ELECTRONIC RESEARCH ARCHIVE
Volume 28 Number 1, March 2020 Pages 311–326
eISSN: 2688-1594 AIMS (2020)
doi:10.3934/era.2020018

LONG-TIME BEHAVIOR OF A CLASS OF VISCOELASTIC PLATE EQUATIONS

YANG LIU*

Abstract. This paper is concerned with the initial-boundary value problem for a class of viscoelastic plate equations on an arbitrary dimensional bounded domain. Under certain assumptions on the memory kernel and the source term, the global well-posedness of solutions and the existence of global attractors are obtained.

1. Introduction

In this paper, we study the following initial-boundary value problem for nonlinear viscoelastic plate equations

$$u_{tt} + \Delta^2 u - \int_{-\infty}^{t} g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \Delta u_t = f(u) + h(x), \quad x \in \Omega,\ t > 0, \tag{1}$$

$$u(x,t) = u_0(x,t),\ u_t(x,0) = u_1(x), \quad x \in \Omega,\ t \le 0, \tag{2}$$

$$u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega,\ t \in \mathbb{R}, \tag{3}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, the memory kernel g and the external forces f, h will be specified later.

Problem 1-3 can be used to describe the vibrations of viscoelastic materials possessing a capacity of storage and dissipation of mechanical energy, see [17] for the details. And $u(x,t)$ represents the displacement at time t of a particle having position x in a given reference configuration with the prescribed past history $u_0 : \Omega \times (-\infty, 0] \to \mathbb{R}$. In view of the main results of [5], we see that the viscoelastic term (namely the memory term) produces the effect of strong dissipation, which prevails the effect of weak damping term on the decay of solutions in time.

There have been many works on the long-time behavior of viscoelastic plate equations, we refer the readers to [1, 2, 4, 6, 18, 20, 21, 25] and the references therein. As for viscoelastic plate equations with past history, Pata [23] studied

$$u_{tt} + \alpha A u - \int_0^{\infty} g(\tau) A u(t-\tau)\,\mathrm{d}\tau + \mu u_t = 0, \quad t > 0,$$
$$u(t) = u_0(t),\ u_t(0) = u_1, \quad t \le 0,$$

Received by the editors December, 2019.

2010 *Mathematics Subject Classification.* Primary: 35A01, 35A02, 35B30, 35B41, 35L35; Secondary: 74D99.

Key words and phrases. Viscoelastic plate equations, global well-posedness, absorbing sets, global attractors, past history approach.

The author is supported by the Fundamental Research Funds for the Central Universities (Grant No. 31920190090).

* Corresponding author: Yang Liu.

where A is a self-adjoint and strictly positive linear operator, α and μ are positive constants. Based on certain assumptions on g, he analyzed the exponential stability of the related semigroup. Guesmia and Messaoudi [13] investigated

$$u_{tt} + Au - \int_0^\infty g(\tau)Au(t-\tau)\,\mathrm{d}\tau = 0, \quad t > 0,$$

$$u(-t) = u_0(t),\ u_t(0) = u_1, \quad t \geq 0.$$

Under some assumptions on A and g, they established a general decay result which depends on the behavior of g. Jorge Silva and Ma [15] considered

$$\begin{aligned} u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^t & g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau \\ & - \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \Delta u_t + f(u) = h(x), \quad x \in \Omega,\ t > 0, \end{aligned}$$

$$u(x,t) = u_0(x,t),\ u_t(x,t) = \partial_t u_0(x,t), \quad x \in \Omega,\ t \leq 0,$$

$$u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega,\ t \in \mathbb{R},$$

where $h \in L^2(\Omega)$, and Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$. Under some assumptions on f and g, they obtained the global well-posedness and regularity of weak solutions. Moreover, they proved the exponential decay of energy. Recently, Conti and Geredeli [9] studied

$$u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau + f_1(u_t) + f_2(u) = h(x), \quad x \in \Omega,\ t > 0,$$

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,-t) = \phi_0(x,t), \quad x \in \Omega,\ t > 0, \end{cases}$$

$$u(x,t) = \frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0,$$

where $h \in L^2(\Omega)$, Ω is a bounded domain of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Under some assumptions on f_1, f_2 and g, they obtained the existence and regularity of global attractors.

In the works mentioned above, authors introduced a variable which reflects the relative displacement history so that the corresponding problem could be turned into an autonomous system. This scheme is so-called the past history approach [12] which suggests to consider some past history variables as additional components of the phase space corresponding to the equation under study.

In the present paper, in order to study the long-time behaviour of solutions of problem 1-3, we employ the past history approach and the operator technique so that Eq. 1 can be transformed into an abstract system in the history phase space. And thus the operator technique combined with the energy estimates becomes a crucial tool for the proof of the existence of global attractors.

This paper is organized as follows. In Section 2 some notations and assumptions on f and g are displayed. Moreover, 1-3 is transformed into a generalized problem, and the main results of this paper are stated. In Section 3 the global well-posedness of regular solutions is obtained. And the global well-posedness of weak solutions is established by the density arguments [6]. In Section 4 the existence of global attractors is derived by means of the existence of an absorbing set and the semigroup decomposition [14, 16, 26].

2. Preliminaries and main results

2.1. **Notations and assumptions.** Throughout the paper, in order to simplify the notations, we denote

$$\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)},\ \|\cdot\| := \|\cdot\|_{L^2(\Omega)}.$$

$(\cdot,\cdot)$ denotes either the L^2-inner product or a duality pairing between a space and its dual space. Moreover, $|\Omega|$ stands for the Lebesgue measure of Ω, C_i, $i = 1, 2, 3, \cdots$ denote some different positive constants, and $\mathfrak{C}_i$, $i = 1, 2, 3$ represent the positive constants for inequalities

$$\|u\| \leq \mathfrak{C}_1\|\nabla u\|,\ \|u\| \leq \mathfrak{C}_2\|\Delta u\|,\ \|\nabla u\| \leq \mathfrak{C}_3\|\Delta u\|.$$

As in [3, 10, 27], we give the following assumptions on f in order to state the main results of this paper.

($\mathbb{A}_1$): $f(0) = 0$, and there exists a constant $b > 0$ such that

$$|f(u) - f(v)| \leq b(|u|^{p-2} + |v|^{p-2})|u - v|, \quad \forall u, v \in \mathbb{R},$$

where

$$2 \leq p < \infty \text{ if } N \leq 4,\ 2 \leq p \leq \frac{2N-4}{N-4} \text{ if } N > 4.$$

Moreover,

$$\limsup_{|u|\to\infty} \frac{F(u)}{|u|^2} \leq 0, \tag{4}$$

and

$$\limsup_{|u|\to\infty} \frac{uf(u) - \varrho F(u)}{|u|^2} \leq 0, \tag{5}$$

where $0 < \varrho < 1$ and

$$F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

In addition, as in [7, 19, 22], we assume that g satisfies the following conditions.

($\mathbb{A}_2$): $g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $g(t) \geq 0$, $g'(t) \leq 0$, $t \in [0, \infty)$, and

$$\kappa := 1 - \int_0^\infty g(t)\,\mathrm{d}t > 0. \tag{6}$$

2.2. **Reformulation of the problem.** As in [2, 3, 8, 10, 27], we define the operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$

$$Au = \Delta^2 u, \quad \forall u \in D(A),$$

where the dense domain

$$D(A) = \{u \in H^4(\Omega) \cap H_0^1(\Omega) | \Delta u \in H^2(\Omega) \cap H_0^1(\Omega)\}.$$

It is easy to verify that A is self-adjoint and strictly positive. Thus A^γ is also self-adjoint and strictly positive for any $\gamma > 0$. Denote $V_{4\gamma} := D(A^\gamma)$ and $V_{-\gamma} := V'_\gamma$. Then, for any $\gamma \in \mathbb{R}$, V_γ and $\mathcal{L}_{g,\gamma}$ are Hilbert spaces equipped with inner products and norms

$$(u, w)_{V_\gamma} = (A^{\frac{\gamma}{4}}u, A^{\frac{\gamma}{4}}w),\ \|u\|_{V_\gamma} = \|A^{\frac{\gamma}{4}}u\|,$$

$$(v, w)_{g,\gamma} = \int_0^\infty g(\tau)(v(\tau), w(\tau))_{V_\gamma}\,\mathrm{d}\tau,\ \|v\|_{g,\gamma}^2 = \int_0^\infty g(\tau)\|v(\tau)\|_{V_\gamma}^2\,\mathrm{d}\tau,$$

where

$$\mathcal{L}_{g,\gamma} := L^2_g(\mathbb{R}^+; V_\gamma) = \left\{ v : \mathbb{R}^+ \to V_\gamma \,\middle|\, \int_0^\infty g(\tau)\|v(\tau)\|^2_{V_\gamma}\,\mathrm{d}\tau < \infty \right\}.$$

Thus

$$V_3 := H_3(\Omega) = \{u \in H^3(\Omega) \cap H^1_0(\Omega) | \Delta u \in H^1_0(\Omega)\},$$

$$V_2 := H^2(\Omega) \cap H^1_0(\Omega),\ V_1 := H^1_0(\Omega),\ V_0 := L^2(\Omega).$$

In this way, problem 1-3 can be seen as

$$u_{tt} + Au - \int_{-\infty}^t g(t-\tau)Au(\tau)\,\mathrm{d}\tau + A^{\frac{1}{2}}u_t = f(u) + h, \quad t > 0, \tag{7}$$

$$u(t) = u_0(t),\ u_t(0) = u_1, \quad t \le 0.$$

Now we are in a position to define the auxiliary function

$$v^t(\tau) = u(t) - u(t-\tau), \quad \tau > 0,\ t \ge 0.$$

Thus the viscoelastic dissipation in 7 can be rewritten as

$$\begin{aligned} -\int_{-\infty}^t g(t-\tau)Au(\tau)\,\mathrm{d}\tau &= -\int_0^\infty g(\tau)Au(t-\tau)\,\mathrm{d}\tau \\ &= -(1-\kappa)Au + \int_0^\infty g(\tau)Av^t(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Therefore, problem 1-3 is transformed into the following system

$$\begin{cases} u_{tt} + \kappa Au + \displaystyle\int_0^\infty g(\tau)Av^t(\tau)\,\mathrm{d}\tau + A^{\frac{1}{2}}u_t = f(u) + h, \quad t > 0, \\ v^t_t(\tau) = u_t(t) - v^t_\tau(\tau), \quad \tau > 0,\ t > 0, \end{cases} \tag{8}$$

with

$$\begin{cases} u(0) = u_0,\ u_t(0) = u_1, \\ v^0(\tau) = v_0(\tau), \end{cases} \tag{9}$$

where

$$\begin{aligned} &u_0 = u_0(0), \\ &v_0(\tau) = u_0(0) - u_0(-\tau), \quad \tau > 0. \end{aligned}$$

Definition 2.1. $(u(t), u_t(t), v^t)$ is called a weak solution of problem 8, 9 if $u \in C([0,T]; V_2)$, $u_t \in C([0,T]; V_0)$, $v^t \in C([0,T]; \mathcal{L}_{g,2})$, $u(0) = u_0$ in V_2, $u_t(0) = u_1$ in V_0, $v^0 = v_0$ in $\mathcal{L}_{g,2}$, and

$$\begin{cases} (u_t, w_1) + \kappa \displaystyle\int_0^t (A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau + \int_0^t (v^s, w_1)_{g,2}\,\mathrm{d}s \\ \quad +(A^{\frac{1}{4}}u, A^{\frac{1}{4}}w_1) = \displaystyle\int_0^t (f(u) + h, w_1)\,\mathrm{d}\tau + (u_1, w_1) + (A^{\frac{1}{4}}u_0, A^{\frac{1}{4}}w_1), \\ (v^t, w_2)_{g,2} = (u, w_2)_{g,2} - (u_0, w_2)_{g,2} - \displaystyle\int_0^t (v^s_\tau, w_2)_{g,2}\,\mathrm{d}s + (v_0, w_2)_{g,2}. \end{cases}$$

for any $w_1 \in V_2$, $w_2 \in \mathcal{L}_{g,2}$ and a.e. $t \in (0,T]$.

Thus, in order to deal with problem 1-3, we study the modified problem 8, 9. In fact, for a solution (u, u_t, v^t) of problem 8, 9, we have

$$(u_{tt}, w_1) + \kappa(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1) + (v^t, w_1)_{g,2} + (A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}w_1) = (f(u) + h, w_1).$$

In view of [17, Chapter 2, Section 4], we see that $g(t) := -G'(t)$, where $G(t)$ is the viscoelastic flexural rigidity. From

$$\begin{aligned}\kappa A^{\frac{1}{2}}u =& \left(1 + \int_0^\infty G'(\tau)\,\mathrm{d}\tau\right) A^{\frac{1}{2}}u \\ =& A^{\frac{1}{2}}u + \left(\lim_{\tau\to\infty} G(\tau) - G(0)\right) A^{\frac{1}{2}}u,\end{aligned}$$

it follows that

$$\begin{aligned}&(u_{tt}, w_1) + (A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1) + \left(\lim_{\tau\to\infty} G(\tau) - G(0)\right)(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1) \\ &\quad + (v^t, w_1)_{g,2} + (A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}w_1) = (f(u) + h, w_1).\end{aligned} \tag{10}$$

Since

$$\begin{aligned}&\left(\lim_{\tau\to\infty} G(\tau) - G(0)\right)(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1) + (v^t, w_1)_{g,2} \\ =&\left(\lim_{\tau\to\infty} G(\tau) - G(0)\right)(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1) + \int_0^t g(\tau)(A^{\frac{1}{2}}v^t(\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau \\ &+ \int_t^\infty g(\tau)(A^{\frac{1}{2}}v^t(\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau,\end{aligned}$$

and

$$v^t(\tau) = \begin{cases} u(t) - u_0(t-\tau), & \tau \geq t, \\ u(t) - u(t-\tau), & \tau < t, \end{cases}$$

we deduce that

$$\begin{aligned}&\left(\lim_{\tau\to\infty} G(\tau) - G(0)\right)(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1) + (v^t, w_1)_{g,2} \\ =&\left(\lim_{\tau\to\infty} G(\tau) - G(0)\right)(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1) - \int_0^t G'(\tau)(A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau \\ &+ \int_0^t G'(\tau)(A^{\frac{1}{2}}u(t-\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau - \int_t^\infty G'(\tau)(A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau \\ &+ \int_t^\infty G'(\tau)(A^{\frac{1}{2}}u_0(t-\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau \\ =&\int_0^t G'(\tau)(A^{\frac{1}{2}}u(t-\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau + \int_t^\infty G'(\tau)(A^{\frac{1}{2}}u_0(t-\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau.\end{aligned}$$

Substituting this into 10, we obtain

$$\begin{aligned}&(u_{tt}, w_1) + (A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1) + \int_0^t G'(\tau)(A^{\frac{1}{2}}u(t-\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau \\ &\quad + \int_t^\infty G'(\tau)(A^{\frac{1}{2}}u_0(t-\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau + (A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}w_1) = (f(u) + h, w_1).\end{aligned}$$

Due to

$$\int_0^t G'(\tau)(A^{\frac{1}{2}}u(t-\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau + \int_t^\infty G'(\tau)(A^{\frac{1}{2}}u_0(t-\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau$$
$$= -\int_{-\infty}^t g(t-\tau)(A^{\frac{1}{2}}u(\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau,$$

we conclude that

$$(u_{tt}, w_1) + (A^{\frac{1}{2}}u, A^{\frac{1}{2}}w_1) - \int_{-\infty}^t g(t-\tau)(A^{\frac{1}{2}}u(\tau), A^{\frac{1}{2}}w_1)\,\mathrm{d}\tau$$
$$+ (A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}w_1) = (f(u)+h, w_1),$$

which shows that (u, u_t) is a solution of problem 1-3.

2.3. **Statement of main results.** The main results of this paper are stated as follows.

Theorem 2.2. *Let* (A_1) *and* (A_2) *be fulfilled. Assume that* $h \in V_0$ *and* $(u_0, u_1, v_0) \in Z := V_2 \times V_0 \times \mathcal{L}_{g,2}$. *Then problem 8, 9 admits a unique solution* $(u, u_t, v^t) \in C([0,\infty); Z)$ *depending continuously on initial data.*

Define the mapping $S(t) : Z \to Z$ by

$$S(t)(u_0, u_1, v_0) = (u(t), u_t(t), v^t).$$

Then it is easy to see from Theorem 2.2 that $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup generated by problem 8, 9.

Theorem 2.3. *Let* (A_1) *and* (A_2) *be fulfilled. And there exists a constant* $\rho > 0$ *such that* $g'(t) + \rho g(t) \leq 0$ *for all* $t \in [0,\infty)$. *Assume that* $h \in V_0$ *and* $(u_0, u_1, v_0) \in Z$. *Then* $S(t)$ *possesses a global attractor in* Z.

3. Proof of Theorem 2.2

Theorem 3.1. *Let* (A_1) *and* (A_2) *be fulfilled. Assume that* $h \in V_0$, $u_0 \in V_3$, $u_1 \in V_1$, $v_0 \in \mathcal{L}_{g,3}$. *Then problem 8, 9 admits a unique solution* $u \in L^\infty(0,\infty; V_3)$, $u_t \in L^\infty(0,\infty; V_1) \cap L^2(0,\infty; V_2)$, $v^t \in L^\infty(0,\infty; \mathcal{L}_{g,3})$, *which depends continuously on initial data.*

Proof. Let $\{\omega_j\}_{j=1}^\infty$ be an orthogonal basis of V_2 and an orthonormal basis of V_0 given by eigenfunctions of A. As in [11, 15], we select $\{e_j\}_{j=1}^\infty$ of the form $\{l_k\tilde{\omega}_j\}_{k,j=1}^\infty$, where $\{l_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2_g(\mathbb{R}^+) \cap C_0^\infty(\mathbb{R}^+)$ and $\tilde{\omega}_j = \dfrac{\omega_j}{\|\omega_j\|_{V_2}}$. Then $\{e_j\}_{j=1}^\infty$ is an orthonormal basis of $\mathcal{L}_{g,2}$.

We construct the approximate solutions of problem 8, 9

$$u_n(t) = \sum_{j=1}^n \xi_{jn}(t)\omega_j,\ \ v_n^t(\tau) = \sum_{j=1}^n \zeta_{jn}(t)e_j(\tau), \quad n = 1, 2, \cdots,$$

which satisfy

$$\begin{cases} (u_{ntt}, \omega_j) + \kappa(A^{\frac{1}{2}}u_n, A^{\frac{1}{2}}\omega_j) + (v_n^t, \omega_j)_{g,2} \\ \quad +(A^{\frac{1}{4}}u_{nt}, A^{\frac{1}{4}}\omega_j) = (f(u_n), \omega_j) + (h, \omega_j), \\ (v_{nt}^t, e_j)_{g,2} = (u_{nt}, e_j)_{g,2} - (v_{n\tau}^t, e_j)_{g,2}, \quad j = 1, 2, \cdots, n, \end{cases} \tag{11}$$

with

$$\begin{cases} u_n(0) = \displaystyle\sum_{j=1}^{n} \xi_{jn}(0)\omega_j \to u_0 \text{ in } V_3, \\ u_{nt}(0) = \displaystyle\sum_{j=1}^{n} \xi'_{jn}(0)\omega_j \to u_1 \text{ in } V_1, \\ v_n^0 = \displaystyle\sum_{j=1}^{n} \zeta_{jn}(0)e_j \to v_0 \text{ in } \mathcal{L}_{g,3}. \end{cases} \tag{12}$$

The approximate problem 11, 12 can be reduced to an ordinary differential system in the variables $\xi_{jn}(t)$ and $\zeta_{jn}(t)$. In terms of standard theory for ODEs, there exists a solution $(u_n(t), u_{nt}(t), v_n^t)$ on some interval $[0, T_n)$ with $T_n \le T$. The following estimates will allow us to extend the local solutions to $[0, T]$ with any $T > 0$.

Replacing ω_j in 11_1 with u_{nt} and e_j in 11_2 with v_n^t, summing for j and adding the two results, we obtain

$$E'_n(t) + \|A^{\frac{1}{4}} u_{nt}\|^2 = -(v_{n\tau}^t, v_n^t)_{g,2}, \tag{13}$$

where

$$E_n(t) = \frac{1}{2}\|u_{nt}\|^2 + \frac{\kappa}{2}\|A^{\frac{1}{2}} u_n\|^2 + \frac{1}{2}\|v_n^t\|_{g,2}^2 - \int_\Omega F(u_n)\,\mathrm{d}x - (h, u_n). \tag{14}$$

Since $\lim\limits_{\tau\to 0} v_n^t(\tau) = 0$, we deduce from (A_2) that

$$\begin{aligned}(v_{n\tau}^t, v_n^t)_{g,2} =& \frac{1}{2}\int_0^\infty \frac{\partial}{\partial\tau}\left(g(\tau)\|A^{\frac{1}{2}} v_n^t(\tau)\|^2\right) \mathrm{d}\tau - \frac{1}{2}\int_0^\infty g'(\tau)\|A^{\frac{1}{2}} v_n^t(\tau)\|^2 \,\mathrm{d}\tau \\ \ge& 0.\end{aligned}$$

Hence, by integrating 13 with respect to t from 0 to t, we get

$$E_n(t) + \int_0^t \|A^{\frac{1}{4}} u_{n\tau}\|^2 \,\mathrm{d}\tau \le E_n(0). \tag{15}$$

It follows from 4 in (A_1) that, for any $\eta > 0$, there exists a constant $C_\eta > 0$ such that

$$\int_\Omega F(u_n)\,\mathrm{d}x \le \eta\|u_n\|^2 + C_\eta|\Omega|.$$

By virtue of Cauchy's inequality with $\epsilon > 0$, we get

$$\begin{aligned}(h, u_n) \le& \|h\|\|u_n\| \\ \le& \epsilon\mathfrak{C}_2^2\|A^{\frac{1}{2}} u_n\|^2 + \frac{1}{4\epsilon}\|h\|^2.\end{aligned}$$

Consequently, taking sufficiently small η and ϵ such that

$$C_1 := \frac{\kappa}{2} - \eta\mathfrak{C}_2^2 - \epsilon\mathfrak{C}_2^2 > 0,$$

we deduce from 14 that

$$E_n(t) \ge \frac{1}{2}\|u_{nt}\|^2 + C_1\|A^{\frac{1}{2}} u_n\|^2 + \frac{1}{2}\|v_n^t\|_{g,2}^2 - C_2(\|h\|^2 + |\Omega|). \tag{16}$$

Hence, from 15, 16 and 12, it follows that

$$\|u_{nt}\|^2 + \|A^{\frac{1}{2}} u_n\|^2 + \|v_n^t\|_{g,2}^2 + \int_0^t \|A^{\frac{1}{4}} u_{n\tau}\|^2 \,\mathrm{d}\tau \le C_3. \tag{17}$$

Replacing ω_j in 11_1 with $A^{\frac{1}{2}}u_{nt}$ and e_j in 11_2 with $A^{\frac{1}{2}}v_n^t$, summing for j and adding the two results, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|A^{\frac{1}{4}}u_{nt}\|^2+\kappa\|A^{\frac{3}{4}}u_n\|^2+\|v_n^t\|_{g,3}^2\right)+\|A^{\frac{1}{2}}u_{nt}\|^2\\=&(f(u_n),A^{\frac{1}{2}}u_{nt})+(h,A^{\frac{1}{2}}u_{nt})-(v_{n\tau}^t,v_n^t)_{g,3}.\end{aligned}$$

Noting that

$$-(v_{n\tau}^t,v_n^t)_{g,3}\le 0,$$

$$(f(u_n),A^{\frac{1}{2}}u_{nt})\le C_4\|A^{\frac{1}{2}}u_n\|^{2p-2}+\frac{1}{4}\|A^{\frac{1}{2}}u_{nt}\|^2,$$

and

$$(h,A^{\frac{1}{2}}u_{nt})\le\|h\|^2+\frac{1}{4}\|A^{\frac{1}{2}}u_{nt}\|^2,$$

we conclude from 17 that

$$\|A^{\frac{1}{4}}u_{nt}\|^2+\|A^{\frac{3}{4}}u_n\|^2+\|v_n^t\|_{g,3}^2+\int_0^t\|A^{\frac{1}{2}}u_{n\tau}\|^2\,\mathrm{d}\tau\le C_5. \tag{18}$$

Therefore, there exist u, v^t and subsequences of $\{u_n\}$, $\{v_n^t\}$, still represented by the same notations and we shall not repeat, such that, as $n\to\infty$,

$$u_n\rightharpoonup u \text{ weakly star in } L^\infty(0,T;V_3), \tag{19}$$

$$u_{nt}\rightharpoonup u_t \text{ weakly star in } L^\infty(0,T;V_1) \text{ and weakly in } L^2(0,T;V_2), \tag{20}$$

$$v_n^t\rightharpoonup v^t \text{ weakly star in } L^\infty(0,T;\mathcal{L}_{g,3}),$$

for any $T>0$. According to the Aubin-Lions lemma, we have

$$u_n\to u \text{ in } L^2(0,T;V_2).$$

Moreover, from 18-20, it follows that

$$u_n\to u \text{ in } C([0,T];V_2). \tag{21}$$

We now claim that for any $t\in[0,T]$ and fixed j,

$$\int_0^t(f(u_n),\omega_j)\,\mathrm{d}\tau\to\int_0^t(f(u),\omega_j)\,\mathrm{d}\tau, \tag{22}$$

as $n\to\infty$.

Indeed, for any $w\in V_2$, we have

$$(f(u_n)-f(u),w)\le b((|u_n|^{p-2}+|u|^{p-2})|u_n-u|,w).$$

If $p>2$, then when $N\le 4$,

$$(f(u_n)-f(u),w)\le b\left(\|u_n\|_{4(p-2)}^{p-2}+\|u\|_{4(p-2)}^{p-2}\right)\|u_n-u\|_4\|w\|,$$

when $N>4$,

$$(f(u_n)-f(u),w)\le b\left(\|u_n\|_{\frac{N(p-2)}{3}}^{p-2}+\|u\|_{\frac{N(p-2)}{3}}^{p-2}\right)\|u_n-u\|_{\frac{2N}{N-2}}\|w\|_{\frac{2N}{N-4}}.$$

Hence

$$\begin{aligned}(f(u_n)-f(u),w)\le&C_6\left(\|u_n\|_{V_2}^{p-2}+\|u\|_{V_2}^{p-2}\right)\|u_n-u\|_{V_1}\|w\|_{V_2}\\\le&C_7\|u_n-u\|_{V_1}\|w\|_{V_2}.\end{aligned} \tag{23}$$

If $p = 2$, then it is clear that 23 remains valid. Therefore,

$$\left| \int_0^t (f(u_n) - f(u), w_j)\, \mathrm{d}\tau \right| \le C_8 \int_0^t \|u_n - u\|_{V_1}\, \mathrm{d}\tau.$$

Thus assertion 22 follows from 21.

For fixed j,

$$(v^t_{n\tau}, e_j)_{g,2} = -\int_0^\infty g'(\tau)(A^{\frac{1}{2}} v^t_n(\tau), A^{\frac{1}{2}} e_j(\tau))\, \mathrm{d}\tau - \int_0^\infty g(\tau)(A^{\frac{1}{2}} v^t_n(\tau), A^{\frac{1}{2}} e_{j\tau}(\tau))\, \mathrm{d}\tau.$$

Hence

$$\lim_{n\to\infty} (v^t_{n\tau}, e_j)_{g,2} = (v^t_\tau, e_j)_{g,2}.$$

Consequently, for fixed j, integrating 11 with respect to t and taking $n \to \infty$, we get

$$\begin{cases} (u_t, \omega_j) + \kappa \displaystyle\int_0^t (A^{\frac{1}{2}} u, A^{\frac{1}{2}} \omega_j)\, \mathrm{d}\tau + \int_0^t (v^s, \omega_j)_{g,2}\, \mathrm{d}s \\ \quad + (A^{\frac{1}{4}} u, A^{\frac{1}{4}} \omega_j) = \displaystyle\int_0^t (f(u) + h, \omega_j)\, \mathrm{d}\tau + (u_1, \omega_j) + (A^{\frac{1}{4}} u_0, A^{\frac{1}{4}} \omega_j), \\ (v^t, e_j)_{g,2} = (u, e_j)_{g,2} - (u_0, e_j)_{g,2} - \displaystyle\int_0^t (v^s_\tau, e_j)_{g,2}\, \mathrm{d}s + (v_0, e_j)_{g,2}. \end{cases}$$

Moreover, it is easy to see from 12 that $u(0) = u_0$ in V_3, $u_t(0) = u_1$ in V_1, $v^0 = v_0$ in $\mathcal{L}_{g,3}$. Therefore, (u, u_t, v^t) is a solution of problem 8, 9.

Next we prove continuous dependence of $(u(t), u_t(t), v^t)$ on (u_0, u_1, v_0). Suppose that (u, u_t, v^t) and $(\bar{u}, \bar{u}_t, \bar{v}^t)$ are two regular solutions of problem 8, 9 with initial data (u_0, u_1, v_0) and $(\bar{u}_0, \bar{u}_1, \bar{v}_0)$, respectively. Set $\tilde{u} = \bar{u} - u$ and $\tilde{v}^t = \bar{v}^t - v^t$. Then

$$\begin{cases} \tilde{u}_{tt} + \kappa A \tilde{u} + \displaystyle\int_0^\infty g(\tau) A \tilde{v}^t(\tau)\, \mathrm{d}\tau + A^{\frac{1}{2}} \tilde{u}_t = f(\bar{u}) - f(u), \\ \tilde{v}^t_t = \tilde{u}_t - \tilde{v}^t_\tau, \end{cases} \tag{24}$$

$$\begin{cases} \tilde{u}(0) = \tilde{u}_0 = \bar{u}_0 - u_0,\ \ \tilde{u}_t(0) = \tilde{u}_1 = \bar{u}_1 - u_1, \\ \tilde{v}^0(\tau) = \tilde{v}_0 = \bar{v}_0 - v_0. \end{cases}$$

By the arguments similar to [7, Lemma 4.9], we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}_t\|^2 + \kappa \|A^{\frac{1}{2}} \tilde{u}\|^2 + \|\tilde{v}^t\|^2_{g,2} \right) + \|A^{\frac{1}{4}} \tilde{u}_t\|^2 \\ =&(f(\bar{u}) - f(u), \tilde{u}_t) - (\tilde{v}^t_\tau, \tilde{v}^t)_{g,2}. \end{aligned} \tag{25}$$

By the arguments similar to the proof of 23, we have

$$\begin{aligned} (f(\bar{u}) - f(u), \tilde{u}_t) \le& C_9 \|A^{\frac{1}{2}} \tilde{u}\| \|A^{\frac{1}{4}} \tilde{u}_t\| \\ \le& \frac{C_9}{4\epsilon} \|A^{\frac{1}{2}} \tilde{u}\|^2 + C_9 \epsilon \|A^{\frac{1}{4}} \tilde{u}_t\|^2. \end{aligned}$$

Note that $-(\tilde{v}^t_\tau, \tilde{v}^t)_{g,2} \le 0$. Hence, by taking $\epsilon = \dfrac{1}{C_9}$, we deduce from 25 that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}_t\|^2 + \kappa \|A^{\frac{1}{2}} \tilde{u}\|^2 + \|\tilde{v}^t\|^2_{g,2} \right) \le \frac{C_9^2}{4} \|A^{\frac{1}{2}} \tilde{u}\|^2.$$

As a consequence, by Gronwall's inequality, we obtain

$$\|\tilde{u}_t\|^2 + \|A^{\frac{1}{2}} \tilde{u}\|^2 + \|\tilde{v}^t\|^2_{g,2} \le C_{10} \left(\|\tilde{u}_1\|^2 + \|A^{\frac{1}{2}} \tilde{u}_0\|^2 + \|\tilde{v}_0\|^2_{g,2} \right). \tag{26}$$

In particular, by taking $(u_0, u_1, v_0) = (\bar{u}_0, \bar{u}_1, \bar{v}_0)$, it is obvious that (u, u_t, v^t) is the unique solution of problem 8, 9. $\square$

Proof of Theorem 2.2. For $u_0 \in V_2$, $u_1 \in V_0$, $v_0 \in \mathcal{L}_{g,2}$, there exist $\{u_{0m}\} \subset V_3$, $\{u_{1m}\} \subset V_1$, $\{v_{0m}\} \subset \mathcal{L}_{g,3}$ such that

$$u_{0m} \to u_0 \text{ in } V_2,\ u_{1m} \to u_1 \text{ in } V_0,\ v_{0m} \to v_0 \text{ in } \mathcal{L}_{g,2}.$$

According to Theorem 3.1, for any $m \in \mathbb{N}^+$, problem 8, 9 admits a unique regular solution (u_m, u_{mt}, v_m^t) satisfying

$$\begin{cases} u_{mtt} + \kappa A u_m + \displaystyle\int_0^\infty g(\tau) A v_m^t(\tau)\,\mathrm{d}\tau + A^{\frac{1}{2}} u_{mt} \\ \quad = f(u_m) + h, \quad t \in (0, \infty), \\ v_{mt}^t(\tau) = u_{mt}(t) - v_{m\tau}^t(\tau), \quad \tau \in (0,\infty),\ t \in (0,\infty), \end{cases}$$

$$\begin{cases} u_m(0) = u_{0m},\ u_{mt}(0) = u_{1m}, \\ v_m^0(\tau) = v_{0m}(\tau). \end{cases}$$

Hence $u_m \in C([0,T]; V_2)$, $u_{mt} \in C([0,T]; V_0)$. Moreover, according to [24, Theorem 3.2], we have $v_m^t \in C([0,T]; \mathcal{L}_{g,2})$.

Set $y_m = u_{m_2} - u_{m_1}$ and $z_m = v_{m_2}^t - v_{m_1}^t$. Then, by the arguments similar to the proof of 26, we get

$$\|y_{mt}\|^2 + \|A^{\frac{1}{2}} y_m\|^2 + \|z_m\|_{g,2}^2 \le C_{11}(\|y_{mt}(0)\|^2 + \|A^{\frac{1}{2}} y_m(0)\|^2 + \|z_m(0)\|_{g,2}^2). \tag{27}$$

By 27 and the arguments similar to the proof of 17, we have

$$u_m \to u \text{ in } C([0,T]; V_2), \tag{28}$$

$$u_{mt} \to u_t \text{ in } C([0,T]; V_0), \tag{29}$$

$$v_m^t \to v^t \text{ in } C([0,T]; \mathcal{L}_{g,2}). \tag{30}$$

Thus (u, u_t, v^t) is a global weak solution of problem 8, 9.

Suppose that (u, u_t, v^t) and $(\bar{u}, \bar{u}_t, \bar{v}^t)$ are two solutions of problem 8, 9 with initial data (u_0, u_1, v_0), $(\bar{u}_0, \bar{u}_1, \bar{v}_0)$, respectively. Then there exist

$$(u_{0m}, u_{1m}, v_{0m}) \in V_3 \times V_1 \times \mathcal{L}_{g,3},\ (\bar{u}_{0m}, \bar{u}_{1m}, \bar{v}_{0m}) \in V_3 \times V_1 \times \mathcal{L}_{g,3},$$

such that

$$(u_{0m}, u_{1m}, v_{0m}) \to (u_0, u_1, v_0) \text{ in } V_2 \times V_0 \times \mathcal{L}_{g,2}, \tag{31}$$

$$(\bar{u}_{0m}, \bar{u}_{1m}, \bar{v}_{0m}) \to (\bar{u}_0, \bar{u}_1, \bar{v}_0) \text{ in } V_2 \times V_0 \times \mathcal{L}_{g,2}. \tag{32}$$

Set $\tilde{u}_m = \bar{u}_m - u_m$ and $\tilde{v}_m^t = \bar{v}_m^t - v_m^t$. Then, on account of 26, we obtain

$$\|\tilde{u}_{mt}\|^2 + \|A^{\frac{1}{2}} \tilde{u}_m\|^2 + \|\tilde{v}_m^t\|_{g,2}^2 \le C_{12}\left(\|\tilde{u}_{1m}\|^2 + \|A^{\frac{1}{2}} \tilde{u}_{0m}\|^2 + \|\tilde{v}_{0m}\|_{g,2}^2\right).$$

Therefore, in terms of 28-32, the conclusions of Theorem 2.2 are derived immediately. $\square$

4. Proof of Theorem 2.3

In this section, for the sake of convenience, we denote

$$\|S(t)(u_0,u_1,v_0)\|_Z^2 := \|u\|_{V_2}^2 + \|u_t\|_{V_0}^2 + \|v^t\|_{g,2}^2.$$

Lemma 4.1. *Under the conditions of Theorem 2.3, $S(t)$ possesses an absorbing set in Z.*

Proof. Let $U = u_t + \varepsilon u$, $t \in [0,\infty)$, where ε is a positive constant to be determined later. Then 8_1 becomes

$$U_t + A^{\frac{1}{2}}U - \varepsilon U - \varepsilon A^{\frac{1}{2}}u + \varepsilon^2 u + \kappa A u + \int_0^\infty g(\tau) A v^t(\tau)\,\mathrm{d}\tau = f(u) + h. \tag{33}$$

Note that

$$(v_\tau^t, v^t)_{g,2} \geq \frac{\rho}{2}\|v^t\|_{g,2}^2.$$

Multiplying 33 by U in V_0 and 8_2 by v^t in $\mathcal{L}_{g,2}$, integrating over Ω and adding the two results, we obtain

$$E_1'(t) + E_2(t) \leq 0, \tag{34}$$

where

$$\begin{aligned} E_1(t) =& \frac{1}{2}\Big(\|U\|^2 + \kappa\|A^{\frac{1}{2}}u\|^2 + \|v^t\|_{g,2}^2 + \varepsilon^2\|u\|^2 - \varepsilon\|A^{\frac{1}{4}}u\|^2 \\ & - 2\int_\Omega F(u)\,\mathrm{d}x - 2(h,u)\Big), \end{aligned}$$

and

$$\begin{aligned} E_2(t) =& \|A^{\frac{1}{4}}U\|^2 - \varepsilon\|U\|^2 - \varepsilon^2\|A^{\frac{1}{4}}u\|^2 + \varepsilon^3\|u\|^2 + \varepsilon\kappa\|A^{\frac{1}{2}}u\|^2 \\ & + \varepsilon\int_0^\infty g(\tau)(A^{\frac{1}{2}}v^t(\tau), A^{\frac{1}{2}}u(t))\,\mathrm{d}\tau - \varepsilon(f(u),u) - \varepsilon(h,u) + \frac{\rho}{2}\|v^t\|_{g,2}^2. \end{aligned}$$

Hence

$$\begin{aligned} E_2(t) - \varepsilon\varrho E_1(t) =& \varepsilon\frac{\kappa(2-\varrho)}{2}\|A^{\frac{1}{2}}u\|^2 - \varepsilon^2\left(1-\frac{\varrho}{2}\right)\|A^{\frac{1}{4}}u\|^2 \\ & + \varepsilon^3\left(1-\frac{\varrho}{2}\right)\|u\|^2 + \sum_{i=1}^4 \Lambda_i, \end{aligned}$$

where

$$\Lambda_1 = \|A^{\frac{1}{4}}U\|^2 - \varepsilon\left(1+\frac{\varrho}{2}\right)\|U\|^2,$$

$$\Lambda_2 = \frac{\rho}{2}\|v^t\|_{g,2}^2 + \varepsilon\int_0^\infty g(\tau)(A^{\frac{1}{2}}v^t(\tau), A^{\frac{1}{2}}u(t))\,\mathrm{d}\tau - \frac{\varrho\varepsilon}{2}\|v^t\|_{g,2}^2,$$

$$\Lambda_3 = \varepsilon\left(\varrho\int_\Omega F(u)\,\mathrm{d}x - (f(u),u)\right),$$

$$\Lambda_4 = -\varepsilon(1-\varrho)(h,u).$$

Applying Cauchy's inequality with $\epsilon_1 > 0$, we get

$$\Lambda_2 \geq \frac{\rho}{2}\|v^t\|_{g,2}^2 - \epsilon_1\varepsilon(1-\kappa)\|A^{\frac{1}{2}}u\|^2 - \frac{\varepsilon}{4\epsilon_1}\|v^t\|_{g,2}^2 - \frac{\varrho\varepsilon}{2}\|v^t\|_{g,2}^2.$$

It follows from 5 that, for any $\eta > 0$, there exists a constant $C_\eta > 0$ such that

$$\begin{aligned}\Lambda_3 \geq & -\varepsilon\left(\eta\|u\|^2 + C_\eta|\Omega|\right)\\ \geq & -\varepsilon\eta\mathfrak{C}_2^2\|A^{\frac{1}{2}}u\|^2 - \varepsilon C_\eta|\Omega|.\end{aligned}$$

Moreover,

$$\Lambda_1 \geq \frac{1}{\mathfrak{C}_1^2}\|U\|^2 - \varepsilon\left(1+\frac{\varrho}{2}\right)\|U\|^2,$$

and

$$\Lambda_4 \geq -\varepsilon(1-\varrho)\left(\epsilon_2\mathfrak{C}_2^2\|A^{\frac{1}{2}}u\|^2 + \frac{1}{4\epsilon_2}\|h\|^2\right).$$

Consequently, by taking sufficiently small ϵ_1, η and ϵ_2 such that

$$C_{13} := \frac{\kappa(2-\varrho)}{2} - \epsilon_1(1-\kappa) - \eta\mathfrak{C}_2^2 - \epsilon_2(1-\varrho)\mathfrak{C}_2^2 > 0,$$

we deduce that

$$\begin{aligned}E_2(t) - \varepsilon\varrho E_1(t) \geq & \varepsilon C_{13}\|A^{\frac{1}{2}}u\|^2 - \varepsilon^2\left(1-\frac{\varrho}{2}\right)\|A^{\frac{1}{4}}u\|^2\\ & + \left[\frac{1}{\mathfrak{C}_1^2} - \varepsilon\left(1+\frac{\varrho}{2}\right)\right]\|U\|^2 + \left(\frac{\rho}{2} - \frac{\varepsilon}{4\epsilon_1} - \frac{\varrho\varepsilon}{2}\right)\|v^t\|_{g,2}^2\\ & - \varepsilon C_\eta|\Omega| - \frac{\varepsilon(1-\varrho)}{4\epsilon_2}\|h\|^2.\end{aligned}$$

Choosing

$$\varepsilon \leq \min\left\{\frac{2C_{13}}{(2-\varrho)\mathfrak{C}_3^2},\ \frac{2}{(2+\varrho)\mathfrak{C}_1^2},\ \frac{2\epsilon_1\rho}{1+2\epsilon_1\varrho}\right\},$$

we obtain

$$E_2(t) - \varepsilon\varrho E_1(t) \geq -C_{14}(\|h\|^2 + |\Omega|) \tag{35}$$

and

$$\kappa\|A^{\frac{1}{2}}u\|^2 - \varepsilon\|A^{\frac{1}{4}}u\|^2 \geq C_{15}\|A^{\frac{1}{2}}u\|^2. \tag{36}$$

Since

$$\|U\|^2 \geq \|u_t\|^2 - \varepsilon^2\|u\|^2,$$

we conclude from 36 and the arguments similar to the proof of 16 that

$$E_1(t) \geq C_{16}\left(\|u_t\|^2 + \|A^{\frac{1}{2}}u\|^2 + \|v^t\|_{g,2}^2\right) - C_{17}(\|h\|^2 + |\Omega|). \tag{37}$$

It follows from 34 and 35 that

$$E_1'(t) + \varepsilon\varrho E_1(t) \leq C_{14}(\|h\|^2 + |\Omega|),$$

which yields

$$E_1(t) \leq E_1(0)e^{-\varepsilon\varrho t} + \frac{C_{14}}{\varepsilon\varrho}(\|h\|^2 + |\Omega|).$$

This, together with 37, gives

$$\|S(t)(u_0,u_1,v_0)\|_Z^2 \leq \frac{E_1(0)}{C_{16}}e^{-\varepsilon\varrho t} + \frac{C_{14}+\varepsilon\varrho C_{17}}{\varepsilon\varrho C_{16}}(\|h\|^2 + |\Omega|).$$

Hence $S(t)$ possesses an absorbing set with the radius $R > \sqrt{\frac{C_{14}+\varepsilon\varrho C_{17}}{\varepsilon\varrho C_{16}}(\|h\|^2 + |\Omega|)}$. □

Proof of Theorem 2.3. We decompose $u = \hat{u} + \check{u}$ and $v^t = \hat{v}^t + \check{v}^t$ satisfying

$$(38)\qquad \begin{cases} \hat{u}_{tt} + \kappa A\hat{u} + A^{\frac{1}{2}}\hat{u}_t + \int_0^\infty g(\tau)A\hat{v}^t(\tau)\,\mathrm{d}\tau = 0, \\ \hat{v}^t_t = \hat{u}_t - \hat{v}^t_\tau, \\ \hat{u}(0) = u_0,\ \hat{u}_t(0) = u_1,\ \check{v}^0(\tau) = v_0(\tau), \end{cases}$$

$$(39)\qquad \begin{cases} \check{u}_{tt} + \kappa A\check{u} + A^{\frac{1}{2}}\check{u}_t + \int_0^\infty g(\tau)A\check{v}^t(\tau)\,\mathrm{d}\tau = \Phi, \\ \check{v}^t_t = \check{u}_t - \check{v}^t_\tau, \\ \check{u}(0) = 0,\ \check{u}_t(0) = 0,\ \check{v}^0(\tau) = 0, \end{cases}$$

where

$$\Phi = f(u) + h.$$

Let $\psi = A^\delta \check{u}$, $\varphi^t = A^\delta \check{v}^t$, $0 < \delta \le \dfrac{1}{4}$. Then it follows from 39 that

$$(40)\qquad \begin{cases} \psi_{tt} + \kappa A\psi + A^{\frac{1}{2}}\psi_t + \int_0^\infty g(\tau)A\varphi^t(\tau)\,\mathrm{d}\tau = A^\delta\Phi, \\ \varphi^t_t = \psi_t - \varphi^t_\tau, \\ \psi(0) = 0,\ \psi_t(0) = 0,\ \varphi^0(\tau) = 0. \end{cases}$$

Let $\Psi = \psi_t + \varepsilon\psi$, where ε is a positive constant to be determined later. Then 40_1 can be written in the form

$$(41)\qquad \Psi_t + A^{\frac{1}{2}}\Psi - \varepsilon\Psi - \varepsilon A^{\frac{1}{2}}\psi + \varepsilon^2\psi + \kappa A\psi + \int_0^\infty g(\tau)A\varphi^t(\tau)\,\mathrm{d}\tau = A^\delta\Phi.$$

Multiplying 41 by Ψ in V_0 and 40_2 by φ^t in $\mathcal{L}_{g,2}$, integrating over Ω and adding the two results, we obtain

$$\begin{aligned} &E_3'(t) + 2\|A^{\frac{1}{4}}\Psi\|^2 - 2\varepsilon\|\Psi\|^2 - 2\varepsilon^2\|A^{\frac{1}{4}}\psi\|^2 + 2\varepsilon^3\|\psi\|^2 + 2\varepsilon\kappa\|A^{\frac{1}{2}}\psi\|^2 \\ &\quad + 2\varepsilon(\varphi^t(\tau), \psi(t))_{g,2} = 2(A^\delta\Phi, \Psi) - 2(\varphi^t_\tau, \varphi^t)_{g,2}, \end{aligned}$$

where

$$E_3(t) = \|\Psi\|^2 + \kappa\|A^{\frac{1}{2}}\psi\|^2 - \varepsilon\|A^{\frac{1}{4}}\psi\|^2 + \varepsilon^2\|\psi\|^2 + \|\varphi^t\|^2_{g,2}.$$

Hence

$$(42)\qquad E_3'(t) + \sum_{i=1}^{2}\Lambda_i - 2\varepsilon^2\|A^{\frac{1}{4}}\psi\|^2 + 2\varepsilon^3\|\psi\|^2 + 2\varepsilon\kappa\|A^{\frac{1}{2}}\psi\|^2 \le 2(A^\delta\Phi, \Psi),$$

where

$$\Lambda_1 = 2\|A^{\frac{1}{4}}\Psi\|^2 - 2\varepsilon\|\Psi\|^2,$$

and

$$\Lambda_2 = 2\varepsilon(\varphi^t(\tau), \psi(t))_{g,2} + \rho\|\varphi^t\|^2_{g,2}.$$

Note that

$$\Lambda_1 \ge \|A^{\frac{1}{4}}\Psi\|^2 + \left(\frac{1}{\mathfrak{C}_1^2} - 2\varepsilon\right)\|\Psi\|^2,$$

and

$$\Lambda_2 \ge \left(\rho - \frac{\varepsilon}{2\epsilon}\right)\|\varphi^t\|^2_{g,2} - 2\varepsilon\epsilon(1-\kappa)\|A^{\frac{1}{2}}\psi\|^2.$$

Consequently, taking

$$\epsilon \le \frac{\kappa(2-\sigma_1)}{2(1-\kappa)},$$

with some constant $0 < \sigma_1 < 2$, we deduce from 42 that

$$E_3'(t) + \|A^{\frac{1}{4}}\Psi\|^2 + \left(\frac{1}{\mathfrak{C}_1^2} - 2\varepsilon\right)\|\Psi\|^2 - 2\varepsilon^2\|A^{\frac{1}{4}}\psi\|^2 + 2\varepsilon^3\|\psi\|^2 + \sigma_1\varepsilon\kappa\|A^{\frac{1}{2}}\psi\|^2 + \left(\rho - \frac{\varepsilon}{2\epsilon}\right)\|\varphi^t\|_{g,2}^2 \le 2(A^\delta\Phi, \Psi).$$

We further choose

$$\varepsilon < \min\left\{\frac{1}{(\sigma_2+2)\mathfrak{C}_1^2}, \frac{2\epsilon\rho}{1+2\epsilon\sigma_2}, \frac{(\sigma_1-\sigma_2)\kappa}{2\mathfrak{C}_3^2}, \frac{\kappa}{\mathfrak{C}_3^2}\right\},$$

with some constant $0 < \sigma_2 < \sigma_1$. Thus

$$E_3'(t) + \sigma_2\varepsilon E_3(t) + \|A^{\frac{1}{4}}\Psi\|^2 \le 2(A^\delta\Phi, \Psi) \tag{43}$$

and

$$E_3(t) \ge \|\psi_t\|^2 + \left(\kappa - \varepsilon\mathfrak{C}_3^2\right)\|A^{\frac{1}{2}}\psi\|^2 + \|\varphi^t\|_{g,2}^2. \tag{44}$$

Applying Hölder's inequality and Cauchy's inequality to the right side of 43, we get

$$E_3'(t) + \sigma_2\varepsilon E_3(t) \le \|\Phi\|_{V_{4\delta-1}}^2. \tag{45}$$

For any $w \in V_{1-4\delta}$, we have

$$\begin{aligned}(f(u), w) \le& b\||u|^{p-1}\|_{\frac{2N}{N+2(1-4\delta)}}\|w\|_{\frac{2N}{N-2(1-4\delta)}}\\ \le& C_{18}\|u\|_{V_2}^{p-1}\|w\|_{V_{1-4\delta}}.\end{aligned}$$

Since

$$\|\Phi\|_{V_{4\delta-1}} \le \|f(u)\|_{V_{4\delta-1}} + \|h\|_{V_{4\delta-1}},$$

we deduce from 45 that

$$E_3(t) \le E_3(0)e^{-\sigma_2\varepsilon t} + C_{19}.$$

Combining this with 40_3 and 44, we obtain

$$\|\check{u}\|_{V_{2+4\delta}}^2 + \|\check{u}_t\|_{V_{4\delta}}^2 + \|\check{v}^t\|_{g,2+4\delta}^2 \le C_{20}, \quad \forall t \in [0,\infty). \tag{46}$$

Taking into account

$$\check{v}^t(\tau) = \begin{cases}\check{u}(t), & \tau \ge t,\\ \check{u}(t) - \check{u}(t-\tau), & 0 < \tau < t,\end{cases}$$

we get

$$\check{v}_\tau^t(\tau) = \begin{cases}0, & \tau \ge t,\\ \check{u}_t(t-\tau), & 0 < \tau < t.\end{cases}$$

Let $\Upsilon = \bigcup_{t\ge 0}\check{v}^t$. Then Υ is bounded in $\mathcal{L}_{g,2+4\delta} \cap H_g^1(\mathbb{R}^+, V_{4\delta})$ due to 46, where $H_g^1(\mathbb{R}^+, V_{4\delta})$ is a Hilbert space of $V_{4\delta}$-valued functions v on $\mathbb{R}^+$ such that both v and v_τ belong to $\mathcal{L}_{g,4\delta}$. Note that $\sup_{v\in\Upsilon}\|v(\tau)\|_{V_2}^2 \in L_g^1(\mathbb{R}^+)$. Hence, in view of [24, Lemma 5.5], we see that Υ is relatively compact in $\mathcal{L}_{g,2}$. Since $V_{2+4\delta} \times V_{4\delta} \hookrightarrow\hookrightarrow V_2 \times V_0$, we conclude from 46 that there exists a $t_0 = t_0(B)$ such that $\bigcup_{t\ge t_0} S_2(t)B$ is relatively compact. Thus the operators $S_2(t)$ are uniformly compact for t large. Furthermore, as for problem 38, it is easy to check that

$$\|\hat{u}\|_{V_2}^2 + \|\hat{u}_t\|_{V_0}^2 + \|\hat{v}^t\|_{g,2}^2 \le C_{21}e^{-\sigma_2\varepsilon t}, \quad \forall t \in [0,\infty).$$

This means that $S_1(t)$ is a continuous mapping from Z into itself such that

$$\sup_{(u_0,u_1,v_0)\in B} \|S_1(t)(u_0,u_1,v_0)\|_Z \to 0,$$

as $t \to \infty$. Therefore, by virtue of [26, Chapter I, Theorem 1.1] and Lemma 4.1, the proof of Theorem 2.3 is complete. □

Acknowledgments

The author would like to thank the editors and the referees for the valuable comments and suggestions.

References

[1] F. Alabau-Boussouira and P. Cannarsa, A general method for proving sharp energy decay rates for memory-dissipative evolution equations, *C. R. Math. Acad. Sci. Paris*, **347** (2009), 867–872. MR 2542886

[2] F. Alabau-Boussouira, P. Cannarsa and D. Sforza, Decay estimates for second order evolution equations with memory, *J. Funct. Anal.*, **254** (2008), 1342–1372. MR 2386941

[3] S. Borini and V. Pata, Uniform attractors for a strongly damped wave equation with linear memory, *Asymptot. Anal.*, **20** (1999), 263–277. MR 1715336

[4] P. Cannarsa and D. Sforza, Integro-differential equations of hyperbolic type with positive definite kernels, *J. Differential Equations*, **250** (2011), 4289–4335. MR 2793256

[5] M. M. Cavalcanti, V. N. D. Cavalcanti, I. Lasiecka and F. A. F. Nascimento, Intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional dissipative effects, *Discrete Contin. Dyn. Syst. Ser. B*, **19** (2014), 1987–2012. MR 3253241

[6] M. M. Cavalcanti, V. N. D. Cavalcanti and T. F. Ma, Exponential decay of the viscoelastic Euler-Bernoulli equation with a nonlocal dissipation in general domains, *Differential Integral Equations*, **17** (2004), 495–510. MR 2054931

[7] V. V. Chepyzhov and A. Miranville, Trajectory and global attractors of dissipative hyperbolic equations with memory, *Commun. Pure Appl. Anal.*, **4** (2005), 115–142. MR 2126281

[8] I. Chueshov and I. Lasiecka, *Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping*, Mem. Amer. Math. Soc., Rhode Island, 2008. MR 2438025

[9] M. Conti and P. G. Geredeli, Existence of smooth global attractors for nonlinear viscoelastic equations with memory, *J. Evol. Equ.*, **15** (2015), 533–558. MR 3394697

[10] J.-M. Ghidaglia and A. Marzocchi, Longtime behaviour of strongly damped wave equations, global attractors and their dimension, *SIAM J. Math. Anal.*, **22** (1991), 879–895. MR 1112054

[11] C. Giorgi, V. Pata and A. Marzocchi, Asymptotic behavior of a semilinear problem in heat conduction with memory, *NoDEA Nonlinear Differential Equations Appl.*, **5** (1998), 333–354. MR 1638908

[12] M. Grasselli and V. Pata, Uniform attractors of nonautonomous dynamical systems with memory, in *Evolution Equations, Semigroups and Functional Analysis*, Birkhäuser, Basel, 2002, 155–178. MR 1944162

[13] A. Guesmia and S. A. Messaoudi, A new approach to the stability of an abstract system in the presence of infinite history, *J. Math. Anal. Appl.*, **416** (2014), 212–228. MR 3182758

[14] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, American Mathematical Society, Providence, RI, 1988. MR 941371

[15] M. A. J. Silva and T. F. Ma, On a viscoelastic plate equation with history setting and perturbation of p-Laplacian type, *IMA J. Appl. Math.*, **78** (2013), 1130–1146. MR 3142322

[16] O. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991. MR 1133627

[17] J. E. Lagnese, *Boundary Stabilization of Thin Plates*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989. MR 1061153

[18] I. Lasiecka and X. Wang, Intrinsic decay rate estimates for semilinear abstract second order equations with memory, *New Prospects in Direct, Inverse and Control Problems for Evolution Equations*, Springer, Cham, 2014, 271–303. MR 3362997

[19] A. Marzocchi and E. Vuk, Global attractor for damped semilinear elastic beam equations with memory, *Z. Angew. Math. Phys.*, **54** (2003), 224–234. MR 1967327

[20] J. E. M. Rivera and L. H. Fatori, Smoothing effect and propagations of singularities for viscoelastic plates, *J. Math. Anal. Appl.*, **206** (1997), 397–427. MR 1433945

[21] J. E. M. Rivera, E. C. Lapa and R. Barreto, Decay rates for viscoelastic plates with memory, *J. Elasticity*, **44** (1996), 61–87. MR 1417809

[22] J. E. M. Rivera and M. G. Naso, Asymptotic stability of semigroups associated with linear weak dissipative systems with memory, *J. Math. Anal. Appl.*, **326** (2007), 691–707. MR 2277813

[23] V. Pata, Exponential stability in linear viscoelasticity, *Quart. Appl. Math.*, **64** (2006), 499–513. MR 2259051

[24] V. Pata and A. Zucchi, Attractors for a damped hyperbolic equation with linear memory, *Adv. Math. Sci. Appl.*, **11** (2001), 505–529. MR 1907454

[25] J. Prüss, Decay properties for the solutions of a partial differential equation with memory, *Arch. Math.*, **92** (2009), 158–173. MR 2481511

[26] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1988. MR 953967

[27] Z. Yang and B. Jin, Global attractor for a class of Kirchhoff models, *J. Math. Phys.*, **50** (2009), 29 pp. MR 2647872

Yang Liu, College of Mathematics and Computer Science, Northwest Minzu University, Lanzhou 730124, China; College of Mathematics, Sichuan University, Chengdu 610065, China

Email address: liuyangnufn@163.com