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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF BOUND STATES FOR A CLASS OF NONAUTONOMOUS SCHRÖDINGER-POISSON SYSTEM

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Abstract. This paper is concerned with the following Schrödinger-Poisson system

$$(P_\mu): -\Delta u + u + K(x)\phi u = |u|^{p-1}u + \mu h(x)u,\ -\Delta\phi = K(x)u^2,\ x \in \mathbb{R}^3,$$

where $p \in (3,5)$, $K(x)$ and $h(x)$ are nonnegative functions, and μ is a positive parameter. Let $\mu_1 > 0$ be an isolated first eigenvalue of the eigenvalue problem $-\Delta u + u = \mu h(x)u,\ u \in H^1(\mathbb{R}^3)$. As $0 < \mu \le \mu_1$, we prove that (P_μ) has at least one nonnegative bound state with positive energy. As $\mu > \mu_1$, there is $\delta > 0$ such that for any $\mu \in (\mu_1, \mu_1 + \delta)$, (P_μ) has a nonnegative ground state $u_{0,\mu}$ with negative energy, and $u_{0,\mu^{(n)}} \to 0$ in $H^1(\mathbb{R}^3)$ as $\mu^{(n)} \downarrow \mu_1$. Besides, (P_μ) has another nonnegative bound state $u_{2,\mu}$ with positive energy, and $u_{2,\mu^{(n)}} \to u_{\mu_1}$ in $H^1(\mathbb{R}^3)$ as $\mu^{(n)} \downarrow \mu_1$, where u_{μ_1} is a bound state of (P_{μ_1}).

1. Introduction

In this paper, we study a class of Schrödinger-Poisson system with the following version

$$\begin{cases} -\Delta u + u + K(x)\phi u = |u|^{p-1}u + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $p \in (3,5)$, $\mu > 0$, $K(x)$ and $h(x)$ are nonnegative functions. System (1) can be looked on as a non-autonomous version of the system

$$\begin{cases} -\Delta u + u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{2}$$

which has been derived from finding standing waves of the Schrödinger-Poisson system

$$\begin{cases} i\psi_t - \Delta\psi + \phi\psi = f(\psi) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |\psi|^2 & \text{in } \mathbb{R}^3. \end{cases}$$

A starting point of studying system (1) is the following fact. For any $u \in H^1(\mathbb{R}^3)$ and $K \in L^\infty(\mathbb{R}^3)$, there is a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ with

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)|u(y)|^2}{|x-y|}dy$$

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such that $-\Delta\phi_u = K(x)u^2$, see e.g. [11, 20]. Inserting this ϕ_u into the first equation of the system (1), we get that

$$-\Delta u + u + K(x)\phi_u u = |u|^{p-1}u + \mu h(x)u, \quad u \in H^1(\mathbb{R}^3). \tag{3}$$

Problem (3) can be also looked on as a usual semilinear elliptic equation with an additional nonlocal perturbation $K(x)\phi_u u$. Our aim here is to prove some new phenomenon of (3) due to the presence of the term $K(x)\phi_u u$. Before giving the main results, we state the following assumptions.

- **(A1):** $h(x) \geq 0$, $h(x) \not\equiv 0$ in $\mathbb{R}^3$ and $h(x) \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.
- **(A2):** There exist $b > 0$ and $H_0 > 0$ such that $h(x) \geq H_0 e^{-b|x|}$ for all $x \in \mathbb{R}^3$.
- **(A3):** $K(x) \geq 0$ and $K(x) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.
- **(A4):** There exist $a > 0$ and $K_0 > 0$ such that $K(x) \leq K_0 e^{-a|x|}$ for all $x \in \mathbb{R}^3$.

From Lemma 2.1, we know that under the condition (A1), the following eigenvalue problem

$$-\Delta u + u = \mu h(x)u, \qquad u \in H^1(\mathbb{R}^3)$$

has a first eigenvalue $\mu_1 > 0$ and μ_1 is simple. Denote

$$F(u) := \int_{\mathbb{R}^3} K(x)\phi_u(x)|u(x)|^2 dx$$

and introduce the energy functional $I_\mu : H^1(\mathbb{R}^3) \to \mathbb{R}$ associated with (3)

$$I_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3} \left(\frac{1}{p+1}|u|^{p+1} + \frac{\mu}{2}h(x)u^2 \right) dx,$$

where $\|u\|^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx$. From [11] and the Sobolev inequality, I_μ is well defined and $I_\mu \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. Moreover, for any $v \in H^1(\mathbb{R}^3)$,

$$\langle I'_\mu(u), v\rangle = \int_{\mathbb{R}^3} \left(\nabla u \nabla v + uv + K(x)\phi_u uv - |u|^{p-1}uv + \mu h(x)uv\right) dx.$$

It is known that there is a one to one correspondence between solutions of (3) and critical points of I_μ in $H^1(\mathbb{R}^3)$. Note that if $u \in H^1(\mathbb{R}^3)$ is a solution of (3), then (u, ϕ_u) is a solution of the system (1). If $u \geq 0$ and u is a solution of (3), then (u, ϕ_u) is a nonnegative solution of (1) since ϕ_u is always nonnegative. We call $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ a bound state of (3) if $I'_\mu(u) = 0$. At this time (u, ϕ_u) is called a bound state of (1). A bound state u is called a ground state of (3) if $I'_\mu(u) = 0$ and $I_\mu(u) \leq I_\mu(w)$ for any bound state w. In this case, we call (u, ϕ_u) a ground state of (1). The first result is about μ less than μ_1.

Theorem 1.1. *Suppose that the assumptions of* (A1) - (A4) *hold and* $0 < b < a < 2$. *If* $0 < \mu \leq \mu_1$, *then problem (3) has at least one nonnegative bound state.*

The second result is about μ in a small right neighborhood of μ_1.

Theorem 1.2. *Under the assumptions of* (A1) - (A4), *if* $0 < b < a < 1$, *then there exists* $\delta > 0$ *such that, for any* $\mu \in (\mu_1, \mu_1 + \delta)$,

(1) *problem (3) has at least one nonnegative ground state* $u_{0,\mu}$ *with* $I_\mu(u_{0,\mu}) < 0$. *Moreover,* $u_{0,\mu^{(n)}} \to 0$ *strongly in* $H^1(\mathbb{R}^3)$ *for any sequence* $\mu^{(n)} > \mu_1$ *and* $\mu^{(n)} \to \mu_1$;

(2) *problem (3) has another nonnegative bound state* $u_{2,\mu}$ *with* $I_\mu(u_{2,\mu}) > 0$. *Moreover,* $u_{2,\mu^{(n)}} \to u_{\mu_1}$ *strongly in* $H^1(\mathbb{R}^3)$ *for any sequence* $\mu^{(n)} > \mu_1$ *and* $\mu^{(n)} \to \mu_1$, *where* u_{μ_1} *satisfies* $I'_{\mu_1}(u_{\mu_1}) = 0$ *and* $I_{\mu_1}(u_{\mu_1}) > 0$.

The proofs of Theorem 1.1 and Theorem 1.2 are based on critical point theory. There are several difficulties in the road of getting critical points of I_μ in $H^1(\mathbb{R}^3)$ since we are dealing with the problem in the whole space $\mathbb{R}^3$, the embedding from $H^1(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ $(2 < q < 6)$ is not compact, the appearance of a nonlocal term $K(x)\phi_u u$ and the non coercive linear part. To explain our strategy, we review some related known results. For the system (2), under various conditions of f, there are a lot of papers dealing with the existence and nonexistence of positive solutions $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, see for example [2, 23] and the references therein. The lack of compactness from $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ $(2 < q < 6)$ was overcome by restricting the problem in $H^1_r(\mathbb{R}^3)$ which is a subspace of $H^1(\mathbb{R}^3)$ containing only radial functions. The existence of multiple radial solutions and non-radial solutions have been obtained in [2, 13]. See also [6, 15, 16, 17, 18, 19, 24, 29, 30] for some other results related to the system (2).

While for nonautonomous version of Schrödinger-Poisson system, only a few results are known in the literature. Jiang et.al.[21] have studied the following Schrödinger-Poisson system with non constant coefficient

$$\begin{cases} -\Delta u + (1 + \lambda g(x))u + \theta\phi(x)u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \quad \lim_{|x|\to\infty} \phi(x) = 0, \end{cases}$$

in which the authors prove the existence of ground state solution and its asymptotic behavior depending on θ and λ. The lack of compactness was overcome by suitable assumptions on $g(x)$ and λ large enough. The Schrödinger-Poisson system with critical nonlinearity of the form

$$\begin{cases} -\Delta u + u + \phi u = V(x)|u|^4 u + \mu P(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \quad 2 < q < 6, \mu > 0 \end{cases}$$

has been studied by Zhao et al. [31]. Besides some other conditions, Zhao et. al. [31] assume that $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $\lim_{|x|\to\infty} V(x) = V_\infty \in (0, \infty)$ and $V(x) \geq V_\infty$ for $x \in \mathbb{R}^3$ and prove the existence of one positive solution for $4 < q < 6$ and each $\mu > 0$. It is also proven the existence of one positive solution for $q = 4$ and μ large enough. Cerami et. al. [11] study the following type of Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + L(x)\phi u = g(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = L(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

Besides some other conditions and the assumption $L(x) \in L^2(\mathbb{R}^3)$, they prove the existence and nonexistence of ground state solutions. We emphasize that $L(x) \in L^2(\mathbb{R}^3)$ will imply suitable compactness property of the coupled term $L(x)\phi u$. Huang et. al. [20] have used this property to prove the existence of multiple solutions of (4) when $g(x, u) = a(x)|u|^{p-2}u + \mu h(x)u$ and μ stays in a right neighborhood of μ_1. The lack of compactness was overcome by suitable assumptions on the sign changing function $a(x)$. While for (3), none of the aboved mentioned properties can be used. We have to analyze the energy level of the functional I_μ such that the Palais-Smale ((PS) for short) condition may hold at suitable interval. Also for (3), another difficulty is to find mountain pass geometry for the functional I_μ in the case of $\mu \geq \mu_1$. We point out that for the semilinear elliptic equation

$$-\Delta u = a(x)|u|^{p-2}u + \tilde{\mu}k(x)u, \quad \text{in } \mathbb{R}^N, \tag{5}$$

Costa et.al.[14] have proven the mountain pass geometry for the related functional of (5) when $\tilde{\mu} \geq \tilde{\mu}_1$, where $\tilde{\mu}_1$ is the first eigenvalue of $-\Delta u = \tilde{\mu}k(x)u$ in $D^{1,2}(\mathbb{R}^N)$. Costa et. al. have managed to do these with the help of the condition

$\int_{\mathbb{R}^N} a(x)\tilde{e}_1^p dx < 0$, where $\tilde{e}_1$ is a positive eigenfunction corresponding to $\tilde{\mu}_1$. In the present paper, it is not possible to use such kind of condition. We will develop further the techniques in [20] to prove the mountain pass geometry. A third difficulty is to look for a ground state of (3). A usual method of getting a ground state is by minimizing the functional I_μ over the Nehari set $\{u \in H^1(\mathbb{R}^3)\backslash\{0\} \,:\, \langle I'_\mu(u), u\rangle = 0\}$. But in the case of $\mu > \mu_1$, one can not do like this because we do not know if 0 belongs to the boundary of this Nehari set. To overcome this trouble, we will minimize the functional over the set $\{u \in H^1(\mathbb{R}^3)\backslash\{0\} \,:\, I'_\mu(u) = 0\}$.

This paper is organized as follows. In Section 2, we give some preliminaries. Special attentions are focused on several lemmas analyzing the Palais-Smale conditions of the functional I_μ, which will play an important role in the proofs of Theorem 1.1 and Theorem 1.2. In Section 3, we prove Theorem 1.1. And Section 4 is devoted to the proof of Theorem 1.2.

Notations. Throughout this paper, $o(1)$ is a generic infinitesimal. The $H^{-1}(\mathbb{R}^3)$ denotes dual space of $H^1(\mathbb{R}^3)$. $L^q(\mathbb{R}^3)$ $(1 \le q \le +\infty)$ is a Lebesgue space with the norm denoted by $\|u\|_{L^q}$. The S_{p+1} is defined by

$$S_{p+1} = \inf_{u\in H^1(\mathbb{R}^3)\backslash\{0\}} \frac{\int_{\mathbb{R}^3}\left(|\nabla u|^2 + |u|^2\right)dx}{\left(\int_{\mathbb{R}^3}|u|^{p+1}dx\right)^{\frac{2}{p+1}}}.$$

For any $\rho > 0$ and $x \in \mathbb{R}^3$, $B_\rho(x)$ denotes the ball of radius ρ centered at x. C or C_j $(j = 1,\ 2,\ \cdots)$ denotes various positive constants, whose exact value is not important.

2. Preliminaries

In this section, we give some preliminary lemmas, which will be helpful to analyze the (PS) conditions for the functional I_μ. Firstly, for any $u \in H^1(\mathbb{R}^3)$ and $K \in L^\infty(\mathbb{R}^3)$, defining the linear functional

$$L_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v dx, \qquad v \in D^{1,2}(\mathbb{R}^3),$$

one may deduce from the Hölder and the Sobolev inequalities that

$$|L_u(v)| \le C\|u\|^2_{L^{\frac{12}{5}}}\|v\|_{L^6} \le C\|u\|^2_{L^{\frac{12}{5}}}\|v\|_{D^{1,2}}. \tag{6}$$

Hence, for any $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi = K(x)u^2$ in $D^{1,2}(\mathbb{R}^3)$. Moreover it holds that

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{K(y)u^2(y)}{|x-y|}dy.$$

Clearly $\phi_u(x) \ge 0$ for any $x \in \mathbb{R}^3$. We also have that

$$\|\phi_u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3}|\nabla\phi_u|^2 dx = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx. \tag{7}$$

Using (6) and (7), we obtain that

$$\|\phi_u\|_{L^6} \le C\|\phi_u\|_{D^{1,2}} \le C\|u\|^2_{L^{\frac{12}{5}}} \le C\|u\|^2. \tag{8}$$

Then we deduce that

$$\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2(x)dx \le C\|u\|^4. \tag{9}$$

Hence on $H^1(\mathbb{R}^3)$, both the functional

$$F(u) = \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2(x)dx \tag{10}$$

and

$$I_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3}\left(\frac{1}{p+1}|u|^{p+1} + \frac{\mu}{2}h(x)u^2\right)dx \tag{11}$$

are well defined and C^1. Moreover, for any $v \in H^1(\mathbb{R}^3)$,

$$\langle I'_\mu(u), v\rangle = \int_{\mathbb{R}^3}\left(\nabla u\nabla v + uv + K(x)\phi_u uv - |u|^{p-1}uv - \mu h(x)uv\right)dx.$$

The following Lemma 2.1 is a direct consequence of [28, Lemma 2.13].

Lemma 2.1. *Assume that the hypothesis (A1) holds. Then the functional $u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} h(x)u^2dx$ is weakly continuous and for each $v \in H^1(\mathbb{R}^3)$, the functional $u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} h(x)uvdx$ is weakly continuous.*

Using the spectral theory of compact symmetric operators on Hilbert space, the above lemma implies the existence of a sequence of eigenvalues $(\mu_n)_{n\in\mathbb{N}}$ of

$$-\Delta u + u = \mu h(x)u, \text{ in } H^1(\mathbb{R}^3)$$

with $\mu_1 < \mu_2 \le \cdots$ and each eigenvalue being of finite multiplicity. The associated normalized eigenfunctions are denoted by $e_1, e_2, \cdots$ with $\|e_i\| = 1$, $i = 1, 2, \cdots$. Moreover, one has $\mu_1 > 0$ with an eigenfunction $e_1 > 0$ in $\mathbb{R}^3$. In addition, we have the following variational characterization of μ_n:

$$\mu_1 = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2dx}, \quad \mu_n = \inf_{u\in S^\perp_{n-1}\setminus\{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2dx},$$

where $S^\perp_{n-1} = \{span\{e_1, e_2, \cdots, e_{n-1}\}\}^\perp$.

Next we analyze the (PS) condition of the functional I_μ in $H^1(\mathbb{R}^3)$. The following definition is standard.

Definition 2.2. For $d \in \mathbb{R}$, the functional I_μ is said to satisfy $(PS)_d$ condition if for any $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ with $I_\mu(u_n) \to d$ and $I'_\mu(u_n) \to 0$, the $(u_n)_{n\in\mathbb{N}}$ contains a convergent subsequence in $H^1(\mathbb{R}^3)$. The functional I_μ is said to satisfy (PS) conditions if I_μ satisfies $(PS)_d$ condition for any $d \in \mathbb{R}$.

Lemma 2.3. *Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be such that $I_\mu(u_n) \to d \in \mathbb{R}$ and $I'_\mu(u_n) \to 0$, then $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$.*

Proof. For n large enough, we have that

$$\begin{aligned} d \quad +1 + o(1)\|u_n\| &= I_\mu(u_n) - \frac{1}{4}\langle I'_\mu(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 - \frac{\mu}{4}\int_{\mathbb{R}^3} h(x)u_n^2dx + \frac{p-3}{4(p+1)}\int_{\mathbb{R}^3}|u_n|^{p+1}dx. \end{aligned} \tag{12}$$

Note that $\frac{p+1}{p-1} > \frac{3}{2}$ for $p \in (3, 5)$. Then for any $\vartheta > 0$, we obtain from $h \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ that

$$\begin{aligned} \int_{\mathbb{R}^3} h(x)u_n^2dx &\le \left(\int_{\mathbb{R}^3}|u_n|^{p+1}dx\right)^{\frac{2}{p+1}}\left(\int_{\mathbb{R}^3}|h(x)|^{\frac{p+1}{p-1}}dx\right)^{\frac{p-1}{p+1}} \\ &\le \frac{2\vartheta}{p+1}\int_{\mathbb{R}^3}|u_n|^{p+1}dx + \frac{p-1}{p+1}\vartheta^{-\frac{2}{p-1}}\int_{\mathbb{R}^3}|h(x)|^{\frac{p+1}{p-1}}dx. \end{aligned}$$

Choosing $\vartheta = \frac{p-3}{2\mu}$, we get

$$d + 1 + o(1)\|u_n\| \geq \frac{1}{4}\|u_n\|^2 - D(p,h)\mu^{\frac{p+1}{p-1}}, \tag{13}$$

where $D(p,h) = \frac{p-1}{4(p+1)} \left(\frac{p-3}{2}\right)^{-\frac{2}{p-1}} \int_{\mathbb{R}^3} |h(x)|^{\frac{p+1}{p-1}} dx$. Hence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. □

The following lemma is a variant of Brezis-Lieb lemma. One may find the proof in [20].

Lemma 2.4. [20] *If a sequence $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ and $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$, then*

$$\lim_{n\to\infty} F(u_n) = F(u_0) + \lim_{n\to\infty} F(u_n - u_0).$$

Lemma 2.5. *There is a $\delta_1 > 0$ such that for any $\mu \in [\mu_1, \mu_1 + \delta_1)$, any solution u of (3) satisfies*

$$I_\mu(u) > -\frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}.$$

Proof. Since u is a solution of (3), we get that

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\left(\|u\|^2 - \mu \int_{\mathbb{R}^3} h(x)u^2 dx\right) + \frac{1}{4}F(u) - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx \\ &= \frac{p-1}{2(p+1)}\left(\|u\|^2 - \mu \int_{\mathbb{R}^3} h(x)u^2 dx\right) + \frac{p-3}{4(p+1)}F(u). \end{aligned}$$

Noticing that $\|u\|^2 \geq \mu_1 \int_{\mathbb{R}^3} h(x)u^2 dx$ for any $u \in H^1(\mathbb{R}^3)$, we deduce that for any $u \neq 0$,

$$I_{\mu_1}(u) \geq \frac{p-3}{4(p+1)}F(u) > 0.$$

Next, we **claim:** there is a $\delta_1 > 0$ such that for any $\mu \in [\mu_1, \mu_1 + \delta_1)$, any solution u of (3) satisfies

$$I_\mu(u) > -\frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}.$$

Suppose this claim is not true, then there is a sequence $\mu^{(n)} > \mu_1$ with $\mu^{(n)} \to \mu_1$ and solutions $u_{\mu^{(n)}}$ of (3) such that

$$I_{\mu^{(n)}}(u_{\mu^{(n)}}) \leq -\frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}.$$

Note that $I'_{\mu^{(n)}}(u_{\mu^{(n)}}) = 0$. Then we deduce that for n large enough,

$$\begin{aligned} I_{\mu^{(n)}}(u_{\mu^{(n)}}) + o(1)\|u_{\mu^{(n)}}\| &\geq I_{\mu^{(n)}}(u_{\mu^{(n)}}) - \frac{1}{4}\langle I'_{\mu^{(n)}}(u_{\mu^{(n)}}), u_{\mu^{(n)}}\rangle \\ &\geq \frac{1}{4}\|u_{\mu^{(n)}}\|^2 - D(p,h)\left(\mu^{(n)}\right)^{\frac{p+1}{p-1}}. \end{aligned}$$

This implies that $(u_{\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Since for any $n \in \mathbb{N}$, $\|u_{\mu^{(n)}}\|^2 \geq \mu_1 \int_{\mathbb{R}^3} h(x)(u_{\mu^{(n)}})^2 dx$, we obtain that as $\mu^{(n)} \to \mu_1$

$$\|u_{\mu^{(n)}}\|^2 - \mu^{(n)} \int_{\mathbb{R}^3} h(x)(u_{\mu^{(n)}})^2 dx \geq \left(1 - \frac{\mu^{(n)}}{\mu_1}\right)\|u_{\mu^{(n)}}\|^2 \to 0$$

because $(u_{\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Noting that

$$\begin{aligned} I_{\mu^{(n)}}(u_{\mu^{(n)}}) &= \frac{p-1}{2(p+1)}\left(\|u_{\mu^{(n)}}\|^2 - \mu^{(n)}\int_{\mathbb{R}^3} h(x)(u_{\mu^{(n)}})^2dx\right) \\ &+\frac{p-3}{4(p+1)}F(u_{\mu^{(n)}}), \end{aligned}$$

we deduce that

$$\liminf_{n\to\infty} I_{\mu^{(n)}}(u_{\mu^{(n)}}) \geq \frac{p-3}{4(p+1)}\liminf_{n\to\infty} F(u_{\mu^{(n)}}) \geq 0,$$

which contradicts to the

$$I_{\mu^{(n)}}(u_{\mu^{(n)}}) \leq -\frac{p-1}{2(p+1)}S_{p+1}^{\frac{p+1}{p-1}}.$$

This proves the **claim** and the proof of Lemma 2.5 is complete. □

Lemma 2.6. *If $\mu\in[\mu_1,\mu_1+\delta_1)$, then I_μ satisfies $(PS)_d$ condition for any $d<0$.*

Proof. Let $(u_n)_{n\in\mathbb{N}}\subset H^1(\mathbb{R}^3)$ be a $(PS)_d$ sequence of I_μ with $d<0$. Then for n large enough,

$$d+o(1)=\frac{1}{2}\|u_n\|^2-\frac{\mu}{2}\int_{\mathbb{R}^3}h(x)u_n^2dx+\frac{1}{4}F(u_n)-\frac{1}{p+1}\int_{\mathbb{R}^3}|u_n|^{p+1}dx$$

and

$$\langle I'_\mu(u_n),u_n\rangle=\|u_n\|^2-\mu\int_{\mathbb{R}^3}h(x)u_n^2dx+F(u_n)-\int_{\mathbb{R}^3}|u_n|^{p+1}dx.$$

Then we can prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Without loss of generality, we may assume that $u_n\rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ and $u_n\to u_0$ a. e. in $\mathbb{R}^3$. Denoting $w_n:=u_n-u_0$, we obtain from Brezis-Lieb lemma and Lemma 2.4 that for n large enough,

$$\begin{aligned} \|u_n\|^2 &= \|u_0\|^2+\|w_n\|^2+o(1), \\ F(u_n) &= F(u_0)+F(w_n)+o(1) \end{aligned}$$

and

$$\|u_n\|_{L^{p+1}}^{p+1}=\|u_0\|_{L^{p+1}}^{p+1}+\|w_n\|_{L^{p+1}}^{p+1}+o(1).$$

Using Lemma 2.1, we also have that $\int_{\mathbb{R}^3}h(x)u_n^2dx\to\int_{\mathbb{R}^3}h(x)u_0^2dx$ as $n\to\infty$. Therefore

$$\begin{aligned} d+o(1) &= I_\mu(u_n)=I_\mu(u_0)+\frac{1}{2}\|w_n\|^2 \\ &+ \frac{1}{4}F(w_n)-\frac{1}{p+1}\int_{\mathbb{R}^3}|w_n|^{p+1}dx. \end{aligned} \tag{14}$$

Noticing $\langle I'_\mu(u_n),\psi\rangle\to 0$ for any $\psi\in H^1(\mathbb{R}^3)$, we obtain that $I'_\mu(u_0)=0$. From which we deduce that

$$\|u_0\|^2-\mu\int_{\mathbb{R}^3}h(x)u_0^2dx+F(u_0)=\int_{\mathbb{R}^3}|u_0|^{p+1}dx. \tag{15}$$

Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, we obtain from $I'_\mu(u_n)\to 0$ that

$$o(1)=\|u_n\|^2-\mu\int_{\mathbb{R}^3}h(x)u_n^2dx+F(u_n)-\int_{\mathbb{R}^3}|u_n|^{p+1}dx.$$

Combining this with (15) as well as Lemma 2.1, we obtain that

$$o(1)=\|w_n\|^2+F(w_n)-\int_{\mathbb{R}^3}|w_n|^{p+1}dx. \tag{16}$$

Recalling the definition of S_{p+1}, we have that $\|u\|^2 \geq S_{p+1}\|u\|^2_{L^{p+1}}$ for any $u \in H^1(\mathbb{R}^3)$. Now we distinguish two cases:

(i): $\int_{\mathbb{R}^3} |w_n|^{p+1} dx \not\to 0$ as $n \to \infty$;

(ii): $\int_{\mathbb{R}^3} |w_n|^{p+1} dx \to 0$ as $n \to \infty$.

Suppose that the case (i) occurs. We may obtain from (16) that

$$\|w_n\|^2 \geq S_{p+1}\left(\|w_n\|^2 + F(w_n) - o(1)\right)^{\frac{2}{p+1}}.$$

Hence we get that for n large enough,

$$\|w_n\|^2 \geq S_{p+1}^{\frac{p+1}{p-1}} + o(1). \tag{17}$$

Therefore using (14), (16) and (17), we deduce that for n large enough,

$$\begin{aligned} d \quad & +o(1) = I_\mu(u_n) \\ = \quad & I_\mu(u_0) + \frac{1}{2}\|w_n\|^2 + \frac{1}{4}F(w_n) - \frac{1}{p+1}\int_{\mathbb{R}^3} |w_n|^{p+1} dx. \\ = \quad & I_\mu(u_0) + \frac{p-1}{2(p+1)}\|w_n\|^2 + \frac{p-3}{4(p+1)}F(w_n) \\ > \quad & -\frac{p-1}{2(p+1)}S_{p+1}^{\frac{p+1}{p-1}} + \frac{p-1}{2(p+1)}\|w_n\|^2 + \frac{p-3}{4(p+1)}F(w_n) \\ > \quad & 0, \end{aligned} \tag{18}$$

which contradicts to the condition $d < 0$. This means that the case (i) does not occur. Therefore the case (ii) occurs. Using (16), we deduce that $\|w_n\|^2 \to 0$ as $n \to \infty$. Hence we have proven that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$. □

Next we give a mountain pass geometry for the functional I_μ.

Lemma 2.7. *There exist $\delta_2 > 0$ with $\delta_2 \leq \delta_1$, $\rho > 0$ and $\alpha > 0$, such that for any $\mu \in [\mu_1, \mu_1 + \delta_2)$, $I_\mu|_{\partial B_\rho} \geq \alpha > 0$.*

Proof. For any $u \in H^1(\mathbb{R}^3)$, there exist $t \in \mathbb{R}$ and $v \in S_1^\perp$ such that

$$u = te_1 + v, \text{ where } \int_{\mathbb{R}^3} (\nabla v \nabla e_1 + v e_1)\, dx = 0. \tag{19}$$

Hence we deduce that

$$\|u\|^2 = \|\nabla(te_1 + v)\|^2_{L^2} + \|te_1 + v\|^2_{L^2} = t^2 + \|v\|^2, \tag{20}$$

$$\mu_2 \int_{\mathbb{R}^3} h(x) v^2 dx \leq \|v\|^2, \quad \mu_1 \int_{\mathbb{R}^3} h(x) e_1^2 dx = \|e_1\|^2 = 1 \tag{21}$$

and

$$\mu_1 \int_{\mathbb{R}^3} h(x) e_1 v dx = \int_{\mathbb{R}^3} (\nabla v \nabla e_1 + v e_1)\, dx = 0. \tag{22}$$

We first consider the case of $\mu = \mu_1$. Denoting $\theta_1 := (\mu_2 - \mu_1)/2\mu_2 > 0$, then by the relations from (19) to (22), we obtain that

$$\begin{aligned} I_{\mu_1}(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx \\ &= \frac{1}{2}\|te_1 + v\|^2 + \frac{1}{4}F(te_1+v) \\ &\quad -\frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)(te_1+v)^2dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|te_1+v|^{p+1}dx \\ &\geq \frac{1}{2}\left(1-\frac{\mu_1}{\mu_2}\right)\|v\|^2 + \frac{1}{4}F(te_1+v) - \frac{1}{p+1}\int_{\mathbb{R}^3}|te_1+v|^{p+1}dx \\ &\geq \theta_1\|v\|^2 + \frac{1}{4}F(te_1+v) - C_1|t|^{p+1} - C_2\|v\|^{p+1}. \end{aligned}$$

Next we estimate the term $F(te_1+v)$. Using the expression of $F(u)$, we have that

$$F(te_1+v) = \frac{1}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{K(x)K(y)(te_1(y)+v(y))^2(te_1(x)+v(x))^2}{|x-y|}dydx.$$

Since

$$\begin{aligned} &(te_1(y)+v(y))^2\,(te_1(x)+v(x))^2 = t^4(e_1(y))^2(e_1(x))^2 + (v(y))^2(v(x))^2 \\ &\quad +2t^3\left(e_1(y)(e_1(x))^2v(y) + e_1(x)(e_1(y))^2v(x)\right) \\ &\quad +2t\left(e_1(x)v(x)(v(y))^2 + e_1(y)v(y)(v(x))^2\right) \\ &\quad +t^2\left((e_1(x))^2(v(y))^2 + 4e_1(y)e_1(x)v(y)v(x) + (e_1(y))^2(v(x))^2\right), \end{aligned}$$

we know that

$$\left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{K(x)K(y)\left(e_1(y)(e_1(x))^2v(y)+e_1(x)(e_1(y))^2v(x)\right)}{|x-y|}dydx\right| \leq C\|v\|; \tag{23}$$

$$\left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{K(x)K(y)\left(2(e_1(x))^2(v(y))^2+4e_1(y)e_1(x)v(y)v(x)\right)}{|x-y|}dydx\right| \leq C\|v\|^2 \tag{24}$$

and

$$\left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{K(x)K(y)\left(e_1(x)v(x)(v(y))^2+e_1(y)v(y)(v(x))^2\right)}{|x-y|}dydx\right| \leq C\|v\|^3. \tag{25}$$

Hence

$$\begin{aligned} I_{\mu_1}(u) &\geq \theta_1\|v\|^2 + \theta_2|t|^4 - C_1|t|^{p+1} - C_2\|v\|^{p+1} \\ &\quad -C_3|t|^3\|v\| - C_4|t|^2\|v\|^2 - C_5|t|\|v\|^3 + \frac{1}{4}F(v), \end{aligned}$$

where $\theta_2 = \frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_{e_1}e_1^2dx$. Note that

$$t^2\|v\|^2 \leq \frac{2}{p+1}|t|^{p+1} + \frac{p-1}{p+1}\|v\|^{\frac{2(p+1)}{p-1}},$$

$$|t|\|v\|^3 \leq \frac{1}{p+1}|t|^{p+1} + \frac{p}{p+1}\|v\|^{\frac{3(p+1)}{p}}$$

and for some q_0 with $2 < q_0 < 4$, we also have that

$$|t|^3\|v\| \leq \frac{1}{q_0}\|v\|^{q_0} + \frac{q_0-1}{q_0}|t|^{\frac{3q_0}{q_0-1}}.$$

Therefore we deduce that

$$
\begin{aligned}
I_{\mu_1}(u) \geq\ & \theta_1\|v\|^2 + \theta_2|t|^4 - \frac{C_3}{q_0}\|v\|^{q_0} - \frac{C_3(q_0-1)}{q_0}|t|^{\frac{3q_0}{q_0-1}} \\
& - \frac{2C_4}{p+1}|t|^{p+1} - \frac{(p-1)C_4}{p+1}\|v\|^{\frac{2(p+1)}{p-1}} - \frac{C_5}{p+1}|t|^{p+1} \\
& - \frac{pC_5}{p+1}\|v\|^{\frac{3(p+1)}{p}} - C|t|^{p+1} - C\|v\|^{p+1}.
\end{aligned} \tag{26}
$$

From $q_0 > 2$ and $\frac{3q_0}{q_0-1} > 4$ (since $q_0 < 4$), we know that there are positive constants θ_3, θ_4 and $\tilde{\theta}_3$, $\tilde{\theta}_4$ such that

$$I_{\mu_1}(u) \geq \theta_3\|v\|^2 + \theta_4|t|^4$$

provided that $\|v\| \leq \tilde{\theta}_3$ and $|t| \leq \tilde{\theta}_4$. Hence there are positive constants θ_5 and $\tilde{\theta}_5$ such that

$$I_{\mu_1}(u) \geq \theta_5\|u\|^4 \qquad \text{for} \quad \|u\|^2 \leq \tilde{\theta}_5^2. \tag{27}$$

Set $\bar{\delta} := \min\{\frac{\mu_1}{2}\theta_5\tilde{\theta}_5^2, \quad \mu_2 - \mu_1\} > 0$ and $\delta_2 := \min\{\bar{\delta}, \delta_1\}$. Then for any $\mu \in [\mu_1, \mu_1 + \delta_2)$, we deduce from (27) that

$$
\begin{aligned}
I_\mu(u) \quad &= I_{\mu_1}(u) + \frac{1}{2}(\mu_1 - \mu)\int_{\mathbb{R}} h(x)u^2dx \\
&\geq \theta_5\|u\|^4 - \frac{\mu-\mu_1}{2\mu_1}\|u\|^2 \\
&= \|u\|^2\left(\theta_5\|u\|^2 - \frac{\mu-\mu_1}{2\mu_1}\right) \\
&\geq \|u\|^2\left(\frac{1}{2}\theta_5\tilde{\theta}_5^2 - \frac{1}{4}\theta_5\tilde{\theta}_5^2\right) = \frac{1}{4}\theta_5\tilde{\theta}_5^2\|u\|^2
\end{aligned}
$$

for $\frac{1}{2}\tilde{\theta}_5^2 \leq \|u\|^2 \leq \tilde{\theta}_5^2$. Choosing $\rho^2 = \frac{1}{2}\tilde{\theta}_5^2$ and $\alpha = \frac{1}{4}\theta_5\tilde{\theta}_5^2\rho^2$, we finish the proof of Lemma 2.7. □

3. Proof of Theorem 1.1

In this section, our aim is to prove Theorem 1.1. For $0 < \mu < \mu_1$, it is standard to prove that the functional I_μ contains mountain pass geometry. For $\mu = \mu_1$, as we have seen in Lemma 2.7, with the help of the competing between the Poisson term $K(x)\phi_u u$ and the nonlinear term, the 0 is a local minimizer of the functional I_{μ_1} and I_{μ_1} contains mountain pass geometry. To get a mountain pass type critical point of the functional I_μ, it suffices to prove the $(PS)_d$ condition by the mountain pass theorem of [3]. In the following we will focus our attention to the case of $\mu = \mu_1$, since the case of $0 < \mu < \mu_1$ is similar.

Proposition 3.1. *Let the assumptions* $(A1)-(A4)$ *hold and* $0 < b < a < 2$. *Define*

$$d_{\mu_1} = \inf_{\gamma\in\Gamma_1}\sup_{t\in[0,1]} I_{\mu_1}(\gamma(t))$$

with

$$\Gamma_1 = \left\{\gamma \in C([0,1], H^1(\mathbb{R}^3)) \ : \ \gamma(0) = 0, \ I_{\mu_1}(\gamma(1)) < 0\right\}.$$

Then d_{μ_1} *is a critical value of* I_{μ_1}.

Before proving Proposition 3.1, we analyze the $(PS)_{d_{\mu_1}}$ condition of I_{μ_1}. Let $U(x)$ be the unique positive solution of $-\Delta u + u = |u|^{p-1}u$ in $H^1(\mathbb{R}^3)$. We know that for any $\varepsilon \in (0,1)$, there is a $C \equiv C(\varepsilon) > 0$ such that $U(x) \leq Ce^{-(1-\varepsilon)|x|}$.

Lemma 3.2. *If the assumptions* $(A1)-(A4)$ *hold and* $0<b<a<2$*, then the* d_{μ_1} *defined in Proposition 3.1 satisfies* $d_{\mu_1} < \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}$.

Proof. It suffices to find a path $\gamma(t)$ starting from 0 such that

$$\sup_{t\in[0,1]} I_{\mu_1}(\gamma(t)) < \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}.$$

Define $U_R(x) = U(x-R\theta)$ with $\theta = (0,0,1)$. Note that for the U_R defined as above, the $I_{\mu_1}(tU_R) \to -\infty$ as $t \to +\infty$ and $I_{\mu_1}(tU_R) \to 0$ as $t \to 0$. We know that there is a unique $T_R > 0$ such that $\frac{\partial}{\partial t} I_{\mu_1}(tU_R)|_{t=T_R} = 0$, which is

$$\|U_R\|^2 - \mu_1 \int h(x) U_R^2 dx + T_R^2 F(U_R) - T_R^{p-1} \int U_R^{p+1} dx = 0.$$

If $T_R \to 0$ as $R \to \infty$, then $\|U_R\|^2 - \mu_1 \int h(x) U_R^2 dx \to 0$ as $R \to \infty$, which is impossible. If $T_R \to \infty$ as $R \to \infty$, then as $R \to \infty$,

$$\frac{1}{T_R^2}\left(\|U_R\|^2 - \mu_1 \int h(x) U_R^2 dx\right) + F(U_R) = T_R^{p-3} \int U_R^{p+1} dx \to \infty,$$

which is impossible either. Hence we only need to estimate $I_{\mu_1}(tU_R)$ for t in a finite interval and we may write

$$I_{\mu_1}(tU_R) \le g(t) + CF(U_R),$$

where

$$g(t) = \frac{t^2}{2}\left(\|U_R\|^2 - \mu_1 \int_{\mathbb{R}^3} h(x) U_R^2 dx\right) - \frac{|t|^{p+1}}{p+1} \int_{\mathbb{R}^3} U_R^{p+1} dx.$$

Noting that under the assumptions $(A1)-(A4)$, we obtain that for R large enough,

$$\begin{aligned} F(U_R) &\le \left(\int_{\mathbb{R}^3} K(x)^{\frac{6}{5}} U_R^{\frac{12}{5}} dx\right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} \phi_{U_R}^6 dx\right)^{\frac{1}{6}} \\ &\le C\left(\int_{\mathbb{R}^3} e^{-\frac{6}{5}a|x+R\theta|} (U(x))^{\frac{12}{5}} dx\right)^{\frac{5}{6}} \\ &\le C\left(\int_{\mathbb{R}^3} e^{-\frac{6}{5}aR} e^{(\frac{6}{5}a - \frac{12}{5}(1-\varepsilon))|x|} dx\right)^{\frac{5}{6}} \le Ce^{-aR} \end{aligned} \tag{28}$$

since $0<a<2$. We can also prove that

$$\begin{aligned} &\int_{\mathbb{R}^3} h(x) U_R^2 dx = \int_{\mathbb{R}^3} h(x+R\theta) U^2(x) dx \\ &\ge C \int_{\mathbb{R}^3} e^{-b|x+R\theta|} U^2(x) dx \ge C \int_{\mathbb{R}^3} e^{-b|x|-bR} U^2(x) dx \\ &\ge Ce^{-bR} \int_{\mathbb{R}^3} e^{-b|x|} U^2(x) dx \ge Ce^{-bR}. \end{aligned} \tag{29}$$

It is now deduced from (28) and (29) that

$$\begin{aligned} &\sup_{t>0} I_{\mu_1}(tU_R) \le \sup_{t>0} g(t) + Ce^{-aR} \\ &\le \tfrac{p-1}{2(p+1)} \left(\|U_R\|^2 - \mu_1 \textstyle\int_{\mathbb{R}^3} h(x) U_R^2 dx\right)^{\frac{p+1}{p-1}} \left(\|U_R\|_{L^{p+1}}^{-2}\right)^{\frac{p+1}{p-1}} + Ce^{-aR} \\ &\le \tfrac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}} - Ce^{-bR} + o(e^{-bR}) + Ce^{-aR} \\ &< \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}} \end{aligned}$$

for R large enough since $0<b<a$. The proof is complete. ∎

Lemma 3.3. *Under the assumptions* $(A1)-(A4)$*,* I_{μ_1} *satisfies* $(PS)_d$ *condition for any* $d < \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}$.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be a $(PS)_d$ sequence of I_{μ_1} with $d < \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}$. Then we have that for n large enough,

$$d + o(1) = \frac{1}{2}\|u_n\|^2 - \frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)u_n^2 dx + \frac{1}{4}F(u_n) - \frac{1}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1}dx$$

and

$$\langle I'_{\mu_1}(u_n), u_n\rangle = \|u_n\|^2 - \mu_1\int_{\mathbb{R}^3} h(x)u_n^2 dx + F(u_n) - \int_{\mathbb{R}^3} |u_n|^{p+1}dx.$$

Hence we can deduce that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, we may assume that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u_0$ a. e. in $\mathbb{R}^3$. Denote $w_n := u_n - u_0$. We then obtain from Brezis-Lieb lemma and Lemma 2.4 that for n large enough,

$$\|u_n\|^2 = \|u_0\|^2 + \|w_n\|^2 + o(1), \quad F(u_n) = F(u_0) + F(w_n) + o(1)$$

and

$$\|u_n\|_{L^{p+1}}^{p+1} = \|u_0\|_{L^{p+1}}^{p+1} + \|w_n\|_{L^{p+1}}^{p+1} + o(1).$$

Since $\int_{\mathbb{R}^3} h(x)u_n^2 dx \to \int_{\mathbb{R}^3} h(x)u_0^2 dx$ as $n \to \infty$, we deduce that

$$d + o(1) = I_{\mu_1}(u_n) = I_{\mu_1}(u_0) + \frac{1}{2}\|w_n\|^2 + \frac{1}{4}F(w_n) - \frac{1}{p+1}\int_{\mathbb{R}^3} |w_n|^{p+1}dx. \tag{30}$$

From $\langle I'_{\mu_1}(u_n), \psi\rangle \to 0$ for any $\psi \in H^1(\mathbb{R}^3)$, one may deduce that $I'_{\mu_1}(u_0) = 0$. Therefore

$$\|u_0\|^2 - \mu_1\int_{\mathbb{R}^3} h(x)u_0^2 dx + F(u_0) = \int_{\mathbb{R}^3} |u_0|^{p+1}dx$$

and then

$$I_{\mu_1}(u_0) \geq \frac{p-1}{2(p+1)}\left(\|u_0\|^2 - \mu_1\int_{\mathbb{R}^3} h(x)u_0^2 dx\right) + \frac{p-3}{4(p+1)}F(u_0) \geq 0.$$

Now using an argument similar to the proof of (16), we obtain that

$$o(1) = \|w_n\|^2 + F(w_n) - \int_{\mathbb{R}^3} |w_n|^{P+1}dx. \tag{31}$$

By the relation $\|u\|^2 \geq S_{p+1}\|u\|_{L^{p+1}}^2$ for any $u \in H^1(\mathbb{R}^3)$, we proceed our discussion according to the following two cases:

(I): $\int_{\mathbb{R}^3} |w_n|^{p+1}dx \not\to 0$ as $n \to \infty$;

(II): $\int_{\mathbb{R}^3} |w_n|^{p+1}dx \to 0$ as $n \to \infty$.

Suppose that the case (I) occurs. Then up to a sbusequence, we may obtain from (31) that

$$\|w_n\|^2 \geq S_{p+1}\left(\|w_n\|^2 + F(w_n) - o(1)\right)^{\frac{2}{p+1}},$$

which implies that for n large enough,

$$\|w_n\|^2 \geq S_{p+1}^{\frac{p+1}{p-1}} + o(1).$$

It is deduced from this and (30) that $d \geq \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}$, which is a contradiction. Therefore the case (II) must occur. This and (31) imply that $\|w_n\| \to 0$. Hence we have proven that I_{μ_1} satisfies $(PS)_d$ condition for any $d < \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}$. □

Proof of Proposition 3.1. Since 0 is a local minimizer of I_{μ_1} and for $v \neq 0$, $I_{\mu_1}(sv) \to -\infty$ as $s \to +\infty$, Lemma 3.2, Lemma 3.3 and the mountain pass theorem [3] imply that d_{μ_1} is a critical value of I_{μ_1}.

Proof of Theorem 1.1. By Proposition 3.1, the d_{μ_1} is a critical value of I_{μ_1} and $d_{\mu_1} > 0$. The proof of nonnegativity for at least one of the corresponding critical point is inspired by the idea of [1]. In fact, since $I_{\mu_1}(u) = I_{\mu_1}(|u|)$ for any $u \in H^1(\mathbb{R}^3)$, for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma_1$ with $\gamma_n(t) \geq 0$ (a.e. in $\mathbb{R}^3$) for all $t \in [0,1]$ such that

$$d_{\mu_1} \leq \max_{t\in[0,1]} I_{\mu_1}(\gamma_n(t)) < d_{\mu_1} + \frac{1}{n}. \tag{32}$$

By Ekeland's variational principle [5], there exists $\gamma_n^* \in \Gamma_1$ satisfying

$$\begin{cases} d_{\mu_1} \leq \max_{t\in[0,1]} I_{\mu_1}(\gamma_n^*(t)) \leq \max_{t\in[0,1]} I_{\mu_1}(\gamma_n(t)) < d_{\mu_1} + \frac{1}{n}; \\ \max_{t\in[0,1]} \|\gamma_n(t) - \gamma_n^*(t)\| < \frac{1}{\sqrt{n}}; \\ \text{there exists } t_n \in [0,1] \text{ such that } z_n = \gamma_n^*(t_n) \text{ satisfies :} \\ I_{\mu_1}(z_n) = \max_{t\in[0,1]} I_{\mu_1}(\gamma_n^*(t)), \text{ and } \|I'_{\mu_1}(z_n)\| \leq \frac{1}{\sqrt{n}}. \end{cases} \tag{33}$$

By Lemma 3.2 and Lemma 3.3 we get a convergent subsequence (still denoted by $(z_n)_{n\in\mathbb{N}}$). We may assume that $z_n \to z$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. On the other hand, by (33), we also arrive at $\gamma_n(t_n) \to z$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Since $\gamma_n(t) \geq 0$, we conclude that $z \geq 0$, $z \not\equiv 0$ in $\mathbb{R}^3$ with $I_{\mu_1}(z) > 0$ and it is a nonnegative bound state of (3) in the case of $\mu = \mu_1$.

4. Ground state and bound states for $\mu > \mu_1$

In this section, we always assume the conditions $(A1) - (A4)$. We will prove the existence of ground state and bound states of (3) as well as their asymptotical behavior with respect to μ. We emphasize that if $0 < \mu < \mu_1$, then one may consider a minimization problem like

$$\inf\{I_\mu(u) \ : \ u \in \mathcal{M}\}, \quad \mathcal{M} = \{u \in H^1(\mathbb{R}^3) \ : \ \langle I'_\mu(u), u\rangle = 0\}$$

to get a ground state solution. But for $\mu \geq \mu_1$, we can not do like this because for $\mu > \mu_1$, we do not know if $0 \notin \partial\mathcal{M}$. To overcome this difficulty, we define the set of all nontrivial critical points of I_μ in $H^1(\mathbb{R}^3)$:

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : I'_\mu(u) = 0\}.$$

And then we consider the following minimization problem

$$c_{0,\mu} = \inf\{I_\mu(u) : u \in \mathcal{N}\}. \tag{34}$$

Lemma 4.1. *Let δ_2 and ρ be as in Lemma 2.7 and $\mu \in (\mu_1, \mu_1 + \delta_2)$. Define the following minimization problem*

$$d_{0,\mu} = \inf_{\|u\|<\rho} I_\mu(u).$$

Then the $d_{0,\mu}$ is achieved by a nonnegative function $w_{0,\mu} \in H^1(\mathbb{R}^3)$. Moreover this $w_{0,\mu}$ is a nonnegative solution of (3).

Proof. Firstly, we prove that $-\infty < d_{0,\mu} < 0$ for $\mu \in (\mu_1, \mu_1 + \delta_2)$. Keeping the expression of $I_\mu(u)$ in mind, we obtain from the Sobolev inequality that

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\|u\|^2 - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)u^2dx + \frac{1}{4}F(u) - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\mu}{2\mu_1}\|u\|^2 - C\|u\|^{p+1} > -\infty \end{aligned}$$

as $\|u\| < \rho$. Next, for any $t > 0$, we have that

$$I_\mu(te_1) = \frac{t^2}{2}\|e_1\|^2 - \frac{\mu t^2}{2}\int_{\mathbb{R}^3} h(x)e_1^2dx + \frac{t^4}{4}F(e_1) - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3}|e_1|^{p+1}dx.$$

It is now deduced from $\mu_1\int_{\mathbb{R}^3} h(x)e_1^2dx = \|e_1\|^2$ that

$$I_\mu(te_1) = \frac{t^2}{2}\left(1 - \frac{\mu}{\mu_1}\right)\|e_1\|^2 + \frac{t^4}{4}F(e_1) - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3}|e_1|^{p+1}dx.$$

Since $\mu > \mu_1$, we obtain that for t small enough, the $I_\mu(te_1) < 0$. Thus we have proven that $-\infty < d_{0,\mu} < 0$ for $\mu \in (\mu_1, \mu_1 + \delta_2)$.

Secondly, let $(v_n)_{n\in\mathbb{N}}$ be a minimizing sequence, that is, $\|v_n\| < \rho$ and $I_\mu(v_n) \to d_{0,\mu}$ as $n \to \infty$. By the Ekeland's variational principle, we can obtain that there is a sequence $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ with $\|u_n\| < \rho$ such that as $n \to \infty$,

$$I_\mu(u_n) \to d_{0,\mu} \qquad \text{and} \qquad I'_\mu(u_n) \to 0.$$

Then we can prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Using Lemma 2.6, we obtain that $(u_n)_{n\in\mathbb{N}}$ contains a convergent subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$. Noticing the fact that if $(v_n)_{n\in\mathbb{N}}$ is a minimizing sequence, then $(|v_n|)_{n\in\mathbb{N}}$ is also a minimizing sequence, we may assume that for each $n \in \mathbb{N}$, the $u_n \geq 0$ in $\mathbb{R}^3$. Therefore we may assume that $u_0 \geq 0$ in $\mathbb{R}^3$. The $I'_\mu(u_n) \to 0$ and $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$ imply that $I'_\mu(u_0) = 0$. Hence choosing $w_{0,\mu} \equiv u_0$, we know that $w_{0,\mu}$ is a nonnegative solution of the (3). □

We emphasize that the above lemma does **NOT** mean that $w_{0,\mu}$ is a ground state of (3). But it does imply that $\mathcal{N} \neq \emptyset$ for any $\mu \in (\mu_1, \mu_1 + \delta_2)$. Now we are in a position to prove that the $c_{0,\mu}$ defined in (34) can be achieved.

Lemma 4.2. *For $\mu \in (\mu_1, \mu_1 + \delta_2)$, the $c_{0,\mu}$ is achieved by a nontrivial $v_{0,\mu} \in H^1(\mathbb{R}^3)$, which is a nontrivial critical point of I_μ and hence a solution of the (3).*

Proof. By Lemma 4.1, we know that $\mathcal{N} \neq \emptyset$ for $\mu \in (\mu_1, \mu_1 + \delta_2)$. Hence we have that $c_{0,\mu} < 0$. Next we prove that the $c_{0,\mu} > -\infty$.

For any $u \in \mathcal{N}$, since $I'_\mu(u) = 0$, then $\langle I'_\mu(u), u\rangle = 0$. Then we can deduce that

$$I_\mu(u) = I_\mu(u) - \frac{1}{4}\langle I'_\mu(u), u\rangle \geq \frac{1}{4}\|u\|^2 - D(p,h)\mu^{\frac{p+1}{p-1}}.$$

Therefore the $c_{0,\mu} > -\infty$.

Now let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ be a sequence such that

$$I_\mu(u_n) \to c_{0,\mu} \qquad \text{and} \qquad I'_\mu(u_n) = 0.$$

Since $-\infty < c_{0,\mu} < 0$, we know from Lemma 2.6 that $(u_n)_{n\in\mathbb{N}}$ contains a convergent subsequence in $H^1(\mathbb{R}^3)$ and then we may assume without loss of generality that $u_n \to v_0$ strongly in $H^1(\mathbb{R}^3)$. Therefore we have that $I_\mu(v_0) = c_{0,\mu}$ and $I'_\mu(v_0) = 0$. Choosing $v_{0,\mu} \equiv v_0$ and we finish the proof of the Lemma 4.2. □

Next, to analyze further the $(PS)_d$ condition of the functional I_μ, we have to prove a relation between the minimizer $w_{0,\mu}$ obtained in Lemma 4.1 and the minimizer $v_{0,\mu}$ obtained in Lemma 4.2.

Lemma 4.3. *There exists $\delta_3 \in (0, \delta_2]$ such that for any $\mu \in (\mu_1, \mu_1 + \delta_3)$, the $v_{0,\mu}$ obtained in Lemma 4.2 can be chosen to coincide the $w_{0,\mu}$ obtained in Lemma 4.1.*

Proof. The proof is divided into two steps. In the first place, for $u \neq 0$ and $I'_{\mu_1}(u) = 0$, we have that

$$\|u\|^2 - \mu_1 \int_{\mathbb{R}^3} h(x)u^2 dx + F(u) = \int_{\mathbb{R}^3} |u|^{p+1} dx$$

and hence

$$I_{\mu_1}(u) = \frac{p-1}{2(p+1)} \left(\|u\|^2 - \mu_1 \int_{\mathbb{R}^3} h(x)u^2 dx \right) + \frac{p-3}{4(p+1)} F(u).$$

Since $\|u\|^2 \geq \mu_1 \int_{\mathbb{R}^3} h(x)u^2 dx$ for any $u \in H^1(\mathbb{R}^3)$, we obtain that

$$I_{\mu_1}(u) \geq \frac{p-3}{4(p+1)} F(u) > 0.$$

In the second place, denoted by $u_{0,\mu}$ a ground state obtained in Lemma 4.2. For any sequence $\mu^{(n)} > \mu_1$ and $\mu^{(n)} \to \mu_1$ as $n \to \infty$, we have that $u_{0,\mu^{(n)}}$ satisfies

$$I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0$$

and we also have that

$$c_{0,\mu^{(n)}} = I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) < 0.$$

Hence we deduce that $(u_{0,\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Since $I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0$, one also has that

$$\begin{aligned} I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) &= \frac{p-1}{2(p+1)} \left(\|u_{0,\mu^{(n)}}\|^2 - \mu^{(n)} \int_{\mathbb{R}^3} h(x)(u_{0,\mu^{(n)}})^2 dx \right) \\ &\quad + \frac{p-3}{4(p+1)} F(u_{0,\mu^{(n)}}). \end{aligned}$$

Using the definition of μ_1, we obtain that, as $n \to \infty$,

$$\|u_{0,\mu^{(n)}}\|^2 - \mu^{(n)} \int_{\mathbb{R}^3} h(x)(u_{0,\mu^{(n)}})^2 dx \geq \left(1 - \frac{\mu^{(n)}}{\mu_1}\right) \|u_{0,\mu^{(n)}}\|^2 \to 0$$

because $(u_{0,\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Next since $(u_{0,\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, we may assume without loss of generality that $u_{0,\mu^{(n)}} \rightharpoonup \tilde{u}_0$ weakly in $H^1(\mathbb{R}^3)$.

Claim. *As $n \to \infty$, the $u_{0,\mu^{(n)}} \to \tilde{u}_0$ strongly in $H^1(\mathbb{R}^3)$ and $\tilde{u}_0 = 0$.*

Proof of the Claim. From $u_{0,\mu^{(n)}} \rightharpoonup \tilde{u}_0$ weakly in $H^1(\mathbb{R}^3)$, we may assume that $u_{0,\mu^{(n)}} \to \tilde{u}_0$ a. e. in $\mathbb{R}^3$. Using these and the fact of $I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0$, we deduce that $I'_{\mu_1}(\tilde{u}_0) = 0$. Then similar to the proof in Lemma 2.6, we obtain that

$$\begin{aligned} o(1) + I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) &= I_{\mu^{(n)}}(\tilde{u}_0) + \frac{1}{2}\|\tilde{w}_n\|^2 \\ &\quad + \frac{1}{4}F(\tilde{w}_n) - \frac{1}{p+1}\int_{\mathbb{R}^3} |\tilde{w}_n|^{p+1} dx, \end{aligned} \tag{35}$$

where $\tilde{w}_n := u_{0,\mu^{(n)}} - \tilde{u}_0$.

Now we distinguish two cases:

(i): $\int_{\mathbb{R}^3} |\tilde{w}_n|^{p+1}dx \not\to 0$ as $n \to \infty$;

(ii): $\int_{\mathbb{R}^3} |\tilde{w}_n|^{p+1}dx \to 0$ as $n \to \infty$.

Suppose that the case (i) occurs. We may deduce from a proof similar to Lemma 2.6 that

$$I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) + o(1) \geq I_{\mu_1}(\tilde{u}_0) + \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}},$$

which is a contradiction because $I_{\mu_1}(\tilde{u}_0) > -\frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}$ by Lemma 2.5 and the fact of $I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) < 0$. Therefore the case (ii) occurs, which implies that $u_{0,\mu^{(n)}} \to \tilde{u}_0$ strongly in $H^1(\mathbb{R}^3)$ (the proof is similar to those in Lemma 2.6). From this we also deduce that $F(\tilde{w}_n) \to F(\tilde{u}_0)$.

Next we prove that $\tilde{u}_0 = 0$. Arguing by a contradiction, if $\tilde{u}_0 \neq 0$, then we know from $I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0$ that

$$\liminf_{n\to\infty} I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) \geq \frac{p-3}{4(p+1)} F(\tilde{u}_0) > 0,$$

which is also a contradiction since $I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) < 0$. Therefore $\tilde{u}_0 = 0$.

Hence there is $\delta_3 \in (0, \delta_2]$ such that for any $\mu \in (\mu_1, \mu_1 + \delta_3)$, $\|u_{0,\mu}\| < \rho$, which implies that $c_{0,\mu} = d_{0,\mu}$. Using Lemma 4.1, we can get a nonnegative ground state of (3), called $w_{0,\mu}$ and $c_{0,\mu} = d_{0,\mu} = I_\mu(w_{0,\mu})$. The proof is complete. □

Remark 4.4. The proof of Lemma 4.3 implies that (1) of Theorem 1.2 holds.

In the following, we are going to prove the existence of another nonnegative bound state solution of (3). To obtain this goal, we have to analyze further the $(PS)_d$ condition of the functional I_μ.

Lemma 4.5. *Under the assumptions of $(A1)-(A4)$, if $\mu \in (\mu_1, \mu_1 + \delta_3)$, then I_μ satisfies $(PS)_d$ condition for any $d < c_{0,\mu} + \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}$.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be a $(PS)_d$ sequence of I_μ with $d < c_{0,\mu} + \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}$. Then we have that for n large enough,

$$d + o(1) = \frac{1}{2}\|u_n\|^2 - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)u_n^2dx + \frac{1}{4}F(u_n) - \frac{1}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1}dx$$

and

$$\langle I'_\mu(u_n), u_n\rangle = \|u_n\|^2 - \mu\int_{\mathbb{R}^3} h(x)u_n^2dx + F(u_n) - \int_{\mathbb{R}^3} |u_n|^{p+1}dx.$$

Similar to the proof in Lemma 2.3, we can deduce that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, we may assume that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u_0$ a. e. in $\mathbb{R}^3$. Denote $w_n := u_n - u_0$. We then obtain from Brezis-Lieb lemma and Lemma 2.4 that for n large enough,

$$\|u_n\|^2 = \|u_0\|^2 + \|w_n\|^2 + o(1),$$

$$F(u_n) = F(u_0) + F(w_n) + o(1)$$

and

$$\|u_n\|_{L^{p+1}}^{p+1} = \|u_0\|_{L^{p+1}}^{p+1} + \|w_n\|_{L^{p+1}}^{p+1} + o(1).$$

Using Lemma 2.1, we also have that $\int_{\mathbb{R}^3} h(x)u_n^2dx \to \int_{\mathbb{R}^3} h(x)u_0^2dx$ as $n \to \infty$. Therefore we deduce that

$$d + o(1) = I_\mu(u_n) = I_\mu(u_0) + \frac{1}{2}\|w_n\|^2 + \frac{1}{4}F(w_n) - \frac{1}{p+1}\int_{\mathbb{R}^3} |w_n|^{p+1}dx. \tag{36}$$

Since $\langle I'_\mu(u_n), \psi\rangle \to 0$ for any $\psi \in H^1(\mathbb{R}^3)$, we know that $I'_\mu(u_0) = 0$. Moreover we have that

$$I_\mu(u_0) \geq c_{0,\mu}$$

and

$$\|u_0\|^2 - \mu\int_{\mathbb{R}^3} h(x)u_0^2dx + \int_{\mathbb{R}^3} \phi_{u_0}u_0^2 = \int_{\mathbb{R}^3} |u_0|^{p+1}dx.$$

Note that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. The Brezis-Lieb lemma, Lemma 2.4 and

$$o(1) = \|u_n\|^2 - \mu\int_{\mathbb{R}^3} h(x)u_n^2dx + F(u_n) - \int_{\mathbb{R}^3} |u_n|^{p+1}dx$$

imply that

$$o(1) = \|w_n\|^2 + F(w_n) - \int_{\mathbb{R}^3} |w_n|^{p+1}dx. \tag{37}$$

Using $\|u\|^2 \geq S_{p+1}\|u\|^2_{L^{p+1}}$ for any $u \in H^1(\mathbb{R}^3)$, we distinguish two cases:

(I): $\displaystyle\int_{\mathbb{R}^3} |w_n|^{p+1}dx \not\to 0$ as $n \to \infty$;

(II): $\displaystyle\int_{\mathbb{R}^3} |w_n|^{p+1}dx \to 0$ as $n \to \infty$.

Suppose (I) occurs. Up to a subsequence, we may obtain from (37) that

$$\|w_n\|^2 \geq S_{p+1}\left(\|w_n\|^2 + F(w_n) - o(1)\right)^{\frac{2}{p+1}}.$$

Hence we get that for n large enough,

$$\|w_n\|^2 \geq S_{p+1}^{\frac{p+1}{p-1}} + o(1). \tag{38}$$

Therefore using (36) and (38), we deduce that for n large enough,

$$\begin{aligned} d \quad &+ o(1) = I_\mu(u_n) \\ &= I_\mu(u_0) + \frac{1}{2}\|w_n\|^2 + \frac{1}{4}F(w_n) - \frac{1}{p+1}\int_{\mathbb{R}^3} |w_n|^{p+1}dx \\ &= I_\mu(u_0) + \frac{p-1}{2(p+1)}\|w_n\|^2 + \frac{p-3}{4(p+1)}F(w_n) \\ &\geq c_{0,\mu} + \frac{p-1}{2(p+1)}\|w_n\|^2 + \frac{p-3}{4(p+1)}F(w_n) \\ &> c_{0,\mu} + \frac{p-1}{2(p+1)}S_{p+1}^{\frac{p+1}{p-1}}, \end{aligned} \tag{39}$$

which contradicts to the assumption $d < c_{0,\mu} + \frac{p-1}{2(p+1)}S_{p+1}^{\frac{p+1}{p-1}}$. Therefore the case (II) must occur, i.e., $\int_{\mathbb{R}^3} |w_n|^{p+1}dx \to 0$ as $n \to \infty$. This and (37) imply that $\|w_n\| \to 0$. Hence we have proven that I_μ satisfies $(PS)_d$ condition for any $d < c_{0,\mu} + \frac{p-1}{2(p+1)}S_{p+1}^{\frac{p+1}{p-1}}$. □

Next, for the $w_{0,\mu}$ obtained in Lemma 4.3, we define

$$d_{2,\mu} = \inf_{\gamma\in\Gamma_2} \sup_{t\in[0,1]} I_\mu(\gamma(t))$$

with

$$\Gamma_2 = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) \ : \ \gamma(0) = w_{0,\mu},\ I_\mu(\gamma(1)) < c_{0,\mu}\}.$$

Lemma 4.6. *Suppose that the conditions* $(A1)-(A4)$ *hold and* $0 < b < a < 1$*. If* $\mu \in (\mu_1, \mu_1 + \delta_3)$*, then*

$$d_{2,\mu} < c_{0,\mu} + \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}.$$

Proof. It suffices to find a path starting from $w_{0,\mu}$ and the maximum of the energy functional over this path is strictly less than $c_{0,\mu} + \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}$. To simplify the notation, we denote $w_0 := w_{0,\mu}$, which corresponds to the critical value $c_{0,\mu}$. We will prove that there is a T_0 such that the path $\gamma(t) = w_0 + tT_0U_R$ is what we need, here $U_R(x) \equiv U(x - R\theta)$ is defined as before. Similar to the discussion in the proof of Lemma 3.2, we only need to estimate $I_\mu(w_0 + tU_R)$ for positive t in a finite interval. By direct calculation, we have that

$$\begin{aligned} I_\mu(w_0+tU_R) &= \frac{1}{2}\left(\|w_0+tU_R\|^2 - \mu\int_{\mathbb{R}^3} h(x)|w_0+tU_R|^2dx\right)\\ &\quad +\frac{1}{4}F(w_0+tU_R) - \frac{1}{p+1}\int_{\mathbb{R}^3}|w_0+tU_R|^{p+1}dx\\ &= I_\mu(w_0) + A_1 + A_2 + A_3 + \frac{t^2}{2}\|U_R\|^2 - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)U_R^2dx, \end{aligned}$$

where

$$A_1 = \langle w_0, tU_R\rangle - \mu t\int_{\mathbb{R}^3} h(x)w_0U_Rdx,$$

$$A_2 = \frac{1}{4}\left(F(w_0+tU_R) - F(w_0)\right)$$

and

$$A_3 = \frac{1}{p+1}\int_{\mathbb{R}^3}\left(|w_0|^{p+1} - |w_0+tU_R|^{p+1}\right)dx.$$

Since w_0 is a solution of (3), we have that

$$A_1 = \int_{\mathbb{R}^3}(w_0)^p tU_Rdx - \int_{\mathbb{R}^3} K(x)\phi_{w_0}w_0tU_Rdx.$$

From an elementary inequality:

$$(a+b)^q - a^q \ge b^q + qa^{q-1}b, \qquad q>1,\quad a>0, b>0,$$

we deduce that

$$|A_3| \le -\frac{1}{p+1}\int_{\mathbb{R}^3}|tU_R|^{p+1}dx - \int_{\mathbb{R}^3}|w_0|^p tU_Rdx.$$

For the estimate of A_2, using the expression of $F(u) = \int_{\mathbb{R}^3} K(x)\phi_u u^2dx$ and the symmetry property of the integral with respect to x and y, we can obtain that

$$\begin{aligned} |A_2| &\le t\int_{\mathbb{R}^3} K(x)\phi_{w_0}w_0U_Rdx + \frac{t^2}{2}\int_{\mathbb{R}^3} K(x)\phi_{w_0}(U_R)^2dx\\ &\quad +\frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{U_R}(U_R)^2dx + t^3\int_{\mathbb{R}^3} K(x)\phi_{U_R}w_0U_Rdx\\ &\quad +t^2\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{K(x)K(y)w_0(x)w_0(y)U_R(x)U_R(y)}{|x-y|}dxdy. \end{aligned}$$

Since w_0 is a nonnegative solution of (3) and $w_0 \in L^\infty(\mathbb{R}^3)$, we obtain from the assumption on $K(x)$ that

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{K(x)K(y)w_0(x)w_0(y)U_R(x)U_R(y)}{|x-y|}dxdy\\
&= \int_{\mathbb{R}^3} K(x)\phi_{\sqrt{w_0U_R}} w_0 U_R dx\\
&\le \|\phi_{\sqrt{w_0U_R}}\|_{L^6}\left(\int_{\mathbb{R}^3} K(x)^{\frac{6}{5}}(w_0U_R)^{\frac{6}{5}}dx\right)^{\frac{5}{6}}\\
&\le C\left(\int_{\mathbb{R}^3} e^{-\frac{6}{5}aR}e^{\left(\frac{6}{5}a-\frac{6}{5}(1-\delta)\right)|x|}dx\right)^{\frac{5}{6}}\\
&\le Ce^{-aR} \quad \text{since} \quad 0<a<1.
\end{aligned}$$

Similarly we can deduce that for R large enough,

$$\int_{\mathbb{R}^3} K(x)\phi_{w_0}w_0U_Rdx \le Ce^{-aR}, \quad \int_{\mathbb{R}^3} K(x)\phi_{w_0}(U_R)^2dx \le Ce^{-aR},$$

$$\int_{\mathbb{R}^3} K(x)\phi_{U_R}(U_R)^2dx \le Ce^{-aR} \ \text{ and } \ \int_{\mathbb{R}^3} K(x)\phi_{U_R}w_0U_Rdx \le Ce^{-aR}.$$

Since $\int_{\mathbb{R}^3} h(x)(U_R)^2dx \ge Ce^{-bR}$ for R large enough, we obtain that

$$\begin{aligned}
I_\mu(w_0+tU_R) &\le I_\mu(w_0) + \frac{t^2}{2}\|U_R\|^2dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)U_R^2dx\\
&\quad -\frac{1}{p+1}\int_{\mathbb{R}^3}|tU_R|^{p+1}dx + Ce^{-aR}\\
&\le I_\mu(w_0) + \frac{p-1}{2(p+1)}S_{p+1}^{\frac{p+1}{p-1}} + Ce^{-aR} - Ce^{-bR} + o(e^{-bR})\\
&< c_{0,\mu} + \frac{p-1}{2(p+1)}S_{p+1}^{\frac{p+1}{p-1}}
\end{aligned}$$

for R large enough since $0<b<a<1$. The proof is complete. □

Proposition 4.7. *Under the conditions (A1)-(A4), if $\mu \in (\mu_1, \mu_1+\delta_3)$ and $w_{0,\mu}$ be the minimizer obtained in Lemma 4.3, then the $d_{2,\mu}$ is a critical value of I_μ.*

Proof. Since for $\mu \in (\mu_1, \mu_1+\delta_3)$, we know from Lemma 4.1 and Lemma 4.3 that the $w_{0,\mu}$ is a local minimizer of I_μ. Moreover, one has that $I_\mu(w_{0,\mu}+sU_R) \to -\infty$ as $s \to +\infty$. Therefore Lemma 4.5, Lemma 4.7 and the mountain pass theorem of [3] imply that $d_{2,\mu}$ is a critical value of I_μ. □

Proof of Theorem 1.2. The conclusion (1) of Theorem 1.2 follows from Lemma 4.3 and Remark 4.4. It remains to prove (2) of Theorem 1.2. By Proposition 4.7, the $d_{2,\mu}$ is a critical value of I_μ and $d_{2,\mu} > 0$. The proof of nonnegativity for at least one of the corresponding critical point is inspired by the idea of [1]. In fact, since $I_\mu(u) = I_\mu(|u|)$ for any $u \in H^1(\mathbb{R}^3)$, for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma_2$ with $\gamma_n(t) \ge 0$ (a.e. in $\mathbb{R}^3$) for all $t \in [0,1]$ such that

$$d_{2,\mu} \le \max_{t\in[0,1]} I_\mu(\gamma_n(t)) < d_{2,\mu} + \frac{1}{n}. \tag{40}$$

By Ekeland's variational principle, there exists $\gamma_n^* \in \Gamma_2$ satisfying

$$\begin{cases} d_{2,\mu} \le \max_{t\in[0,1]} I_\mu(\gamma_n^*(t)) \le \max_{t\in[0,1]} I_\mu(\gamma_n(t)) < d_{2,\mu} + \frac{1}{n}; \\ \max_{t\in[0,1]} \|\gamma_n(t) - \gamma_n^*(t)\| < \frac{1}{\sqrt{n}}; \\ \text{there exists } t_n \in [0,1] \text{ such that } z_n := \gamma_n^*(t_n) \text{ satisfies :} \\ I_\mu(z_n) = \max_{t\in[0,1]} I_\mu(\gamma_n^*(t)), \text{ and } \|I'_\mu(z_n)\| \le \frac{1}{\sqrt{n}}. \end{cases} \tag{41}$$

By Lemma 4.6 we get a convergent subsequence (still denoted by $(z_n)_{n\in\mathbb{N}}$). We may assume that $z_n \to z$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$. On the other hand, by (41), we also arrive at $\gamma_n(t_n) \to z$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Since $\gamma_n(t) \ge 0$, we conclude that $z \ge 0$, $z \not\equiv 0$ in $\mathbb{R}^3$ with $I_\mu(z) > 0$ and it is a nonnegative solution of problem (3).

Next, let $u_{2,\mu}$ be the nonnegative solution given by the above proof, that is, $I'_\mu(u_{2,\mu}) = 0$ and $I_\mu(u_{2,\mu}) = d_{2,\mu}$. We claim that for any sequence $\mu^{(n)} > \mu_1$ and $\mu^{(n)} \to \mu_1$, there exist a sequence of solution $u_{2,\mu^{(n)}}$ of (3) with $\mu = \mu^{(n)}$ and a u_{μ_1} with $I'_{\mu_1}(u_{\mu_1}) = 0$ such that $u_{2,\mu^{(n)}} \to u_{\mu_1}$ strongly in $H^1(\mathbb{R}^3)$. In fact, denoted by $w_{0,\mu^{(n)}}$ the minimizer corresponding to $d_{0,\mu^{(n)}}$, according to the definition of $d_{2,\mu}$ and the proof of Lemma 4.6, we deduce that for n large enough,

$$0 < \alpha \le d_{2,\mu^{(n)}} \le \max_{s>0} I_{\mu^{(n)}}(w_{0,\mu^{(n)}} + sU_R)$$

and

$$I_{\mu^{(n)}}(w_{0,\mu^{(n)}} + sU_R) \le \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}} + Ce^{-aR} - Ce^{-bR} + o(e^{-bR}),$$

$$\limsup_{n\to\infty} d_{2,\mu^{(n)}} \le \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}}. \tag{42}$$

Next, similar to the proof in Lemma 2.3, we can deduce that $(u_{2,\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, we may assume that $u_{2,\mu^{(n)}} \rightharpoonup \tilde{u}_2$ weakly in $H^1(\mathbb{R}^3)$ and $u_{2,\mu^{(n)}} \to \tilde{u}_2$ a. e. in $\mathbb{R}^3$. Then we have that $I'_{\mu_1}(\tilde{u}_2) = 0$. Moreover $I_{\mu_1}(\tilde{u}_2) \ge 0$. If $(u_{2,\mu^{(n)}})_{n\in\mathbb{N}}$ does not converge strongly to $\tilde{u}_2$ in $H^1(\mathbb{R}^3)$, then using an argument similar to the proof of Lemma 4.5, we may deduce that

$$I_{\mu^{(n)}}(u_{2,\mu^{(n)}}) \ge I_{\mu_1}(\tilde{u}_2) + \frac{p-1}{2(p+1)} S_{p+1}^{\frac{p+1}{p-1}},$$

which contradicts to (42). Hence $u_{2,\mu^{(n)}} \to \tilde{u}_2$ strongly in $H^1(\mathbb{R}^3)$ and hence $I_{\mu_1}(\tilde{u}_2) > 0$. The proof is complete by choosing $u_{\mu_1} = \tilde{u}_2$.

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