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ON THE MUTUAL SINGULARITY OF MULTIFRACTAL MEASURES

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Abstract. The aim of this article is to show that the multifractal Hausdorff and packing measures are mutually singular, which in particular provides an answer to Olsen's questions.

1. Introduction

The notion of singularity exponents or spectrum and generalized dimensions are the major components of the multifractal analysis. They were introduced with a view of characterizing the geometry of measure and to be linked with the multifractal spectrum which is the map which affects the Hausdorff or packing dimension of the iso-Hölder set

$$E(\alpha) = \left\{ x \in \operatorname{supp}\mu;\ \lim_{r\to 0} \frac{\log\Big(\mu\big(B(x,r)\big)\Big)}{\log r} = \alpha \right\}$$

for a given $\alpha \geq 0$ and $\operatorname{supp}\mu$ is the topological support of probability measure μ on $\mathbb{R}^n$, $B(x,r)$ is the closed ball of center x and radius r. It unifies the multifractal spectra to the multifractal Hausdorff (packing) function $b_\mu(q)$ ($B_\mu(q)$) via the Legendre transform [3, 6, 7], i.e.,

$$f_\mu(\alpha) := \dim_H \Big(E(\alpha)\Big) = \inf_{q\in\mathbb{R}} \Big\{ q\alpha + b_\mu(q) \Big\}$$

or

$$F_\mu(\alpha) := \dim_P \Big(E(\alpha)\Big) = \inf_{q\in\mathbb{R}} \Big\{ q\alpha + B_\mu(q) \Big\}.$$

In the last decay, there has been a great interest in understanding the fractal dimensions of the iso-Hölder sets and measures.

In the following we aim to introduce the general tools that will be applied next. We will review in brief the notion of multifractal Hausdorff and packing measures already introduced in [6]. The key ideas behind the fine multifractal formalism in [6] are certain measures of Hausdorff-packing type which are tailored to see only the multifractal decomposition sets $E(\alpha)$. These measures are natural multifractal generalizations of the centered Hausdorff measure and the packing measure and are motivated by the τ_μ-function which appears in the multifractal formalism. We first recall the definition of the multifractal Hausdorff measure and the multifractal packing measure. We start by introducing the multifractal Hausdorff and packing

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measures. Let μ be a compactly supported probability measure on $\mathbb{R}^n$. For $q, t \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$, we define

$$\overline{\mathscr{P}}^{q,t}_{\mu,\delta}(E) = \sup\left\{\sum_i \mu\big(B(x_i, r_i)\big)^q \big(2r_i\big)^t\right\}, \quad E \neq \emptyset,$$

where the supremum is taken over all centered δ-packing of E.

Moreover we can set $\overline{\mathscr{P}}^{q,t}_{\mu,\delta}(\emptyset) = 0$. The packing pre-measure is then given by

$$\overline{\mathscr{P}}^{q,t}_{\mu}(E) = \inf_{\delta>0} \overline{\mathscr{P}}^{q,t}_{\mu,\delta}(E).$$

In a similar way, we define

$$\overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E) = \inf\left\{\sum_i \mu\big(B(x_i, r_i)\big)^q \big(2r_i\big)^t\right\}, \quad E \neq \emptyset,$$

where the infinimum is taken over all centered δ-covering of E.

Moreover we can set $\overline{\mathscr{H}}^{q,t}_{\mu,\delta}(\emptyset) = 0$. The Hausdorff pre-measure is defined by

$$\overline{\mathscr{H}}^{q,t}_{\mu}(E) = \sup_{\delta>0} \overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E).$$

Especially, we have the conventions $0^q = \infty$ for $q \leq 0$ and $0^q = 0$ for $q > 0$.

$\overline{\mathscr{H}}^{q,t}_{\mu}$ is σ-subadditive but not increasing and $\overline{\mathscr{P}}^{q,t}_{\mu}$ is increasing but not σ-subadditive. That's why Olsen introduced the following modifications on the multifractal Hausdorff and packing measures $\mathscr{H}^{q,t}_{\mu}$ and $\mathscr{P}^{q,t}_{\mu}$,

$$\mathscr{H}^{q,t}_{\mu}(E) = \sup_{F\subseteq E} \overline{\mathscr{H}}^{q,t}_{\mu}(F) \quad \text{and} \quad \mathscr{P}^{q,t}_{\mu}(E) = \inf_{E\subseteq \bigcup_i E_i} \sum_i \overline{\mathscr{P}}^{q,t}_{\mu}(E_i).$$

In follows that $\mathscr{H}^{q,t}_{\mu}$ and $\mathscr{P}^{q,t}_{\mu}$ are metric outer measures and thus measures on the Borel family of subsets of $\mathbb{R}^n$. An important feature of the Hausdorff and packing measures is that $\mathscr{P}^{q,t}_{\mu} \leq \overline{\mathscr{P}}^{q,t}_{\mu}$. Moreover, there exists an integer $\xi \in \mathbb{N}$, such that $\mathscr{H}^{q,t}_{\mu} \leq \xi \mathscr{P}^{q,t}_{\mu}$. The measure $\mathscr{H}^{q,t}_{\mu}$ is a multifractal generalization of the centered Hausdorff measure, whereas $\mathscr{P}^{q,t}_{\mu}$ is a multifractal generalization of the packing measure. In fact, it is easily seen that if $t \geq 0$, then $\mathscr{H}^{0,t}_{\mu} = \mathscr{H}^t$ and $\mathscr{P}^{0,t}_{\mu} = \mathscr{P}^t$, where $\mathscr{H}^t$ denotes the t-dimensional centered Hausdorff measure and $\mathscr{P}^t$ denotes the t-dimensional packing measure.

The measures $\mathscr{H}^{q,t}_{\mu}$ and $\mathscr{P}^{q,t}_{\mu}$ and the pre-measure $\overline{\mathscr{P}}^{q,t}_{\mu}$ assign in the usual way a multifractal dimension to each subset E of $\mathbb{R}^n$. They are respectively denoted by $b^q_{\mu}(E)$, $B^q_{\mu}(E)$ and $\Lambda^q_{\mu}(E)$ and satisfy

$$b^q_{\mu}(E) = \inf\Big\{t \in \mathbb{R}; \quad \mathscr{H}^{q,t}_{\mu}(E) = 0\Big\}, \quad B^q_{\mu}(E) = \inf\Big\{t \in \mathbb{R}; \quad \mathscr{P}^{q,t}_{\mu}(E) = 0\Big\},$$

$$\Lambda^q_{\mu}(E) = \inf\Big\{t \in \mathbb{R}; \quad \overline{\mathscr{P}}^{q,t}_{\mu}(E) = 0\Big\}.$$

The number $b^q_{\mu}(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim_H(E)$ of E whereas $B^q_{\mu}(E)$ and $\Lambda^q_{\mu}(E)$ are obvious multifractal analogues of the packing dimension $\dim_P(E)$ and the pre-packing dimension $\Delta(E)$ of E respectively. In fact, it follows immediately from the definitions that

$$\dim_H(E) = b^0_{\mu}(E), \quad \dim_P(E) = B^0_{\mu}(E) \quad \text{and} \quad \Delta(E) = \Lambda^0_{\mu}(E).$$

Next, for $q \in \mathbb{R}$, we define the separator functions b_μ, B_μ and Λ_μ by

$$b_\mu(q) = b^q_\mu\big(\operatorname{supp}\mu\big), \quad B_\mu(q) = B^q_\mu\big(\operatorname{supp}\mu\big) \ \text{ and } \ \Lambda_\mu(q) = \Lambda^q_\mu\big(\operatorname{supp}\mu\big).$$

It is well known that the functions b_μ, B_μ and Λ_μ are decreasing and B_μ, Λ_μ are convex and satisfying $b_\mu \leq B_\mu \leq \Lambda_\mu$.

The multifractal formalism based on the measures $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ and the dimension functions b_μ, B_μ and Λ_μ provides a natural, unifying and very general multifractal theory which includes all the hitherto introduced multifractal parameters, i.e., the multifractal spectra functions f_μ and F_μ, the multifractal box dimensions. The dimension functions b_μ and B_μ are intimately related to the spectra functions f_μ and F_μ, whereas the dimension function Λ_μ is closely related to the upper box spectrum (more precisely, to the upper multifractal box dimension function $\overline{C}_\mu$, see [6, Propositions 2.19 and 2.22]).

It should be noted that the interest of mathematicians in singularly continuous measures and probability distributions were fairly weak, which can be explained, on the one hand, by the absence of adequate analytic apparatus for specification and investigation of these measures, and, on the other hand, by a widespread opinion about the absence of applications of these measures. Due to the fractal explosion and a deep connection between the theory of fractals and singular measures, the situation has radically changed in the last years. The multifractal and the fractal analysis allows one to perform a certain classification of these measures. Therefore, Olsen in [6, Questions 7.1 and 7.2], posed the following two questions: Let $p, q \in \mathbb{R}$.

(1) Assume that b_μ is differentiable at p and q with $b'_\mu(p) \neq b'_\mu(q)$. Then, the following problem remains open:
$$\mathscr{H}_\mu^{p,b_\mu(p)}{}_{\llcorner \operatorname{supp}\mu} \perp \mathscr{H}_\mu^{q,b_\mu(q)}{}_{\llcorner \operatorname{supp}\mu}.$$

(2) Assume that B_μ is differentiable at p and q with $B'_\mu(p) \neq B'_\mu(q)$. Then, the following problem remains open:
$$\mathscr{P}_\mu^{p,B_\mu(p)}{}_{\llcorner \operatorname{supp}\mu} \perp \mathscr{P}_\mu^{q,B_\mu(q)}{}_{\llcorner \operatorname{supp}\mu}.$$

The aim of this paper is to focus on the above questions relying on these multifractal measures and functions. More precisely, we study the mutual singularity of multifractal Hausdorff and packing measures on the homogeneous Moran sets and this result completely differ to Olsen's main theorems [6, Theorems 5.1 and 6.1] which are based on graph directed self-similar measures in $\mathbb{R}^n$ with totally disconnected support, cookie-cutter measures and self-similar measures satisfying the significantly weaker open set condition [4, 5].

2. Main result

Before we set our main result, let us recall the class of homogeneous Moran sets. We denote by $\{n_k\}_{k\geq 1}$ a sequence of positive integers and $\{c_k\}_{k\geq 1}$ a sequence of positive numbers satisfying
$$n_k \geq 2, \quad 0 < c_k < 1, \quad n_k c_k \leq 1 \quad \text{for } k \geq 1.$$
Let $D_0 = \emptyset$, and for any $k \geq 1$, set
$$D_{m,k} = \{(i_m, i_{m+1}, \dots, i_k)\,; \quad 1 \leqslant i_j \leqslant n_j, \quad m \leqslant j \leqslant k\}$$
and $D_k = D_{1,k}$. Define $D = \bigcup_{k\geqslant 1} D_k$. If $\sigma = (\sigma_1, \dots, \sigma_k) \in D_k, \tau = (\tau_1, \dots, \tau_m) \in D_{k+1,m}$, we denote $\sigma * \tau = (\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_m)$.

Definition 2.1. Let J be a closed interval such that $|J| = 1$. We say the collection $\mathscr{F} = \{J_\sigma, \sigma \in D\}$ of closed subsets of J fulfills the Moran structure if it satisfies the

following conditions:
(a) $J_{\emptyset} = J$.
(b) For all $k \geq 0$ and $\sigma \in D_k, J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*n_{k+1}}$ are subintervals of J_σ, and satisfy that $J^\circ_{\sigma*i} \cap J^\circ_{\sigma*j} = \emptyset$ $(i \neq j)$, where A° denotes the interior of A.
(c) For any $k \geq 1, \sigma \in D_{k-1}, \quad c_k = \dfrac{|J_{\sigma*j}|}{|J_\sigma|}, \quad 1 \leqslant j \leqslant n_k$ where $|A|$ denotes the diameter of A.

Let $\mathscr{F}$ be a collection of closed subintervals of J having homogeneous Moran structure. The set $E(\mathscr{F}) = \bigcap_{k \geqslant 1} \bigcup_{\sigma \in D_k} J_\sigma$ is called an homogeneous Moran set determined by $\mathscr{F}$. It is convenient to denote $M\left(J, \{n_k\}, \{c_k\}\right)$ for the collection of homogeneous Moran sets determined by J, $\{n_k\}$ and $\{c_k\}$.

Remark 1. If $\lim\limits_{n\to+\infty} \sup\limits_{\sigma\in D_n} |J_\sigma| > 0$, then E contains interior points. Thus the measure and dimension properties will be trivial. We assume therefore $\lim\limits_{n\to+\infty} \sup\limits_{\sigma\in D_n} |J_\sigma| = 0$.

Let $A = \{a, b\}$ be a two-letter alphabet, and A^* the free monoid generated by A. Let F be the homomorphism on A^*, defined by $F(a) = ab$ and $F(b) = a$. It is easy to see that $F^n(a) = F^{n-1}(a)F^{n-2}(a)$. We denote by $|F^n(a)|$ the length of the word $F^n(a)$, thus

$$F^n(a) = s_1 s_2 \cdots s_{|F^n(a)|}, \quad s_i \in A.$$

Therefore, as $n \to +\infty$, we get the infinite sequence

$$\omega = \lim_{n\to\infty} F^n(a) = s_1 s_2 s_3 \cdots s_n \cdots \in \{a, b\}^{\mathbb{N}}$$

which is called the Fibonacci sequence. For any $n \geqslant 1$, write $\omega_n = \omega|_n = s_1 s_2 \cdots s_n$. We denote by $|\omega_n|_a$ the number of the occurrence of the letter a in ω_n, and $|\omega_n|_b$ the number of occurrence of b. Then $|\omega_n|_a + |\omega_n|_b = n$. It follows from [7, pp. 143], [8, pp. 271] that $\lim\limits_{n\to+\infty} \dfrac{|\omega_n|_a}{n} = \eta$, where $\eta^2 + \eta = 1$.

Let $0 < r_a < \frac{1}{2}, 0 < r_b < \frac{1}{3}, r_a, r_b \in \mathbb{R}$. In the Moran construction above, let

$$|J| = 1, \quad n_k = \begin{cases} 2, & \text{if } s_k = a \\ 3, & \text{if } s_k = b, \end{cases}$$

$$c_{k_j} = c_k = \begin{cases} r_a, & \text{if } s_k = a \\ r_b, & \text{if } s_k = b \end{cases}, \quad 1 \leqslant j \leqslant n_k.$$

Here, we consider a class of homogeneous Moran sets E witch satisfy a special property called the strong separation condition (SSC), i.e., take $J_\sigma \in \mathscr{F}$. Let $J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*n_{k+1}}$ be the n_{k+1} basic intervals of order $k+1$ contained in J_σ arranged from the left to the right, then we assume that for all $1 \leq i \leq n_{k+1} - 1$, $\text{dist}(J_{\sigma*i}, J_{\sigma*(i+1)}) \geq \Delta_k |J_\sigma|$, where $(\Delta_k)_k$ is a sequence of positive real numbers, such that $0 < \Delta = \inf\limits_k \Delta_k < 1$. Then we construct the homogeneous Moran set relating to the Fibonacci sequence and denote it by $E := E(\omega) = (J, \{n_k\}, \{c_k\})$. By the construction of E, we have

$$|J_\sigma| = r_a^{|\omega_k|_a} r_b^{|\omega_k|_b}, \quad \forall \sigma \in D_k.$$

Let $P_a = (P_{a_1}, P_{a_2}), P_b = (P_{b_1}, P_{b_2}, P_{b_3})$ be probability vectors i.e., $P_{a_i} > 0, P_{b_i} > 0$, and $\sum_{i=1}^{2} P_{a_i} = 1, \sum_{i=1}^{3} P_{b_i} = 1$. For any $k \geqslant 1$ and any $\sigma \in D_k$, we know $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ where

$$\sigma_k \in \begin{cases} \{1,2\}, & \text{if } s_k = a \\ \{1,2,3\}, & \text{if } s_k = b. \end{cases}$$

For $\sigma = \sigma_1\sigma_2\cdots\sigma_k$, we define $\sigma(a)$ as follows: let $\omega_k = s_1s_2\cdots s_k$ and $e_1 < e_2 < \cdots < e_{|\omega_k|_a}$ be the occurrences of the letter a in ω_k, then $\sigma(a) = \sigma_{e_1}\sigma_{e_2}\cdots\sigma_{e_{|\omega_k|_a}}$. Similarly, let $\delta_1 < \delta_2 < \cdots < \delta_{|\omega_k|_b}$ be the occurrences of the letter b in ω_k, then $\sigma(b) = \sigma_{\delta_1}\sigma_{\delta_2}\cdots\sigma_{\delta_{|\omega_k|_b}}$.

Let

$$P_{\sigma(a)} = P_{\sigma_{e_1}}P_{\sigma_{e_2}}\cdots P_{\sigma_{e_{|\omega_k|_a}}} \quad \text{and} \quad P_{\sigma(b)} = P_{\sigma_{\delta_1}}P_{\sigma_{\delta_2}}\cdots P_{\sigma_{\delta_{|\omega_k|_b}}}.$$

Obviously

$$\sum_{\sigma\in D_k} P_{\sigma(a)}P_{\sigma(b)} = 1.$$

Let μ be a mass distribution on E, such that for any $\sigma \in D_k$,

$$\mu(J_\sigma) = P_{\sigma(a)}P_{\sigma(b)}.$$

Now we define an auxiliary function $\beta(q)$ as follows: For each $q \in \mathbb{R}$ and $k \geq 1$, there is a unique number $\beta_k(q)$ such that

$$\sum_{\sigma\in D_k} \left(P_{\sigma(a)}P_{\sigma(b)}\right)^q |J_\sigma|^{\beta_k(q)} = 1.$$

By a simple calculation, we get

$$\beta_k(q) = \frac{-\log\left(\sum_{i=1}^{2} P_{a_i}^q\right) - \frac{k-|\omega_k|_a}{|\omega_k|_a}\log\left(\sum_{i=1}^{3} P_{b_i}^q\right)}{\log r_a + \frac{k-|\omega_k|_a}{|\omega_k|_a}\log r_b}.$$

Clearly, for any $k \geqslant 1$ we have $\beta_k(1) = 0$. Thus $\beta_k'(q) < 0$ for all q and $\beta_k(q)$ is a strictly decreasing function. Our auxiliary function is

$$\beta(q) = \lim_{k\to+\infty}\beta_k(q) = \frac{-\log\left(\sum_{i=1}^{2} P_{a_i}^q\right) - \eta\log\left(\sum_{j=1}^{3} P_{b_j}^q\right)}{\log r_a + \eta\log r_b},$$

where $\eta^2 + \eta = 1$. The function β is strictly decreasing and differentiable at q, $\lim_{q\to\mp\infty}\beta(q) = \pm\infty$ and $\beta(1) = 0$. Note that in [7, Theorem B] it is shown that the dimension of the level sets of the local Hölder exponent $E(-\beta'(q))$ is given by

$$\dim_H E(-\beta'(q)) = \dim_P E(-\beta'(q)) = -q\beta'(q) + \beta(q).$$

Definition 2.2. Let μ, ν be two Borel probability measures on $\mathbb{R}^n$. μ and ν are said to be mutually singular and we write $\mu \perp \nu$ if there exists a set $A \subset \mathbb{R}^n$, such that

$$\mu(A) = 0 = \nu(\mathbb{R}^n \setminus A).$$

In the following we show that the Olsen's multifractal Hausdorff and packing are mutually singular, which in particular provides an answer to Olsen's questions [6, Questions 7.1 and 7.2].

Theorem 2.3. *Suppose that E is a homogeneous Moran set satisfying (SSC) and μ is the Moran measure on E. Then, for all $p,q\in\mathbb{R}$ where $\beta'(p)\neq\beta'(q)$ we have*

$$\mathscr{H}_\mu^{p,\beta(p)} \perp \mathscr{H}_\mu^{q,\beta(q)} \quad \text{and} \quad \mathscr{P}_\mu^{p,\beta(p)} \perp \mathscr{P}_\mu^{q,\beta(q)} \quad \text{on} \quad E.$$

Remark 2. The results of Theorem 2.3 hold if we replace the multifractal Hausdorff and packing measures by the multifractal Hewitt-Stromberg measures (see [1, 2] for the precise definitions).

3. Proof of the main result

In this section, we give a proof of the main theorem. Given $q\in\mathbb{R}$, it follows from [7, Proposition 3.1] that there exists a probability measure ν_q supported by E such that for any $k\geq 1$ and $\sigma_0\in D_k$,

$$\nu_q\left(J_{\sigma_0}\right)=\frac{\mu\left(J_{\sigma_0}\right)^q\left|J_{\sigma_0}\right|^{\beta(q)}}{\sum\limits_{\sigma\in D_k}\mu\left(J_\sigma\right)^q\left|J_\sigma\right|^{\beta(q)}}.$$

However, in [7] it is shown that

$$\limsup_{r\downarrow 0}\frac{\log\mu\big(B(x,r)\big)}{\log r}=-\beta'(q),\quad \nu_q-\text{a.s}$$

which implies that $\nu_q\Big(E\big(-\beta'(q)\big)\Big)=1$. We therefore infer that if $p,q\in\mathbb{R}$ with $\beta'(p)\neq\beta'(q)$, then

$$\nu_p\perp\nu_q. \tag{1}$$

We now prove the following three claims.

Claim 1. *We have*

$$0<\liminf_{k\to+\infty}\sum_{\sigma\in D_k}\mu\left(J_\sigma\right)^q\left|J_\sigma\right|^{\beta(q)}\leq\limsup_{k\to+\infty}\sum_{\sigma\in D_k}\mu\left(J_\sigma\right)^q\left|J_\sigma\right|^{\beta(q)}<+\infty.$$

Proof of Claim 1. By a simple calculation, we can get $\beta(q)-\beta_k(q)=O(\frac{1}{k})$. Then,

$$\sum_{\sigma\in D_k}\mu\left(J_\sigma\right)^q\left|J_\sigma\right|^{\beta(q)}=\left|J_\sigma\right|^{\beta(q)-\beta_k(q)}\geq\left(\min\left\{r_a,r_b\right\}\right)^{k(\beta(q)-\beta_k(q))},$$

which implies that

$$\liminf_{k\to+\infty}\sum_{\sigma\in D_k}\mu\left(J_\sigma\right)^q\left|J_\sigma\right|^{\beta(q)}>0.$$

The proof of the

$$\liminf_{k\to+\infty}\sum_{\sigma\in D_k}\mu\left(J_\sigma\right)^q\left|J_\sigma\right|^{\beta(q)}<+\infty.$$

is identical to the proof of the statement in the first part and is therefore omitted. □

Claim 2. *There exists a constant $\underline{K}>0$ such that for any $q\in\mathbb{R}$*

$$\underline{K}\nu_q(E)\leq\mathscr{H}_\mu^{q,\beta(q)}(E).$$

Proof of Claim 2. For convenience of presentation let $J_n(x)$ be the nth-level basic set of E containing the point x. Let $\delta > 0$ and $\Big(B\left(x_i, r_i\right)\Big)_{i\in\mathbb{N}}$ be a centered δ-covering of E. For each i choose $\sigma(i) \in D_n$, for any $n \geqslant 1$ such that $x_i \in J_{\sigma(i)}$. For each $i \in \mathbb{N}$ choose $k_i, \ell_i \in \mathbb{N}$ such that

$$\left|J_{\sigma(i)|k_i+1}\right| \leqslant r_i < \left|J_{\sigma(i)|k_i}\right| \quad \text{and} \quad \Delta\left|J_{\sigma(i)|\ell_i+1}\right| \leqslant r_i < \Delta\left|J_{\sigma(i)|\ell_i}\right|,$$

which implies that

$$J_{\sigma(i)|k_i+1}\left(x_i\right) \subseteq B\left(x_i, r_i\right) \quad \text{and} \quad E \cap B\left(x_i, r_i\right) \subseteq J_{\sigma(i)|\ell_i+1}(x_i). \tag{2}$$

Then we have

$$\begin{aligned}
\nu_q(E) &\leq \sum_i \nu_q\left(B\left(x_i, r_i\right)\right) \\
&\leq \sum_i \nu_q\left(J_{\sigma(i)|\ell_i+1}\left(x_i\right)\right) \\
&= \sum_i \frac{\mu\left(J_{\sigma(i)|\ell_i+1}\left(x_i\right)\right)^q \left|J_{\sigma(i)|\ell_i+1}\left(x_i\right)\right|^{\beta(q)}}{\sum\limits_{\sigma\in D_{l_i+1}} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)}} \\
&\leq C_1 \sum_i \mu\left(J_{\sigma(i)|\ell_i+1}\left(x_i\right)\right)^q \left|J_{\sigma(i)|\ell_i+1}\left(x_i\right)\right|^{\beta(q)}. \qquad (3)
\end{aligned}$$

If $\beta(q) \geqslant 0$, then

$$\left|J_{\sigma(i)|\ell_i+1}\right|^{\beta(q)} \leqslant (2\Delta)^{\beta(q)} \left(2r_i\right)^{\beta(q)}.$$

If $\beta(q) < 0$, then

$$\left|J_{\sigma(i)|\ell_i+1}\right| = \begin{cases} r_a\left|J_{\sigma(i)|\ell_i}\right|, & s_{\ell_i+1} = a \\ r_b\left|J_{\sigma(i)|\ell_i}\right|, & s_{\ell_i+1} = b, \end{cases}$$

which implies that

$$\left|J_{\sigma(i)|\ell_{i+1}}\right| \geq \min\left\{r_a, r_b\right\} \cdot |J_{\sigma(i)|\ell_i}|,$$

thus we deduce that

$$2r_i \leqslant 2\Delta\left|J_{\sigma(i)|\ell_i}\right| \leqslant \frac{2\Delta}{\min\left\{r_a, r_b\right\}}\left|J_{\sigma(i)|\ell_i+1}\right|.$$

And this gives us

$$\left|J_{\sigma(i)|\ell_i+1}\right|^{\beta(q)} \leqslant \left(\frac{\min\left\{r_a, r_b\right\}}{2\Delta}\right)^{\beta(q)} \left(2r_i\right)^{\beta(q)}.$$

Which leads to the following inequality

$$\left|J_{\sigma(i)|\ell_i+1}\right|^{\beta(q)} \leqslant k_1\left(2r_i\right)^{\beta(q)} \tag{4}$$

where k_1 is a suitable constant. If $q < 0$, it follows from (2) that

$$\mu\left(J_{\sigma(i)|\ell_i+1}(x_i)\right)^q \leq \mu\left(B\left(x_i, r_i\right)\right)^q. \tag{5}$$

Since the measure μ satisfies the doubling condition (see [7, Proposition 3.2]) then for all $q \geq 0$, there exists a constant $A > 0$ such that

$$\mu\left(J_{\sigma(i)|\ell_i+1}(x_i)\right)^q \leq \left(\frac{\mu\left(B\left(x_i, \frac{r_i}{\Delta}\right)\right)}{\mu\left(B\left(x_i, r_i\right)\right)}\right)^q \mu\left(B\left(x_i, r_i\right)\right)^q \leq A^q \mu\left(B\left(x_i, r_i\right)\right)^q. \tag{6}$$

It follows from (5) and (6) that there exists a constant C_2 such that

$$\mu\left(J_{\sigma(i)|\ell_i+1}(x_i)\right)^q \le C_2\mu\left(B\left(x_i,r_i\right)\right)^q. \tag{7}$$

Now combining (3), (4) and (7) shows that

$$\nu_q(E) \le k_1C_1C_2\sum_i \mu\left(B(x_i,r_i)\right)^q (2r_i)^{\beta(q)}.$$

Finally, this yields

$$\underline{K}\nu_q(E) \le \overline{\mathscr{H}}^{q,\beta(q)}_{\mu,\delta}(E) \le \overline{\mathscr{H}}^{q,\beta(q)}_{\mu}(E) \le \mathscr{H}^{q,\beta(q)}_{\mu}(E)$$

where $\underline{K} = \frac{1}{k_1C_1C_2}$. □

Claim 3. *There exists a constant $\overline{K} > 0$ such that for any $q \in \mathbb{R}$*

$$\overline{\mathscr{P}}^{q,\beta(q)}_{\mu}(E) \le \overline{K}\nu_q(E).$$

Proof of Claim 3. Let F be a closed subset E and $D_\delta(F) = \big\{x \in E \;\big|\; \operatorname{dist}(x,F) \le \delta\big\}$. Recall that, if $\delta \searrow 0$, then $D_\delta(F) \searrow F$. So, for all $\varepsilon > 0$ there exists δ_0 satisfying

$$\nu_q\big(D_\delta(F)\big) \le \nu_q(F) + \varepsilon, \qquad \forall\, 0 < \delta < \delta_0.$$

Let $\Big(B(x_i,r_i)\Big)_{i\in\mathbb{N}}$ be a centered δ-packing of F. For each i choose $\sigma(i) \in D_n$, for any $n \geqslant 1$ such that $x_i \in J_{\sigma(i)}$. For each $i \in \mathbb{N}$ choose $k_i, \ell_i \in \mathbb{N}$ such that

$$\left|J_{\sigma(i)|k_i+1}\right| \leqslant r_i < \left|J_{\sigma(i)|k_i}\right| \quad \text{and} \quad \Delta\left|J_{\sigma(i)|\ell_i+1}\right| \leqslant r_i < \Delta\left|J_{\sigma(i)|\ell_i}\right|.$$

Notice that

$$J_{\sigma(i)|k_i+1}\left(x_i\right) \subseteq B\left(x_i,r_i\right) \quad \text{and} \quad E \cap B\left(x_i,r_i\right) \subseteq J_{\sigma(i)|\ell_i+1}(x_i).$$

Using a similar argument as that in Claim 2. There exist constants $K_1, K_2 > 0$ such that

$$(2r_i)^{\beta(q)} \le K_1\left|J_{\sigma(i)|k_i+1}\right|^{\beta(q)}$$

and

$$\mu\left(B\left(x_i,r_i\right)\right)^q \le K_2\mu\left(J_{\sigma(i)|k_i+1}(x_i)\right)^q,$$

which implies that

$$\begin{aligned}
\sum_i \mu\left(B(x_i,r_i)\right)^q (2r_i)^{\beta(q)} &\le K_1K_2\sum_i \mu\left(J_{\sigma(i)|k_i+1}(x_i)\right)^q \left|J_{\sigma(i)|k_i+1}\right|^{\beta(q)} \\
&\le K_1K_2\sum_i \left(\frac{\mu\left(J_{\sigma(i)|k_i+1}(x_i)\right)^q \left|J_{\sigma(i)|k_i+1}\right|^{\beta(q)}}{\displaystyle\sum_{\sigma\in D_{k_i+1}} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)}}\right) \\
&\qquad \times \sum_{\sigma\in D_{k_i+1}} \mu\left(J_\sigma\right)^q \left|J_\sigma\right|^{\beta(q)} \\
&\le CK_1K_2\sum_i \nu_q\big(J_{\sigma(i)|k_i+1}(x_i)\big) \\
&\le CK_1K_2\sum_i \nu_q\big(B\left(x_i,r_i\right)\big) \\
&\le CK_1K_2\nu_q\big(D_\delta(F)\big) \le CK_1K_2\Big(\nu_q(E)+\varepsilon\Big).
\end{aligned}$$

Which leads to the following inequality

$$\overline{\mathscr{P}}_{\mu}^{q,\beta(q)}(F) \leq \overline{K} \Big(\nu_q(E) + \varepsilon\Big), \quad \text{where} \quad \overline{K} = CK_1K_2.$$

Tending ε to 0 now yields

$$\overline{\mathscr{P}}_{\mu}^{q,\beta(q)}(E) \leq \overline{K}\ \nu_q(E).$$

This complete the proof of Claim 3. □

Proof of Theorem 2.3. It follows from Claim 2 and Claim 3 and since μ satisfies the doubling condition that

$$\underline{K}\ \nu_q \leq \mathscr{H}_{\mu}^{q,\beta(q)} \leq \mathscr{P}_{\mu}^{q,\beta(q)} \leq \overline{\mathscr{P}}_{\mu}^{q,\beta(q)} \leq \overline{K}\ \nu_q \quad \text{on} \quad E.$$

Which implies that

$$\frac{1}{\overline{K}}\ \mathscr{H}_{\mu}^{q,\beta(q)} \leq \nu_q \leq \frac{1}{\underline{K}}\ \mathscr{H}_{\mu}^{q,\beta(q)} \quad \text{on} \quad E$$

and

$$\frac{1}{\overline{K}}\ \mathscr{P}_{\mu}^{q,\beta(q)} \leq \nu_q \leq \frac{1}{\underline{K}}\ \mathscr{P}_{\mu}^{q,\beta(q)} \quad \text{on} \quad E.$$

The desired result now follows from (1). □

Remark 3. It follows from Claim 2 and Claim 3 and since $0 < \nu_q(E) \leq 1$ that

$$b_{\mu}^{q}(E) = B_{\mu}^{q}(E) = \Lambda_{\mu}^{q}(E) = \beta(q), \quad \forall\, q \in \mathbb{R}.$$

It is also instructive to consider the special case $q = 0$. In particular, we have

$$\dim_H(E) = \dim_P(E) = \Delta(E) = \beta(0) = \frac{-\log 2 - \eta \log 3}{\log r_a + \eta \log r_b},$$

where $\eta^2 + \eta = 1$.

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