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BLOW-UP CRITERION FOR THE 3D VISCOUS POLYTROPIC FLUIDS WITH DEGENERATE VISCOSITIES

YUE CAO*

ABSTRACT. In this paper, the Cauchy problem of the 3D compressible Navier-Stokes equations with degenerate viscosities and far field vacuum is considered. We prove that the L^∞ norm of the deformation tensor $D(u)$ (u: the velocity of fluids) and the L^6 norm of $\nabla \log \rho$ (ρ: the mass density) control the possible blow-up of regular solutions. This conclusion means that if a solution with far field vacuum to the Cauchy problem of the compressible Navier-Stokes equations with degenerate viscosities is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by losing the bound of $D(u)$ or $\nabla \log \rho$ as the critical time approaches; equivalently, if both $D(u)$ and $\nabla \log \rho$ remain bounded, a regular solution persists.

1. Introduction

We consider the compressible isentropic Navier-Stokes equations with degenerate viscosities in $\mathbb{R}^3$, which gives the conservation laws of mass and momentum of fluids. This model comes from the Boltzmann equations through the Chapman-Enskog expansion to the second order, and the viscosities depend on the density $\rho \geq 0$ by the laws of Boyle and Gay-Lussac for ideal gas. This system can be written as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div}\mathbb{T}, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^3$ is the spatial coordinate; $t \geq 0$ is the time; ρ is the density of the fluid; $u = (u^{(1)}, u^{(2)}, u^{(3)})^\top \in \mathbb{R}^3$ is the velocity of the fluid; P is the pressure, and for the polytropic fluid

$$P = A\rho^\gamma, \quad \gamma > 1, \tag{2}$$

where A is a positive constant, γ is the adiabatic index; $\mathbb{T}$ is the stress tensor given by

$$\mathbb{T} = \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho)\operatorname{div}u\mathbb{I}_3, \tag{3}$$

where $\mathbb{I}_3$ is the 3×3 unit matrix, $\mu(\rho)$ is the shear viscosity, $\lambda(\rho)$ is the second viscosity, and

$$\mu(\rho) = \alpha\rho, \quad \lambda(\rho) = \beta\rho, \tag{4}$$

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* Corresponding author.

where the constants α and β satisfy

$$\alpha > 0, \quad 2\alpha + 3\beta \geq 0.$$

Here, the initial data are given by

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^3, \tag{5}$$

and the far field behavior is given by

$$(\rho, u) \to (0, 0) \quad \text{as} \quad |x| \to \infty, \quad t \geq 0. \tag{6}$$

The aim of this paper is to prove a blow-up criterion for the regular solution to the Cauchy problem (1) with (5)-(6).

Throughout the paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

$$D^{k,r} = \{f \in L^1_{loc}(\mathbb{R}^3) : |f|_{D^{k,r}} = |\nabla^k f|_{L^r} < +\infty\},$$

$$D^k = D^{k,2}(k \geq 2), \quad D^1 = \{f \in L^6(\mathbb{R}^3) : |f|_{D^1} = |\nabla f|_{L^2} < \infty\},$$

$$\|f\|_{X \cap Y} = \|f\|_X + \|f\|_Y, \quad \|f\|_s = \|f\|_{H^s(\mathbb{R}^3)},$$

$$|f|_p = \|f\|_{L^p(\mathbb{R}^3)}, \quad |f|_{D^k} = \|f\|_{D^k(\mathbb{R}^3)}.$$

A detailed study of homogeneous Sobolev space can be found in [5].

The compressible isentropic Navier-Stokes system is a well-known mathematical model, which has attracted great attention from the researchers, and some significant processes have been made in the well-posedness for this system.

When (μ, λ) are both constants, with the assumption that there is no vacuum, the local existence of the classical solutions to system (1) follows from a standard Banach fixed point argument. For the existence results with vacuum and general data, the main breakthrough is due to Lions [16]. He established the global existence of weak solutions in $\mathbb{R}^3$, periodic domains or bounded domains, under the homogenous Dirichlet boundary conditions and the restriction $\gamma > 9/5$. Later, the restriction on γ was improved to $\gamma > 3/2$ by Feireisl-Novotný-Petzeltová [4]. Recently, Cho-Choe-Kim [2] introduced the following initial layer compatibility condition

$$-\text{div}\mathbb{T}_0 + \nabla P(\rho_0) = \sqrt{\rho_0} g$$

for some $g \in L^2$ to deal with the vacuum. They proved the local existence of the strong solutions in $\mathbb{R}^3$ or bounded domains with homogenous Dirichlet boundary conditions. Moreover, Huang-Li-Xin proved the global existence of the classical solutions to the Cauchy problem of the isentropic system with small energy and vacuum in [8].

When (μ, λ) depend on density in the following form

$$\mu(\rho) = \alpha \rho^{\delta_1}, \quad \lambda(\rho) = \beta \rho^{\delta_2}, \tag{7}$$

where $\delta_1 > 0$, $\delta_2 \geq 0$, $\alpha > 0$ and β are all real constants, system (1) has received a lot of attention. However, except for the 1D problems, there are few results on the strong solutions for the multi-dimensional problems, since the possible degeneracy of the Lamé operator caused by initial vacuum. This degeneracy gives rise to some difficulties in the regularity estimates because of the less regularizing effect of the viscosity on the solutions. Recently, Li-Pan-Zhu [11] have obtained the local existence of the classical solutions to system (1) in 2D space under

$$\delta_1 = 1, \quad \delta_2 = 0 \text{ or } 1, \quad \alpha > 0, \quad \alpha + \beta \geq 0, \tag{8}$$

and (6), where the vacuum cannot appear in any local point. They [12] also prove the same existence result in 3D space under

$$(\rho, u) \to (\bar{\rho}, 0) \quad \text{as} \quad |x| \to \infty, \tag{9}$$

with initial vacuum appearing in some open set or the far field, the constant $\bar{\rho} \geq 0$ and

$$1 < \delta_1 = \delta_2 \leq \min\Big(3, \frac{\gamma+1}{2}\Big), \quad \alpha > 0, \quad \alpha + \beta \geq 0. \tag{10}$$

We also refer readers to [3], [6], [10], [13], [18], [26] and references therein for other interesting progress for this compressible degenerate system, corresponding radiation hydrodynamic equations and magnetohydrodynamic equations.

It should be noted that one should not always expect the global existence of solutions with better regularities or general initial data because of Xin's results [23] and Rozanova's results [20]. It was proved that there is no global smooth solutions to (1), if the initial density has nontrivial compact support (1D) or the solutions are highly decreasing at infinity(dD, $d \geq 1$). These motivate us to find the blow-up mechanisms and singularity structures of the solutions.

For constant viscosity, Beale-Kato-Majda [1] first proved that the maximum norm of the vorticity controls the blow-up of the smooth solutions to 3D incompressible Euler equations

$$\lim_{T \to T^*} \int_0^T |\mathrm{curl} u|_\infty dt = \infty, \tag{11}$$

where T^* is the maximum existence time. Later, for the same problem, Ponce [19] proved that the maximum norm of the deformation tensor controls the blow-up of the smooth solutions

$$\lim_{T \to T^*} \int_0^T |D(u)|_\infty dt = \infty, \tag{12}$$

where the deformation tensor $D(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^\top\big)$. Huang-Li-Xin [7] proved that the criterion (12) holds for the strong solutions to the system (1). Sun-Wang-Zhang [22] proved that the upper bound of the density controls the blowup of the strong solution to the system (1). There are some other interesting results about infinite time blowup and finite time blowup results on the nonlinear wave equation with different initial energy levels, refer to [14], [15], [24] and references therein for detailed study.

When the viscosities depend on density in the form of (4), S. Zhu [25] introduced the regular solutions, which can be defined as

Definition 1.1. [25] Let $T > 0$ be a finite constant, (ρ, u) is called a regular solution to the Cauchy problem (1) with (5)-(6) on $[0, T] \times \mathbb{R}^3$ if (ρ, u) satisfies

(A) (ρ, u) in $[0,T]\times\mathbb{R}^3$ satisfies the Cauchy problem (1)with (5) − (6) in the sense of distributions;

(B) $\rho \geq 0, \quad \rho^{\frac{\gamma-1}{2}} \in C([0,T];H^2),\ (\rho^{\frac{\gamma-1}{2}})_t \in C([0,T];H^1);$

(C) $\nabla \log\rho \in C([0,T];D^1),\ (\nabla\log\rho)_t \in C([0,T];L^2);$

(D) $u \in C([0,T];H^2)\cap L^2([0,T];D^3),\ u_t \in C([0,T];L^2)\cap L^2([0,T];D^1).$

The local existence of the regular solutions has been obtained by Zhu [25].

Theorem 1.2. [25] *Let $1<\gamma\leq 2$ or $\gamma=3$. If the initial data (ρ_0,u_0) satisfies the regularity conditions*

$$\rho_0^{\frac{\gamma-1}{2}} \geq 0, \quad (\rho_0^{\frac{\gamma-1}{2}}, u_0)\in H^2, \quad \nabla\log\rho_0 \in D^1, \tag{13}$$

then there exist a small time T_* and a unique regular solution (ρ,u) to the Cauchy problem (1) with (5)-(6). Moreover, we also have $\rho(t,x)\in C([0,T_*]\times\mathbb{R}^3)$ and

$$\rho \in C([0,T_*];H^2), \quad \rho_t \in C([0,T_*];H^1).$$

Based on Theorem 1.2, we establish the blow-up criterion for the regular solution in terms of $\nabla\log\rho$ and the deformation tensor $D(u)$, which is similar to the Beale-Kato-Majda criterion for the ideal incompressible Euler equations and the compressible Navier-Stokes equations.

Theorem 1.3. *Let (ρ,u) be a regular solution obtained in Theorem 1.2. Then if $\overline{T}<+\infty$ is the maximal existence time, one has both*

$$\lim_{T\mapsto\overline{T}}\left(\sup_{0\leq t\leq T}|\nabla\log\rho|_6 + \int_0^T |D(u)|_\infty\ dt\right) = +\infty, \tag{14}$$

and

$$\limsup_{T\mapsto\overline{T}}\int_0^T \|D(u)\|_{L^\infty\cap D^{1,6}}\ dt = +\infty. \tag{15}$$

The rest of the paper can be organized as follows. In Section 2, we will give the proof for the criterion (14). Section 3 is an appendix which will present some important lemmas which are frequently used in our proof, and also give the detail derivation for the desired system used in our following proof.

2. Blow-up criterion

In this section, we give the proof for Theorem 1.3. We use a contradiction argument to prove (14), let (ρ,u) be the unique regular solution to the Cauchy problem (1) with (5)-(6) and the maximal existence time $\overline{T}$. We assume that $\overline{T}<+\infty$ and

$$\lim_{T\mapsto\overline{T}}\left(\sup_{0\leq t\leq T}|\nabla\log\rho|_6 + \int_0^T |D(u)|_\infty dt\right) = C_0 < +\infty \tag{16}$$

for some constant $0<C_0<\infty$. If we prove that under assumption (16), $\overline{T}$ is actually not the maximal existence time for the regular solution, there will be a contradiction, thus (14) holds.

Notice that, one can also prove (15) by contradiction argument. Assume that

$$\limsup_{T\mapsto\overline{T}}\int_0^T \|D(u)\|_{L^\infty\cap D^{1,6}}\,dt = C_0' < +\infty \tag{17}$$

for some constant $0 < C_0' < \infty$. Combing (17) with the mass equation, we know that

$$\lim_{T\mapsto\overline{T}}\sup_{0\le t\le T}|\nabla\log\rho|_6 \le CC_0',$$

which implies that under assumption (17), we have (16). Thus, if we prove that (14) holds, then (15) holds immediately.

In the rest part of this section, based on the assumption (16), we will prove that $\bar{T}$ is not the maximal existence time for the regular solution.

From the definition of the regular solution, we know for

$$\phi = \rho^{\frac{\gamma-1}{2}}, \quad \psi = \frac{2}{\gamma-1}\nabla\log\phi, \tag{18}$$

(ϕ,ψ,u) satisfies

$$\begin{cases} \phi_t + u\cdot\nabla\phi + \frac{\gamma-1}{2}\phi\mathrm{div}u = 0,\\ \psi_t + \nabla(u\cdot\psi) + \nabla\mathrm{div}u = 0,\\ u_t + u\cdot\nabla u + 2\theta\phi\nabla\phi + Lu = \psi\cdot Q(u), \end{cases} \tag{19}$$

where L is the so-called Lamé operator given by

$$Lu = -\mathrm{div}(\alpha(\nabla u + (\nabla u)^\top) + \beta\mathrm{div}u\mathbb{I}_3), \tag{20}$$

and terms $\big(Q(u),\theta\big)$ are given by

$$Q(u) = \alpha(\nabla u + (\nabla u)^\top) + \beta\mathrm{div}u\mathbb{I}_3, \quad \theta = \frac{A\gamma}{\gamma-1}. \tag{21}$$

See our appendix for the detailed process of the reformulation.

For $(19)_2$, we have the equivalent form

$$\psi_t + \sum_{l=1}^{3} A_l\partial_l\psi + B\psi + \nabla\mathrm{div}u = 0. \tag{22}$$

Here $A_l = (a_{ij}^{(l)})_{3\times3}$ $(i,j,l = 1,2,3)$ are symmetric with $a_{ij}^{(l)} = u^{(l)}$ when $i = j$; and $a_{ij}^{(l)} = 0$, otherwise. $B = (\nabla u)^\top$, so (22) is a positive symmetric hyperbolic system. By direct computation, one knows

$$\psi = \frac{2}{\gamma-1}\frac{\nabla\phi}{\phi} = \frac{2}{\gamma-1}\frac{\nabla\rho^{\frac{\gamma-1}{2}}}{\rho^{\frac{\gamma-1}{2}}} = \frac{\nabla\rho}{\rho} = \nabla\log\rho, \tag{23}$$

combing this with (16), one has $\psi \in L^\infty([0,T];L^6)$.

Under (16) and (19), we first show that the density ρ is uniformly bounded.

Lemma 2.1. *Let (ρ,u) be the unique regular solution to the Cauchy problem (1) with (5)-(6) on $[0,\overline{T})\times\mathbb{R}^3$ satisfying (16). Then*

$$\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3)} + \|\phi\|_{L^\infty([0,T];L^q)} \le C, \quad 0 \le T < \overline{T},$$

where $C > 0$ depends on C_0, constant $q \in [2,+\infty]$ and $\overline{T}$.

Proof. First, it is obvious that ϕ can be represented by

$$\phi(t,x) = \phi_0\big(W(0,t,x)\big)\exp\Big(-\frac{\gamma-1}{2}\int_0^t \mathrm{div} u(s,W(s,t,x))ds\Big), \tag{24}$$

where $W \in C^1([0,T]\times[0,T]\times\mathbb{R}^3)$ is the solution to the initial value problem

$$\begin{cases} \dfrac{d}{dt}W(t,s,x) = u\big(t,W(t,s,x)\big), & 0 \le t \le T, \\ W(s,s,x) = x, & 0 \le s \le T,\ x\in\mathbb{R}^3. \end{cases}$$

Then it is clear that

$$\|\phi\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le |\phi_0|_\infty \exp(CC_0) \le C. \tag{25}$$

Similarly,

$$\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le C. \tag{26}$$

Next, multiplying $(19)_1$ by 2ϕ and integrating over $\mathbb{R}^3$, we get

$$\frac{d}{dt}|\phi|_2^2 \le C|\mathrm{div} u|_\infty|\phi|_2^2, \tag{27}$$

from (16), (27) and the Gronwall's inequality, we immediately obtain

$$\|\phi\|_{L^\infty([0,T];L^2)} \le C. \tag{28}$$

Combing (25)-(28) together, one has

$$\|\phi\|_{L^\infty([0,T];L^q)} \le C, \quad q\in[2,+\infty].$$

We complete the proof of this lemma. □

Before go further, notice that

$$\begin{aligned} |\nabla\phi|_6 = |\phi\nabla\log\phi|_6 &= \frac{2}{\gamma-1}|\phi\nabla\log\rho|_6 \\ &\le C|\phi|_\infty|\nabla\log\rho|_6 \le C, \end{aligned} \tag{29}$$

where we have used (16) and Lemma 2.1. Next, we give the basic energy estimates on u.

Lemma 2.2. *Let (ρ,u) be the unique regular solution to the Cauchy problem (1) with (5)-(6) on $[0,\overline{T})\times\mathbb{R}^3$ satisfying (16). Then*

$$\sup_{0\le t\le T}|u(t)|_2^2 + \int_0^T |\nabla u(t)|_2^2 dt \le C, \quad 0\le T<\overline{T},$$

where C only depends on C_0 and $\overline{T}$.

Proof. Multiplying $(19)_3$ by $2u$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} &\frac{d}{dt}|u|_2^2 + 2\int_{\mathbb{R}^3}\big(\alpha|\nabla u|^2 + (\alpha+\beta)(\mathrm{div} u)^2\big)dx \\ =&\int_{\mathbb{R}^3} 2\Big(-u\cdot\nabla u\cdot u - \theta\nabla\phi^2\cdot u + \psi\cdot Q(u)\cdot u\Big)dx \\ \equiv:& L_1+L_2+L_3. \end{aligned} \tag{30}$$

The right-hand side terms can be estimated as follows.

$$
\begin{aligned}
L_1 =& -\int_{\mathbb{R}^3} 2u\cdot\nabla u\cdot u \mathrm{d}x \le C|\mathrm{div}u|_\infty |u|_2^2,\\
L_2 =& 2\int_{\mathbb{R}^3} \theta\phi^2 \mathrm{div}u\mathrm{d}x \le C|\phi|_2^2|\mathrm{div}u|_\infty \le C|\mathrm{div}u|_\infty,\\
L_3 =& \int_{\mathbb{R}^3} 2\psi\cdot Q(u)\cdot u\mathrm{d}x \le C|\psi|_6|\nabla u|_2|u|_3\\
\le& C|\nabla u|_2|u|_2^{\frac{1}{2}}|\nabla u|_2^{\frac{1}{2}} \le \frac{\alpha}{2}|\nabla u|_2^2 + C|u|_2|\nabla u|_2\\
\le& \alpha|\nabla u|_2^2 + C|u|_2^2,
\end{aligned} \tag{31}
$$

where we have used (16), (23) and the facts

$$
|u|_3 \le C|u|_2^{\frac{1}{2}}|\nabla u|_2^{\frac{1}{2}}. \tag{32}
$$

Thus (30) and (31) yield

$$
\frac{d}{dt}|u|_2^2 + \alpha|\nabla u|_2^2 \le C(|\mathrm{div}u|_\infty + 1)|u|_2^2 + C|\mathrm{div}u|_\infty. \tag{33}
$$

By the Gronwall's inequality, (16) and (33), we have

$$
|u(t)|_2^2 + \int_0^t |\nabla u(s)|_2^2 ds \le C, \quad 0\le t\le T. \tag{34}
$$

This completes the proof of this lemma. □

The next lemma provides the key estimates on $\nabla\phi$ and ∇u.

Lemma 2.3. *Let (ρ,u) be the unique regular solution to the Cauchy problem (1) with (5)-(6) on $[0,\overline{T})\times\mathbb{R}^3$ satisfying (16). Then*

$$
\sup_{0\le t\le T}\big(|\nabla u(t)|_2^2 + |\nabla\phi(t)|_2^2\big) + \int_0^T (|\nabla^2 u|_2^2 + |u_t|_2^2)dt \le C, \quad 0\le T<\overline{T},
$$

where C only depends on C_0 and $\overline{T}$.

Proof. Multiplying $(19)_3$ by $-Lu-\theta\nabla\phi^2$ and integrating over $\mathbb{R}^3$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\alpha|\nabla u|^2 + (\alpha+\beta)|\mathrm{div}u|^2\Big)dx + \int_{\mathbb{R}^3}(-Lu-\theta\nabla\phi^2)^2\, dx\\
=& -\alpha\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\nabla\times\mathrm{curl}u\, dx + \int_{\mathbb{R}^3}(2\alpha+\beta)(u\cdot\nabla u)\cdot\nabla\mathrm{div}u\, dx\\
&+\theta\int_{\mathbb{R}^3}(\psi\cdot Q(u))\cdot\nabla\phi^2 dx - \theta\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\nabla\phi^2\, dx\\
&+\int_{\mathbb{R}^3}(\psi\cdot Q(u))\cdot Lu\, dx - \theta\int_{\mathbb{R}^3} u_t\cdot\nabla\phi^2\, dx \equiv: \sum_{i=4}^{9} L_i,
\end{aligned} \tag{35}
$$

where we have used the fact that

$$
-\triangle u + \nabla\mathrm{div}u = \mathrm{curl}(\mathrm{curl}u) = \nabla\times\mathrm{curl}u.
$$

First, from the standard elliptic estimate shown in Lemma 3.3, we have

$$\begin{aligned}
&|\nabla^2 u|_2^2 - C|\theta\nabla\phi^2|_2^2\\
\le& C\big|\mathrm{div}\big(\alpha(\nabla u+(\nabla u)^\top)+\beta\mathrm{div}u\mathbb{I}_3\big)\big|_2^2 - C|\theta\nabla\phi^2|_2^2\\
\le& C\big|\mathrm{div}(\alpha(\nabla u+(\nabla u)^\top)+\beta\mathrm{div}u\mathbb{I}_3)-\theta\nabla\phi^2\big|_2^2\\
=& C\int_{\mathbb{R}^3}(-Lu-\theta\nabla\phi^2)^2dx.
\end{aligned} \tag{36}$$

Second, we estimate the right-hand side of (35) term by term. According to

$$\begin{cases}
u\times \mathrm{curl}u=\dfrac{1}{2}\nabla(|u|^2)-u\cdot\nabla u,\\
\nabla\times(a\times b)=(b\cdot\nabla)a-(a\cdot\nabla)b+(\mathrm{div}b)a-(\mathrm{div}a)b,
\end{cases}$$

Hölder's inequality, Young's inequality, (16), (23), (29), Lemma 2.2, Lemma 3.1 and $(19)_1$, one can obtain that

$$\begin{aligned}
|L_4|=&\alpha\Big|\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\nabla\times\mathrm{curl}u\ dx\Big|\\
=&\alpha\Big|\int_{\mathbb{R}^3}\Big(\mathrm{curl}u\cdot\nabla\times((u\cdot\nabla)u)\Big)dx\Big|\\
=&\alpha\Big|\int_{\mathbb{R}^3}\Big(\mathrm{curl}u\cdot\nabla\times(u\times\mathrm{curl}u)\Big)dx\Big|\\
=&\alpha\Big|\int_{\mathbb{R}^3}\Big(\frac{1}{2}|\mathrm{curl}u|^2\mathrm{div}u-\mathrm{curl}u\cdot D(u)\cdot\mathrm{curl}u\Big)dx\Big|\\
\le& C|\nabla u|_\infty|\nabla u|_2^2,
\end{aligned}$$

$$\begin{aligned}
|L_5|=&\Big|\int_{\mathbb{R}^3}(2\alpha+\beta)(u\cdot\nabla)u\cdot\nabla\mathrm{div}u\ dx\Big|\\
\le&\Big|\int_{\mathbb{R}^3}(2\alpha+\beta)\big(-\nabla u:\nabla u^\top\mathrm{div}u+\frac{1}{2}(\mathrm{div}u)^3\big)dx\Big|\\
&+C|\nabla\phi|_6|u|_3|\nabla u|_2|\mathrm{div}u|_\infty\\
\le& C\big(|\nabla u|_2^2|\mathrm{div}u|_\infty+|u|_2^{\frac12}|\nabla u|_2^{\frac12}|\nabla u|_2|\mathrm{div}u|_\infty\big)\\
\le& C\big(|\nabla u|_2^2+|u|_2|\nabla u|_2\big)|\mathrm{div}u|_\infty\\
\le& C|\mathrm{div}u|_\infty(|\nabla u|_2^2+1),
\end{aligned} \tag{37}$$

$$\begin{aligned}
L_6=&\theta\int_{\mathbb{R}^3}(\psi\cdot Q(u))\cdot\nabla\phi^2\ dx\le C|\psi|_6|\nabla u|_3|\nabla\phi^2|_2\\
\le& C|\nabla u|_2^{\frac12}|\nabla^2 u|_2^{\frac12}|\nabla\phi|_2|\phi|_\infty\\
\le& C|\nabla\phi|_2^2+C(\epsilon)|\nabla u|_2^2+\epsilon|\nabla^2 u|_2^2,
\end{aligned}$$

$$\begin{aligned}
|L_7|=&\theta\Big|\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\nabla\phi^2dx\Big|\\
=&\theta\Big|-\int_{\mathbb{R}^3}\nabla u:(\nabla u)^\top\phi^2dx-\int_{\mathbb{R}^3}\phi^2u\cdot\nabla(\mathrm{div}u)dx\Big|
\end{aligned}$$

$$
\begin{aligned}
=&\theta\Big|-\int_{\mathbb{R}^3}\nabla u:(\nabla u)^\top\phi^2dx+\int_{\mathbb{R}^3}(\text{div}u)^2\phi^2dx\\
&+\int_{\mathbb{R}^2}u\cdot\nabla\phi^2\text{div}u\ dx\Big|\\
\le&C\big(|\nabla u|_2^2|\phi^2|_\infty+|u|_2|\nabla\phi|_2|\phi|_\infty|\text{div}u|_\infty\big)\\
\le&C\big(|\nabla u|_2^2+|\text{div}u|_\infty|\nabla\phi|_2\big)\\
\le&C(|\nabla u|_2^2+|\text{div}u|_\infty+|\text{div}u|_\infty|\nabla\phi|_2^2),
\end{aligned}
$$

$$
\begin{aligned}
L_8=&\int_{\mathbb{R}^3}(\psi\cdot Q(u))\cdot Lu\ dx\\
\le&C|\psi|_6|\nabla u|_3|\nabla^2u|_2\\
\le&C|\nabla^2u|_2^{\frac{3}{2}}|\nabla u|_2^{\frac{1}{2}}\le C(\epsilon)|\nabla u|_2^2+\epsilon|\nabla^2u|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
L_9=&-\theta\int_{\mathbb{R}^3}u_t\cdot\nabla\phi^2dx=\theta\int_{\mathbb{R}^3}\phi^2\text{div}u_t\ dx\\
=&\theta\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2\text{div}u\ dx-\theta\int_{\mathbb{R}^3}(\phi^2)_t\text{div}u\ dx\\
=&\theta\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2\text{div}u\ dx-\theta\int_{\mathbb{R}^3}2\phi\phi_t\text{div}u\ dx\\
=&\theta\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2\text{div}u\ dx+\theta\int_{\mathbb{R}^3}u\cdot\nabla\phi^2\text{div}u\ dx\\
&+\theta(\gamma-1)\int_{\mathbb{R}^3}\phi^2(\text{div}u)^2dx\\
\le&\theta\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2\text{div}u\ dx+C\big(|u|_2|\nabla\phi|_2|\phi|_\infty|\text{div}u|_\infty+|\nabla u|_2^2|\phi^2|_\infty\big)\\
\le&\theta\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2\text{div}u\ dx+C\big(|\nabla\phi|_2|\text{div}u|_\infty+|\nabla u|_2^2\big)\\
\le&\theta\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2\text{div}u\ dx+C(|\nabla u|_2^2+|\text{div}u|_\infty+|\text{div}u|_\infty|\nabla\phi|_2^2),
\end{aligned}
\tag{38}
$$

where $\epsilon>0$ is a sufficiently small constant. Thus (35)-(38) imply

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\alpha|\nabla u|^2+(\alpha+\beta)|\text{div}u|^2-2\theta\phi^2\text{div}u\Big)dx+C|\nabla^2u|_2^2\\
\le&C\big((|\nabla u|_2^2+|\nabla\phi|_2^2)(|\text{div}u|_\infty+1)+|\text{div}u|_\infty\big).
\end{aligned}
\tag{39}
$$

Third, applying ∇ to $(19)_1$ and multiplying by $(\nabla\phi)^\top$, we have

$$
\begin{aligned}
&(|\nabla\phi|^2)_t+\text{div}(|\nabla\phi|^2u)+(\gamma-2)|\nabla\phi|^2\text{div}u\\
=&-2(\nabla\phi)^\top\cdot\nabla u\cdot(\nabla\phi)-(\gamma-1)\phi\nabla\phi\cdot\nabla\text{div}u\\
=&-2(\nabla\phi)^\top\cdot D(u)\cdot(\nabla\phi)-(\gamma-1)\phi\nabla\phi\cdot\nabla\text{div}u.
\end{aligned}
\tag{40}
$$

Integrating (40) over $\mathbb{R}^3$, we get

$$
\frac{d}{dt}|\nabla\phi|_2^2\le C(\epsilon)(|D(u)|_\infty+1)|\nabla\phi|_2^2+\epsilon|\nabla^2u|_2^2.
\tag{41}
$$

Adding (41) to (39), from the Gronwall's inequality and (16), we immediately obtain

$$\alpha|\nabla u(t)|_2^2 - 2\theta \int_{\mathbb{R}^3} \phi^2 \text{div} u \, dx + C|\nabla \phi(t)|_2^2 + \int_0^t |\nabla^2 u(s)|_2^2 ds \leq C,$$

that is

$$\begin{aligned} &\alpha|\nabla u(t)|_2^2 + C|\nabla \phi(t)|_2^2 + \int_0^t |\nabla^2 u(s)|_2^2 ds \\ \leq& C + 2\theta \int_{\mathbb{R}^3} \phi^2 \text{div} u \, dx \leq C(1 + |\nabla u|_2 |\phi|_2 |\phi|_\infty) \\ \leq& C + \frac{\alpha}{4}|\nabla u(t)|_2^2, \end{aligned}$$

which implies

$$|\nabla u(t)|_2^2 + |\nabla \phi(t)|_2^2 + \int_0^t |\nabla^2 u(s)|_2^2 ds \leq C, \quad 0 \leq t \leq T.$$

Finally, due to $u_t = -Lu - u \cdot \nabla u - 2\theta \phi \nabla \phi + \psi \cdot Q(u)$, we deduce that

$$\begin{aligned} \int_0^t |u_t|_2^2 ds \leq& C \int_0^t (|\nabla^2 u|_2^2 + |\nabla u|_3^2 |u|_6^2 + |\phi|_\infty^2 |\nabla \phi|_2^2 + |\nabla u|_3^2 |\psi|_6^2) ds \\ \leq& C. \end{aligned}$$

Thus we complete the proof of this lemma. □

Next, we proceed to improve the regularity of ϕ, ψ and u. First, we start with the estimates on the velocity.

Lemma 2.4. *Let (ρ, u) be the unique regular solution to the Cauchy problem (1) with (5)-(6) on $[0, \overline{T}) \times \mathbb{R}^3$ satisfying (16). Then*

$$\sup_{0 \leq t \leq T} \big(|u_t(t)|_2^2 + |u(t)|_{D^2}\big) + \int_0^T |\nabla u_t|_2^2 dt \leq C, \quad 0 \leq T < \overline{T}, \tag{42}$$

where C only depends on C_0 and $\overline{T}$.

Proof. From the standard elliptic estimate shown in Lemma 3.3 and

$$Lu = -u_t - u \cdot \nabla u - 2\theta \phi \nabla \phi + \psi \cdot Q(u), \tag{43}$$

one has

$$\begin{aligned} |u|_{D^2} \leq& C\big(|u_t|_2 + |u \cdot \nabla u|_2 + |\phi \nabla \phi|_2 + |\psi \cdot Q(u)|_2\big) \\ \leq& C\big(|u_t|_2 + |u|_6 |\nabla u|_3 + |\phi|_3 |\nabla \phi|_6 + |\psi|_6 |\nabla u|_3\big) \\ \leq& C\big(1 + |u_t|_2 + |u|_6 |\nabla u|_2^{\frac{1}{2}} |\nabla^2 u|_2^{\frac{1}{2}} + |\nabla u|_2^{\frac{1}{2}} |\nabla^2 u|_2^{\frac{1}{2}}\big) \\ \leq& C\big(1 + |u_t|_2 + |\nabla^2 u|_2^{\frac{1}{2}}\big) \\ \leq& C\big(1 + |u_t|_2\big) + \frac{1}{2}|u|_{D^2}, \end{aligned} \tag{44}$$

where we have used Sobolev inequalities, (16), (23), (29) and Lemmas 2.1-2.3. Then we immediately obtain that

$$|u|_{D^2} \leq C(1 + |u_t|_2). \tag{45}$$

Next, differentiating $(19)_3$ with respect to t, it reads

$$u_{tt} + Lu_t = -(u \cdot \nabla u)_t - 2\theta(\phi \nabla \phi)_t + (\psi \cdot Q(u))_t. \tag{46}$$

Multiplying (46) by u_t and integrating over $\mathbb{R}^3$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_t|_2^2+\alpha|\nabla u_t|_2^2\\
\le&\frac{1}{2}\frac{d}{dt}|u_t|_2^2+\int_{\mathbb{R}^3}\Big(\alpha|\nabla u_t|^2+(\alpha+\beta)|\text{div}u_t|^2\Big)dx\\
=&\int_{\mathbb{R}^3}\Big(-(u\cdot\nabla u)_t\cdot u_t+(\psi\cdot Q(u))_t\cdot u_t-2\theta(\phi\nabla\phi)_t\cdot u_t\Big)dx\\
\equiv:&\sum_{i=10}^{12}L_i.
\end{aligned}
\tag{47}
$$

Similarly, based on (16), (23), (29) and Lemmas 2.1-2.3, we estimate the right-hand side of (47) term by term as follows.

$$
\begin{aligned}
L_{10}=&-\int_{\mathbb{R}^3}(u\cdot\nabla u)_t\cdot u_t\ dx\\
=&-\int_{\mathbb{R}^3}\Big(\big(u_t\cdot\nabla u\big)\cdot u_t+\big(u\cdot\nabla u_t\big)\cdot u_t\Big)dx\\
=&-\int_{\mathbb{R}^3}\Big(u_t\cdot D(u)\cdot u_t-\frac{1}{2}(u_t)^2\text{div}u\Big)dx\\
\le&C|D(u)|_\infty|u_t|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
L_{11}=&\int_{\mathbb{R}^3}(\psi\cdot Q(u))_t\cdot u_t\ dx\\
=&\int_{\mathbb{R}^3}\psi\cdot Q(u)_t\cdot u_t\ dx+\int_{\mathbb{R}^3}\psi_t\cdot Q(u)\cdot u_t\ dx\\
=&\int_{\mathbb{R}^3}\psi\cdot Q(u)_t\cdot u_t\ dx-\int_{\mathbb{R}^3}\nabla\text{div}u\cdot Q(u)\cdot u_t\ dx\\
&+\int_{\mathbb{R}^3}u\cdot\psi\text{div}(Q(u)\cdot u_t)dx\\
\le&C\big(|\psi|_6|\nabla u_t|_2|u_t|_3+|\nabla^2u|_2|Q(u)|_\infty|u_t|_2\\
&+|\psi|_6|u|_6|\nabla^2u|_2|u_t|_6+|\psi|_6|u|_6|Q(u)|_6|\nabla u_t|_2\big)\\
\le&C\big(|\nabla u_t|_2|u_t|_2^{\frac{1}{2}}|\nabla u_t|_2^{\frac{1}{2}}+|\nabla^2u|_2|Q(u)|_\infty|u_t|_2\\
&+|\nabla u|_6|\nabla u_t|_2+|\nabla^2u|_2|\nabla u_t|_2\big)\\
\le&\frac{\alpha}{8}|\nabla u_t|_2^2+C(1+|D(u)|_\infty)(|u_t|_2^2+|u|_{D^2}^2),
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
L_{12}=&-\int_{\mathbb{R}^3}2\theta(\phi\nabla\phi)_t\cdot u_t\ dx\\
=&\theta\int_{\mathbb{R}^3}(\phi^2)_t\text{div}u_tdx=2\theta\int_{\mathbb{R}^3}\phi\phi_t\text{div}u_tdx\\
=&-2\theta\int_{\mathbb{R}^3}\phi(u\cdot\nabla\phi+\frac{\gamma-1}{2}\phi\text{div}u)\text{div}u_tdx\\
=&-\theta\int_{\mathbb{R}^3}(u\cdot\nabla\phi^2+(\gamma-1)\phi^2\text{div}u)\text{div}u_tdx
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\theta(\gamma-1)}{2}\int_{\mathbb{R}^3}\phi^2(\mathrm{div}u)_t^2dx-\theta\int_{\mathbb{R}^3}u\cdot\nabla\phi^2\mathrm{div}u_tdx\\
&= -\frac{\theta(\gamma-1)}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2(\mathrm{div}u)^2dx+\theta(\gamma-1)\int_{\mathbb{R}^3}u\phi\phi_t(\mathrm{div}u)^2dx\\
&\quad-\theta\int_{\mathbb{R}^3}u\cdot\nabla\phi^2\mathrm{div}u_tdx\\
&= -\frac{\theta(\gamma-1)}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2(\mathrm{div}u)^2dx-\theta(\gamma-1)\int_{\mathbb{R}^3}u\phi(u\cdot\nabla\phi)(\mathrm{div}u)^2dx\\
&\quad-\frac{\theta(\gamma-1)^2}{2}\int_{\mathbb{R}^3}u\phi^2(\mathrm{div}u)^3dx-\theta\int_{\mathbb{R}^3}u\cdot\nabla\phi^2\mathrm{div}u_tdx\\
&= -\frac{\theta(\gamma-1)}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2(\mathrm{div}u)^2dx+\frac{\theta(\gamma-1)}{2}\int_{\mathbb{R}^3}u\phi^2\nabla(\mathrm{div}u)^2dx\\
&\quad+\frac{\theta(\gamma-1)(3-\gamma)}{2}\int_{\mathbb{R}^3}\phi^2(\mathrm{div}u)^3dx-\theta\int_{\mathbb{R}^3}u\cdot\nabla\phi^2\mathrm{div}u_tdx\\
&\le -\frac{\theta(\gamma-1)}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2(\mathrm{div}u)^2dx+C\big(|u|_\infty|\phi|_\infty^2|\nabla u|_2|\nabla^2u|_2\\
&\quad+|\phi|_\infty^2|D(u)|_\infty|\nabla u|_2^2+|\phi|_\infty|\nabla\phi|_2|u|_\infty|\nabla u_t|_2\big)\\
&\le -\frac{\theta(\gamma-1)}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2(\mathrm{div}u)^2dx\\
&\quad+C\big(|u|_\infty||\nabla^2u|_2+|D(u)|_\infty+|u|_\infty|\nabla u_t|_2\big)
\end{aligned}\tag{49}
$$

$$
\begin{aligned}
&\le -\frac{\theta(\gamma-1)}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\phi^2(\mathrm{div}u)^2dx+\frac{\alpha}{4}|\nabla u_t|_2^2\\
&\quad+C(1+|D(u)|_\infty+|u|_{D^2}^2),
\end{aligned}\tag{50}
$$

where we also used Hölder's inequality, Young's inequality and

$$
|u|_\infty\le C|u|_{W^{1,3}}\le C(|u|_2^{\frac12}|\nabla u|_2^{\frac12}+|\nabla u|^{\frac12}|\nabla^2u|^{\frac12}).\tag{51}
$$

It is clear from (47)-(50) and (45) that

$$
\frac{d}{dt}(|u_t|_2^2+|\phi\mathrm{div}u|_2^2)+|\nabla u_t|_2^2\le C(1+|\nabla^2u|_2+|D(u)|_\infty)|u_t|_2^2.\tag{52}
$$

Integrating (52) over (τ,t) ($\tau\in(0,t)$), we have

$$
\begin{aligned}
&|u_t(t)|_2^2+|\phi\mathrm{div}u(t)|_2^2+\int_\tau^t|\nabla u_t(s)|_2^2ds\\
\le&|u_t(\tau)|_2^2+|\phi\mathrm{div}u(\tau)|_2^2+C\int_\tau^t\Big(\big(1+|\nabla^2u|_2+|D(u)|_\infty\big)|u_t|_2^2\Big)(s)ds.
\end{aligned}\tag{53}
$$

From the momentum equations $(19)_3$, we obtain

$$
\begin{aligned}
|u_t(\tau)|_2\le&C\big(|u\cdot\nabla u|_2+|\phi\nabla\phi|_2+|Lu|_2+|\psi\cdot Q(u)|_2\big)(\tau)\\
\le&C\big(|u|_\infty|\nabla u|_2+|\phi|_\infty|\nabla\phi|_2+|u|_{D^2}+|\psi|_6|\nabla u|_3\big)(\tau),
\end{aligned}\tag{54}
$$

which, together with the definition of regular solution, gives

$$
\begin{aligned}
&\limsup_{\tau\to0}|u_t(\tau)|_2\\
\le&C\big(|u_0|_\infty|\nabla u_0|_2+|\phi_0|_\infty|\nabla\phi_0|_2+|u_0|_{D^2}+|\psi_0|_6|\nabla u_0|_3\big)\le C_0.
\end{aligned}\tag{55}
$$

Letting $\tau \to 0$ in (53), applying the Gronwall's inequality, (16) and Lemma 2.3, we arrive at

$$|u_t(t)|_2^2 + |u(t)|_{D^2}^2 + \int_0^t |\nabla u_t(s)|_2^2 ds \le C, \quad 0 \le t \le T. \tag{56}$$

This completes the proof of this lemma. □

The following lemma gives bounds of $\nabla\phi$ and $\nabla^2 u$.

Lemma 2.5. *Let (ρ, u) be the unique regular solution to the Cauchy problem (1) with (5)-(6) on $[0, \overline{T}) \times \mathbb{R}^3$ satisfying (16). Then*

$$\sup_{0 \le t \le T} \big(\|\phi(t)\|_{W^{1,6}} + |\phi_t(t)|_6 \big) + \int_0^T |u(t)|_{D^{2,6}}^2 dt \le C, \quad 0 \le T < \overline{T}, \tag{57}$$

where C only depends on C_0 and $\overline{T}$.

Proof. First, taking $q = 6$ in Lemma 2.1, combing with (29) we have

$$\sup_{0 \le t \le T} \|\phi(t)\|_{W^{1,6}} \le C, \quad 0 \le T < \overline{T}.$$

Second, one has

$$\begin{aligned} |\phi_t|_6 =& \Big| u \cdot \nabla\phi + \frac{\gamma - 1}{2}\phi \mathrm{div} u \Big|_6 \\ \le& C(|\nabla\phi|_6 |u|_\infty + |\phi|_\infty |\mathrm{div} u|_6) \le C, \end{aligned} \tag{58}$$

where we have used Lemmas 2.3-2.4 and (29).

Third, according to

$$Lu = -u_t - u \cdot \nabla u - 2\theta\phi\nabla\phi + \psi \cdot Q(u), \tag{59}$$

and the standard elliptic estimate shown in Lemma 3.3, one has

$$\begin{aligned} |\nabla^2 u|_6 \le& C(|u_t|_6 + |u \cdot \nabla u|_6 + |\phi\nabla\phi|_6 + |\psi \cdot Q(u)|_6) \\ \le& C(|\nabla u_t|_2 + |u|_\infty |\nabla u|_6 + |\phi|_\infty |\nabla\phi|_6 + |\psi|_6 |Q(u)|_\infty) \\ \le& C(1 + |\nabla u_t|_2 + |D(u)|_2^{\frac{1}{4}} |\nabla D(u)|_6^{\frac{3}{4}}) \\ \le& C(1 + |\nabla u_t|_2 + |\nabla^2 u|_6^{\frac{3}{4}}) \\ \le& C(1 + |\nabla u_t|_2) + \frac{1}{2}|\nabla^2 u|_6, \end{aligned} \tag{60}$$

where we have used (16), (23), (29), Lemmas 2.1-2.4 and

$$\begin{aligned} &|\mathrm{div} u|_\infty \le C|D(u)|_\infty, \\ &|D(u)|_\infty \le C|D(u)|_2^{\frac{1}{4}} |\nabla D(u)|_6^{\frac{3}{4}}. \end{aligned} \tag{61}$$

Thus, (60) implies that

$$|\nabla^2 u|_6 \le C(1 + |\nabla u_t|_2). \tag{62}$$

Combing (62) with Lemma 2.4, one has

$$\int_0^t |u(s)|_{D^{2,6}}^2 ds \le C \int_0^t (1 + |\nabla u_t(s)|_2^2) ds \le C, \quad 0 \le t \le T. \tag{63}$$

The proof of this lemma is completed. □

Lemma 2.5 implies that

$$\int_0^t |\nabla u(\cdot,s)|_\infty ds \leq C, \tag{64}$$

for any $t \in [0,\overline{T})$ with $C > 0$ a finite number. Noting that (19) is essentially a parabolic-hyperbolic system, it is then standard to derive other higher order estimates for the regularity of the regular solutions. We will show this fact in the following lemma.

Lemma 2.6. *Let (ρ,u) be the unique regular solution to the Cauchy problem (1) with (5)-(6) on $[0,\overline{T})\times\mathbb{R}^3$ satisfying (16). Then*

$$\sup_{0\leq t\leq T}\big(|\phi(t)|^2_{D^2}+|\psi(t)|^2_{D^1}+\|\phi_t(t)\|_1^2+|\psi_t(t)|_2^2\big)$$
$$+\int_0^T\Big(|u(t)|^2_{D^3}+|\phi_{tt}(t)|_2^2\Big)dt\leq C,\quad 0\leq T<\overline{T},$$

where C only depends on C_0 and $\overline{T}$.

Proof. From $(19)_3$ and Lemma 3.3, we have

$$\begin{aligned}
|u|_{D^3}\leq& C\big(|u_t|_{D^1}+|u\cdot\nabla u|_{D^1}+|\phi\nabla\phi|_{D^1}+|\psi\cdot Q(u)|_{D^1}\big)\\
\leq& C\big(|u_t|_{D^1}+|u|_\infty|\nabla^2u|_2+|\nabla u|_6|\nabla u|_3+|\psi|_6|\nabla^2u|_3\\
&+|\nabla\phi|_6|\nabla\phi|_3+|\phi|_\infty|\nabla^2\phi|_2+|\nabla\psi|_2|D(u)|_\infty\big)\\
\leq& C\big(1+|u_t|_{D^1}+|\phi|_{D^2}+|u|^{\frac12}_{D^3}+|\psi|_{D^1}|D(u)|_\infty\big)\\
\leq& C\big(1+|u_t|_{D^1}+|\phi|_{D^2}+|\psi|_{D^1}|D(u)|_\infty\big)+\frac12|u|_{D^3},
\end{aligned}\tag{65}$$

where we have used Young's inequality, Lemma 2.5, (16), (23), (29) and (61). Thus (65) offers that

$$|u|_{D^3}\leq C\big(1+|u_t|_{D^1}+|\phi|_{D^2}+|D(u)|_\infty|\psi|_{D^1}\big). \tag{66}$$

Next, applying ∂_i $(i=1,2,3)$ to $(19)_2$ with respect to x, we obtain

$$\begin{aligned}
&(\partial_i\psi)_t+\sum_{l=1}^3A_l\partial_l\partial_i\psi+B\partial_i\psi+\partial_i\nabla\mathrm{div}u\\
=&\big(-\partial_i(B\psi)+B\partial_i\psi\big)+\sum_{l=1}^3\big(-\partial_i(A_l)\partial_l\psi\big).
\end{aligned}\tag{67}$$

Multiplying (67) by $2(\partial_i\psi)^\top$, integrating over $\mathbb{R}^3$, and then summing over i, noting that A_l $(l=1,2,3)$ are symmetric, it is not difficult to show that

$$\begin{aligned}
\frac{d}{dt}|\nabla\psi|_2^2\leq& C\int_{\mathbb{R}^3}\Big(|\mathrm{div}A||\nabla\psi|^2+|\nabla^3u||\nabla\psi|+|\nabla\psi|^2|\nabla u|\\
&+|\partial_i(B\psi)-B\partial_i\psi||\nabla\psi|\Big)dx\\
\leq& C\big(\big|\mathrm{div}A\big|_\infty|\nabla\psi|_2^2+|\nabla^3u|_2|\nabla\psi|_2+|\nabla\psi|_2^2|\nabla u|_\infty\\
&+|\partial_i(B\psi)-B\partial_i\psi|_2|\nabla\psi|_2\big),
\end{aligned}\tag{68}$$

where $\mathrm{div}A=\sum_{l=1}^{3}\partial_l A_l$. When $|\zeta|=1$, choosing $r=2,\ a=3,\ b=6$ in (83), we have

$$
\begin{aligned}
|\partial_i(B\psi)-B\partial_i\psi|_2=&|D^\zeta(B\psi)-BD^\zeta\psi|_2\le C|\nabla^2u|_3|\psi|_6\\
&\le C|\nabla^2 u|_2^{\frac12}|\nabla^3 u|_2^{\frac12}\le C|\nabla^3 u|_2^{\frac12}.
\end{aligned}
\tag{69}
$$

Thus

$$
\frac{d}{dt}|\nabla\psi|_2^2\le C\big(|\nabla u|_\infty|\nabla\psi|_2^2+|\nabla^3u|_2|\nabla\psi|_2+|\nabla^3u|_2^{\frac12}|\nabla\psi|_2\big).
\tag{70}
$$

Combining (70) with (66) and Lemma 3.1, we have

$$
\begin{aligned}
\frac{d}{dt}|\psi|_{D^1}^2\le&C(1+|\nabla u|_\infty)|\psi|_{D^1}^2+C(1+|\nabla^3u|_2)|\psi|_{D^1}\\
\le&C(1+|\nabla u|_\infty)|\psi|_{D^1}^2+C(1+|\phi|_{D^2}^2+|\nabla u_t|_2^2).
\end{aligned}
\tag{71}
$$

On the other hand, let $\nabla\phi=G=(G^{(1)},G^{(2)},G^{(3)})^\top$. Applying ∇^2 to $(19)_1$, we have

$$
0=(\nabla G)_t+\nabla(\nabla u\cdot G)+\nabla(\nabla G\cdot u)+\frac{\gamma-1}{2}\nabla\big(G\mathrm{div}u+\phi\nabla\mathrm{div}u\big),
\tag{72}
$$

similarly to the previous step, we multiply (72) by $2\nabla G$ and integrate it over $\mathbb{R}^3$ to derive

$$
\begin{aligned}
\frac{d}{dt}|G|_{D^1}^2\le&C\int_{\mathbb{R}^3}\big(|\nabla^2u||G|+|\nabla u||\nabla G|+|\nabla\phi||\nabla^2u|+|\phi||\nabla^3u|\big)|\nabla G|dx\\
\le&C\big(|G|_6|\nabla^2u|_3+|\nabla u|_\infty|\nabla G|_2+|\phi|_\infty|\nabla^3u|_2\big)|\nabla G|_2\\
\le&C\big(|\nabla^2u|_2^{\frac12}|\nabla^3u|_2^{\frac12}+|\nabla u|_\infty|G|_{D^1}+|u|_{D^3}\big)|G|_{D^1}\\
\le&C\big(|u|_{D^3}^{\frac12}+|u|_{D^3}\big)|G|_{D^1}+C|\nabla u|_\infty|G|_{D^1}^2\\
\le&C(1+|u|_{D^3})|G|_{D^1}+C|\nabla u|_\infty|G|_{D^1}^2\\
\le&C(1+|u_t|_{D^1}+|\phi|_{D^2}+|D(u)|_\infty|\nabla\psi|_2)|G|_{D^1}+C|\nabla u|_\infty|G|_{D^1}^2\\
\le&C(1+|u_t|_{D^1}+|\nabla u|_\infty|\psi|_{D^1})|G|_{D^1}+C(1+|\nabla u|_\infty)|G|_{D^1}^2\\
\le&C(1+|\nabla u|_\infty)(|G|_{D^1}^2+|\psi|_{D^1}^2)+C(1+|\nabla u_t|_2^2),
\end{aligned}
\tag{73}
$$

where we have used the Young's inequality, (29) and (66). This estimate, together with (71), gives that

$$
\frac{d}{dt}(|G|_{D^1}^2+|\psi|_{D^1}^2)\le C(1+|\nabla u|_\infty)(|G|_{D^1}^2+|\psi|_{D^1}^2)+C(1+|\nabla u_t|_2^2).
\tag{74}
$$

Then the Gronwall's inequality, (42), (64) and (74) imply

$$
|\phi(t)|_{D^2}^2+|\psi(t)|_{D^1}^2\le C,\quad 0\le t\le T.
\tag{75}
$$

Combing (75) with (66) and Lemma 2.4, one has

$$
\int_0^t|u(s)|_{D^3}^2ds\le C\int_0^t(1+|\nabla u_t(s)|_2^2)ds\le C,\quad 0\le t\le T.
\tag{76}
$$

Finally, using the following relations

$$\begin{aligned}\psi_t =& -\nabla(u\cdot\psi)-\nabla\mathrm{div}u, \quad \phi_t=-u\cdot\nabla\phi-\frac{\gamma-1}{2}\phi\mathrm{div}u,\\ \phi_{tt}=&-u_t\cdot\nabla\phi-u\cdot\nabla\phi_t-\frac{\gamma-1}{2}\phi_t\mathrm{div}u-\frac{\gamma-1}{2}\phi\mathrm{div}u_t,\end{aligned}\tag{77}$$

according to Hölder's inequality, (16), (29), Lemmas 2.1-2.5, one has

$$\begin{aligned}|\psi_t|_2 \le& C(|\nabla u\cdot\psi|_2+|u\cdot\nabla\psi|_2+|\nabla\mathrm{div}u|_2)\\ \le& C(|\nabla u|_3|\psi|_6+|u|_\infty|\nabla\psi|_2+|\nabla^2u|_2)\le C,\\ |\phi_t|_2\le& C(|u\cdot\nabla\phi|_2+|\phi\mathrm{div}u|_2)\\ \le& C(|u|_\infty|\nabla\phi|_2+|\phi|_\infty|\nabla u|_2)\le C,\\ |\nabla\phi_t|_2\le& C(|\nabla(u\cdot\nabla\phi)|_2+|\nabla(\phi\mathrm{div}u)|_2)\\ \le& C(|\nabla u\cdot\nabla\phi|_2+|\nabla^2\phi\cdot u|_2+|\nabla\phi\mathrm{div}u|_2+|\phi\nabla\mathrm{div}u|_2)\\ \le& C(|\nabla u|_3|\nabla\phi|_6+|u|_\infty|\nabla^2\phi|_2+|\nabla\phi|_6|\nabla u|_3+|\phi|_\infty|\nabla^2u|_2)\\ \le& C,\\ |\phi_{tt}|_2\le& C(|u_t\cdot\nabla\phi|_2+|u\cdot\nabla\phi_t|_2+|\phi_t\mathrm{div}u|_2+|\phi\mathrm{div}u_t|_2)\\ \le& C(|u_t|_6|\nabla\phi|_3+|u|_\infty|\nabla\phi_t|_2+|\phi_t|_6|\nabla u|_3+|\phi|_\infty|\nabla u_t|_2)\\ \le& C(1+|\nabla u_t|_2).\end{aligned}\tag{78}$$

Thus

$$\sup_{0\le t\le T}\left(\|\phi_t(t)\|_1^2+|\psi_t(t)|_2^2\right)\le C,$$

and according to (42), one has

$$\int_0^T|\phi_{tt}(t)|_2^2dt\le\int_0^T\left(1+|\nabla u_t(t)|_2^2\right)dt\le C.$$

The proof of this lemma is completed. □

Now we know from Lemmas 2.1-2.6 that, if the regular solution $(\rho,u)(x,t)$ exists up to the time $\overline{T}>0$, with the maximal time $\overline{T}<+\infty$ such that the assumption (16) holds, then

$$(\rho^{\frac{\gamma-1}{2}},\nabla\log\rho,u)|_{t=\overline{T}}=\lim_{t\to\overline{T}}(\rho^{\frac{\gamma-1}{2}},\nabla\log\rho,u)$$

satisfies the conditions imposed on the initial data (13). If we solve the system (1) with the initial time $\overline{T}$, then Theorem 1.1 ensures that $(\rho,u)(x,t)$ extends beyond $\overline{T}$ as the unique regular solution. This contradicts to the fact that $\overline{T}$ is the maximal existence time. We thus complete the proof of Theorem 1.3.

3. Appendix

3.1. **Some important lemmas.** In this subsection, we present some important lemmas which are frequently used in our previous proof. The first one is the well-known Gagliardo-Nirenberg inequality, which can be found in [9].

Lemma 3.1. [9] *Let $r\in(1,+\infty)$ and $h\in W^{1,p}(\mathbb{R}^3)\cap L^r(\mathbb{R}^3)$. Then the following inequality holds for some constant $C(c,p,r)$*

$$|h|_q\le C|\nabla h|_p^c|h|_r^{1-c},\tag{79}$$

where

$$c=\Big(\frac{1}{r}-\frac{1}{q}\Big)\Big(\frac{1}{r}-\frac{1}{p}+\frac{1}{3}\Big)^{-1},\quad 0\leq c\leq 1. \tag{80}$$

If $p<3$, then $q\in[r,\frac{3p}{3-p}]$ when $r<\frac{3p}{3-p}$; and $q\in[\frac{3p}{3-p},r]$ when $r\geq\frac{3p}{3-p}$. If $p=3$, then $q\in[r,+\infty)$. If $p>3$, then $q\in[r,+\infty]$.

Some common versions of this inequality can be written as

$$|f|_3\leq C|f|_2^{\frac{1}{2}}|\nabla f|_2^{\frac{1}{2}},\quad |f|_6\leq C|\nabla f|_2,\quad |f|_\infty\leq C|f|_2^{\frac{1}{4}}|\nabla f|_6^{\frac{3}{4}}, \tag{81}$$

which have be used frequently in our previous proof.

The second one can be found in Majda [17], and we omit its proof.

Lemma 3.2. [17] *Let positive constants r, a and b satisfy the relation*

$$\frac{1}{r}=\frac{1}{a}+\frac{1}{b}$$

and $1\leq a,\ b,\ r\leq+\infty$. $\forall s\geq 1$, if $f,g\in W^{s,a}(\mathbb{R}^3)\cap W^{s,b}(\mathbb{R}^3)$, then we have

$$|D^s(fg)-fD^sg|_r\leq C_s\big(|\nabla f|_a|D^{s-1}g|_b+|D^sf|_b|g|_a\big), \tag{82}$$

$$|D^s(fg)-fD^sg|_r\leq C_s\big(|\nabla f|_a|D^{s-1}g|_b+|D^sf|_a|g|_b\big), \tag{83}$$

where $C_s>0$ is a constant depending only on s, and $\nabla^s f$ means that the set of all elements of $\partial^\zeta f$ with $|\zeta|=s$.

The third one is on the regularity estimates for Lamé operator. For the elliptic problem

$$\begin{cases}-\alpha\Delta u-(\alpha+\beta)\nabla\mathrm{div}u=f,\\ u\to 0\quad\text{as } |x|\to+\infty,\end{cases} \tag{84}$$

one has

Lemma 3.3. [21] *If $u\in D^{k,q}$ with $1<q<+\infty$ is a weak solution to the problem (84), then*

$$|u|_{D^{k+2,q}}\leq C|f|_{D^{k,q}}, \tag{85}$$

where k is an integer and the constant $C>0$ depend on α,β and q. Moreover, if $u\in D^{k,q}$ is a weak solution to the following problem

$$-\Delta u=f,\quad u\to 0\quad \text{as } |x|\to+\infty, \tag{86}$$

then (85) holds and if $f=divg$, we also have

$$|u|_{D^{1,q}}\leq C|g|_{L^q}. \tag{87}$$

The proof can be obtained via the classical estimates from harmonic analysis, which can be found in [21] or [22]. We omit it here.

3.2. **Reformulation of the system** (1)**.** Now we show that, via introducing new variables

$$\phi=\rho^{\frac{\gamma-1}{2}},\ \psi=\nabla\log\rho=\frac{2}{\gamma-1}\nabla\phi/\phi, \tag{88}$$

the system (1) can be rewritten as

$$\begin{cases}\phi_t+\dfrac{\gamma-1}{2}\phi\mathrm{div}u+u\cdot\nabla\phi=0,\\[2ex] \psi_t+\nabla(u\cdot\psi)+\nabla\mathrm{div}u=0,\\[2ex] u_t+u\cdot\nabla u+2\theta\phi\nabla\phi+Lu=\psi\cdot Q(u).\end{cases} \tag{89}$$

Proof. First, from the momentum equation, one has

$$\begin{aligned}\rho u_t+\rho u\cdot\nabla u+\nabla P-\rho\mathrm{div}\big(\alpha(\nabla u{+}\nabla u^\top)+\beta\mathrm{div}u\mathbb{I}_3\big)&\\ =\nabla\rho\cdot[\alpha(\nabla u+\nabla u^\top)+\beta\mathrm{div}u\mathbb{I}_3],&\end{aligned}$$

where $P=A\rho^\gamma$, divide both side by ρ, one has

$$\begin{aligned}u_t+u\cdot\nabla u+A\gamma\rho^{\gamma-2}\nabla\rho-\mathrm{div}\big(\alpha(\nabla u+\nabla u^\top)+\beta\mathrm{div}u\mathbb{I}_3\big)&\\ =\frac{\nabla\rho}{\rho}\cdot[\alpha(\nabla u+\nabla u^\top)+\beta\mathrm{div}u\mathbb{I}_3].&\end{aligned}$$

Denote

$$\begin{aligned}Lu&=-\mathrm{div}\big(\alpha(\nabla u+\nabla u^\top)+\beta\mathrm{div}u\mathbb{I}_3\big),\\ Q(u)&=\alpha(\nabla u+\nabla u^\top)+\beta\mathrm{div}u\mathbb{I}_3,\quad \theta=\frac{A\gamma}{\gamma-1},\end{aligned}$$

we have

$$u_t+u\cdot\nabla u+2\theta\phi\nabla\phi+Lu=\psi\cdot Q(u). \tag{90}$$

Second, for $\psi=\nabla\log\rho$, one has

$$\begin{aligned}\psi_t&=(\nabla\log\rho)_t=\nabla(\log\rho)_t=\nabla\Big(\frac{\rho_t}{\rho}\Big)=\nabla\Big(\frac{-\mathrm{div}(\rho u)}{\rho}\Big)\\ &=\nabla\big(\frac{-\nabla\rho\cdot u-\rho\mathrm{div}u}{\rho}\big)=-\nabla(\nabla\log\rho\cdot u+\mathrm{div}u)\\ &=-\nabla\mathrm{div}u-u\cdot\nabla(\nabla\log\rho)-\nabla\log\rho\cdot\nabla u^\top\\ &=-\nabla\mathrm{div}u-u\cdot\nabla\psi-\psi\cdot\nabla u^\top\\ &=-\nabla\mathrm{div}u-\nabla(u\cdot\psi).\end{aligned} \tag{91}$$

Third, for $\phi=\rho^{\frac{\gamma-1}{2}}$, one has

$$\begin{aligned}\phi_t&=(\rho^{\frac{\gamma-1}{2}})_t=\frac{\gamma-1}{2}\rho^{\frac{\gamma-3}{2}}\rho_t\\ &=\frac{\gamma-1}{2}\rho^{\frac{\gamma-1}{2}}\frac{\rho_t}{\rho}=\frac{\gamma-1}{2}\rho^{\frac{\gamma-1}{2}}\frac{-\mathrm{div}(\rho u)}{\rho}\\ &=\frac{\gamma-1}{2}\phi\frac{-\rho\mathrm{div}u-\nabla\rho\cdot u}{\rho}\\ &=-\frac{\gamma-1}{2}\phi\mathrm{div}u-u\cdot\nabla\phi.\end{aligned} \tag{92}$$

Combing (90)-(92) together, we complete the proof of the transformation. □

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Yue Cao, School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China and School of Mathematics, Georgia Institute of Technology, Atlanta 30332, USA

Email address: cao_yue712@sjtu.edu.cn