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NORMALIZED SOLUTIONS FOR CHOQUARD EQUATIONS WITH GENERAL NONLINEARITIES

SHUAI YUAN, SITONG CHEN* AND XIANHUA TANG

ABSTRACT. In this paper, we prove the existence of positive solutions with prescribed L^2-norm to the following Choquard equation:

$$-\Delta u - \lambda u = (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^3,$$

where $\lambda \in \mathbb{R}, \alpha \in (0,3)$ and $I_\alpha : \mathbb{R}^3 \to \mathbb{R}$ is the Riesz potential. Under the weaker conditions, by using a minimax procedure and some new analytical techniques, we show that for any $c > 0$, the above equation possesses at least a couple of weak solution $(\bar{u}_c, \bar{\lambda}_c) \in \mathcal{S}_c \times \mathbb{R}^-$ such that $\|\bar{u}_c\|_2^2 = c$.

1. Introduction

This paper is dedicated to deal with the existence of normalized solutions to the generalized Choquard equation as follows:

$$-\Delta u - \lambda u = (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^3, \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $\alpha \in (0,3)$, $I_\alpha : \mathbb{R}^3 \to \mathbb{R}$ is the Riesz potential. Problem (1.1) is a nonlocal one due to the existence of the nonlocal nonlinearity. When $\lambda \in \mathbb{R}$ is a fixed and assigned a parameter or even with an additional external, the existence of (1.1) has been studied during the last decade.

For example, when $\lambda = -1$, $\alpha = 2$ and $F(u) = u^2$, (1.1) comes back to the description of the quantum theory of a polaron at rest by Pekar [22] and the modeling of an electron trapped in its own hole (in the work of Choquard in 1976), in a certain approximation to Hartree-Fock theory of one-component plasma [17]. The equation is also known as the Schrödinger-Newton equation, which was proposed by Penrose [23] in 1996 as a model of self-gravitating matter. Under this condition, the existence of nontrivial solutions was investigated by various variational methods by Lieb and Menzala [17, 19] and also by ordinary differential equations methods [11, 21, 27]. There are also many papers investigating the Choquard equation under the general pure nonlinearity condition,

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $N \geq 3$ and $\alpha \in (0, N)$, We can refer to [5, 15, 17, 20]. In [20], Moroz and Van Schaftingen obtained that problem (1.2) has a nontrivial solution when $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$.

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* Corresponding author: Sitong Chen.

Nowadays, since physicist are more and more interested in the normalized solutions, like [1, 2, 3, 13, 14, 30], mathematical researchers are committed to investigate the solutions with prescribed L^2-norm, that is, solutions which satisfy $\|u\|_2^2 = c > 0$ for a priori given constant. Such prescribed L^2-norm solutions of (1.2) can be obtained by looking for critical points of the following functional

$$I_N(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\mathrm{d}x \tag{1.3}$$

on the constraint

$$\mathcal{S}'_c = \left\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c\right\}, \tag{1.4}$$

where $F(u) = \int_0^u f(t)\mathrm{d}t$. In this sense, the parameter $\lambda \in \mathbb{R}$ cannot be fixed but regarded as a Lagrange multiplier, and each critical point $u_c \in \mathcal{S}'_c$ of $I_N|_{\mathcal{S}'_c}$, corresponds a Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that (u_c, λ_c) solves (weakly) (1.2). In particular, if $u_c \in \mathcal{S}'_c$ is a minimizer of problem

$$\sigma(c) := \inf_{u \in \mathcal{S}'_c} I_N(u), \tag{1.5}$$

then there exists $\lambda_c \in \mathbb{R}$ such that $I'_N(u_c) = \lambda_c u_c$, hence, (u_c, λ_c) is a solution of (1.2).

In [12], Jeanjean proved the existence of normalized solutions of the following Schrödinger equation

$$-\Delta u - f(u) = \lambda u \text{ in } \mathbb{R}^N, \tag{1.6}$$

where $N \geq 1$, $f : \mathbb{R} \to \mathbb{R}$ satisfies the following cases:

(f1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and f is odd;

(f2) $\exists\, \alpha, \beta \in \mathbb{R}$ with $\frac{2N+4}{N} < \alpha \leq \beta < 2^*$ such that

$$\alpha G(s) \leq g(s)s \leq \beta G(s),$$

where $G(s) = \int_0^s g(t)dt$ and $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 1, 2$;

(f3) let $\bar{G}(s) = g(s)s - 2G(s)$. Then $\bar{G}'$ exists and

$$\bar{G}'(s)s > \frac{2N+4}{N}\bar{G}(s).$$

Jeanjean deduced the existence of normalized solutions by dealing with the minimization problem

$$\inf_{u \in H^1(\mathbb{R}^N), \|u\|_2 = c} \int_{\mathbb{R}^N} \left[\frac{1}{2}|\nabla u|^2 - F(u)\right] \mathrm{d}x,$$

and the author verified the existence of the mountain pass structure on the constraint defined by $\mathcal{S}'_c$. Moreover, one of the highlights in the proof is that the auxiliary functional $\tilde{I} : H^1(\mathbb{R}^N) \times \mathbb{R} \to \mathbb{R}$ is introduced, defined by:

$$\tilde{I}(u, t) = \frac{e^{2t}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{1}{e^{Nt}}\int_{\mathbb{R}^N} G(e^{\frac{Nt}{2}}u)\mathrm{d}x.$$

By applying the new functional $\tilde{I}$, Jeanjean proved that for any fixed $c > 0$, problem (1.6) has a couple of weak solution $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}^-$ such that $\|u_c\|_2 = c$ under the conditions (f1), (f2) and (f3).

Bellazzini, Jeanjean and Luo [4] verified the existence of standing waves with prescribed L^2-norm for the following Schrödinger-Poisson equation:

$$-\Delta u + \left(|x|^{-1} * |u|^2\right) u - |u|^{q-2}u = \lambda u, \quad x \in \mathbb{R}^3, \tag{1.7}$$

where $q \in (\frac{10}{3}, 6)$, and which different from [12] is that the function defined by

$$\hat{I}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q \mathrm{d}x, \tag{1.8}$$

is no more bounded from below on the constraint:

$$\mathcal{S}_c = \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = c\right\}. \tag{1.9}$$

To overcome this difficulty, they first investigated the mountain-pass structure of $\hat{I}$ on the constraint $\mathcal{S}_c$, and then they show the existence of special bounded Palais-Smale sequence $\{u_n\}$ at the level $\gamma(c)$ which surrounds around the constraint set

$$\mathcal{M}'_c = \left\{u \in \mathcal{S}'_c : \hat{J}(u) := \frac{d}{dt}\hat{I}(u^t)|_{t=1} = 0\right\}, \tag{1.10}$$

that is $\hat{J}(u_n) = o(1)$, where $u^t(x) = t^{3/2}u(tx)$. In particular, $\mathcal{M}'_c$ used in [4] acts as a natural restriction and $\gamma(c)$ equals numerically to

$$\hat{m}(c) = \inf_{u \in \mathcal{M}'_c} \hat{I}(u). \tag{1.11}$$

As far as we know, there seems to be only one paper [16] dealing with the Choquard equation in the sense of prescribed L^2-norm, Li and Ye considered the Choquard equation (1.1) in the $N-$dimension space under the following conditions:

(F1′) $f(s) = 0$ for $s \leq 0$ and there exists $r \in (\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2})$ such that

$$\lim_{|s|\to 0} \frac{f(s)}{|s|^{r-2}s} = 0 \quad and \quad \lim_{|s|\to +\infty} \frac{F(s)}{|s|^r} = +\infty;$$

(F2′) $\lim_{|s|\to+\infty} \frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0$;

(F3′) there exists $\theta_1 \geq 1$ such that $\theta_1 \tilde{F}(s) \geq \tilde{F}(ts)$ for $s \in \mathbb{R}$ and $t \in [0,1]$, where $\tilde{F}(s) = f(s)s - \frac{N+\alpha+2}{N}F(s)$;

(F4′) $f(s)s < \frac{N+\alpha}{N-2}F(s)$ for all $s > 0$;

(F5′) let $\bar{F}(s) := f(s)s - \frac{N+\alpha}{N}F(s)$. $\bar{F}'(s)$ exists and

$$\bar{F}'(s)s > \frac{N+\alpha+2}{N}\bar{F}(s);$$

(F6′) there exists $0 < \theta_2 < 1$ and t_0 such that for all $s \in \mathbb{R}$ and $|t| \leq t_0$,

$$F(ts) \leq \theta_2 |t|^{\frac{N+\alpha+2}{N}} F(s).$$

In fact, the nonlinearity term in paper [16] needs the assumption $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$. A natural question is whether the above result in [16] on the existence of normalized solutions to (1.1) can be generalized to more general $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. The purpose of the present paper is to address this question. To this end, we introduce the following assumptions:

(F1) $f(s) = 0$ for $s \leq 0$ and there exists $r \in (\frac{3+\alpha+2}{3}, 3+\alpha)$ such that

$$\lim_{|s|\to 0} \frac{f(s)}{|s|^{r-2}s} = 0, \quad and \quad \lim_{|s|\to +\infty} \frac{F(s)}{|s|^r} = +\infty;$$

(F2) $\lim_{|s|\to+\infty} \frac{F(s)}{|s|^{3+\alpha}} = 0$ and $\lim_{|s|\to 0} \frac{F(s)}{|s|^2} = 0$;

(F3) $f(s)s < (3+\alpha)F(s)$ for all $s > 0$;

(F4) there exists $\theta_1 \geq 1$ such that $\theta_1 \tilde{F}(s) \geq \tilde{F}(ts)$ for $s \in \mathbb{R}$ and $t \in [0,1]$, where $\tilde{F}(s) = f(s)s - \frac{9+2\alpha}{6}F(s)$;

(F5) $[f(t)t - \frac{3+\alpha}{3}F(t)]/|t|^{\frac{6+2\alpha}{3}}t$ is nondecreasing on $(-\infty, 0)$ and $(0, +\infty)$.

In this paper, we define

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\mathrm{d}x \tag{1.12}$$

and

$$\mathcal{M}_c = \left\{ u \in \mathcal{S}_c : J(u) := \frac{d}{dt} I(u^t)|_{t=1} = 0 \right\},$$

where the definition of $\mathcal{S}_c$ is given by (1.9). Our main result is as follows:

Theorem 1.1. *Assume that* (F1)-(F5) *hold. Then for any constant* $c > 0$, (1.1) *has a couple of solutions* $(\bar{v}_c, \bar{\lambda}_c) \in \mathcal{S}_c \times \mathbb{R}^-$ *such that* $\bar{v}_c > 0$ *and*

$$I(\bar{v}_c) = \inf_{v \in \mathcal{M}_c} I(v) = \inf_{v \in \mathcal{S}_c} \max_{t>0} I(v^t) > 0.$$

Notice that, we proved the existence of normalized solution of problem (1.1) under the assumptions (F1)-(F5). Compared to [16], case (F5) plays an important role to overcome the difficulty caused by the absence of condition (F5′), that is, we generalized the problem (1.1) concerning the prescribe L^2-norm solutions to fit on more general nonlinearity term. But also, the absence of (F5′) in [16] causes new difficulties. In the proof, we present a new and more general approach to overcome this difficulty.

Now, we give our main idea for the proof of Theorem 1.1.

By (F1) and (F2), there exists some $C > 0$ such that

$$|F(s)| \leq C(|s|^r + |s|^{3+\alpha}). \tag{1.13}$$

By Hardy-Littlewood-Sobolev inequality: $\forall f \in L^p(\mathbb{R}^3)$, $g \in L^q(\mathbb{R}^3)$, if $0 < \alpha < 3$, $1 < p, q < +\infty$, and $\frac{1}{p} + \frac{1}{q} + \frac{3-\alpha}{3} = 2$, then

$$\int_{\mathbb{R}^3} (I_\alpha * f) g \mathrm{d}x \leq C\|f\|_p \|g\|_q,$$

we see that $F(u) \in L^{\frac{6}{3+\alpha}}(\mathbb{R}^3)$ for each $u \in H^1(\mathbb{R}^3)$ and $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$.

Inspired by [16], (F4) implies that

$$f(s)s - \frac{9+2\alpha}{6}F(s) \geq 0 \text{ for any } s \in \mathbb{R}. \tag{1.14}$$

Then for any $s > 0$, $\frac{F(s)}{s^{\frac{9+2\alpha}{6}}}$ is nondecreasing in $s > 0$. By (F1), we conclude that:

$$F(s) \geq 0 \text{ for any } s \in \mathbb{R}. \tag{1.15}$$

Then by (1.14), we see that $f(s) \geq 0$ for all $s \in \mathbb{R}$ and $F(s)$ is nondecreasing in $s \in \mathbb{R}$.

As in [4, 12], I is no more bounded from below on $\mathcal{S}_c$ by (F1), similarly we shall seek for a critical point satisfying a minimax characterization, i.e., we try to prove, I possesses a mountain pass geometry on the constrain $\mathcal{S}_c$.

Definition 1.2. For given $c > 0$, we say that $I(u)$ possesses a mountain pass geometry on $\mathcal{S}_c$ if there exists $\rho_c > 0$ such that

$$\gamma(c) = \inf_{g \in \Gamma_c} \max_{\tau \in [0,1]} I(g(\tau)) > \max_{g \in \Gamma_c} \max\{I(g(0)), I(g(1))\}, \tag{1.16}$$

where $\Gamma_c = \{g \in \mathcal{C}([0,1], \mathcal{S}_c) : \|\nabla g(0)\|_2^2 \leq \rho_c, I(g(1)) < 0\}$.

Let us recall this, to obtain this conclusion, the authors in [4] constructed some sequence of paths $\{g_n\} \subset \Gamma_c$ which have nice 'shape' properties, and by Taylor's formula which is relies on $I \in \mathcal{C}^2(H^1(\mathbb{R}^3), \mathbb{R})$, the author deduced a localization lemma concerning the specific (PS) sequence. Different from his work, in the present paper we shall investigate the following auxiliary functional:

$$\tilde{I}(v,t) = I(\beta(v,t)) = \frac{e^{2t}}{2}\|\nabla v\|_2^2 - \frac{1}{2e^{(3+\alpha)t}}\int_{\mathbb{R}^3}(I_\alpha * F(e^{\frac{3t}{2}}v))F(e^{\frac{3t}{2}}v)\mathrm{d}x,$$

and also we shall know the fact that $\tilde{I}$ possesses the same mountain pass structure on $\mathcal{S}_c \times \mathbb{R}$ as the functional $I\big|_{\mathcal{S}_c}$. Based on this fact, in Lemma 2.3, we find a $(\mathrm{PS})_{\gamma(c)}$ sequence $\{u_n\}$ with the additional property $J(u_n) \to 0$, and then prove the convergence of $\{u_n\}$, this idea comes from [12] in which the classical Schrödinger equation (1.6) was studied.

Since we have obtained the boundness of $\{u_n\}$, next we using scaling tramsform to verify the convergence of $\{u_n\}$. Because the nonlocal term and the gradient term in I scale differently, we overcome this difficulty by verifying whether $\gamma(c)$ is nonincreasing. As in [8], we first prove that $\gamma(c)$ is nonincreasing and then combining with the fact $\gamma(c) = m(c)$ which is verified in Lemma 2.10, then we can prove the convergence of $\{u_n\}$.

In [4] the fact I may be not $\mathcal{C}^2$ prevents us using the Implicit Function Theorem which influence above approach, then there needs new techniques and more subtle analyses to apply to more general $f \notin \mathcal{C}^1$. To deduce the convergence of $(\mathrm{PS})_{\gamma(c)}$ sequence $\{u_n\}$, we shall establish a new key inequality with the help of (F5), see Lemma 2.4, which is also inspired by [6, 7, 9, 10, 24, 26, 28]. In particular, we present a new and more general approach to recover the compactness of minimizing sequence.

Throughout the paper we use the following notations:

- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + uv)\mathrm{d}x, \quad \|u\| = (u,u)^{1/2}, \quad \forall\, u,v \in H^1(\mathbb{R}^3);$$

- $L^s(\mathbb{R}^3)$ $(1 \leq s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^3}|u|^s\mathrm{d}x\right)^{1/s}$;
- for any $u \in H^1(\mathbb{R}^3)$, $u^t(x) := t^{3/2}u(tx)$;
- for any $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$;
- $S = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \|\nabla u\|_2^2/\|u\|_6^2$;
- $C, C_1, C_2, \cdots$ denote positive constants possibly different in different places.

2. Preliminary results

To prove Theorem 1.1, recalling the Gagliardo-Nirenberg inequality, that is, let $p \in [2,6)$,

$$\|u\|_{L^p} \leq C_p\|\nabla u\|_{L^2}^{\beta}\|u\|_{L^2}^{1-\beta},$$

where $\beta = 3(\frac{1}{2} - \frac{1}{p})$.

In the following lemma, we show that I possesses the mountain pass geometry on the constraint $\mathcal{S}_c$.

Lemma 2.1. *Assume that* (F1), (F2) *and* (F4) *hold. Then for any* $c > 0$, *there exist* $0 < k_1 < k_2$ *and* $u_1, u_2 \in \mathcal{S}_c$ *such that* $u_1 \in \mathcal{A}_{k_1}$ *and* $u_2 \in \mathcal{A}^{k_2}$, *where*

$$\mathcal{A}_{k_1} = \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 \le k_1, I(u) > 0\right\} \tag{2.1}$$

and

$$\mathcal{A}^{k_2} = \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 > k_2, I(u) < 0\right\}. \tag{2.2}$$

Moreover, I has a mountain pass geometry on the constraint $\mathcal{S}_c$.

Proof. Given any $k > 0$, let

$$\mathcal{B}_k = \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 \le k\right\}. \tag{2.3}$$

We shall check that there exist $0 < k_1 < k_2$ such that

$$I(u) > 0, \quad \forall\ u \in \mathcal{B}_{k_2} \text{ and } \sup_{u \in \mathcal{B}_{k_1}} I(u) < \inf_{u \in \partial \mathcal{B}_{k_2}} I(u). \tag{2.4}$$

We have known that $F(u) \in L^{\frac{6}{3+\alpha}}(\mathbb{R}^3)$, then by the Hardy-Littlewood-Sobolev inequality, Sobolev embedding inequality and the Gagliardo-Nirenberg inequality, we can find

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\mathrm{d}x\right| &\le C\left(\int_{\mathbb{R}^3} |F(u)|^{\frac{6}{3+\alpha}}\mathrm{d}x\right)^{\frac{3+\alpha}{3}} \\
&\le C\left(\int_{\mathbb{R}^3} |u|^{\frac{6r}{3+\alpha}}\mathrm{d}x + \int_{\mathbb{R}^3} |u|^{2^*}\mathrm{d}x\right)^{\frac{3+\alpha}{3}} \\
&\le C\left[\|u\|_{\frac{6r}{3+\alpha}}^{2r} + \left(\int_{\mathbb{R}^3} |u|^{2^*}\mathrm{d}x\right)^{\frac{3+\alpha}{3}}\right] \\
&\le C\left(\|\nabla u\|_2^{3r-(3+\alpha)} + \|\nabla u\|_2^{6+2\alpha}\right).
\end{aligned} \tag{2.5}$$

Hence, we have that

$$I(u) \ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{C}{2}\|\nabla u\|_2^{3r-(3+\alpha)} - \frac{C}{2}\|\nabla u\|_2^{6+2\alpha}. \tag{2.6}$$

Since $3r - 3 - \alpha > 2$ and $6 + 2\alpha > 2$, it follows from (2.6) that there exist $k_2 > 0$ small and $\rho > 0$ such that

$$\inf_{u \in \partial\mathcal{B}_{k_2}} I(u) \ge \rho > 0 \text{ and } I(u) > 0 \text{ for } u \in \mathcal{B}_{k_2}. \tag{2.7}$$

On the other hand, use (2.5) again, we have

$$\begin{aligned}
|I(u)| &\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\left|\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\mathrm{d}x\right| \\
&\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{C}{2}\|\nabla u\|_2^{3r-(3+\alpha)} + \frac{C}{2}\|\nabla u\|_2^{6+2\alpha}.
\end{aligned} \tag{2.8}$$

which implies

$$\sup_{u \in \mathcal{B}_k} |I(u)| \to 0 \text{ as } k \to 0. \tag{2.9}$$

Combining (2.7) with (2.9), there exists $k_1 \in (0, k_2)$ small such that

$$\sup_{u \in \mathcal{B}_{k_1}} I(u) < \rho \le \inf_{u \in \partial\mathcal{B}_{k_2}} I(u).$$

and (2.4) follows.

Let

$$u^t(x) = t^{3/2}u(tx), \quad \forall\ t > 0, u \in H^1(\mathbb{R}^3). \tag{2.10}$$

Then $\|u^t\|_2 = \|u\|_2$, and so $u^t \in \mathcal{S}_c$ for any $u \in \mathcal{S}_c$ and $t > 0$. Note that

$$I(u^t) = \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{1}{2t^{3+\alpha}}\int_{\mathbb{R}^3}(I_\alpha * F(t^{3/2}u))F(t^{3/2}u)\mathrm{d}x. \tag{2.11}$$

Using (F1), (2.10) and Fatou's Lemma, which is inspired by [16], we can see that

$$\begin{aligned}&\liminf_{t\to\infty}\int_{\mathbb{R}^3}\left(I_\alpha * \frac{F(t^{3/2}u)}{|t^{3/2}u|}|u|^r\right)\frac{F(t^{3/2}u)}{|t^{3/2}u|}|u|^r\\ &\geq \int_{\mathbb{R}^3}\liminf_{t\to\infty}\left[\left(\frac{F(t^{3/2}u)}{|t^{3/2}u|}|u|^r\right)(x)\frac{F(t^{3/2}u)}{|t^{3/2}u|}|u|^r\right]\\ &\to +\infty.\end{aligned} \tag{2.12}$$

Hence, we have

$$\begin{aligned}\frac{I(u^t)}{t^{3r-3-\alpha}} &= \frac{\|\nabla u\|_2^2}{2t^{3r-(3+\alpha+2)}} - \frac{1}{2}\int_{\mathbb{R}^3}\left(I_\alpha * \frac{F(t^{3/2}u)}{|t^{3/2}u|}|u|^r\right)\frac{F(t^{3/2}u)}{|t^{3/2}u|}|u|^r\\ &\to -\infty \quad as \quad t \to +\infty.\end{aligned} \tag{2.13}$$

So $I(u^t) \to -\infty$ as $t \to +\infty$. For any $u \in \mathcal{S}_c$, there exist $t_1 > 0$ small and $t_2 > 1$ large such that

$$\|\nabla u^{t_1}\|_2^2 = t_1^2\|\nabla u\|_2^2 \leq k_1, \quad \|\nabla u^{t_2}\|_2^2 = t_2^2\|\nabla u\|_2^2 > k_2 \text{ and } I(u^{t_2}) < 0. \tag{2.14}$$

Set $u_1 = u^{t_1}$ and $u_2 = u^{t_2}$. Then (2.14) yields

$$\|\nabla u_1\|_2^2 \leq k_1, \qquad \|\nabla u_2\|_2^2 > k_2.$$

This fact indicates that $u_1 \in \mathcal{A}_{k_1}$ and $u_2 \in \mathcal{A}^{k_2}$.

We next claim that I possesses a mountain pass geometry on $\mathcal{S}_c$. For

$$\Gamma_c := \left\{g \in \mathcal{C}([0,1], \mathcal{S}_c) : \|\nabla g(0)\|_2^2 \leq k_1, I(g(1)) < 0\right\},$$

if $\Gamma_c \neq \emptyset$, then for any $g \in \Gamma_c$, (2.4) implies $\|\nabla g(0)\|_2^2 \leq k_1 < k_2 < \|\nabla g(1)\|_2^2$. Thus, by the intermediate value theorem, there exists $\tau_0 \in (0,1)$ such that $\|\nabla g(\tau_0)\|_2^2 = k_2$, i.e., $g(\tau_0) \in \partial\mathcal{B}_{k_2}$. It follows from (2.4) that

$$\max_{t\in[0,1]} I(g(t)) \geq I(g(\tau_0)) \geq \inf_{u\in\partial\mathcal{B}_{k_2}} I(u) > \sup_{u\in\mathcal{B}_{k_1}} I(u), \qquad \forall\ g \in \Gamma_c,$$

which, together with the arbitrariness of $g \in \Gamma_c$, implies

$$\gamma(c) = \inf_{g\in\Gamma_c}\max_{t\in[0,1]} I(g(t)) > \max_{g\in\Gamma_c}\max\{I(g(0)), I(g(1))\}. \tag{2.15}$$

Indeed, to obtain the desired conclusion, it suffices to check that $\Gamma_c \neq \emptyset$. For any $u \in \mathcal{S}_c$, set

$$g_0(\tau) = u^{(1-\tau)t_1+\tau t_2}, \qquad \forall\ \tau \in [0,1].$$

It follows from (2.14) that $g_0 \in \gamma(c)$. Hence, $\Gamma_c \neq \emptyset$ and the proof is completed. □

Next, inspired by [6, 12], we will show the existence of a (PS) sequence for the functional I on the constraint $\mathcal{S}_c$ attaching the property $J(u_n) \to 0$, where

$$J(u) = \|\nabla u\|_2^2 - \frac{3}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))\left[f(u)u - \frac{3+\alpha}{3}F(u)\right]\mathrm{d}x, \qquad \forall\ u \in H^1(\mathbb{R}^3). \tag{2.16}$$

To achieve this, we define a continuous map $\beta : H := H^1(\mathbb{R}^3) \times \mathbb{R} \to H^1(\mathbb{R}^3)$ by

$$\beta(v,t)(x) = e^{\frac{3t}{2}} v(e^t x) \ \text{ for } v \in H^1(\mathbb{R}^N),\ t \in \mathbb{R}, \text{ and } x \in \mathbb{R}^3, \tag{2.17}$$

where H is a Banach space equipped with the product norm $\|(v,t)\|_H := \left(\|v\|^2 + |t|^2\right)^{1/2}$. We introduce the following auxiliary functional:

$$\tilde{I}(v,t) = I(\beta(v,t)) = \frac{e^{2t}}{2}\|\nabla v\|_2^2 - \frac{1}{2e^{(3+\alpha)t}} \int_{\mathbb{R}^3} (I_\alpha * F(e^{\frac{3t}{2}} v)) F(e^{\frac{3t}{2}} v) \mathrm{d}x. \tag{2.18}$$

It is easy to see that $\tilde{I} \in \mathcal{C}^1(H,\mathbb{R})$, and for any $(w,s) \in H$,

$$\begin{aligned} \left\langle \tilde{I}'(v,t),(w,s) \right\rangle &= e^{2t} \int_{\mathbb{R}^3} \nabla v \cdot \nabla w \mathrm{d}x - \frac{1}{2e^{(3+\alpha)t}} \int_{\mathbb{R}^3} (I_\alpha * F(e^{\frac{3t}{2}} v)) f(e^{\frac{3t}{2}} v) e^{\frac{3t}{2}} w \mathrm{d}x \\ &\quad + e^{2t} s \|\nabla v\|_2^2 + \frac{(3+\alpha)s}{2e^{(3+\alpha)t}} \int_{\mathbb{R}^3} (I_\alpha * F(e^{\frac{3t}{2}} v)) F(e^{\frac{3t}{2}} v) \mathrm{d}x \\ &\quad - \frac{3s}{2e^{(3+\alpha)t}} \int_{\mathbb{R}^3} (I_\alpha * F(e^{\frac{3t}{2}} v)) f(e^{\frac{3t}{2}} v) e^{\frac{3t}{2}} v \mathrm{d}x. \end{aligned} \tag{2.19}$$

Set

$$\tilde{\gamma}(c) := \inf_{\tilde{g} \in \tilde{\Gamma}_c} \max_{\tau \in [0,1]} \tilde{I}(\tilde{g}(\tau)), \tag{2.20}$$

where

$$\tilde{\Gamma}_c = \{\tilde{g} \in \mathcal{C}([0,1], \mathcal{S}_c \times \mathbb{R}) : \tilde{g}(0) \in \mathcal{A}_{k_1} \times \{0\}, \tilde{g}(1) \in \mathcal{A}^{k_2} \times \{0\}\}.$$

and the sets $\mathcal{A}_{k_1}$ and $\mathcal{A}^{k_2}$ are defined in Lemma 2.1. For any $g \in \Gamma_c$, let $\tilde{g}_0(\tau) = (g(\tau), 0)$ for $\tau \in [0,1]$. It is easy to see that $\tilde{g}_0 \in \tilde{\Gamma}_c$, and then $\tilde{\Gamma}_c \neq \emptyset$. Since $\Gamma_c = \left\{\beta \circ \tilde{g} : \tilde{g} \in \tilde{\Gamma}_c\right\}$, then we know the minimax value of I agrees to $\tilde{I}$, i.e. $\gamma(c) = \tilde{\gamma}(c)$, moreover, (2.15) leads to

$$\tilde{\gamma}(c) = \gamma(c) > \max_{g \in \Gamma_c} \max\{I(g(0)), I(g(1))\} = \max_{\tilde{g} \in \tilde{\Gamma}_c} \max\{\tilde{I}(\tilde{g}(0)), \tilde{I}(\tilde{g}(1))\}. \tag{2.21}$$

Following by [29], we recall that for any $c > 0$, $\mathcal{S}_c$ is a submanifold of $H^1(\mathbb{R}^3)$ with codimension 1 and the tangent space at $\mathcal{S}_c$ is given

$$T_u = \left\{ v \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} uv \mathrm{d}x = 0 \right\}. \tag{2.22}$$

The norm of the $\mathcal{C}^1$ restriction functional $I|_{\mathcal{S}_c}$ is defined by

$$\|I|'_{\mathcal{S}_c}(u)\| = \sup_{v \in T_u, \|v\|=1} \langle I'(u), v \rangle. \tag{2.23}$$

And the tangent space at $(u,t) \in \mathcal{S}_c \times \mathbb{R}$ is given as

$$\tilde{T}_{u,t} = \left\{ (v,s) \in H : \int_{\mathbb{R}^3} uv \mathrm{d}x = 0 \right\}. \tag{2.24}$$

The norm of the derivative of the $\mathcal{C}^1$ restriction functional $\tilde{I}|_{\mathcal{S}_c \times \mathbb{R}}$ is defined by

$$\|\tilde{I}|'_{\mathcal{S}_c \times \mathbb{R}}(u,t)\| = \sup_{(v,s) \in \tilde{T}_{u,t}, \|(v,s)\|_H = 1} \left\langle \tilde{I}|'_{\mathcal{S}_c \times \mathbb{R}}(u,t), (v,s) \right\rangle. \tag{2.25}$$

Learning from [12, Proposition 2.2], we have the following proposition.

Proposition 1. *Assume that $\tilde{I}$ has a mountain pass geometry on the constraint $\mathcal{S}_c \times \mathbb{R}$. Let $\tilde{g}_n \in \tilde{\Gamma}_c$ be such that*

$$\max_{\tau\in[0,1]} \tilde{I}(\tilde{g}_n(\tau)) \le \tilde{\gamma}(c) + \frac{1}{n}. \tag{2.26}$$

Then there exists a sequence $(u_n, t_n) \in \mathcal{S}_c \times \mathbb{R}$ such that

(i) $\tilde{I}(u_n, t_n) \in \left[\tilde{\gamma}(c) - \frac{1}{n}, \tilde{\gamma}(c) - \frac{1}{n}\right]$;
(i) $\min_{\tau\in[0,1]} \|(u_n, t_n) - \tilde{g}_n(\tau)\|_H \le \frac{1}{\sqrt{n}}$;
(i) $\|\tilde{I}|'_{\mathcal{S}_c\times\mathbb{R}}(u_n, t_n)\| \le \frac{2}{\sqrt{n}}$, *i.e.,*

$$|\langle \tilde{I}'(u_n, t_n), (v, s)\rangle| \le \frac{2}{\sqrt{n}}\|(v, s)\|_H, \qquad \forall\ (v, s) \in \tilde{T}_{u_n,t_n}.$$

Applying proposition 1 to $\tilde{I}$ and also by [8], we conclude the following key lemma.

Lemma 2.2. *Assume that* (F1), (F2) *and* (F4) *hold. Then for any $c > 0$, there exists a sequence $\{v_n\} \subset \mathcal{S}_c$ such that*

$$I(v_n) \to \gamma(c) > 0, \quad I|'_{\mathcal{S}_c}(v_n) \to 0 \quad \textit{and} \quad J(v_n) \to 0. \tag{2.27}$$

Proof. Given $\{g_n\} \subset \Gamma_c$ satisfy

$$\max_{\tau\in[0,1]} I(g_n(\tau)) \le \gamma(c) + \frac{1}{n}. \tag{2.28}$$

In order to obtain the desired sequence, we first apply proposition 1 to $\tilde{I}$. We define

$$\tilde{g}_n(\tau) = (g_n(\tau), 0), \qquad \forall\ \tau \in [0, 1].$$

It is easy to know that $\tilde{g}_n \in \tilde{\Gamma}_c$ and $\tilde{I}(\tilde{g}_n(\tau)) = I(g_n(\tau))$. Since $\tilde{\gamma}(c) = \gamma(c)$, it follows from (2.28) that

$$\max_{\tau\in[0,1]} \tilde{I}(\tilde{g}_n(\tau)) \le \tilde{\gamma}(c) + \frac{1}{n}. \tag{2.29}$$

From the preceding proposition 1, there exists a sequence $\{(u_n, t_n)\} \subset \mathcal{S}_c \times \mathbb{R}$ such that

(i) $\tilde{I}(u_n, t_n) \to \tilde{\gamma}(c)$;
(i) $\min_{\tau\in[0,1]} \|(u_n, t_n) - (g_n(\tau), 0)\|_H \to 0$;
(i) $\|\tilde{I}|'_{\mathcal{S}_c\times\mathbb{R}}(u_n, t_n)\| \le \frac{2}{\sqrt{n}}$.

Set $v_n := \beta(u_n, t_n)$, and the definition of β is given in (2.17). Since $v_n \in \mathcal{S}_c$ and $\tilde{\gamma}(c) = \gamma(c)$, it follows from (i) that

$$I(v_n) \to \gamma(c). \tag{2.30}$$

Accoring to (2.19) and (ii), we derive

$$\langle I'(v_n), w\rangle = \left\langle \tilde{I}'(u_n, t_n), (\beta(w, -t_n), 0)\right\rangle \le \frac{2}{\sqrt{n}}\|(\beta(w, -t_n), 0)\|_H, \qquad \forall\ w \in T_{v_n}. \tag{2.31}$$

To prove $I|'_{\mathcal{S}_c}(v_n) \to 0$, by (2.31), it suffices to prove that $\{(\beta(w, -t_n), 0)\}$ is uniformly bounded in H and $\{(\beta(w, -t_n), 0)\} \subset \tilde{\Gamma}_{u_n,t_n}$. For any $w \in T_{v_n}$, i.e.,

$$\int_{\mathbb{R}^3} v_n w \mathrm{d}x = \int_{\mathbb{R}^3} e^{\frac{3t_n}{2}} u_n(e^{t_n}x)w(x)\mathrm{d}x = 0,$$

we can see that

$$\int_{\mathbb{R}^3} u_n(x)\beta(w,-t_n)\mathrm{d}x = \int_{\mathbb{R}^3} u_n(x)e^{\frac{-3t_n}{2}} w(e^{-t_n}x)\mathrm{d}x = \int_{\mathbb{R}^3} e^{\frac{3t_n}{2}} u_n(e^{t_n}x)w(x)\mathrm{d}x = 0,$$

follows

$$(\beta(w,-t_n),0) \in \tilde{\Gamma}_{u_n,t_n}. \tag{2.32}$$

Then by (ii), we have

$$|t_n| \le \min_{\tau\in[0,1]} \|(u_n,t_n) - \tilde{g}_n(\tau)\|_H \le 1 \text{ for large } n \in \mathbb{N},$$

which leads to

$$\|(\beta(w,-t_n),0)\|_H^2 = \|\beta(w,-t_n)\|^2 = e^{-2t_n}\|\nabla w\|_2^2 + \|w\|_2^2 \le e^2\|w\|^2 \text{ for large } n \in \mathbb{N}. \tag{2.33}$$

This shows that $\{(\beta(w,-t_n),0)\} \subset \tilde{\Gamma}_{u_n,t_n}$ is uniformly bounded in H, and so $I|'_{\mathcal{S}_c}(v_n) \to 0$. In the end, by (iii), we obtain

$$\left|\langle \tilde{I}'(u_n,t_n),(0,1)\rangle\right| = J(\beta(u_n,t_n)) = J(v_n) = o(1). \tag{2.34}$$

Hence, $\{v_n\}$ satisfies (2.27). □

In connection with the additional minimax characterization of $\gamma(c)$, we have the following Lemma 2.10. To achieve this goal, we have to establish some new inequalities, which is the crucial procedure for our convenience to obtain our final conclusion of this paper.

Lemma 2.3. *Assume that* (F1), (F2), (F4) *and* (F5) *hold. Then*

$$\begin{aligned} h(t) :=& \frac{(3+\alpha)t^{\frac{3}{2}}}{N}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)\mathrm{d}x + \frac{1}{2(t^{3+\alpha})}\int_{\mathbb{R}^3}(I_\alpha * F(t^{\frac{3}{2}}u))F(t^{\frac{3}{2}}u)\mathrm{d}x \\ &- t^{\frac{3}{2}}\int_{\mathbb{R}^3}(I_\alpha * F(u))f(u)u\mathrm{d}x \ge 0, \qquad \forall\ t>0. \end{aligned} \tag{2.35}$$

Proof. For any $t\in\mathbb{R}$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}h(t) =& \frac{(3+\alpha)t^{\frac{1}{2}}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)\mathrm{d}x - \frac{3+\alpha}{2t^{3+\alpha+1}}\int_{\mathbb{R}^3}(I_\alpha * F(t^{\frac{3}{2}}u))F(t^{\frac{3}{2}}u)\mathrm{d}x \\ &+ \frac{3}{2t^{3+\alpha+1}}\int_{\mathbb{R}^3}(I_\alpha * F(t^{\frac{3}{2}}u))f(t^{\frac{3}{2}}u)(t^{\frac{3}{2}}u)\mathrm{d}x - \frac{3t^{\frac{1}{2}}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))f(u)u\mathrm{d}x \end{aligned} \tag{2.36}$$

Now we only need to study $q(t,\tau_1,\tau_2)$ which is defined by the following form:

$$\begin{aligned} q(t,\tau_1,\tau_2) =& F(t^{\frac{3}{2}}\tau_1)\left[\frac{3}{2t^{3+\alpha+1}}f(t^{\frac{3}{2}}\tau_2)t^{\frac{3}{2}}\tau_2 - \frac{3+\alpha}{2t^{3+\alpha+1}}F(t^{\frac{3}{2}}\tau_2)\right] \\ &- F(\tau_1)\left[\frac{3t}{2}f(\tau_2)\tau_2 - \frac{(3+\alpha)t^{\frac{1}{2}}}{2}F(\tau_2)\right] \end{aligned}$$

$$= \frac{3}{2}|\tau_2|^{\frac{9+2\alpha}{3}} t^{\frac{1}{2}} F(t^{\frac{3}{2}}\tau_1) \left[\frac{f(t^{\frac{3}{2}}\tau_2)t^{\frac{3}{2}}\tau_2 - \frac{3+\alpha}{3}F(t^{\frac{3}{2}}\tau_2)}{|t^{\frac{3}{2}}\tau_2|^{\frac{9+2\alpha}{3}}} \right]$$

$$- \frac{3}{2}|\tau_2|^{\frac{9+2\alpha}{3}} t^{\frac{1}{2}} F(\tau_1) \left[\frac{f(\tau_2)\tau_2 - \frac{3+\alpha}{3}F(\tau_2)}{|\tau_2|^{\frac{9+2\alpha}{3}}} \right]$$

$$\begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1, \end{cases}$$

By (F5) and (1.15), we can easily get the above conclusion, which implies that $h(t) \geq h(1) = 0$ for all $t > 0$, This shows that (2.35) holds. □

By the preceding scaling (2.10), we have

$$I(u^t) = \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{1}{2t^{3+\alpha}} \int_{\mathbb{R}^3} (I_\alpha * F(t^{3/2}u))F(t^{3/2}u)\mathrm{d}x. \tag{2.37}$$

It can be easily checked that $J(u) = \frac{\mathrm{d}}{\mathrm{d}t} I(u^t)\Big|_{t=1}$, where the definition of J is given in (2.16). Set

$$h_1(t) := 4t^{\frac{3}{2}} - 3t^2 - 1, \quad t \geq 0. \tag{2.38}$$

After basic calculations, we can see

$$h_1(1) = 0, \qquad h_1(t) > 0, \qquad \forall\ t \in [0,1) \cup (1,+\infty). \tag{2.39}$$

Inspired by [7, 25], we obtain the following key inequality.

Lemma 2.4. *Assume that* (F1), (F2), (F4) *and* (F5) *hold. Then*

$$I(u) \geq I\left(u^t\right) + \frac{2\left(1 - t^{\frac{3}{2}}\right)}{3} J(u) + \frac{h_1(t)}{6}\|\nabla u\|_2^2, \quad \forall\ u \in H^1(\mathbb{R}^3),\ t > 0 \tag{2.40}$$

and

$$I(u) \geq \frac{2}{3} J(u) - \frac{1}{6}\|\nabla u\|_2^2, \qquad \forall\ u \in H^1(\mathbb{R}^3). \tag{2.41}$$

Proof. By (1.12), (2.16), (2.35), (2.37), (2.38) and (2.39), we have

$$\begin{aligned}
& I(u) - I\left(u^t\right) \\
&= \frac{1-t^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\mathrm{d}x \\
&+ \frac{1}{2t^{3+\alpha}} \int_{\mathbb{R}^3} (I_\alpha * F(t^{\frac{3}{2}}u))F(t^{\frac{3}{2}}u)\mathrm{d}x \\
&= \frac{2(1-t^{\frac{3}{2}})}{3} J(u) + (1-t^{\frac{3}{2}}) \int_{\mathbb{R}^3} (I_\alpha * F(u))f(u)u\mathrm{d}x \\
&- \frac{3 + 2(3+\alpha)(1-t^{\frac{3}{2}})}{6} \int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\mathrm{d}x \\
&+ \frac{4t^{\frac{3}{2}} - 3t^2 - 1}{6}\|\nabla u\|_2^2 + \frac{1}{2t^{3+\alpha}} \int_{\mathbb{R}^3} (I_\alpha * F(t^{\frac{3}{2}}u))F(t^{\frac{3}{2}}u)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(1-t^{\frac{3}{2}})}{3}J(u) + \frac{h_1(t)}{6}\|\nabla u\|_2^2 + h(t) \\
&+ \int_{\mathbb{R}^3}(I_\alpha * F(u))\left[f(u)u - \frac{9+2\alpha}{6}F(u)\right] \\
&\geq \frac{2(1-t^{\frac{3}{2}})}{3}J(u) + \frac{h_1(t)}{6}\|\nabla u\|_2^2, \qquad \forall\ u \in H^1(\mathbb{R}^3),\quad t>0.
\end{aligned} \tag{2.42}
$$

This shows that (2.40) holds. Letting $t \to 0$ in (2.40), we derive that (2.41) holds. □

Following the Lemma 2.4 naturally, we obtain the following corollary.

Corollary 1. *Assume that* (F1), (F2), (F4) *and* (F5) *hold. Then*

$$
I(u) = \max_{t>0} I\left(u^t\right), \qquad \forall\ u \in \mathcal{M}_c. \tag{2.43}
$$

Lemma 2.5. *Assume that* (F1), (F2), (F4) *and* (F5) *hold. Then for any* $u \in H^1(\mathbb{R}^3)\setminus\{0\}$*, there exists a unique* $t_u > 0$ *such that* $u^{t_u} \in \mathcal{M}_c$*.*

Proof. Let $u \in H^1(\mathbb{R}^3)\setminus\{0\}$ be fixed and define a function $\zeta(t) := I\left(u^t\right)$ on $(0,\infty)$. Clearly, by (1.12) and (2.16), we have

(2.44)

$$
\begin{aligned}
&\zeta'(t) = 0 \\
&\Leftrightarrow\quad t\|\nabla u\|_2^2 = \frac{3}{2t^{3+\alpha+1}}\int_{\mathbb{R}^3} I_\alpha * F(t^{3/2}u))\left[f(t^{3/2}u)t^{3/2}u - \frac{N+\alpha}{3}F(t^{3/2}u)\right]\mathrm{d}x \\
&\Leftrightarrow\quad \frac{1}{t}J\left(u^t\right) = 0 \quad\Leftrightarrow\quad u^t \in \mathcal{M}_c.
\end{aligned}
$$

Note that (F1) leads to

$$
|F(s)| \leq |s|^r \ \ for \quad \forall\ |s| \leq \delta, \tag{2.45}
$$

From (2.37) and (2.45), we infer that

$$
I(u^t) \geq \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{1}{2}t^{3r-3-\alpha}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)\mathrm{d}x, \tag{2.46}
$$

which, together with $2 < 3r-3-\alpha$ implies that $\zeta(t) > 0$ for $t > 0$ small enough. Moreover, by (2.37) and (2.13), it is easy to verify that $\lim_{t\to 0}\zeta(t) = 0$ and $\zeta(t) < 0$ for t large enough. We conclude that $\max_{t\in(0,\infty)}\zeta(t)$ is achieved at $t_u > 0$ so that $\zeta'(t_u) = 0$ and $u^{t_u} \in \mathcal{M}_c$.

In order to finish this proof, it is suffices to show that t_u is unique for any $u \in H^1(\mathbb{R}^3)\setminus\{0\}$. Otherwise, for any given $u \in H^1(\mathbb{R}^3)\setminus\{0\}$, there exist positive constants $t_1 \neq t_2$ such that $u^{t_1}, u^{t_2} \in \mathcal{M}_c$, i.e. $J\left(u^{t_1}\right) = J\left(u^{t_2}\right) = 0$, then (2.39) and (2.40) lead to

$$
\begin{aligned}
I\left(u^{t_1}\right) \quad &> \quad I\left(u^{t_2}\right) + \frac{2\left[t_1^{\frac{3}{2}} - t_2^{\frac{3}{2}}\right]}{3t_1^{\frac{3}{2}}}J\left(u^{t_1}\right) = I\left(u^{t_2}\right) \\
&> \quad I\left(u^{t_1}\right) + \frac{2\left[t_2^{\frac{3}{2}} - t_1^{\frac{3}{2}}\right]}{3t_2^{\frac{3}{2}}}J\left(u^{t_2}\right) = I\left(u^{t_1}\right).
\end{aligned} \tag{2.47}
$$

This contradiction shows us that $t_u > 0$ is unique for any $u \in H^1(\mathbb{R}^3)\setminus\{0\}$. □

Combining the Corollary 1 and Lemma 2.5, we can easily obtain the following facts.

Lemma 2.6. *Assume that* (F1), (F2), (F4) *and* (F5) *hold. Then*

$$\inf_{u\in\mathcal{M}_c} I(u) = m(c) = \inf_{u\in\mathcal{S}_c}\max_{t>0} I\left(u^t\right).$$

Lemma 2.7. *Assume that* (F1), (F2), (F4) *and* (F5) *hold. The function* $c\mapsto m(c)$ *is nonincreasing on* $(0,\infty)$.

Proof. To achieve this purpose, it is sufficient to verify whether in the condition that for any $c_1 < c_2$ and $\varepsilon > 0$ arbitrary, we have

$$m(c_2) \le m(c_1) + \varepsilon \tag{2.48}$$

By the definition of $m(c_1)$, there exists $u \in \mathcal{M}_{c_1}$ such that $I(u) \le m(c_1) + \varepsilon/4$. Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ satisfies

$$\eta(x) = \begin{cases} 1, & |x| \le 1, \\ \in [0,1], & 1 \le |x| < 2, \\ 0, & |x| \ge 2. \end{cases}$$

For any small $\delta \in (0,1]$, let

$$u_\delta(x) = \eta(\delta x)\cdot u(x). \tag{2.49}$$

It is easy to obtain that $u_\delta \to u$ in $H^1(\mathbb{R}^3)$ as $\delta \to 0$. Then we have

$$I(u_\delta) \to I(u) \le m(c_1) + \frac{\varepsilon}{4}, \qquad J(u_\delta) \to J(u) = 0. \tag{2.50}$$

From Lemma 2.5, for any $\delta > 0$, there exists $t_\delta > 0$ such that ${u_\delta}^{t_\delta} \in \mathcal{M}_c$ for some $c > 0$. Next we show that $\{t_\delta\}$ is bounded. Actually, if $t_\delta \to \infty$ as $\delta \to 0$, since $u_\delta \to u \ne 0$ in $H^1(\mathbb{R}^3)$ as $\delta \to 0$, in view of (F1), we infer that

$$\begin{aligned} 0 = \lim_{\delta\to 0}\frac{I(u_\delta^{t_\delta})}{t_\delta^2} &= \frac{1}{2}\|\nabla u\|_2^2 + \lim_{\delta\to 0}\int_{\mathbb{R}^3}\left| I_\alpha * \frac{F(t_\delta^{3/2}u)}{t_\delta^{\frac{5+\alpha}{2}}}\right| \frac{F(t_\delta^{3/2})u}{t_\delta^{\frac{5+\alpha}{2}}}\mathrm{d}x \\ &= -\infty, \end{aligned}$$

which is a contradiction. So we may assume that up to a subsequence, $t_\delta \to \bar t$ as $\delta \to 0$, and so $J({u_\delta}^{t_\delta}) \to J(u^{\bar t})$, which jointly with $J(u) = 0$ implies that $\bar t = 1$. By (2.40), we have

$$I(u_\delta^{t_\delta}) \le I(u_\delta) - \frac{2(1-t_\delta^{\frac{3}{2}})}{3}J(u_\delta) + \frac{h_1(t_\delta)}{6}\|\nabla u_\delta^{t_\delta}\|_2^2,$$

which, together with (2.50), implies that there exists $\delta_0 \in (0,1)$ small enough such that

$$I\left(u_{\delta_0}^{t_{\delta_0}}\right) \le I(u_{\delta_0}) + \frac{\varepsilon}{8} \le I(u) + \frac{\varepsilon}{4} \le m(c_1) + \frac{\varepsilon}{2}. \tag{2.51}$$

Let $v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ be such that $\mathrm{supp} v \subset B_{2R_{\delta_0}} \setminus B_{R_{\delta_0}}$ with $R_{\delta_0} = 2/\delta_0$. Set

$$v_0 = \frac{c_2 - \|u_{\delta_0}\|_2^2}{\|v\|_2^2}v,$$

for which we have $\|v_0\|_2^2 = c_2 - \|u_{\delta_0}\|_2^2$. For $\lambda \in (0,1)$, we define $w_\lambda = u_{\delta_0} + v_0^\lambda$ with $\|v_0^\lambda\|_2 = \|v_0\|_2$. Observing that

$$\mathrm{dist}\left\{\mathrm{supp} u_{\delta_0}, \mathrm{supp} v_0^\lambda\right\} \ge \frac{2R_{\delta_0}}{\lambda} - R_{\delta_0} = \frac{2}{\delta_0}\left(\frac{2}{\lambda} - 1\right) > 0, \tag{2.52}$$

following which we can easily obtain

$$|w_\lambda(x)|^2 = |u_{\delta_0}(x) + v_0^\lambda(x)|^2 = |u_{\delta_0}(x)|^2 + |v_0^\lambda(x)|^2, \tag{2.53}$$

$$\|w_\lambda\|_2^2 = \|u_{\delta_0} + v_0^\lambda\|_2^2 = \|u_{\delta_0}\|^2 + \|v_0^\lambda\|_2^2 = \|u_{\delta_0}\|^2 + \|v_0\|_2^2, \tag{2.54}$$

$$\|\nabla w_\lambda\|_2^2 = \|\nabla u_{\delta_0} + \nabla v_0^\lambda\|_2^2 = \|\nabla u_{\delta_0}\|^2 + \|\nabla v_0^\lambda\|_2^2 = \|\nabla u_{\delta_0}\|^2 + \lambda^2\|\nabla v_0\|_2^2, \tag{2.55}$$

$$\begin{aligned} \int_{\mathbb{R}^3} F(w_\lambda)\mathrm{d}x &= \int_{\mathbb{R}^3} F(u_{\delta_0} + v_0^\lambda)\mathrm{d}x = \int_{\mathbb{R}^3} F(u_{\delta_0})\mathrm{d}x + \int_{\mathbb{R}^3} F(v_0^\lambda)\mathrm{d}x \\ &= \int_{\mathbb{R}^3} F(u_{\delta_0})\mathrm{d}x + \lambda^{-3}\int_{\mathbb{R}^3} F(\lambda^{\frac{3}{2}}v_0)\mathrm{d}x \end{aligned} \tag{2.56}$$

Then (2.55), (2.56) and (F2) imply that as $\lambda \to 0$,

$$\|\nabla w_\lambda\|_2^2 \to \|\nabla u_{\delta_0}\|^2, \qquad \int_{\mathbb{R}^3} F(w_\lambda)\mathrm{d}x \to \int_{\mathbb{R}^3} F(u_{\delta_0})\mathrm{d}x \tag{2.57}$$

and by (2.57), we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{F(w_\lambda(x))F(w_\lambda(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{F(u_{\delta_0}(x)F(u_{\delta_0}(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y, \tag{2.58}$$

which lead to

$$I(w_\lambda) \to I(u_{\delta_0}) \ \text{ and } \ J(w_\lambda) \to J(u_{\delta_0}). \tag{2.59}$$

By (2.54), we have $w_\lambda \in \mathcal{S}_{c_2}$. Using Lemma 2.5, there always exists $t_\lambda > 0$ such that ${w_\lambda}^{t_\lambda} \in \mathcal{M}_{c_2}$. As the preceding proof, the sequence $\{t_\lambda\}$ is bounded. Then assume that up to a subsequence, $t_\lambda \to \hat{t}$ as $\lambda \to 0$. Combining the convergence (2.57) with the Hardy-Littlewood-Sobolev inequality, a standard argument can be used to show that as $\lambda \to 0$,

$$\int_{\mathbb{R}^3} F\left({w_\lambda}^{t_\lambda}\right)\mathrm{d}x \to \int_{\mathbb{R}^3} F\left({u_{\delta_0}}^{\hat{t}}\right)\mathrm{d}x. \tag{2.60}$$

and

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{F(w_\lambda^{t_\lambda}(x))F(w_\lambda^{t_\lambda}(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{F(u_{\delta_0}^{\hat{t}}(x)F(u_{\delta_0}^{\hat{t}}(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \tag{2.61}$$

Deduced by (2.60) and (2.61), there exists $\lambda_0 \in (0,1)$ small enough such $I\left({w_\lambda}^{t_\lambda}\right) \le I\left({u_{\delta_0}}^{\hat{t}}\right) + \varepsilon/2$. And then it follows from (2.43) and (2.51) that

$$\begin{aligned} m(c_2) &\le I\left({w_\lambda}^{t_\lambda}\right) \le I\left({u_{\delta_0}}^{\hat{t}}\right) + \frac{\varepsilon}{2} \\ &\le \max_{t>0} I({u_{\delta_0}}^{t}) + \frac{\varepsilon}{2} = I({u_{\delta_0}}^{t_{\delta_0}}) + \frac{\varepsilon}{2} \\ &\le m(c_1) + \varepsilon. \end{aligned} \tag{2.62}$$

The proof is completed. □

Inspired by the above works, we have established the additional minimax characterization of $\gamma(c)$, which can be summarized as the following lemma.

Lemma 2.8. *Assume that* (F1), (F2), (F4) *and* (F5) *hold. Then* $\gamma(c) = m(c)$ *for any* $c > 0$.

Proof. By (2.14), for any $u \in \mathcal{M}_c$, there exist $t_1 < 0$ small and $t_2 > 1$ large such that $u^{t_1} \in \mathcal{A}_{k_1}$ and $u^{t_2} \in \mathcal{A}^{k_2}$. Set

$$\bar{g}(\tau) = u^{(1-\tau)t_1+\tau t_2}, \qquad \forall\, \tau \in [0,1],$$

we have $\bar{g} \in \Gamma_c$. By (2.43), we have

$$\gamma(c) \leq \max_{\tau\in[0,1]} I(\bar{g}(\tau)) = I(u),$$

and so $\gamma(c) \leq \inf_{u\in\mathcal{M}_c} I(u) = m(c)$ for any $c > 0$.

On the other hand, by (2.41), we have

$$J(u) \leq \frac{3}{2}I(u) + \frac{1}{4}\|\nabla u\|_2^2, \qquad \forall\, u \in \mathcal{S}_c.$$

which implies

$$J(g(1)) \leq \frac{3}{2}I(g(1)) < 0, \qquad \forall\, g \in \Gamma_c.$$

Moreover, it is easy to verify that there exists $u_0 \in \mathcal{B}_{k_1}$ such that $J(u_0) > 0$. Hence, any path in Γ_c has to go through $\mathcal{M}_c$. We deduce that

$$\max_{\tau\in[0,1]} I(g(\tau)) \geq \inf_{u\in\mathcal{M}_c} I(u) = m(c), \qquad \forall\, g \in \Gamma_c,$$

and so $\gamma(c) \geq m(c)$ for any $c > 0$. Therefore, $\gamma(c) = m(c)$ for any $c > 0$. □

Let H be a real Hilbert space, we define its norm and scalar products as $\|\cdot\|_H$ and $(\cdot,\cdot)_H$ respectively. Let $(X, \|\cdot\|_X)$ be a real Banach space, and devoted its dual space by X^* satisfying $X \hookrightarrow H \hookrightarrow X^*$ and $M = \{x \in X \mid \|x\|_H = 1\}$ be a submanifold of X of codimension 1.

Lemma 2.9. *Let $J : X \to \mathbb{R}$ be a C^1 functional and $J|_M$ be a C^1 functional restricted to M, assume that $\{x_n\} \in M$ is a bounded sequence in X. Then the following are equivalent:*

(1) $\|J|'_M(x_n)\| \to 0$ *as* $n \to +\infty$*;*
(1) $J'(x_n) - \langle J'(x_n), x_n\rangle x_n \to 0$ *in* X^* *as* $n \to +\infty$*.*

Lemma 2.10. *Let $\{v_n\} \in S_c$ be a bounded $(PS)_{\gamma_c}$ sequence of $I|_{S_{(c)}}$. Then there exists a sequence $\{\lambda_n\} \in \mathbb{R}$ and $\lambda_c \in \mathbb{R}$, $v_c \in H^1(\mathbb{R}^3)$ such that*

(1) $v_n \rightharpoonup v_c$ *in* $H^1(\mathbb{R}^3)$*;*
(1) $\lambda_n \to \lambda_c$ *in* $\mathbb{R}$*;*
(1) $I'(v_n) - \lambda_c v_n \to 0$ *in* $H^{-1}(\mathbb{R}^3)$*.*

Proof. (1) since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$, we have $v_n \rightharpoonup v_c$ in $H^1(\mathbb{R}^3)$.

(2) Since $I|'_{S_c}(v_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$, by preceding Lemma 2.11, we obtain that

$$I'(v_n) - \langle I'(v_n), v_n\rangle v_n \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^3).$$

It means that for any $\omega \in H^1(\mathbb{R}^3)$,

$$\langle I'(v_n) - \langle I'(v_n), v_n\rangle v_n, \omega\rangle = \int_{\mathbb{R}^3} \nabla v_n \nabla\omega - \int_{\mathbb{R}^3} (I_\alpha * F(v_n))f(v_n)v_n\omega - \lambda_n \int_{\mathbb{R}^3} v_n w \to 0,$$

where

$$\lambda_n = \frac{\|\nabla v_n\|_2^2 - \int_{\mathbb{R}^3}(I_\alpha * F(v_n))f(v_n)v_n}{\|v_n\|_2^2},$$

Then

$$I'(v_n) - \lambda_n v_n \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^3) \tag{2.63}$$

and λ_n is bounded which is deduced by the boundedness of $\{v_n\}$ and the Hardy-Littewood-Sobolev inequality. Finally, there exists $\lambda_c \in \mathbb{R}$ such that $\lambda_n \to \lambda_c$.

(3) follows immediately (1),(2) and (2.63). □

Lemma 2.11. *Assume that $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies the following condition:*

$$\exists\, C > 0 \text{ such that for every } s \in \mathbb{R}, |sf(s)| \leq C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}).$$

If $(u.\lambda) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solves problem (1.1), then

$$\frac{1}{2}\|\nabla u\|_2^2 - \frac{3}{2}\lambda\|u\|_2^2 - \frac{3+\alpha}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u) = 0. \tag{2.64}$$

Next Lemma can also be found in [16], for the sake of completeness and convenience for reading, we show it here again.

Lemma 2.12. *Assume that* (F1)-(F3) *hold and $(\bar{v}_c, \bar{\lambda}_c) \in S(c)\times\mathbb{R}$ is a weak solution of problem (1.1), then $\bar{v}_c \in \mathcal{M}_c$ and $\lambda < 0$.*

Proof. Since $(\bar{v}_c, \bar{\lambda}_c) \in S(c)\times\mathbb{R}$ is a weak solution of (1.1), by Lemma 2.13, we infer that

$$\begin{aligned}\frac{1}{2}\|\nabla \bar{v}_c\|_2^2 &= \frac{3}{2}\bar{\lambda}_c\|\bar{v}_c\|_2^2 + \frac{3+\alpha}{2}\int_{\mathbb{R}^3}(I_\alpha * F(\bar{v}_c))F(\bar{v}_c)\\ &= \frac{3}{2}\|\nabla \bar{v}_c\|_2^2 - \frac{3}{2}\int_{\mathbb{R}^3}(I_\alpha * F(\bar{v}_c))f(\bar{v}_c)\bar{v}_c + \frac{3+\alpha}{2}\int_{\mathbb{R}^3}(I_\alpha * F(\bar{v}_c))F(\bar{v}_c),\end{aligned} \tag{2.65}$$

where we use that

$$\bar{\lambda}_c = \frac{\|\nabla \bar{v}_c\|_2^2 - \int_{\mathbb{R}^3}(I_\alpha * F(\bar{v}_c))f(\bar{v}_c)\bar{v}_c}{\|\bar{v}_c\|_2^2}. \tag{2.66}$$

Then, we have

$$\|\nabla \bar{v}_c\|_2^2 + \frac{3+\alpha}{2}\int_{\mathbb{R}^3}(I_\alpha * F(\bar{v}_c))F(\bar{v}_c) - \frac{3}{2}\int_{\mathbb{R}^3}(I_\alpha * F(\bar{v}_c))f(\bar{v}_c)\bar{v}_c = 0. \tag{2.67}$$

i.e. $\bar{v}_c \in \mathcal{M}_c$.

By (2.66),

$$\begin{aligned}\bar{\lambda}_c c &= \|\nabla \bar{v}_c\|_2^2 - \int_{\mathbb{R}^3}(I_\alpha * F(\bar{v}_c))f(\bar{v}_c)\bar{v}_c\\ &= \frac{1}{2}\int_{\mathbb{R}^3}(I_\alpha * F(\bar{v}_c))f(\bar{v}_c)\bar{v}_c - \frac{3+\alpha}{2}\int_{\mathbb{R}^3}(I_\alpha * F(\bar{v}_c))F(\bar{v}_c) \leq 0.\end{aligned} \tag{2.68}$$

hence $\bar{\lambda}_c < 0$. Just suppose $\bar{\lambda}_c = 0$, then (F3) and (2.68) imply $F(\bar{\lambda}_c) = 0$. If $F(\bar{\lambda}_c) = 0$, then by (3.1) again we have that $\|\nabla \bar{v}_c\|_2 = 0$, hence $I(\bar{v}_c) = 0$, which is a contradiction. Then question (1.1) has no nontrivial solution in $H^1(\mathbb{R}^3)$, hence $\bar{\lambda}_c$ must be negative for that $\bar{v}_c$ is a nontrivial solution of (1.1). □

3. Proof of Theorem 1.1

In view of Lemmas 2.8 and 2.11, for each $c > 0$, there exists a sequence $\{v_n\} \subset \mathcal{S}_c$ such that

$$I(v_n) \to m(c) > 0, \quad I|'_{\mathcal{S}_c}(v_n) \to 0 \ \text{ and } \ J(v_n) \to 0. \tag{3.1}$$

By (2.41) and (3.1), we have

$$m(c) + o(1) = I(v_n) - \frac{2}{3}J(v_n) \geq -\frac{1}{6}\|\nabla v_n\|_2^2, \tag{3.2}$$

which, combining with $\|v_n\|_2^2 = c$, implies $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Then there exists $v \in H^1(\mathbb{R}^3)$ such that up to a subsequence, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, $v_n \to v$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for $2 \le s < 6$ and $v_n \to v$ a.e. in $\mathbb{R}^3$. Since $m(c) = \gamma(c) > 0$, by Lions' concentration compactness principle [29, Lemma 1.21] and a standard procedure, we can obtain that $\{v_n\}$ is non-vanishing, and so there exist $\delta > 0$ and $\{y_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(y_n)} |v_n|^2 \mathrm{d}x > \delta$. Let $\bar{v}_n(x) = v_n(x + y_n)$. Then we have $\|\bar{v}_n\| = \|v_n\|$ and

$$I(\bar{v}_n) \to m(c), \qquad J(\bar{v}_n) = o(1), \qquad \int_{B_1(0)} |\bar{v}_n|^2 \mathrm{d}x > \delta. \tag{3.3}$$

Therefore, there exists $\bar{v} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \bar{v}_n \rightharpoonup \bar{v}, & \text{in } H^1(\mathbb{R}^3); \\ \bar{v}_n \to \bar{v}, & \text{in } L^s_{\text{loc}}(\mathbb{R}^3), \ \forall\, s \in [1, 6); \\ \bar{v}_n \to \bar{v}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{3.4}$$

Let $w_n = \bar{v}_n - \bar{v}$. Then (3.4) and the Brezis-Lieb type Lemma yield

$$\|\bar{v}\|_2^2 := \bar{c} \le c, \qquad \|w_n\|_2^2 := \bar{c}_n \le c \text{ for large } n \in \mathbb{N} \tag{3.5}$$

and

$$I(\bar{v}_n) = I(\bar{v}) + I(w_n) + o(1) \ \text{ and } \ J(\bar{v}_n) = J(\bar{v}) + J(w_n) + o(1). \tag{3.6}$$

Let

$$\begin{aligned} \Psi(u) &:= I(u) - \frac{2}{3} J(u) \\ &= -\frac{1}{6}\|\nabla u\|_2^2 + \int_{\mathbb{R}^3} \left[f(u)u - \frac{9 + 2\alpha}{6} F(u) \right] \mathrm{d}x, \qquad \forall\, u \in H^1(\mathbb{R}^3). \end{aligned} \tag{3.7}$$

Then $\Psi(u) > 0$ for all $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. Moreover, it follows from (3.3), (3.6) and (3.7) that

$$\Psi(w_n) = m(c) - \Psi(\bar{v}) + o(1), \qquad J(w_n) = -J(\bar{v}) + o(1). \tag{3.8}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, by (F4), (3.7), (3.8), the Fatou's lemma and the weak lower semicontinuity of norm, we can deduce that $\|\nabla \bar{v}_n - \nabla \bar{v}\|_2 \to 0$. Next, we verify that this still be true for $w_n \neq 0$. Assume that $w_n \neq 0$. We claim that $J(\bar{v}) \le 0$. Otherwise, if $J(\bar{v}) > 0$, then (3.8) implies $J(w_n) < 0$ for large n. According to the Lemma 2.5, there exists $t_n > 0$ such that $(w_n)^{t_n} \in \mathcal{M}_{\bar{c}_n}$. Then we can know from (1.12), (2.16), (2.40), (3.7), (3.8), Lemmas 2.7 and 2.8 that

$$\begin{aligned} m(c) - \Psi(\bar{v}) + o(1) &\ge \Psi(w_n) = I(w_n) - \frac{2}{3} J(w_n) \\ &\ge I\left((w_n)^{t_n}\right) - \frac{t_n^{\frac{3}{2}}}{3} J(w_n) \\ &\ge m(\bar{c}_n) - \frac{t_n^{\frac{3}{2}}}{3} J(w_n) \\ &\ge m(c) + o(1), \end{aligned}$$

which contradicts $\Psi(\bar{v}) > 0$. This indicates that $J(\bar{v}) \le 0$. In view of Lemma 2.5, there exists $\bar{t} > 0$ such that $\bar{v}^{\bar{t}} \in \mathcal{M}_{\bar{c}}$. Then we can know from (2.40), (3.7), the weak semicontinuity of norm, Fatou's lemma and Lemma 2.7 that

$$\begin{aligned} m(c) &= \lim_{n\to\infty}\left[I(\bar{v}_n) - \frac{2}{3}J(\bar{v}_n)\right] = \lim_{n\to\infty}\Psi(\bar{v}_n) \\ &\geq \Psi(\bar{v}) = I(\bar{v}) - \frac{2}{3}J(\bar{v}) \\ &\geq I\left(\bar{v}^{\bar{t}}\right) - \frac{\bar{t}^{\frac{3}{2}}}{3}J(\bar{v}) \geq m(\bar{c}) \geq m(c), \end{aligned}$$

which implies $\|\nabla\bar{v}_n - \nabla\bar{v}\|_2 \to 0$ for $w_n \neq 0$. In the end, we prove that $\|\bar{v}_n - \bar{v}\|_2 \to 0$. Using Lemma 2.12, there exists $\bar{\lambda}_c \in \mathbb{R}$ such that

$$\langle I'(\bar{v}_n), \bar{v}_n\rangle = \bar{\lambda}_c\|\bar{v}_n\|_2^2 + o(1) \text{ and } \langle I'(\bar{v}), \bar{v}\rangle = \bar{\lambda}_c\|\bar{v}\|_2^2. \tag{3.9}$$

Since $\|\nabla\bar{v}_n - \nabla\bar{v}\|_2 \to 0$, a standard procedure can be used to show that

$$\langle I'(\bar{v}_n), \bar{v}_n\rangle = \langle I'(\bar{v}), \bar{v}\rangle + o(1). \tag{3.10}$$

Combining (3.9) with (3.10), we have $\|\bar{v}_n - \bar{v}\|_2 \to 0$. Hence, for any $c > 0$, (1.1) has a couple of solutions $(\bar{v}_c, \bar{\lambda}_c) \in \mathcal{S}_c \times \mathbb{R}^-$ such that

$$I(\bar{v}_c) = \inf_{v\in\mathcal{M}_c} I(v) = \inf_{v\in\mathcal{S}_c}\max_{t>0} I(v^t) > 0.$$

And by condition (F1) and the strong maximum principle, we conclude that $u(x) > 0$ for all $x \in \mathbb{R}^3$. This completes the proof.

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Shuai Yuan, School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China
Email address: ys950526@163.com

Sitong Chen, School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China
Email address: mathsitongchen@163.com

Xianhua Tang, School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China
Email address: tangxh@mail.csu.edu.cn