrightarrow L^2(0,T;H(\Omega))$ is compact, using Aubin-Lions theorem [21], we can extract a subsequence $(u^n)_{n\in N}$ of $(u^m)_{m\in N}$ such that

$$u^{n}\to u \quad strongly \quad in \quad L^{2}(0,T;H(\Omega)).$$

Therefore,

$$u^{n}\to u \quad strongly \quad and \quad a.e. \quad on \quad (0,T)\times\Omega.$$

The proof now can be completed arguing as in [21].

Step 3. Uniqueness

It is sufficient to show that the only weak solution of (1.7)-(1.8) with $u_0=u_1=u_2=u_3=0$ is

$$u\equiv 0. \tag{3.10}$$

According to the energy estimate (3.8) and noting that $E(u(0))=0$, we obtain

$$E(u(t))=0, \qquad \forall t\in[0,T].$$

So, we have

$$\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2=\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}+\alpha\mathcal{A}^{\frac{1}{2}}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u\right\|^2$$
$$=\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2=\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2=\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2=\|u_{tt}\|^2=0,\qquad\forall t\in[0,T].$$

And this implies (3.10). Thus, we conclude that problem (1.7)-(1.8) has at most one solution. □

4. Technical lemmas

In this section, we state and prove some lemmas needed to establish our general decay result.

Lemma 4.1. *Let (u,u_t,u_{tt},u_{ttt}) be the solution of (1.7). Assume that (A1)-(A3) hold. Then, we have*

$$\begin{aligned}\frac{d}{dt}E(t)\leq&-\alpha\left(\beta-\frac{\alpha\varrho}{\delta}\right)\|u_{tt}\|^2-\left[\alpha\left(\gamma-\frac{\delta}{\alpha}\right)-\frac{\delta g(0)}{2(\alpha-\varepsilon)\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2\\&+\frac{\alpha g'(t)}{2}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2-\left[\frac{\delta g(t)}{2(\alpha-\varepsilon)\varrho}-\frac{\delta g(t)}{2\alpha\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2\\&-\frac{\delta}{2\alpha\varrho}g(t)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u\right\|^2-\frac{\alpha}{2}\left(g''\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)+\frac{\varepsilon\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u\right)\\\leq&0.\end{aligned}$$

Proof. Multiplying (1.7) by $u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t$ and integrating over Ω yield

$$\begin{aligned}\frac{d}{dt}E(t)=&-\alpha\left(\beta-\frac{\alpha\varrho}{\delta}\right)\|u_{tt}\|^2-\left[\alpha\left(\gamma-\frac{\delta}{\alpha}\right)-\frac{\delta g(t)}{2\alpha\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2\\&+\frac{\alpha g'(t)}{2}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2-\frac{\delta}{2\alpha\varrho}g(t)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u\right\|^2\\&+\int_0^t g'(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(s),\mathcal{A}^{\frac{1}{2}}u_{tt}\right)ds\\&-\frac{\alpha}{2}\left(g''\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)+\frac{\alpha\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u\right)(t).\end{aligned}\tag{4.1}$$

We proceed to show that, for a constant $\varepsilon\in(0,\alpha)$,

$$\begin{aligned}&\left|\int_0^t g'(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(s),\mathcal{A}^{\frac{1}{2}}u_{tt}\right)ds\right|\\\leq&-\frac{(\alpha-\varepsilon)\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)+\frac{\delta(g(0)-g(t))}{2(\alpha-\varepsilon)\varrho}\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2.\end{aligned}\tag{4.2}$$

Then, combining (4.1) and (4.2), we can obtain

$$\begin{aligned}\frac{d}{dt}E(t)\leq&-\alpha\left(\beta-\frac{\alpha\varrho}{\delta}\right)\|u_{tt}\|^2-\left[\alpha\left(\gamma-\frac{\delta}{\alpha}\right)-\frac{\delta g(0)}{2(\alpha-\varepsilon)\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2\\&+\frac{\alpha g'(t)}{2}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2-\left[\frac{\delta g(t)}{2(\alpha-\varepsilon)\varrho}-\frac{\delta g(t)}{2\alpha\varrho}\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2\\&-\frac{\delta}{2\alpha\varrho}g(t)\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u\right\|^2-\frac{\alpha}{2}\left(g''\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)+\frac{\varepsilon\varrho}{2\delta}\left(g'\circ\mathcal{A}^{\frac{1}{2}}u\right).\end{aligned}$$

According to (A1)-(A3) and Lemma 3.1, we complete the proof of lemma. □

Lemma 4.2. *Assume that (A1)-(A5) hold. Then, the functional $F_1(t)$ defined by*

$$F_1(t)=\int_\Omega \left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)\left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)dx$$

satisfies the estimate

$$\begin{aligned}F_1'(t)\leq&-\frac{\delta}{2\alpha}\left\|\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right\|^2+\frac{2\alpha\lambda_0}{\delta}\left(\beta-\frac{\alpha\varrho}{\delta}\right)^2\|u_{tt}\|^2\\&+\frac{2\alpha}{\delta}\left(\gamma-\frac{\delta}{\alpha}\right)^2\left\|\mathcal{A}^{\frac12}u_{tt}\right\|^2+\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2\\&+\frac{2\alpha(\varrho-l)^2}{\delta}\left\|\mathcal{A}^{\frac12}u\right\|^2+\frac{2\alpha}{\delta}C_\nu\left(h\circ\mathcal{A}^{\frac12}u\right)(t).\end{aligned}\tag{4.3}$$

Proof. Taking the derivative of $F_1(t)$ with respect to t, exploiting (1.7) and integrating by parts, we get

$$\begin{aligned}F_1'(t)=&\int_\Omega\left[-\left(\beta-\frac{\alpha\varrho}{\delta}\right)u_{tt}\right]\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)dx\\&-\frac{\delta}{\alpha}\left\|\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right\|^2+\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2\\&-\int_\Omega\left(\gamma-\frac{\delta}{\alpha}\right)\mathcal{A}^{\frac12}u_{tt}\left(\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right)dx\\&+\int_\Omega\left(\int_0^t g(t-s)\mathcal{A}^{\frac12}u(s)ds\right)\left(\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right)dx.\end{aligned}$$

Using Young's inequality, Lemma 2.1, $(A5)$ and the fact $\gamma-\frac{\delta}{\alpha}>0$ and $\beta-\frac{\alpha\varrho}{\delta}>0$, we have

$$\begin{aligned}&\int_\Omega\left[-\left(\beta-\frac{\alpha\varrho}{\delta}\right)u_{tt}\right]\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)dx\\\leq&\frac{2\alpha\lambda_0}{\delta}\left(\beta-\frac{\alpha\varrho}{\delta}\right)^2\|u_{tt}\|^2+\frac{\delta}{8\alpha\lambda_0}\left\|u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right\|^2\\\leq&\frac{2\alpha\lambda_0}{\delta}\left(\beta-\frac{\alpha\varrho}{\delta}\right)^2\|u_{tt}\|^2+\frac{\delta}{8\alpha}\left\|\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right\|^2\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\left(\int_0^t g(t-s)\mathcal{A}^{\frac12}u(s)ds\right)\left(\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right)dx\\=&\int_\Omega\left(\int_0^t g(t-s)\left(\mathcal{A}^{\frac12}u(s)-\mathcal{A}^{\frac12}u(t)\right)ds\right)\left(\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right)dx\\&+\int_\Omega\left(\int_0^t g(t-s)\mathcal{A}^{\frac12}u(t)ds\right)\left(\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right)dx\\\leq&\frac{2\alpha}{\delta}C_\nu\left(h\circ\mathcal{A}^{\frac12}u\right)(t)+\frac{\delta}{4\alpha}\left\|\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right\|^2+\frac{2\alpha(\varrho-l)^2}{\delta}\left\|\mathcal{A}^{\frac12}u\right\|^2.\end{aligned}$$

Also, we have

$$\begin{aligned}&-\int_\Omega\left(\gamma-\frac{\delta}{\alpha}\right)\mathcal{A}^{\frac12}u_{tt}\left(\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right)dx\\\leq&\frac{2\alpha}{\delta}\left(\gamma-\frac{\delta}{\alpha}\right)^2\left\|\mathcal{A}^{\frac12}u_{tt}\right\|^2+\frac{\delta}{8\alpha}\left\|\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right\|^2.\end{aligned}$$

Then, combining the above inequalities, we complete the proof of (4.3). □

Lemma 4.3. *Assume that (A1)-(A5) hold. Then the functional $F_2(t)$ defined by*

$$\begin{aligned}&F_2(t)\\&=-\int_\Omega\left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)\int_0^t g(t-s)\left[\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)(t)-\frac{\alpha\varrho}{\delta}u(s)\right]dsdx\end{aligned}$$

satisfies the estimate

$$\begin{aligned}&F_2'(t)\\&\le-\frac{G(t)}{4}\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2+\left[\frac{(\varrho-l)^2\alpha^2}{2\varepsilon_1}+\frac{4\lambda_0 g^2(0)\alpha^2}{G(t)}\right]\left\|\mathcal{A}^{\frac12}u_t\right\|^2\\&\quad+\left[\frac{\lambda_0(\varrho-l)^2}{2}+\left(\frac{\delta^2}{\alpha^2}+\varrho^2\right)+3(\varrho-l)^2\right]\varepsilon_1\left\|\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right\|^2\\&\quad+\left[\frac{\alpha^2\varrho^2(\varrho-l)^2\lambda_0\varepsilon_1}{2\delta^2}+\frac{(\varrho-l)^2}{2\varepsilon_1}\right.\\&\quad\left.+3(\varrho-l)^2\left(\frac{\alpha\varrho}{\delta}\right)^2\varepsilon_1+\frac{(\varrho-l)^2}{2}\left(\frac{\alpha\varrho}{\delta}\right)^2\right]\left\|\mathcal{A}^{\frac12}u\right\|^2\\&\quad+\left[\frac{(\varrho-l)^2}{2\varepsilon_1}+\frac{1}{2\varepsilon_1}\left(\gamma-\frac{\delta}{\alpha}\right)^2+\frac12\left(\frac{\alpha\varrho}{\delta}\right)^2\left(\gamma-\frac{\delta}{\alpha}\right)^2\right]\left\|\mathcal{A}^{\frac12}u_{tt}\right\|^2\\&\quad+\left[\left(\frac{\varepsilon_1\alpha^2\varrho^2\lambda_0}{4\delta^2}+\frac{3}{4\varepsilon_1}+1+\frac{\alpha\varrho}{\delta}+\frac{2\alpha^4\varrho^2\lambda_0}{G(t)\delta^2}\right)C_\nu+\frac{2\alpha^2\varrho^2\lambda_0}{G(t)\delta^2}\right]\left(h\circ\mathcal{A}^{\frac12}u\right)(t)\\&\quad+\left[\frac{2\left(\beta-\frac{\alpha\varrho}{\delta}\right)^2}{\varepsilon_1}+\frac{4g^2(0)}{G(t)}\right]\|u_{tt}\|^2,\end{aligned}\tag{4.4}$$

where $0<\varepsilon_1<1$.

Proof. By differentiating $F_2(t)$ with respect to t, using (1.7) and integrating by parts, we obtain

$$\begin{aligned}&F_2'(t)\\&=\int_\Omega\left[\beta u_{tt}+\gamma\mathcal{A}u_{tt}+\delta\mathcal{A}u_t+\varrho\mathcal{A}u-\int_0^t g(t-s)\mathcal{A}u(s)ds-\frac{\alpha\varrho}{\delta}u_{tt}\right]\\&\quad\times\int_0^t g(t-s)\left[\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)(t)-\frac{\alpha\varrho}{\delta}u(s)\right]dsdx\\&\quad-g(0)\int_\Omega\left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)(u_{tt}+\alpha u_t)\,dx\\&\quad-\int_\Omega\left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)\int_0^t g'(t-s)\left[\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)(t)\right.\\&\quad\left.-\frac{\alpha\varrho}{\delta}u(s)\right]dsdx-\int_0^t g(s)ds\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2\\&=\int_\Omega\left[\left(\beta-\frac{\alpha\varrho}{\delta}\right)u_{tt}+\frac{\delta}{\alpha}\left(\mathcal{A}u_{tt}+\alpha\mathcal{A}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}u\right)+\left(\gamma-\frac{\delta}{\alpha}\right)\mathcal{A}u_{tt}\right.\\&\quad\left.-\int_0^t g(s)ds\mathcal{A}u(t)\right]\int_0^t g(t-s)\left[\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)(t)-\frac{\alpha\varrho}{\delta}u(s)\right]dsdx\\&\quad-\int_0^t g(s)ds\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2\end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)\int_0^t g'(t-s)\left[\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)(t)-\frac{\alpha\varrho}{\delta}u(s)\right]dsdx\\
&+\left(\int_0^t g(s)ds\right)\int_\Omega\int_0^t g(t-s)\left(\mathcal{A}u(t)-\mathcal{A}u(s)\right)ds(u_{tt}+\alpha u_t)dx\\
&+\frac{\alpha\varrho}{\delta}\int_\Omega\left(\int_0^t g(t-s)\left(\mathcal{A}^{\frac12}u(t)-\mathcal{A}^{\frac12}u(s)\right)ds\right)^2dx\\
&-g(0)\int_\Omega\left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)(u_{tt}+\alpha u_t)\,dx.
\end{aligned}
$$

Now, we estimate the terms in the right-hand side of the above identity.

Using Young's inequality, we obtain, for $0<\varepsilon_1<1$,

$$
\begin{aligned}
&\int_\Omega\left[\left(\beta-\frac{\alpha\varrho}{\delta}\right)u_{tt}+\frac{\delta}{\alpha}\left(\mathcal{A}u_{tt}+\alpha\mathcal{A}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}u\right)+\left(\gamma-\frac{\delta}{\alpha}\right)\mathcal{A}u_{tt}\right.\\
&\left.-\int_0^t g(s)ds\mathcal{A}u(t)\right]\int_0^t g(t-s)\left[\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)-\frac{\alpha\varrho}{\delta}u(s)\right]dsdx\\
\le&\left[\frac{\lambda_0(\varrho-l)^2}{2}+\left(\frac{\delta^2}{\alpha^2}+\varrho^2\right)+2(\varrho-l)^2\right]\varepsilon_1\left\|\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right\|^2\\
&+\frac{2\left(\beta-\frac{\alpha\varrho}{\delta}\right)^2}{\varepsilon_1}\|u_{tt}\|^2+\left[\frac{\alpha^2\varrho^2(\varrho-l)^2\lambda_0\varepsilon_1}{2\delta^2}+\frac{(\varrho-l)^2}{2\varepsilon_1}\right.\\
&\left.+2(\varrho-l)^2\left(\frac{\alpha\varrho}{\delta}\right)^2\varepsilon_1+\frac{(\varrho-l)^2}{2}\left(\frac{\alpha\varrho}{\delta}\right)^2\right]\left\|\mathcal{A}^{\frac12}u\right\|^2\\
&+\left[\frac{(\varrho-l)^2}{2\varepsilon_1}+\frac{1}{2\varepsilon_1}\left(\gamma-\frac{\delta}{\alpha}\right)^2+\frac12\left(\frac{\alpha\varrho}{\delta}\right)^2\left(\gamma-\frac{\delta}{\alpha}\right)^2\right]\left\|\mathcal{A}^{\frac12}u_{tt}\right\|^2\\
&+\frac{(\varrho-l)^2\alpha^2}{2\varepsilon_1}\left\|\mathcal{A}^{\frac12}u_t\right\|^2+\left(\frac{\varepsilon_1\alpha^2\varrho^2\lambda_0}{4\delta^2}+\frac{1}{4\varepsilon_1}+1\right)C_\nu\left(h\circ\mathcal{A}^{\frac12}u\right)(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\int_0^t g(s)ds\right)\int_\Omega\int_0^t g(t-s)\left(\mathcal{A}u(t)-\mathcal{A}u(s)\right)ds(u_{tt}+\alpha u_t)dx\\
=&\left(\int_0^t g(s)ds\right)\int_\Omega\int_0^t g(t-s)\left(\mathcal{A}^{\frac12}u(t)-\mathcal{A}^{\frac12}u(s)\right)ds(\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t)dx\\
\le&\frac{1}{2\varepsilon_1}C_\nu\left(h\circ\mathcal{A}^{\frac12}u\right)(t)+\frac{(\varrho-l)^2}{2}\varepsilon_1\left\|\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t\right\|^2\\
\le&\frac{1}{2\varepsilon_1}C_\nu\left(h\circ\mathcal{A}^{\frac12}u\right)(t)+(\varrho-l)^2\varepsilon_1\left\|\mathcal{A}^{\frac12}u_{tt}+\alpha\mathcal{A}^{\frac12}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac12}u\right\|^2\\
&+(\varrho-l)^2\left(\frac{\alpha\varrho}{\delta}\right)^2\varepsilon_1\left\|\mathcal{A}^{\frac12}u\right\|^2.
\end{aligned}
$$

Also, we have

$$
\frac{\alpha\varrho}{\delta}\int_\Omega\left(\int_0^t g(t-s)\left(\mathcal{A}^{\frac12}u(t)-\mathcal{A}^{\frac12}u(s)\right)ds\right)^2dx\le\frac{\alpha\varrho}{\delta}C_\nu\left(h\circ\mathcal{A}^{\frac12}u\right)(t)
$$

and

$$
-g(0)\int_\Omega\left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)(u_{tt}+\alpha u_t)\,dx
$$

$$\leq \frac{G(t)}{4}\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2+\frac{g^2(0)}{G(t)}\left\|u_{tt}+\alpha u_t\right\|^2$$
$$\leq \frac{G(t)}{4}\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2+\frac{2g^2(0)}{G(t)}\left\|u_{tt}\right\|^2+\frac{2\lambda_0 g^2(0)\alpha^2}{G(t)}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2.$$

Exploiting Young's inequality and $(A5)$, we get

$$-\int_\Omega\left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)\int_0^t g'(t-s)\left[\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)(t)-\frac{\alpha\varrho}{\delta}u(s)\right]dsdx$$
$$=-\int_\Omega\left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)\int_0^t g'(t-s)\left(u_{tt}+\alpha u_t\right)(t)dsdx$$
$$-\frac{\alpha\varrho}{\delta}\int_\Omega\left(u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right)\int_0^t g'(t-s)(u(t)-u(s))dsdx$$
$$\leq\frac{G(t)}{2}\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2+\frac{2g^2(0)}{G(t)}\|u_{tt}\|^2+\frac{2\lambda_0\alpha^2 g^2(0)}{G(t)}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2$$
$$+\frac{2\alpha^2\varrho^2\lambda_0}{G(t)\delta^2}\left(\alpha^2C_\nu+1\right)\left(h\circ\mathcal{A}^{\frac{1}{2}}u\right)(t).$$

A combination of all the above estimates gives the desired result. □

As in [11], we introduce the following auxiliary functional

$$F_3(t)=\int_\Omega\left(u_{ttt}+\alpha u_{tt}\right)u_t dx+\frac{\varrho}{2}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2.$$

Lemma 4.4. *Assume that $(A1)$-$(A5)$ hold. Then the functional $F_3(t)$ satisfies the estimate*

$$\begin{aligned}F_3'(t)\leq&-\left(\frac{3\delta}{8}-\frac{\varepsilon_2\delta}{4}\right)\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2+\frac{\varepsilon_2\delta^3}{8\alpha^2\varrho^2\lambda_0}\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2\\&+\frac{2\gamma^2}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2+\left(\frac{\varepsilon_2\delta^3}{4\varrho^2\lambda_0}+\frac{4\alpha^2\varrho^2\lambda_0}{\varepsilon_2\delta^3}+\frac{2\beta^2\lambda_0}{\delta}\right)\|u_{tt}\|^2\\&+\frac{1}{\delta}C_\nu\left(h\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)+\frac{2(\varrho-l)^2}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2,\end{aligned}\tag{4.5}$$

where $0<\varepsilon_2<1$.

Proof. Using the equation (1.7), a direct computation leads to the following identity

$$\begin{aligned}F_3'(t)=&\int_\Omega(u_{ttt}+\alpha u_{tt})u_{tt}dx+\int_\Omega(u_{tttt}+\alpha u_{ttt})u_t dx+\varrho\left(\mathcal{A}^{\frac{1}{2}}u,\mathcal{A}^{\frac{1}{2}}u_t\right)\\=&(u_{ttt},u_{tt})+\alpha\|u_{tt}\|^2-\beta(u_{tt},u_t)-\gamma\left(\mathcal{A}^{\frac{1}{2}}u_{tt},\mathcal{A}^{\frac{1}{2}}u_t\right)-\delta\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2\\&+\left(\int_0^t g(t-s)\mathcal{A}^{\frac{1}{2}}u(s)ds,\mathcal{A}^{\frac{1}{2}}u_t\right).\end{aligned}\tag{4.6}$$

Now, the first and third terms in the right-hand side of (4.6) can be estimated as follows:

$$(u_{ttt},u_{tt})$$
$$\leq\frac{\varepsilon_2\delta^3}{16\alpha^2\varrho^2\lambda_0}\|u_{ttt}\|^2+\frac{4\alpha^2\varrho^2\lambda_0}{\varepsilon_2\delta^3}\|u_{tt}\|^2$$
$$\leq\frac{\varepsilon_2\delta^3}{8\alpha^2\varrho^2\lambda_0}\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2+\frac{\varepsilon_2\delta^3}{8\alpha^2\varrho^2\lambda_0}\left\|\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2+\frac{4\alpha^2\varrho^2\lambda_0}{\varepsilon_2\delta^3}\|u_{tt}\|^2$$

$$\leq \frac{\varepsilon_2\delta^3}{8\alpha^2\varrho^2\lambda_0}\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2+\alpha^2\left(\frac{\varepsilon_2\delta^3}{4\alpha^2\varrho^2\lambda_0}+\frac{4\varrho^2\lambda_0}{\varepsilon_2\delta^3}\right)\|u_{tt}\|^2+\frac{\varepsilon_2\delta}{4}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2$$

and

$$-\beta(u_{tt},u_t)\leq\frac{2\beta^2\lambda_0}{\delta}\|u_{tt}\|^2+\frac{\delta}{8\lambda_0}\|u_t\|^2\leq\frac{2\beta^2\lambda_0}{\delta}\|u_{tt}\|^2+\frac{\delta}{8}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2,$$

where $0<\varepsilon_2<1$.

Using Young's inequality and Lemma 2.1, we get

$$-\gamma\left(\mathcal{A}^{\frac{1}{2}}u_{tt},\mathcal{A}^{\frac{1}{2}}u_t\right)\leq\frac{2\gamma^2}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2+\frac{\delta}{8}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2$$

and

$$\begin{aligned}
&\left(\int_0^t g(t-s)\mathcal{A}^{\frac{1}{2}}u(s)ds,\mathcal{A}^{\frac{1}{2}}u_t\right)\\
=&\left(\int_0^t g(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(s)-\mathcal{A}^{\frac{1}{2}}u(t)+\mathcal{A}^{\frac{1}{2}}u(t)\right)ds,\mathcal{A}^{\frac{1}{2}}u_t\right)\\
\leq&\frac{1}{\delta}\int_\Omega\left(\int_0^t g(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(s)\right)ds\right)^2dx\\
&+\frac{\delta}{4}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2+\frac{2(\varrho-l)^2}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2+\frac{\delta}{8}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2\\
\leq&\frac{1}{\delta}C_\nu\left(h\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)+\frac{3\delta}{8}\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2+\frac{2(\varrho-l)^2}{\delta}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2.
\end{aligned}$$

Then, combining the above inequalities, we obtain the desired result. □

Lemma 4.5. *Assume that (A1)-(A5) hold. Then the functional $F_4(t)$ defined by*

$$F_4(t)=\int_\Omega\int_0^t f(t-s)\left|\mathcal{A}^{\frac{1}{2}}u(s)\right|^2dsdx$$

satisfies the estimate

$$F_4'(t)\leq-\frac{1}{2}\left(g\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)+3(\varrho-l)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2,\tag{4.7}$$

where $f(t)=\int_t^\infty g(s)ds$.

Proof. Noting that $f'(t)=-g(t)$, we see that

$$\begin{aligned}
&F_4'(t)\\
=&f(0)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2-\int_\Omega\int_0^t g(t-s)\left|\mathcal{A}^{\frac{1}{2}}u(s)\right|^2dsdx\\
=&f(0)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2-\int_\Omega\int_0^t g(t-s)\left|\mathcal{A}^{\frac{1}{2}}u(s)-\mathcal{A}^{\frac{1}{2}}u(t)\right|^2dsdx\\
&-2\int_\Omega\mathcal{A}^{\frac{1}{2}}u\int_0^t g(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(s)-\mathcal{A}^{\frac{1}{2}}u(t)\right)dsdx-\left(\int_0^t g(s)ds\right)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2\\
=&-\left(g\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)-2\int_\Omega\mathcal{A}^{\frac{1}{2}}u\int_0^t g(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(s)-\mathcal{A}^{\frac{1}{2}}u(t)\right)dsdx+f(t)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2.
\end{aligned}$$

Exploiting Young's inequality and the fact $\int_0^t g(s)ds\leq\varrho-l$, we obtain

$$-2\int_\Omega\mathcal{A}^{\frac{1}{2}}u\int_0^t g(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(s)-\mathcal{A}^{\frac{1}{2}}u(t)\right)dsdx$$

$$
\begin{aligned}
\leq & 2(\varrho-l)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2 \\
& +\frac{1}{2(\varrho-l)}\left(\int_0^t g(t-s)ds\right)\int_\Omega\int_0^t g(t-s)\left(\mathcal{A}^{\frac{1}{2}}u(s)-\mathcal{A}^{\frac{1}{2}}u(t)\right)^2 dsdx \\
\leq & 2(\varrho-l)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2+\frac{1}{2}\left(g\circ\mathcal{A}^{\frac{1}{2}}u\right)(t).
\end{aligned}
$$

Moreover, taking account of $f(t)\leq f(0)=\varrho-l$, we have

$$
f(t)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2\leq(\varrho-l)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2.
$$

Combining the above estimates, we arrive at the desired result. □

Lemma 4.6. *Assume that* $(A1)-(A5)$ *hold. The functional* $\mathcal{L}(t)$ *defined by*

$$
\mathcal{L}(t)=NE(t)+F_1(t)+N_2F_2(t)+N_3F_3(t)
$$

satisfies, for a suitable choice of N,N_2,N_3,

$$
\mathcal{L}(t)\sim E(t)
$$

and the estimate, for all $t\geq t_0$,

$$
\begin{aligned}
\mathcal{L}'(t)\leq & -c\left[\|u_{tt}\|^2+\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2+\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2\right. \\
& \left.+\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}+\alpha\mathcal{A}^{\frac{1}{2}}u_t+\frac{\alpha\varrho}{\delta}u\right\|^2\right]-4(\varrho-l)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2+\frac{1}{8}\left(g\circ\mathcal{A}^{\frac{1}{2}}u\right)(t),
\end{aligned}
\tag{4.8}
$$

where t_0 *has been introduced in Remark* 2.1.

Proof. Combining Lemmas 4.1-4.4 and recalling that $g'=\nu g-h$, we obtain, for all $t\geq t_0$,

$$
\begin{aligned}
& \mathcal{L}'(t) \\
\leq & -\left[\alpha\left(\beta-\frac{\alpha\varrho}{\delta}\right)N-\frac{2\alpha\lambda_0}{\delta}\left(\beta-\frac{\alpha\varrho}{\delta}\right)^2-\left(\frac{2\left(\beta-\frac{\alpha\varrho}{\delta}\right)^2}{\varepsilon_1}+\frac{4g^2(0)}{G(t)}\right)N_2\right. \\
& \left.-\left(\frac{\varepsilon_2\delta^3}{4\varrho^2\lambda_0}+\frac{4\alpha^2\varrho^2\lambda_0}{\varepsilon_2\delta^3}+\frac{2\beta^2\lambda_0}{\delta}\right)N_3\right]\|u_{tt}\|^2-\left[\left(\alpha\left(\gamma-\frac{\delta}{\alpha}\right)-\frac{\delta g(0)}{2(\alpha-\varepsilon)\varrho}\right)N\right. \\
& +\left(\frac{\delta g(t)}{2(\alpha-\varepsilon)\varrho}-\frac{\delta g(t)}{2\alpha\varrho}\right)N-\frac{2\alpha}{\delta}\left(\gamma-\frac{\delta}{\alpha}\right)^2-\left(\frac{(\varrho-l)^2}{2\varepsilon_1}+\frac{1}{2\varepsilon_1}\left(\gamma-\frac{\delta}{\alpha}\right)^2\right. \\
& \left.\left.+\frac{1}{2}\left(\frac{\alpha\varrho}{\delta}\right)^2\left(\gamma-\frac{\delta}{\alpha}\right)^2\right)N_2-\frac{2\gamma^2}{\delta}N_3\right]\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2-\frac{\alpha}{2}N\left(g''\circ\mathcal{A}^{\frac{1}{2}}u\right)(t) \\
& -\left[\left(\frac{3\delta}{8}-\frac{\varepsilon_2\delta}{4}\right)N_3-\left(\frac{(\varrho-l)^2\alpha^2}{2\varepsilon_1}+\frac{4\lambda_0g^2(0)\alpha^2}{G(t)}\right)N_2\right]\left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2 \\
& -\left(\frac{G(t)}{4}N_2-1-\frac{\varepsilon_2\delta^3N_3}{8\alpha^2\varrho^2\lambda_0}\right)\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2+\frac{\varepsilon\varrho\nu}{2\delta}N\left(g\circ\mathcal{A}^{\frac{1}{2}}u\right)(t) \\
& -\left[\frac{\delta}{2\alpha}-\left(\frac{\lambda_0(\varrho-l)^2}{2}+\left(\frac{\delta^2}{\alpha^2}+\varrho^2\right)+3(\varrho-l)^2\right)\varepsilon_1N_2\right] \\
& \times\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}+\alpha\mathcal{A}^{\frac{1}{2}}u_t+\frac{\alpha\varrho}{\delta}\mathcal{A}^{\frac{1}{2}}u\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- \left[-\frac{\alpha g'(t)}{2} N - \frac{2\alpha(\varrho - l)^2}{\delta} - \left(\frac{\alpha^2 \varrho^2 (\varrho - l)^2 \lambda_0 \varepsilon_1}{2\delta^2} + \frac{(\varrho - l)^2}{2\varepsilon_1} \right.\right. \\
&\left.\left. + 3(\varrho - l)^2 \left(\frac{\alpha \varrho}{\delta} \right)^2 \varepsilon_1 + \frac{(\varrho - l)^2}{2} \left(\frac{\alpha \varrho}{\delta} \right)^2 \right) N_2 - \frac{2(\varrho - l)^2}{\delta} N_3 \right] \left\| \mathcal{A}^{\frac{1}{2}} u \right\|^2 \\
&- \left[\frac{\varepsilon \varrho}{2\delta} N - \frac{2\alpha}{\delta} C_\nu - \left(\left(\frac{\varepsilon_1 \alpha^2 \varrho^2 \lambda_0}{4\delta^2} + \frac{3}{4\varepsilon_1} + 1 + \frac{\alpha \varrho}{\delta} + \frac{2\alpha^4 \varrho^2 \lambda_0}{G(t)\delta^2} \right) C_\nu \right.\right. \\
&\left.\left. + \frac{2\alpha^2 \varrho^2 \lambda_0}{G(t)\delta^2} \right) N_2 - \frac{N_3}{\delta} C_\nu \right] \left(h \circ \mathcal{A}^{\frac{1}{2}} u \right)(t).
\end{aligned}
$$

At this point, we need to choose our constants very carefully. First, we choose

$$
\varepsilon_1 = \frac{\alpha \delta}{2N_2 \left[\lambda_0 \alpha^2 (\varrho - l)^2 + 2(\delta^2 + \alpha^2 \varrho^2) + 6\alpha^2 (\varrho - l)^2 \right]} \qquad \text{and} \qquad \varepsilon_2 = \frac{1}{N_3}.
$$

The above choice yields

$$
\begin{aligned}
&\mathcal{L}'(t) \\
&\leq - \left[\alpha \left(\beta - \frac{\alpha \varrho}{\delta} \right) N - \frac{2\alpha \lambda_0}{\delta} \left(\beta - \frac{\alpha \varrho}{\delta} \right)^2 - \left(\frac{2\left(\beta - \frac{\alpha \varrho}{\delta} \right)^2}{\varepsilon_1} + \frac{4g^2(0)}{G(t)} \right) N_2 \right. \\
&\left. - \frac{\delta^3}{4\varrho^2 \lambda_0} - \frac{4\alpha^2 \varrho^2 \lambda_0}{\delta^3} N_3^2 - \frac{2\beta^2 \lambda_0}{\delta} N_3 \right] \|u_{tt}\|^2 - \left[\left(\alpha \left(\gamma - \frac{\delta}{\alpha} \right) - \frac{\delta g(0)}{2(\alpha - \varepsilon)\varrho} \right) N \right. \\
&+ \left(\frac{\delta g(t)}{2(\alpha - \varepsilon)\varrho} - \frac{\delta g(t)}{2\alpha \varrho} \right) N - \frac{2\alpha}{\delta} \left(\gamma - \frac{\delta}{\alpha} \right)^2 - \left(\frac{(\varrho - l)^2}{2\varepsilon_1} + \frac{1}{2\varepsilon_1} \left(\gamma - \frac{\delta}{\alpha} \right)^2 \right. \\
&\left.\left. + \frac{1}{2} \left(\frac{\alpha \varrho}{\delta} \right)^2 \left(\gamma - \frac{\delta}{\alpha} \right)^2 \right) N_2 - \frac{2\gamma^2}{\delta} N_3 \right] \left\| \mathcal{A}^{\frac{1}{2}} u_{tt} \right\|^2 - \frac{\alpha}{2} N \left(g'' \circ \mathcal{A}^{\frac{1}{2}} u \right)(t) \\
&- \left[\frac{3\delta}{8} N_3 - \frac{\delta}{4} - \left(\frac{(\varrho - l)^2 \alpha^2}{2\varepsilon_1} + \frac{4\lambda_0 g^2(0) \alpha^2}{G(t)} \right) N_2 \right] \left\| \mathcal{A}^{\frac{1}{2}} u_t \right\|^2 \\
&+ \frac{\varepsilon \varrho \nu}{2\delta} N \left(g \circ \mathcal{A}^{\frac{1}{2}} u \right)(t) - \left(\frac{G(t)}{4} N_2 - 1 - \frac{\delta^3}{8\alpha^2 \varrho^2 \lambda_0} \right) \left\| u_{ttt} + \alpha u_{tt} + \frac{\alpha \varrho}{\delta} u_t \right\|^2 \\
&- \frac{\delta}{4\alpha} \left\| \mathcal{A}^{\frac{1}{2}} u_{tt} + \alpha \mathcal{A}^{\frac{1}{2}} u_t + \frac{\alpha \varrho}{\delta} \mathcal{A}^{\frac{1}{2}} u \right\|^2 \\
&- \left[-\frac{\alpha g'(t)}{2} N - \frac{2\alpha(\varrho - l)^2}{\delta} - \left(\frac{\alpha^2 \varrho^2 (\varrho - l)^2 \lambda_0 \varepsilon_1}{2\delta^2} + \frac{(\varrho - l)^2}{2\varepsilon_1} + 3(\varrho - l)^2 \left(\frac{\alpha \varrho}{\delta} \right)^2 \varepsilon_1 \right.\right. \\
&\left.\left. + \frac{(\varrho - l)^2}{2} \left(\frac{\alpha \varrho}{\delta} \right)^2 \right) N_2 - \frac{2(\varrho - l)^2}{\delta} N_3 \right] \left\| \mathcal{A}^{\frac{1}{2}} u \right\|^2 - \left[\frac{\varepsilon \varrho}{2\delta} N - \frac{2\alpha}{\delta} C_\nu \right. \\
&- \left(\left(\frac{\varepsilon_1 \alpha^2 \varrho^2 \lambda_0}{4\delta^2} + \frac{3}{4\varepsilon_1} + 1 + \frac{\alpha \varrho}{\delta} + \frac{2\alpha^4 \varrho^2 \lambda_0}{G(t)\delta^2} \right) C_\nu + \frac{2\alpha^2 \varrho^2 \lambda_0}{G(t)\delta^2} \right) N_2 \\
&\left. - \frac{N_3}{\delta} C_\nu \right] \left(h \circ \mathcal{A}^{\frac{1}{2}} u \right)(t).
\end{aligned}
$$

Then, we choose N_2 large enough so that

$$
\frac{G(t)}{4} N_2 - 1 - \frac{\delta^3}{8\alpha^2 \varrho^2 \lambda_0} > 0.
$$

Next, we choose N_3 large enough so that

$$
\frac{3\delta}{8} N_3 - \frac{\delta}{4} - \left(\frac{(\varrho - l)^2 \alpha^2}{2\varepsilon_1} + \frac{4\lambda_0 g^2(0) \alpha^2}{G(t)} \right) N_2 > 0.
$$

Now, as $\frac{\nu^2 g(s)}{\nu g(s)-g'(s)} < g(s)$, it is easy to show, using the Lebesgue dominated convergence theorem, that

$$\nu C_\nu = \int_0^\infty \frac{\nu^2 g(s)}{\nu g(s) - g'(s)} ds \to 0, \qquad as \qquad \nu \to 0.$$

Hence, there is $0 < \nu_0 < 1$ such that if $\nu < \nu_0$, then

$$\nu C_\nu < \frac{1}{16\left(\frac{2\alpha}{\delta} + \left(\frac{\varepsilon_1 \alpha^2 \varrho^2 \lambda_0}{4\delta^2} + \frac{3}{4\varepsilon_1} + 1 + \frac{\alpha\varrho}{\delta} + \frac{2\alpha^4 \varrho^2 \lambda_0}{G(t)\delta^2}\right) N_2 + \frac{N_3}{\delta}\right)}.$$

Now, let us choose N large enough and choose ν satisfying

$$\frac{\varepsilon\varrho}{4\delta} N - \frac{2\alpha^2 \varrho^2 \lambda_0}{G(t)\delta^2} N_2 > 0 \qquad and \qquad \nu = \frac{\delta}{4\varepsilon\varrho N} < \nu_0,$$

which means

$$\begin{aligned}
&\frac{\varepsilon\varrho}{2\delta} N - \frac{2\alpha^2 \varrho^2 \lambda_0}{G(t)\delta^2} N_2 \\
&- C_\nu \left(\frac{2\alpha}{\delta} + \left(\frac{\varepsilon_1 \alpha^2 \varrho^2 \lambda_0}{4\delta^2} + \frac{3}{4\varepsilon_1} + 1 + \frac{\alpha\varrho}{\delta} + \frac{2\alpha^4 \varrho^2 \lambda_0}{G(t)\delta^2}\right) N_2 + \frac{N_3}{\delta}\right) \\
>&\frac{\varepsilon\varrho}{2\delta} N - \frac{2\alpha^2 \varrho^2 \lambda_0}{G(t)\delta^2} N_2 - \frac{1}{16\nu} = \frac{\varepsilon\varrho}{4\delta} N - \frac{2\alpha^2 \varrho^2 \lambda_0}{G(t)\delta^2} N_2 > 0
\end{aligned}$$

and

$$\begin{aligned}
&\alpha\left(\beta - \frac{\alpha\varrho}{\delta}\right) N - \frac{2\alpha\lambda_0}{\delta}\left(\beta - \frac{\alpha\varrho}{\delta}\right)^2 - \left(\frac{2\left(\beta - \frac{\alpha\varrho}{\delta}\right)^2}{\varepsilon_1} + \frac{4g^2(0)}{G(t)}\right) N_2 \\
&- \frac{\delta^3}{4\varrho^2\lambda_0} - \frac{4\alpha^2\varrho^2\lambda_0}{\delta^3} N_3^2 - \frac{2\beta^2\lambda_0}{\delta} N_3 > 0, \\
&\left(\alpha\left(\gamma - \frac{\delta}{\alpha}\right) - \frac{\delta g(0)}{2(\alpha-\varepsilon)\varrho}\right) N + \left(\frac{\delta g(t)}{2(\alpha-\varepsilon)\varrho} - \frac{\delta g(t)}{2\alpha\varrho}\right) N - \frac{2\alpha}{\delta}\left(\gamma - \frac{\delta}{\alpha}\right)^2 \\
&- \left(\frac{(\varrho-l)^2}{2\varepsilon_1} + \frac{1}{2\varepsilon_1}\left(\gamma - \frac{\delta}{\alpha}\right)^2 + \frac{1}{2}\left(\frac{\alpha\varrho}{\delta}\right)^2\left(\gamma - \frac{\delta}{\alpha}\right)^2\right) N_2 - \frac{2\gamma^2}{\delta} N_3 > 0, \\
&- \frac{\alpha g'(t)}{2} N - \frac{2\alpha(\varrho-l)^2}{\delta} - \left(\frac{\alpha^2\varrho^2(\varrho-l)^2\lambda_0\varepsilon_1}{2\delta^2} + \frac{(\varrho-l)^2}{2\varepsilon_1} + 3(\varrho-l)^2\left(\frac{\alpha\varrho}{\delta}\right)^2\varepsilon_1\right. \\
&\left. + \frac{(\varrho-l)^2}{2}\left(\frac{\alpha\varrho}{\delta}\right)^2\right) N_2 - \frac{2(\varrho-l)^2}{\delta} N_3 > 4(\varrho-l).
\end{aligned}$$

So we arrive at, for positive constant c,

$$\begin{aligned}
\mathcal{L}'(t) \le &- c\left[\|u_{tt}\|^2 + \left\|\mathcal{A}^{\frac{1}{2}} u_{tt}\right\|^2 + \left\|u_{ttt} + \alpha u_{tt} + \frac{\alpha\varrho}{\delta} u_t\right\|^2\right. \\
&\left. + \left\|\mathcal{A}^{\frac{1}{2}} u_{tt} + \alpha\mathcal{A}^{\frac{1}{2}} u_t + \frac{\alpha\varrho}{\delta} u\right\|^2\right] - 4(\varrho-l)\left\|\mathcal{A}^{\frac{1}{2}} u\right\|^2 + \frac{1}{8}\left(g \circ \mathcal{A}^{\frac{1}{2}} u\right)(t).
\end{aligned}$$

On the other hand, from Lemma 3.2, we find that

$$\begin{aligned}
&|\mathcal{L}(t) - NE(t)| \\
&\le \int_\Omega \left|u_{tt} + \alpha u_t + \frac{\alpha\varrho}{\delta} u\right| \left|u_{ttt} + \alpha u_{tt} + \frac{\alpha\varrho}{\delta} u_t\right| dx
\end{aligned}$$

$$
\begin{aligned}
&+N_2\int_\Omega\left|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right|\int_0^t g(t-s)\\
&\times\left|\left(u_{tt}+\alpha u_t+\frac{\alpha\varrho}{\delta}u\right)(t)-\frac{\alpha\varrho}{\delta}u(s)\right|dsdx\\
&+N_3\int_\Omega\left|u_{ttt}+\alpha u_{tt}\right|\left|u_t\right|dx+N_3\frac{\varrho}{2}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2\\
\leq& cE(t).
\end{aligned}
$$

Therefore, we can choose N even large (if needed) so that (4.8) is satisfied. □

5. Proof of the general decay result

In this section, we will give an estimate to the decay rate for the problem (1.7)-(1.8).

Proof of Theorem 2.2. Our proof starts with the observation that, for any $t\geq t_0$,

$$
\begin{aligned}
&\int_0^{t_0}g(s)\int_\Omega\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2dxds\\
\leq&-\frac{g(0)}{a}\int_0^{t_0}g'(s)\int_\Omega\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2dxds\\
\leq&-\frac{g(0)}{a}\int_0^{t}g'(s)\int_\Omega\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2dxds\\
\leq&-cE'(t),
\end{aligned}
$$

which are derived from (2.2) and Lemma 4.1 and can be used in (4.8).

Taking $\mathcal{F}(t)=\mathcal{L}(t)+cE(t)$, which is obviously equivalent to $E(t)$, we get, for all $t\geq t_0$,

$$
\begin{aligned}
&\mathcal{L}'(t)\\
\leq&-c\left[\|u_{tt}\|^2+\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2+\left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2+\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}+\alpha\mathcal{A}^{\frac{1}{2}}u_t+\frac{\alpha\varrho}{\delta}u\right\|^2\right]\\
&-4(\varrho-l)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2+\frac{1}{8}\left(g\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)\\
\leq&-mE(t)+c\left(g\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)\\
\leq&-mE(t)-cE'(t)+c\int_{t_0}^t g(s)\int_\Omega\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2dxds,
\end{aligned}
$$

where m is a positive constant. Then, we obtain that

$$
\begin{aligned}
\mathcal{F}'(t)&=\mathcal{L}'(t)+cE'(t)\\
&\leq-mE(t)+c\int_{t_0}^t g(s)\int_\Omega\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2dxds. \qquad (5.1)
\end{aligned}
$$

We consider the following two cases relying on the ideas presented in [31].

(i) $M(t)$ is linear.

We multiply (5.1) by $\xi(t)$, then on account of $(A1)$-$(A4)$ and Lemma 4.1, we obtain, for all $t\geq t_0$,

$$
\xi(t)\mathcal{F}'(t)\leq-m\xi(t)E(t)+c\xi(t)\int_{t_0}^t g(s)\int_\Omega\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2dxds
$$

$$\begin{aligned}
&\leq - m\xi(t)E(t) + c\int_{t_0}^{t}\xi(s)g(s)\int_{\Omega}\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2 dxds\\
&\leq - m\xi(t)E(t) - c\int_{t_0}^{t}g'(s)\int_{\Omega}\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2 dxds\\
&\leq - m\xi(t)E(t) - c\int_{0}^{t}g'(s)\int_{\Omega}\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2 dxds\\
&\leq - m\xi(t)E(t) - cE'(t).
\end{aligned}$$

Therefore,

$$\xi(t)\mathcal{F}'(t) + cE'(t) \leq -m\xi(t)E(t).$$

As $\xi(t)$ is non-increasing and $\mathcal{F}(t)\sim E(t)$, we have

$$\xi(t)\mathcal{F}(t) + cE(t) \sim E(t)$$

and

$$(\xi\mathcal{F}+cE)'(t) \leq -m\xi(t)E(t), \qquad \forall\ t\geq t_0.$$

It follows immediately that

$$E'(t) \leq -m\xi(t)E(t), \qquad \forall\ t\geq t_0.$$

We may now integrate over (t_0,t) to conclude that, for two positive constants k_1 and k_2

$$E(t) \leq k_2\exp\left(-k_1\int_{t_0}^{t}\xi(s)ds\right), \qquad \forall\ t\geq t_0.$$

By the continuity of $E(t)$, we have

$$E(t) \leq k_2\exp\left(-k_1\int_{0}^{t}\xi(s)ds\right), \qquad \forall\ t> 0.$$

(ii) M is nonlinear.

First, we define the functional

$$L(t) = \mathcal{L}(t) + F_4(t).$$

Obviously, $L(t)$ is nonnegative. And, by Lemma 4.5 and Lemma 4.6, there exists $b>0$ such that

$$\begin{aligned}
L'(t) =&\mathcal{L}'(t) + F_4'(t)\\
\leq& - c\left[\|u_{tt}\|^2 + \left\|\mathcal{A}^{\frac{1}{2}}u_{tt}\right\|^2 + \left\|u_{ttt}+\alpha u_{tt}+\frac{\alpha\varrho}{\delta}u_t\right\|^2\right.\\
&\left.+\left\|\mathcal{A}^{\frac{1}{2}}u_{tt}+\alpha\mathcal{A}^{\frac{1}{2}}u_t+\frac{\alpha\varrho}{\delta}u\right\|^2\right] - (\varrho-l)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2 - \frac{3}{8}\left(g\circ\mathcal{A}^{\frac{1}{2}}u\right)(t)\\
\leq& - bE(t).
\end{aligned}$$

Therefore, integrating the above inequality over (t_0,t), we see at once that

$$-L(t_0) \leq L(t) - L(t_0) \leq -b\int_{t_0}^{t}E(s)\mathrm{d}s.$$

It is sufficient to show that

$$\int_{0}^{\infty}E(s)\mathrm{d}s < \infty \tag{5.2}$$

and

$$E(t) \leq \frac{c}{t-t_0}, \qquad \forall t > t_0.$$

Now, we define a functional $\lambda(t)$ by

$$\lambda(t) := -\int_{t_0}^{t} g'(s) \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \mathrm{d}s.$$

Clearly, we have

$$\begin{aligned} \lambda(t) \leq & -\int_{0}^{t} g'(s) \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 ds \\ & \leq -cE'(t), \qquad \forall\, t \geq t_0. \end{aligned} \tag{5.3}$$

After that, we define another functional $I(t)$ by

$$I(t) := q \int_{t_0}^{t} \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \mathrm{d}s.$$

Now, the following inequality holds under Lemma 4.1 and (5.2) that

$$\begin{aligned} \int_{t_0}^{t} \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \mathrm{d}s \leq & 2 \int_{t_0}^{t} \left(\left\| \mathcal{A}^{\frac{1}{2}} u(t) \right\|^2 + \left\| \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \right) ds \\ \leq & 4 \int_{t_0}^{t} (E(t) + E(t-s))\, ds \\ \leq & 8 \int_{t_0}^{t} E(0) ds \\ < & \infty. \end{aligned} \tag{5.4}$$

Then (5.4) allows for a constant $0 < q < 1$ chosen so that, for all $t \geq t_0$,

$$0 < I(t) < 1; \tag{5.5}$$

otherwise we get an exponential decay from (5.1). Moreover, recalling that M is strict convex on $(0, r]$ and $M(0) = 0$, then

$$M(\theta x) \leq \theta M(x), \qquad for \qquad 0 \leq \theta \leq 1 \qquad and \qquad x \in (0, r].$$

From assumptions $(A2)$ and $(A4)$, (5.5) and Lemma 2.2, it follows that

$$\begin{aligned} \lambda(t) = & -\int_{t_0}^{t} g'(s) \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \mathrm{d}s \\ = & \frac{1}{qI(t)} \int_{t_0}^{t} I(t)(-g'(s)) q \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \mathrm{d}s \\ \geq & \frac{1}{qI(t)} \int_{t_0}^{t} I(t)\xi(s) M(g(s)) q \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \mathrm{d}s \\ \geq & \frac{\xi(t)}{qI(t)} \int_{t_0}^{t} M(I(t) g(s)) q \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \mathrm{d}s \\ \geq & \frac{\xi(t)}{q} M \left(\frac{1}{I(t)} \int_{t_0}^{t} I(t) g(s) q \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \mathrm{d}s \right) \\ = & \frac{\xi(t)}{q} M \left(q \int_{t_0}^{t} g(s) \left\| \mathcal{A}^{\frac{1}{2}} u(t) - \mathcal{A}^{\frac{1}{2}} u(t-s) \right\|^2 \mathrm{d}s \right). \end{aligned}$$

According to $\overline{M}$ is an extension of M (see Remark 2.1(2)), we also have

$$\lambda(t) \geq \frac{\xi(t)}{q}\overline{M}\left(q\int_{t_0}^{t} g(s)\left\|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right\|^2 \mathrm{d}s\right).$$

In this way,

$$\int_{t_0}^{t} g(s)\left\|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right\|^2 \mathrm{d}s \leq \frac{1}{q}\overline{M}^{-1}\left(\frac{q\lambda(t)}{\xi(t)}\right)$$

and (5.1) becomes

$$\begin{aligned}\mathcal{F}'(t) \leq & -mE(t)+c\int_{t_0}^{t} g(s)\int_{\Omega}\left|\mathcal{A}^{\frac{1}{2}}u(t)-\mathcal{A}^{\frac{1}{2}}u(t-s)\right|^2 dxds\\ \leq & -mE(t)+c\overline{M}^{-1}\left(\frac{q\lambda(t)}{\xi(t)}\right), \qquad \forall\ t\geq t_0. \end{aligned} \tag{5.6}$$

Let $0<\varepsilon_0<r$, we define the functional $\mathcal{F}_1(t)$ by

$$\mathcal{F}_1(t) := \overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{F}(t)+E(t), \qquad \forall\ t\geq 0.$$

Then, recalling that $E'(t)\leq 0$, $\overline{M}'>0$ and $\overline{M}''>0$ as well as making use of estimate (5.6), we deduce that $\mathcal{F}_1(t)\sim E(t)$ and also, for any $t\geq t_0$, we have

$$\mathcal{F}_1'(t) \leq -mE(t)\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+c\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{M}^{-1}\left(\frac{q\lambda(t)}{\xi(t)}\right)+E'(t). \tag{5.7}$$

Taking account of Lemma 2.3, we obtain

$$\begin{aligned}&\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{M}^{-1}\left(\frac{q\lambda(t)}{\xi(t)}\right)\\ \leq&\overline{M}^{*}\left(\overline{H}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right)+\overline{M}\left(\overline{M}^{-1}\left(\frac{q\lambda(t)}{\xi(t)}\right)\right)\\ =&\overline{M}^{*}\left(\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right)+\frac{q\lambda(t)}{\xi(t)} \end{aligned} \tag{5.8}$$

where

$$\begin{aligned}&\overline{M}^{*}\left(\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right)\\ =&\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\left(\overline{M}'\right)^{-1}\left(\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right)-\overline{M}\left[\left(\overline{M}'\right)^{-1}\left(\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right)\right]\\ =&\varepsilon_0\frac{E(t)}{E(0)}\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)-\overline{M}\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\ \leq&\varepsilon_0\frac{E(t)}{E(0)}\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right). \end{aligned} \tag{5.9}$$

So, combining (5.7), (5.8) and (5.9), we obtain

$$\mathcal{F}_1'(t) \leq -(mE(0)-c\varepsilon_0)\frac{E(t)}{E(0)}\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+c\frac{q\lambda(t)}{\xi(t)}+E'(t).$$

From this, we multiply the above inequality by $\xi(t)$ to get

$$\xi(t)\mathcal{F}_1'(t) \leq -(mE(0)-c\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}\overline{M}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+cq\lambda(t)+\xi(t)E'(t).$$

Then, using the fact that, as $\varepsilon_0 \frac{E(t)}{E(0)} < r$, $\overline{M}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = M'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and (5.3), we get

$$\xi(t)\mathcal{F}_1'(t) \leq -(mE(0) - c\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}M'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t).$$

Consequently, defining $\mathcal{F}_2(t) = \xi(t)\mathcal{F}_1(t) + cE(t)$, then since $\mathcal{F}_1(t) \sim E(t)$, we arrive at

$$\mathcal{F}_2(t) \sim E(t), \tag{5.10}$$

and with a suitable choice of ε_0, we get, for some positive constant k and for any $t \geq t_0$,

$$\mathcal{F}_2'(t) \leq -k\xi(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right). \tag{5.11}$$

Define

$$R(t) = \frac{\lambda_1 \mathcal{F}_2(t)}{E(0)}, \qquad \lambda_1 > 0 \qquad \textit{and} \qquad M_2(t) = tM'(\varepsilon_0 t).$$

Moreover, it suffices to show that $M_2'(t), M_2(t) > 0$ on $(0,1]$ by the strict convexity of M on $(0,r]$. And, it is easily seen that

$$\mathcal{F}_2'(t) \leq -k\xi(t)M_2\left(\frac{E(t)}{E(0)}\right). \tag{5.12}$$

According to (5.10) and (5.12), there exist $\lambda_2, \lambda_3 > 0$ such that

$$\lambda_2 R(t) \leq E(t) \leq \lambda_3 R(t). \tag{5.13}$$

Then, it follows that there exists $k_1 > 0$ such that

$$k_1\xi(t) \leq -\frac{R'(t)}{M_2(R(t))}, \qquad \forall\ t \geq t_0. \tag{5.14}$$

Next, we define

$$M_1(t) := \int_t^r \frac{1}{sM'(s)}\mathrm{d}s.$$

And based on the properties of M, we know that M_1 is strictly decreasing function on $(0,r]$ and $\lim\limits_{t\to 0} M_1(t) = +\infty$.

Now, we integrate (5.14) over (t_0, t) to obtain

$$-\int_{t_0}^t \frac{R'(s)}{M_2(R(s))}\mathrm{d}s \geq k_1 \int_{t_0}^t \xi(s)\mathrm{d}s$$

so

$$k_1 \int_{t_0}^t \xi(s)\mathrm{d}s \leq M_1\left(\varepsilon_0 R(t)\right) - M_1\left(\varepsilon_0 R(t_0)\right),$$

which implies that

$$M_1\left(\varepsilon_0 R(t)\right) \geq k_1 \int_{t_0}^t \xi(s)\mathrm{d}s.$$

It is easy to obtain that

$$R(t) \leq \frac{1}{\varepsilon_0} M_1^{-1}\left(k_1 \int_{t_0}^t \xi(s)\mathrm{d}s\right), \qquad \forall\ t \geq t_0. \tag{5.15}$$

A combining of (5.13) and (5.15) gives the proof. □

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