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THE EXISTENCE, GENERAL DECAY AND BLOW-UP FOR A PLATE EQUATION WITH NONLINEAR DAMPING AND A LOGARITHMIC SOURCE TERM

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ABSTRACT. In this paper, we consider a plate equation with nonlinear damping and logarithmic source term. By the contraction mapping principle, we establish the local existence. The global existence and decay estimate of the solution at subcritical initial energy are obtained. We also prove that the solution with negative initial energy blows up in finite time under suitable conditions. Moreover, we also give the blow-up in finite time of solution at the arbitrarily high initial energy for linear damping (i.e. $m = 2$).

1. INTRODUCTION

In this paper, we deal with the following plate equation with nonlinear damping and a logarithmic source term

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u \log |u|^k, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with sufficiently smooth boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$ and k is a positive real number, $u_0(x)$, and $u_1(x)$ are given initial data. The parameter $m \geq 2$ and p satisfies

$$2 < p < \frac{2(n-2)}{n-4} \text{ if } n \geq 5; \quad 2 < p < +\infty \text{ if } n \leq 4. \tag{2}$$

The logarithmic nonlinearity is of much interest in many branches of physics such as nuclear physics, optics and geophysics (see [5, 6, 15] and references therein). It has also been applied in quantum field theory, where this kind of nonlinearity appears naturally in cosmological inflation and in super symmetric field theories [4, 13].

Let us review somework with logarithmic term which is closely related to the problem (1). Birula and Mycielski[6, 7] studied the following problem

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$$
(3) \qquad \begin{cases} u_{tt} - u_{xx} + u - \varepsilon u \log |u|^2 = 0, & (x,t) \in [a,b] \times (0,T), \\ u(a,t) = u(b,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in [a,b], \end{cases}
$$

which is a relativistic version of logarithmic quantum mechanics and can also be obtained by taking the limit $p \to 1$ for the p-adic string equation [16, 36]. Cazenave and Haraux [8] established the existence and uniqueness of a solution to the Cauchy problem for the following equation

$$
(4) \qquad u_{tt} - \Delta u = u \log |u|^k,
$$

in $\mathbb{R}^3$. Using some compactness method, Gorka [15] established the global existence of weak solutions for all $(u_0, u_1) \in H_0^1 \times L^2$ to the initial boundary value problem of equation (4) in the one-dimensional case. In [5], Bartkowski and Gorka obtained the existence of classical solutions and investigated weak solutions for the corresponding Cauchy problem of equation (4) in the one-dimensional case. Recently, using potential well combined with logarithmic Sobolev inequality, Lian et al. [25] derived the global existence and infinite time blow up of the solution to the initial boundary value problem of (4) in finite dimensional case under suitable assumptions on initial data. Similar results were obtained by Lian et al. [26] for nonlinear wave equation with weak and strong damping terms and logarithmic source term. Hiramatsu et al. [19] also introduced the following equation

$$
(5) \qquad u_{tt} - \Delta u + u + u_t + |u|^2 u = u \log |u|
$$

to study the dynamics of Q-ball in theoretical physics. A numerical research was given in that work, while, there was no theoretical analysis for this problem. For the initial boundary value problem of(5), Han [17] obtained the global existence of weak solution in $\mathbb{R}^3$, and Zhang et al. [40] obtained the decay estimate of energy for the problem (5) in finite dimensional case. Later, the authors in [20] considered the initial boundary problem of (5) in $\Omega \subset \mathbb{R}^3$, they proved that the solution will grow exponentially as time goes to infinity if the solution lies in unstable set or the initial energy is negative; the decay rate of the energy was also obtained if the solution lies in a smaller set compared with the stable set. Peyravi[35] extended the results obtained in [20] to the following logarithmic wave equation

$$
u_{tt} - \Delta u + u + (g * \Delta u)(t) + h(u_t) u_t + |u|^2 u = u \log |u|^k.
$$

Recently, Al-Gharabli and Messaoudi [1] considered the following plate equation with logarithmic source term

$$
(6) \qquad \begin{cases} u_{tt} + \Delta^2 u + u + h(u_t) = u \log |u|^k, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^2$, and obtained the global existence and decay rate of the solution using the multiplier method. As the special case , i.e. $h(u_t) = u_t$ in (6), the same authors [2] established the global existence and the decay estimate by constructing a Lyapunov function. Moreover, Al-Gharabli et al. [3] considered the following initial boundary value problem of viscoelastic plate equation with with logarithmic source term

$$u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(s)ds = u \log |u|^k, (x,t) \in \Omega \times \mathbb{R}^+, \tag{7}$$

they established the existence of solutions and proved an explicit and general decay rate result. However, there is no information on the finite or infinite blow up results in these researches [1, 2, 3].

At the same time, there are many results concerning the existence and nonexistence on evolution equation with polynomial source term. For example, for plate equation with polynomial source term $|u|^{p-2}u$, Messaoudi [31] considered the following problem

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

established an existence result and showed that the solution continues to exist globally if $m \geq p$ and blows up in finite time if $m < p$ and the initial energy is negative. This result was later improved by Chen and Zhou [12], see also Wu and Tsai [37]. Here, we also mention that there are a lot of results on the global well-posedness of solutions to the initial boundary value problem of nonlinear wave equations can be found [30, 39] and papers cited therein by using of potential well method.

To the best of our knowledge, there are few results on the evolution equation with the nonlinear logarithmic source term $|u|^{p-2}u \log |u|^k$ $(p > 2)$. Kafini and Messaoudi [22] studied the following delayed wave equation with nonlinear logarithmic source

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = |u|^{p-2}u \log |u|^k, \tag{8}$$

obtained the local existence by using the semigroup theory and proved a finite time blow-up result when the initial energy is negative. Of course, these results also hold for the equation (8) without delay term (i.e. $\mu_2 = 0$). However, there are no results on general decay and blow-up for positive initial energy. Chen et al. [9, 10] studied parabolic type equations with logarithmic nonlinearity $u \log |u|^k$, and obtained the global existence of solution and the solutions cannot blow up in finite time. Recently, Chen and Xu [11] study the initial-boundary value problem for infinitely degenerate semilinear pseudo-parabolic equations with logarithmic nonlinearity and obtain the similar results. Nhan and Truong studied parabolic p-Laplacian equation [23] and pseudo-parabolic p-Laplacian equation [24] with logarithmic nonlinearity $|u|^{p-2}u \log |u|^k$ where they need the p-Laplacian term to control the logarithmic nonlinearity. We also refer to [18], where pseudo-parabolic p-Laplacian equation with logarithmic nonlinearity $|u|^{q-2}u \log |u|^k$ was considered.

Motivated by the above mentioned papers, our purpose in this research is to investigative the existence, energy decay and finite time blow-up of the solution to the initial boundary value problem (1). We note here that (i) the term u plays an important role in the studying the problem (6)(see [1, 2]) and (7) (see [3]) when the Logarithmic Sobolev inequality is used, while we do not care the term u in this paper; (ii) The constant k in (6) and (7) should satisfy $0 < k < k_0$, where k_0 is defined by $\frac{2\pi}{k_0 c_p} = e^{-3-\frac{2}{k_0}}$ (see details in [1, 2, 3]), while we only need $k > 0$.

The rest of this article is organized as follows: Section 2 is concerned with some notation and some properties of the potential well. In Sect. 3, we present the existence and uniqueness of local solutions to (1) by using the contraction mapping principle. In Sect. 4, we prove the global existence and energy decay results. The proof of global existence result is based on the potential well theory and the continuous principle; while for energy decay result, the proof is based on the Nakao's inequality and some techniques on logarithmic nonlinearity. In Sect. 5, we prove the the finite time blow-up when the initial energy is negative. In Sect. 6, we establish the finite time blow-up result for problem (1) with $m = 2$ under the arbitrarily high initial energy level ($E(0) > 0$).

2. Preliminaries

We give some material needed in the proof of our results. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms. In particular, we denote $\|.\| = \|.\|_2$. By Poincaré 's inequality [27], we have that $\|\Delta \cdot\|$ is equivalent to $\|\cdot\|_{H_0^2}$ and we will use $\|\Delta \cdot\|$ as the norm of $\|\cdot\|_{H_0^2}$, the corresponding duality between H_0^2 and H^{-2} is denote by $\langle\cdot,\cdot\rangle$. We also use C and C_i to denote various positive constant that may have different values in different places.

Firstly, we introduce the Sobolev's embedding inequality : assume that p be a constant such that $1 \le p \le \frac{2n}{n-4}$ if $n \ge 5$; $p \ge 1$ if $n \le 4$, then $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$ continuously, and

$$\|u\|_p \le C_p \|\Delta u\|_2, \quad \text{for } u \in H_0^2(\Omega) \tag{9}$$

where C_p denotes the best embedding constant.

Suppose (2) holds, we define

$$\alpha^* := \begin{cases} \frac{2n}{n-4} - p & \text{if } n \ge 5, \\ +\infty & \text{if } n \le 4 \end{cases}$$

for any $\alpha \in [0, \alpha^*)$, then $H_0^2(\Omega) \hookrightarrow L^{p+\alpha}(\Omega)$ continuously. And we denote $C_{p+\alpha}$ by C_*.

Definition 2.1. A function $u \in C\left([0,T], H_0^2(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right) \cap C^2([0,T], H^{-2}(\Omega))$ with $u_t \in L^m([0,T], L^m(\Omega))$ is called a weak solution to (1) if the following conditions hold

$$u(0) = u_0, \quad u_t(0) = u_1$$

and

$$\langle u_{tt}, v\rangle + (\Delta u, \Delta v) + \int_\Omega |u_t|^{m-2} u_t v \mathrm{d}x = \int_\Omega (|u|^{p-2} u \log |u|^k) v \mathrm{d}x \tag{10}$$

for any $v \in H_0^2(\Omega)$ and a.e. $t \in [0,T]$.

Now, we introduce the energy functional J and the Nehari functional I defined on $H_0^2(\Omega) \setminus \{0\}$ by

$$J(u) = J(u(t)) = J(t) = \frac{1}{2}\|\Delta u\|^2 - \frac{1}{p}\int_\Omega |u|^p \log |u|^k \mathrm{d}x + \frac{k}{p^2}\|u\|_p^p, \tag{11}$$

and

$$I(u) = I(u(t)) = I(t) = \|\Delta u\|^2 - \int_\Omega |u|^p \log |u|^k \mathrm{d}x. \tag{12}$$

From the definitions (11) and (12), we have

$$J(u)=\frac{1}{p}I(u)+\left(\frac{1}{2}-\frac{1}{p}\right)\|\Delta u\|^2+\frac{k}{p^2}\|u\|_p^p. \tag{13}$$

The following lemmas play an important role in the studying the properties of the potential well.

Lemma 2.2. *Let $u\in H_0^2(\Omega)\setminus\{0\}$ and $g(\lambda)=J(\lambda u)$. Then we have*

- **(i):** $\lim_{\lambda\to 0^+}J(\lambda u)=0,\lim_{\lambda\to+\infty}J(\lambda u)=-\infty$;
- **(ii):** *there exists a unique $\lambda^*=\lambda^*(u)>0$ such that $\frac{d}{d\lambda}J(\lambda u)\mid_{\lambda=\lambda^*}=0$, and $J(\lambda u)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*,+\infty)$ and taking the maximum at λ^*.*
- **(iii):** $I(\lambda u)>0$ *for* $0<\lambda<\lambda^*, I(\lambda u)<0$ *for* $\lambda^*<\lambda<+\infty$ *and* $I(\lambda^*u)=0$.

Proof. We know

$$\begin{aligned}g(\lambda)&=J(\lambda u)\\&==\frac{1}{2}\lambda^2\|\Delta u\|^2-\frac{k}{p}\lambda^p\int_\Omega|u|^p\log|u|\mathrm{d}x-\frac{k}{p}\lambda^p\log\lambda\ \|u\|_p^p+\frac{k}{p^2}\lambda^p\|u\|_p^p.\end{aligned}$$

It is obvious that (i) holds due to $p\geq 2$ and $\|u\|_p\neq 0$. Taking derivative of $g(\lambda)$ we obtain

$$g'(\lambda)=\lambda\left(\|\Delta u\|^2-k\lambda^{p-2}\int_\Omega|u|^p\log|u|\mathrm{d}x-k\lambda^{p-2}\log\lambda\ \|u\|_p^p\right) \tag{14}$$

and

$$g''(\lambda)=\|\Delta u\|^2-k(p-1)\lambda^{p-2}\int_\Omega|u|^p\log|u|\mathrm{d}x-k(p-1)\lambda^{p-2}\log\lambda\ \|u\|_p^p-k\lambda^{p-2}\ \|u\|_p^p.$$

From (14) and $p\geq 2$, we see that there exists a unique positive λ^* such that

$$g'(\lambda)|_{\lambda=\lambda^*}=0,$$

then we obtain

$$\|\Delta u\|^2=k\lambda^{*p-2}\int_\Omega u^2\log|u|dx+k\lambda^{*p-2}\log\lambda^*\ \|u\|_p^p.$$

Substituting the above equation into $g''(\lambda)$, we have

$$\begin{aligned}g''(\lambda^*)=&-k(p-2)\lambda^{*p-2}\int_\Omega|u|^p\log|u|\mathrm{d}x\\&-k(p-2)\lambda^{*p-2}\log\lambda^*\ \|u\|_p^p-k\lambda^{*p-2}\ \|u\|_p^p\\=&-(p-2)\|\Delta u\|^2-k\lambda^{*p-2}\ \|u\|_p^p<0.\end{aligned}$$

From these and (i), we can yield that $g(\lambda)$ has a maximum value at $\lambda=\lambda^*$ and $J(\lambda u)$ increasing on $0<\lambda\leq\lambda^*$ and decreasing on $\lambda^*\leq\lambda<+\infty$. So (ii) holds.

From (12) and (14), we have

$$I(\lambda u)=\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)=\lambda g'(\lambda)\begin{cases}>0,0<\lambda<\lambda,\\=0,\lambda=\lambda^*,\\<0,\lambda^*<\lambda<+\infty.\end{cases}$$

□

Then, we could define the potential well depth of the functional J (also known as mountain pass level) by

$$d = \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u) | u \in H_0^2(\Omega) \backslash \{0\} \right\}. \tag{15}$$

We also define the well-known Nehari manifold

$$\mathcal{N} = \left\{ u | u \in H_0^2(\Omega) \backslash \{0\}, I(u) = 0 \right\}.$$

As in [29, 34], that the mountain pass level d defined in (15) can also be characterized as

$$d = \inf_{u \in \mathcal{N}} J(u).$$

It is easy to see that $d \geq 0$ from (13). Now, we will prove that d is strictly positive.

Lemma 2.3. *Assume that $p > 2$ holds. Let $\alpha \in (0, \alpha^*)$, and*

$$r(\alpha) := \left(\frac{\alpha}{k C_*^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}}.$$

Then, for any $u \in H_0^2(\Omega) \backslash \{0\}$, we have

- **(i):** *if $\|\nabla u\|_2 \leq r(\alpha)$, then $I(u) > 0$*
- **(ii):** *if $I(u) \leq 0$, then $\|\Delta u\|_2 > r(\alpha)$.*

Proof. Since $\log y < y$ for any constant $y > 0$, using (9) and the definition of $I(u)$ in (12), we obtain that

$$\begin{aligned} I(u) &= \|\nabla u\|_2^2 - k \int_\Omega |u|^p \log |u| dx \\ &> \|\nabla u\|_2^2 - \frac{k}{\alpha} \|u\|_{p+\alpha}^{p+\alpha} \\ &\geq \|\nabla u\|_2^2 - \frac{k C_*^{p+\alpha}}{\alpha} \|\nabla u\|_2^{p+\alpha} \\ &= \frac{k C_*^{p+\alpha}}{\alpha} \|\nabla u\|_2^2 \left(r^{p+\alpha-2}(\alpha) - \|\nabla u\|_2^{p+\alpha-2} \right) \end{aligned}. \tag{16}$$

Obviously, the results can be obtained from the above inequality (16). □

Lemma 2.4. *Assume the notations in Lemma 2.2 hold, we have*

$$\begin{aligned} 0 < r_* := \sup_{\alpha \in (0, \alpha^*)} &= \left(\frac{\alpha}{k C_*^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} \\ &\leq r^* := \sup_{\alpha \in (0, \alpha^*)} \left(\frac{\alpha}{k B^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} |\Omega|^{\frac{\alpha}{p(p+\alpha-2)}} \\ &< +\infty, \end{aligned}$$

where $B = C_p$ as in (9) and $|\Omega|$ is the measure of Ω.

Proof. It is obvious that $r_* > 0$ (if exists), hence, we only need to prove $r(\alpha) \leq \gamma(\alpha)$, r^* exists and $r^* < +\infty$, where

$$\gamma(\alpha) = \left(\frac{\alpha}{k B^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} |\Omega|^{\frac{\alpha}{p(p+\alpha-2)}}, \ \alpha \in (0, +\infty).$$

For any $u \in H_0^2(\Omega)$, using the Hölder's inequality, we have

$$\|u\|_p \leq |\Omega|^{\frac{\alpha}{p(p+\alpha)}} \|u\|_{p+\alpha}.$$

Then, noticing $C_* = C_{p+\alpha}$ and $B = C_p$, we deduce

$$\begin{aligned} C_* &= \sup_{u \in H_0^2 \setminus \{0\}} \frac{\|u\|_{p+\alpha}}{\|\Delta u\|_2} \\ &\geq |\Omega|^{\frac{-\alpha}{p(p+\alpha)}} \sup_{u \in H_0^2 \setminus \{0\}} \frac{\|u\|_p}{\|\Delta u\|_2}, \\ &\geq |\Omega|^{\frac{-\alpha}{P(p+\alpha)}} B \end{aligned}$$

which implies

$$\left(\frac{\alpha}{k C_*^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} \leq \left(\frac{\alpha}{k B^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} |\Omega|^{\frac{\alpha}{p(p+\alpha-2)}},$$

that is $r(\alpha) \leq \gamma(\alpha)$.

Now, we will prove r^* exists and $r^* < +\infty$. For this purpose, we divide the proof into two cases.

Case a. If $n \geq 5$, we see that $\alpha \in (0, \alpha^*) = \left(0, \frac{2n}{n-4} - p\right)$ and $\gamma(\alpha)$ is continuous on closed interval $[0, \frac{2n}{n-4} - p]$. Hence, we have r^* exists and

$$r^* = \sup_{\alpha \in \left(0, \frac{2n}{n-4} - p\right)} \gamma(\alpha) \leq \max_{\alpha \in \left[0, \frac{2n}{n-4} - p\right]} \gamma(\alpha) < +\infty$$

Case b. If $n \leq 4$, we define the following auxiliary function

$$h(\alpha) := \log[\gamma(\alpha)] = \frac{1}{p+\alpha-2}[\log \alpha - \log k - (p+\alpha)\log B] + \frac{\alpha}{p(p+\alpha-2)} \log |\Omega|.$$

Hence

$$h'(\alpha) = \frac{p^2 + p\alpha - 2p + p\alpha \log k - p\alpha \log \alpha + 2p\alpha \log B + p\alpha \log |\Omega| - 2\alpha \log |\Omega|}{p\alpha(p+\alpha-2)^2}.$$

For simplicity, we set

$$g(\alpha) := p^2 + p\alpha - 2p + p\alpha \log k - p\alpha \log \alpha + 2p\alpha \log B + p\alpha \log |\Omega| - 2\alpha \log |\Omega|,$$

then

$$\begin{aligned} g'(\alpha) &= p + p\log k - p\log \alpha - p + 2p\log B + p\log |\Omega| - 2\log |\Omega| \\ &= p \log \frac{kB^2 |\Omega|^{1-\frac{2}{p}}}{\alpha}, \end{aligned}$$

which yields that the function $g(\alpha)$ is strictly increasing on $\left(0, kB^2|\Omega|^{1-\frac{2}{p}}\right)$ and strictly decreasing on $\left(kB^2|\Omega|^{1-\frac{2}{p}}, +\infty\right)$.

On the one hand, due to $p > 2$, it is easy to see that

$$\lim_{\alpha \to 0^+} g(\alpha) = p^2 - 2p > 0$$

which implies that $g(\alpha) > 0$ for $\alpha \in \left(0, kB^2|\Omega|^{1-\frac{2}{p}}\right)$ by $g(\alpha)$ is strictly increasing on $\left(0, kB^2|\Omega|^{1-\frac{2}{p}}\right)$.

While on the other hand, we can deduce that

$$\lim_{\alpha\to+\infty} g(\alpha) = \lim_{\alpha\to+\infty}\left(p^2-2p+p\alpha[1+\log(kB^2|\Omega|^{1-\frac{2}{p}})]-\log\alpha\right) = -\infty,$$

which together with $g\left(kB^2|\Omega|^{1-\frac{2}{p}}\right) > 0$ and $g(\alpha)$ is strictly decreasing on $\left(kB^2|\Omega|^{1-\frac{2}{p}},+\infty\right)$, implies that there exists a unique $\alpha_* \in \left(kB^2|\Omega|^{1-\frac{2}{p}},+\infty\right)$ such that $g(\alpha_*)=0$.

Noting the relation between $h'(\alpha)$ and $g(\alpha)$, we deduce that $h'(\alpha)>0$ for $\alpha \in (0,\alpha_*)$, and $h'(\alpha)<0$ for $\alpha\in(\alpha_*,+\infty)$. Therefore, $h(\alpha)$ achieves its maximum value at $\alpha=\alpha_*$, that is

$$r^* = \sup_{\alpha\in(0,+\infty)} \sigma(\alpha) = e^{h(\alpha_*)} < +\infty.$$

□

Making using of the Lemmas 2.2 and 2.3, we obtain the following corollary.

Corollary 1. *Assume that $p>2$ holds. Then, we have*

(i): *if $\|\nabla u\|_2 < r_*$, then $I(u)>0$;*
(ii): *if $I(u)\le 0$, then $\|\nabla u\|_2 \ge r_*$*

for any $u\in H_0^2(\Omega)\setminus\{0\}$, where r_* is the positive constant defined in Lemma 2.3.

Lemma 2.5. *Assume that $p\ge 2$ holds. Then the constant d defined in (15) is strictly positive.*

Proof. (i) For the case $p=2$, we have $d\ge \frac{k}{4}\left(\frac{2\pi}{k}\right)^{\frac{n}{2}} e^n$(see [9, 20, 25] for details).

(ii) For the case $p>2$, by (ii) of Corollary 2.1, we get that $\|\nabla u\|_2\ge r_*$ if $u\in\mathcal{N}$. Then, making using of (13) with $I(u)>0$, we obtian

$$J(u) = \left(\frac{1}{2}-\frac{1}{p}\right)\|\Delta u\|_2^2 + \frac{k}{p^2}\|u\|_p^p \ge \left(\frac{p-2}{2p}\right) r_*^2 > 0.$$

□

We define energy for the problem (1), which obeys the following energy equality of the weak solution u

$$E(t) + \int_0^t \|u_\tau\|_m^m \mathrm{d}\tau = E(0), \qquad \text{for all } t\in[0,T) \tag{17}$$

where

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 - \frac{1}{p}\int_\Omega |u|^p\log|u|^k \mathrm{d}x + \frac{k}{p^2}\|u\|_p^p,$$

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 - \frac{1}{p}\int_\Omega |u_0|^p\log|u_0|^k \mathrm{d}x + \frac{k}{p^2}\|u_0\|_p^p.$$

It is obvious that

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u).$$

Taking $v=u_t$ in (10), after a simple calculation, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|u_t\|_m^m. \tag{18}$$

Now, we define the subsets of $H_0^2(\Omega)$ related to problem (1). Set

$$W = \left\{u \in H_0^1(\Omega) | J(u) < d, I(u) > 0\right\},$$
$$V = \left\{u \in H_0^1(\Omega) | J(u) < d, I(u) < 0\right\}, \tag{19}$$

where W and V are called the stable and unstable set, respectively [21] .

In order to establish the global existence and blow-up results of solution, we have to prove the following invariance sets of W and V.

Lemma 2.6. *If $u_0 \in H_0^2, u_1 \in L^2, p \geq 2, E(0) < d$, and u is a weak solution of problem (1) on $[0,T)$, where T is the maximal existence time of weak solution, then*

(i): *$u \in W$ if $I(u_0) > 0$;*
(ii): *$u \in V$ if $I(u_0) < 0$.*

Proof. It follows from the definition of weak solution and (17) that

$$\frac{1}{2}\|u_t\|^2 + J(u) \leq \frac{1}{2}\|u_1\|^2 + J(u_0) < d, \quad \text{for any} \quad t \in [0,T). \tag{20}$$

(i) Arguing by contradiction, we assume that there exists a number $t_0 \in (0,T)$ such that $u(t) \in W$ on $[0,t_0)$ and $u(t_0) \notin W$. Then, by the continuity of $J(u(t))$ and $I(u(t))$ with respect to t, we have either (a)$J(u(t_0)) = d$ or (b) $I(u(t_0)) = 0$ and $\|u(t_0)\| \neq 0$.

It follows from (20) that (a) is impossible. If (b) holds, then by the definition of d, we have $J(u(t_0)) \geq d$, which contradicts (20) again. Thus, we have $u(t) \in W$ for all $t \in [0,T)$.

(ii) The proof is similar to the proof of (i). We omit it. □

3. Local existence

In this section, we are concerned with the local existence and uniqueness for the solution of the problem (1). The idea comes from [14, 28, 38], where the source term is polynomial. First, we give a technical lemma given in [22] which plays an important role in the uniqueness of the solution.

Lemma 3.1. *([22]) For every $\varepsilon > 0$, there exists $A > 0$, such that the real function*

$$j(s) = |s|^{p-2}\log|s|, \quad p > 2$$

satisfies

$$|j(s)| \leq A + |s|^{p-2+\varepsilon}.$$

Theorem 3.2. *Suppose that $u_0 \in H_0^2(\Omega), u_1 \in L^2(\Omega)$, and $p > 2$, then there is $T > 0$, such that the problem (1) admits a unique local weak solution on $[0,T]$.*

Proof. For every $T > 0$, we consider the space

$$\mathcal{H} := C([0,T]; H_0^2(\Omega)) \cap C^1\left([0,T]; L^2(\Omega)\right)$$

endowed with the norm

$$\|u(t)\|_{\mathcal{H}} = \left(\max_{t\in[0,T]}\left(\|\Delta u(t)\|_2^2 + \|u_t(t)\|_2^2\right)\right)^{\frac{1}{2}}.$$

For every given $u \in \mathcal{H}$, we consider the following initial boundary value problem

$$(21)\qquad \begin{cases} v_{tt} + \Delta^2 v + |v_t|^{m-2} v_t = |u|^{p-2} u \log |u|^k, & (x,t) \in \Omega \times \mathbb{R}^+, \\ v = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ v(x,0) = u_0(x), \;\; v_t(x,0) = u_1(x), & x \in \Omega. \end{cases}$$

We shall prove that the problem (21) admits a unique solution $v \in \mathcal{H} \cap C^2(|0,T|, H^{-2}(\Omega))$ with $v_t \in L^m(|0,T|, L^m(\Omega))$.

Let $W_h = \mathrm{Span}\,\{w_1, \ldots, w_h\}$, where $\{\omega_i\}_{i=1}^{\infty}$ is the orthogonal complete system of eigenfunctions of Δ^2 in $H_0^2(\Omega)$ with $\|w_i\| = 1$ for all i. Then, $\{w_i\}$ is orthogonal and complete in $L^2(\Omega)$ and in $H_0^2(\Omega)$; denote by $\{\lambda_i\}$ the related eigenvalues to their multiplicity. Let

$$u_{0h} = \sum_{i=1}^{h} \left(\int_\Omega \Delta u_0 \Delta w_i \right) w_i \quad \text{and} \quad u_{1h} = \sum_{i=1}^{h} \left(\int_\Omega u_1 w_i \right) w_i$$

such that $u_{0h} \in W_h, u_{1k} \in W_h, u_{0h} \to u_0$ in $H_0^2(\Omega)$ and $u_{1h} \to u_1$ in $L^2(\Omega)$ as $h \to \infty$. For each $h > 0$ we seek h functions $\gamma_{1h}, \ldots, \gamma_{hh} \in C^2[0,T]$ such that

$$(22)\qquad v_h(t) = \sum_{i=1}^{h} \gamma_{ih}(t) \omega_i,$$

solves the following problem

$$(23)\qquad \begin{cases} \int_\Omega \left(v_h'' + \Delta^2 v_h + |v_h'|^{m-2} v_h' - |u|^{p-2} u \log |u|^k \right) \eta \mathrm{d}x = 0, \\ v_k(0) = u_{0h}, \quad v_h'(0) = u_{1h}. \end{cases}$$

For $i = 1, \ldots, h$, taking $\eta = w_i$ in (23) yields the following Cauchy problem for a nonlinear ordinary differential equation with unknown γ_{ih}

$$\begin{cases} \gamma_{ih}''(t) + \lambda_i \gamma_{ih}(t) + c_i |\gamma_{ih}'(t)|^{m-2} \gamma_{ih}'(t) = \psi_i(t), \\ \gamma_{ih}(0) = \int_\Omega u_0 \omega_i, \quad \gamma_{ih}'(0) = \int_\Omega u_1 \omega_i, \end{cases}$$

where

$$c_i = \|\omega_i\|_m^m, \quad \psi_i(t) = \int_\Omega |u(t)^{p-2} u(t) \log |u|^k \omega_i \mathrm{d}x \in C[0,T].$$

Then the above problem admits a unique local solution $\gamma_{ih} \in C^2[0,T]$ for all i, which in turn implies a unique v_h defined by (22) satisfying (23).

Taking $\eta = v_h'(t)$ into (23) and integrating over $[0,t] \subset [0,T]$, we have

$$(24)\qquad \begin{aligned} &\|v_h'(t)\|^2 + \|\Delta v_h(t)\|^2 + 2 \int_0^t \|v_h'(\tau)\|_m^m \, \mathrm{d}\tau \\ &= \|v_{1h}\|^2 + \|\Delta v_{0h}\|^2 + 2 \int_0^t \int_\Omega |u|^{p-2} u \log |u|^k v_h' \end{aligned}$$

for every $h \geq 1$. We estimate the last term in the right-hand side of (24) thanks to Young's and Sobolev's inequalities

$$2 \int_0^t \int_\Omega |u|^{p-2} u \log |u|^k v_h'$$

$$
\begin{aligned}
&\leq 2\int_0^t\int_\Omega \left||u|^{p-1}\log|u|^k\right||v_h'| \\
&\leq \int_0^t\int_\Omega \left(C\left||u|^{p-1}\log|u|\right|^{\frac{m}{m-1}}+\int_0^t \|v_h'\|_m^m\right).
\end{aligned} \tag{25}
$$

In order to estimate (25), we focus on the logarithmic term. Here we denote $\Omega := \Omega_1\cup\Omega_2$, where $\Omega_1=\{x\in\Omega||u(x)|<1\}$ and $\Omega_2=\{x\in\Omega||u(x)|\geq 1\}$. Then we have

$$
\int_\Omega \|\, u|^{p-1}\log|u||^{\frac{m}{m-1}}\,\mathrm{d}x=\int_{\Omega_1}\|\, u|^{p-1}\log|u||^{\frac{m}{m-1}}\,\mathrm{d}x+\int_{\Omega_2}\|\, u|^{p-1}\log|u||^{\frac{m}{m-1}}\,\mathrm{d}x.
$$

By a simple calculation, we obtain

$$
\inf_{s\in(0,1)} s^{p-1}\log s=-\frac{1}{e(p-1)},
$$

which implies

$$
\int_{\Omega_1}\|\, u|^{p-1}\log|u||^{\frac{m}{m-1}}\,\mathrm{d}x\leq [e(p-1)]^{-\frac{m}{m-1}}|\Omega|.
$$

Let

$$
\rho=\frac{2n}{n-4}\cdot\frac{m-1}{m}-p+1>0 \text{ for } n\geq 5;\qquad \text{any positive } \rho,\ n\leq 4.
$$

By the Sobolev embedding from $H_0^2(\Omega)$ to $L^{\frac{2n}{n-4}}(\Omega)$ if $n\geq 5$ and to $L^q(\Omega)$ for any $q\geq 1$ if $n\leq 4$, recalling $u\in\mathcal{H}:=C([0,T];H_0^2(\Omega))$, we have

$$
\begin{aligned}
&\int_{\Omega_2}\|\, u|^{p-1}\log|u||^{\frac{m}{m-1}}\,\mathrm{d}x \\
\leq&\rho^{-\frac{m-1}{m}}\int_{\Omega_2}\left(|u|^{p-1+\rho}\right)^{\frac{m-1}{m}}\mathrm{d}x \\
\leq&\rho^{-\frac{m-1}{m}}\int_{\Omega_2}|u|^{\frac{2n}{n-4}}\mathrm{d}x \\
\leq&\rho^{-\frac{m-1}{m}}\int_{\Omega}|u|^{\frac{2n}{n-4}}\mathrm{d}x \\
=&\rho^{-\frac{m-1}{m}}\|u\|_{\frac{2n}{n-4}}^{\frac{2n}{n-4}} \\
\leq& C\|u\|_{H_0^2}^{\frac{2n}{n-4}}\leq C.
\end{aligned}
$$

The proof of the case $n\leq 4$ is similar. From the above discussion, (25) yields

$$
2\int_0^t\int_\Omega |u|^{p-2}u\log|u|^k v_h'\leq CT+\int_0^t\|v_h'\|_m^m. \tag{26}
$$

Substituting this inequality into (24), we obtain

$$
\|v_h'(t)\|^2+\|\Delta v_h(t)\|^2+\int_0^t\|v_h'(\tau)\|_m^m\,\mathrm{d}\tau\leq C, \tag{27}
$$

where $C>0$ is independent of h. It follows from (27) that

$$\begin{aligned} &v_h(t) \text{ is bounded in } L^\infty\left([0,T],H_0^2(\Omega)\right),\\ &v_h'(t) \text{ is bounded in } L^m\left[(0,T],L^m(\Omega)\right)\cap L^\infty\left([0,T],L^2(\Omega)\right),\\ &{v_h}''(t) \text{ is bounded in } L^2([0,T],H^{-2}(\Omega)). \end{aligned} \tag{28}$$

Hence, up to a subsequence, we could pass to the limit in (23) and obtain a weak solution v of (21) with regularity (28). Then, we have $v\in C\left([0,T],H_0^2(\Omega)\right)\cap C^1\left([0,T],L^2(\Omega)\right)$ with $v_t\in L^m([0,T],L^m(\Omega))$. Finally, from (21), we obtain $v''\in C^0([0,T],H^{-2}(\Omega))$. Then the weak local solution of problem (21) has been obtained.

To prove the uniqueness, arguing by contradiction: if w and v were two solutions of (21) which have the same initial data. Subtracting these two equations and testing the result by w_t-v_t, we could obtain

$$\|w_t-v_t\|^2+\|\Delta w-\Delta v\|^2+2\int_0^t\int_\Omega\left(|w_\tau|^{m-2}w_\tau-|v_\tau|^{m-2}v_\tau\right)(w_\tau-v_\tau)=0. \tag{29}$$

It follows from the following element inequality

$$\left(|\varphi|^{m-2}\varphi-|\psi|^{m-2}\psi\right)(\varphi-\psi)\ge C|\varphi-\psi|^m \text{ for } m\ge 2,$$

that (29) can make to be

$$\|w_t-v_t\|^2+\|v-w\|_{H_0^2}^2+C\int_0^T\|w_\tau-v_\tau\|_m^m\le 0.$$

Therefore, we have $w=v$, i.e. the problem (21) obeys a unique weak solution.

Now, we are in the position to prove Theorem 3.1. For $u_0\in H_0^2(\Omega), u_1\in L^2(\Omega)$, we denote

$$R^2:=2\left(\|u_1\|^2+\|\Delta u_0\|^2\right),$$

and

$$B_{RT}:=\{u\in\mathcal{H}|u(0,x)=u_0(x),u_t(0,x)=u_1(x),\|u\|_{\mathcal{H}}\le R\}$$

for every $T>0$. From the above discussion, for any $u\in B_{RT}$, we could introduce a map $\Phi:\mathcal{H}\to\mathcal{H}$ defined by $v=\Phi(u)$, where v is the unique solution to (21).

Claim. Φ *is a contract map satisfying* $\Phi(B_{RT})\subseteq B_{RT}$, *for small* $T>0$.

In fact, assume that $u\in B_{RT}$, the corresponding solution $v=\Phi(u)$ satisfies (21) for all $t\in[0,T]$. Thus, as did the proof (26) and (27), we have

$$\begin{aligned}\|v_t(t)\|^2+\|\Delta v(t)\|^2&\le\|u_1\|^2+\|\Delta u_0\|^2+CR^{\frac{2n}{n-4}}T\\ &\le\frac{R^2}{2}+CR^{\frac{2n}{n-4}}T\end{aligned}$$

for $n\ge 5$, For the case $n\le 4$, the index $\frac{2n}{n-4}$ of R in the last inequality can be replaced by any fixed positive number. If T is sufficiently small, then $\|v\|_{\mathcal{H}}\le R$, which implies that $\Phi(B_{RT})\subseteq B_{RT}$

Next we show that Φ is contractive in B_{RT}. We set $v_1=\Phi(w_1), v_2=\Phi(w_2)$ with $w_1,w_2\in B_{RT}$, and $v=v_1-v_2$, then, v satisfies

$$\begin{aligned}&\langle v_{tt},\eta\rangle+(\Delta v,\Delta\eta)+\int_\Omega\left(|v_{1t}|^{m-2}v_{1t}-|v_{2t}|^{m-2}v_{2t}\right)\eta\mathrm{d}x\\ &=\int_\Omega\left(|w_1|^{p-2}w_1\log|w_1|^k-|w_2|^{p-2}w_2\log|w_2|^k\right)\eta\mathrm{d}x,\end{aligned}\tag{30}$$

for any $\eta\in H_0^2(\Omega)$ and a.e. $t\in[0,T]$.

Taking $\eta = v_t = v_{1t} - v_{2t}$, noticing

$$\int_\Omega \left(|v_{1t}|^{m-2} v_{1t} - |v_{2t}|^{m-2} v_{2t} \right) (v_{1t} - v_{2t}) \, \mathrm{d}x \geq 0,$$

and integrating both sides of (30) over $(0, t)$, we have

$$\begin{aligned} &\|v_t\|^2 + \|\Delta v\|^2 \\ \leq& 2k \left\| |w_1|^{p-2} w_1 \log |w_1| - |w_2|^{p-2} w_2 \log |w_2| \right\| \; \|v_t\|, \end{aligned} \tag{31}$$

We need estimating the logarithmic term in (31) by using Lemma 3.1. By the similar argument as [22], we give the sketch of the proof.

Making use of mean value theorem, we have, for $0 < \theta < 1$,

$$\begin{aligned} &\left| |w_1|^{p-2} w_1 \log |w_1| - |w_2|^{p-2} w_2 \log |w_2| \right| \\ =& k \left| 1 + (p-1) \log |\theta w_1 + (1-\theta) w_2| \right| |\theta w_1 + (1-\theta)\, w_2|^{p-2} |w_1 - w_2|. \end{aligned}$$

Then, it follows from Lemma 3.1 that

$$\begin{aligned} &\left| |w_1|^{p-2} w_1 \log |w_1| - |w_2|^{p-2} w_2 \log |w_2| \right| \\ \leq& k |\theta w_1 + (1-\theta)\, w_2|^{p-2} |w_1 - w_2| + k(p-1) A |w_1 - w_2| \\ &+ k(p-1) |\theta w_1 + (1-\theta) w_2|^{p-2+\varepsilon} |w_1 - w_2| \\ \leq& k (|w_1| + |w_2|)^{p-2} |w_1 - w_2| + k(p-1) A |w_1 - w_2| \\ &+ k(p-1) (|w_1| + |w_2|)^{p-2+\varepsilon} |w_1 - w_2|. \end{aligned}$$

Since $w_1, w_2 \in B_{RT}$, using Hölder's inequality and the Sobolev embedding, we can obtain

$$\begin{aligned} &\int_\Omega \left[(|w_1| + |w_2|)^{p-2} |w_1 - w_2| \right]^2 \mathrm{d}x \\ \leq& C \left(\int_\Omega (|w_1| + |w_2|)^{2(p-1)} \mathrm{d}x \right)^{(p-2)/(p-1)} \times \left(\int_\Omega |w_1 - w_2|^{2(p-1)} \mathrm{d}x \right)^{1/(p-1)} \\ \leq& C \left[\|w_1\|_{L^{2(p-1)}}^{2(p-1)} + \|w_2\|_{L^{2(p-1)}}^{2(p-1)} \right]^{(p-2)/(p-1)} \|w_1 - w_2\|_{L^{2(p-1)}}^2 \\ \leq& C \left[\|w_1\|_{H_0^2(\Omega)}^{2(p-1)} + \|w_2\|_{H_0^2(\Omega)}^{2(p-1)} \right]^{(p-2)/(p-1)} \|w_1 - w_2\|_{H_0^2(\Omega)}^2 \\ \leq& C R^{2(p-2)} \|w_1 - w_2\|_{H_0^2(\Omega)}^2. \end{aligned}$$

By the similar argument, we have

$$\begin{aligned} &\int_\Omega \left[(|w_1| + |w_2|)^{p-2+\varepsilon} |w_1 - w_2| \right]^2 \mathrm{d}x \\ \leq& C \left(\int_\Omega (|w_1| + |w_2|)^{2(p-2+\varepsilon)(p-1)/(p-2)} \, \mathrm{d}x \right)^{(p-2)/(p-1)} \end{aligned}$$

$$\times \left(\int_\Omega |w_1 - w_2|^{2(p-1)}\,\mathrm{d}x\right)^{1/(p-1)}$$

$$\leq \left(\int_\Omega (|w_1| + |w_2|)^{2(p-1)+2\varepsilon(p-1)/(p-2)}\mathrm{d}x\right)^{(p-2)/(p-1)} \|w_1 - w_2\|^2_{L^{2(p-1)}}.$$

Using (2), we can choose sufficiently small $\varepsilon > 0$ such that

$$\bar{p} = 2(p-1) + \frac{2\varepsilon(p-1)}{p-2} \leq \frac{2n}{n-4},$$

which yields that

$$\int_\Omega \left[(|w_1| + |w_2|)^{p-2+\varepsilon} |w_1 - w_2|\right]^2 \mathrm{d}x$$
$$\leq C\left[\|w_1\|^{\bar{p}}_{L^{\bar{p}}(\Omega)} + \|w_2\|^{\bar{p}}_{L^{\bar{p}}(\Omega)}\right]^{(p-2)/(p-1)} \|w_1 - w_2\|^2_{L^{2(p-1)}}$$
$$\leq C R^{\bar{p}(p-2)/(p-1)} \|w_1 - w_2\|^2_{H_0^2(\Omega)}.$$

Noticing $\|w_1 - w_2\| \leq C\|w_1 - w_2\|_{H_0^2(\Omega)}$, from the above discussions, we can deduce that

$$\left\| |w_1|^{p-2} w_1 \log|w_1| - |w_2|^{p-2} w_2 \log|w_2| \right\|$$
$$\leq C\left(R^{p-2} + 1 + R^{\bar{p}(p-2)/2(p-1)}\right) \|w_1 - w_2\|_{H_0^2(\Omega)}.$$

Thus, it follows from (31) that

$$\begin{aligned}\|\Phi(w_1) - \Phi(w_2)\|_{\mathcal{H}} &= \|v_1 - v_2\|_{\mathcal{H}} \\ &\leq C\left(R^{p-2} + 1 + R^{\bar{p}(p-2)/2(p-1)}\right) T\|w_1 - w_2\|_{\mathcal{H}}.\end{aligned} \tag{32}$$

We choose T sufficiently small such that $C\left(R^{p-2} + 1 + R^{\bar{p}(p-2)/2(p-1)}\right) T < 1$. Thus from (32) we obtain Φ is a contract map in B_{RT}. The contraction mapping principle then shows that there exists a unique $u \in B_{RT}$ satisfying $u = \Phi(u)$ which is a solution to problem (1). The proof is complete. □

4. Global existence and energy decay

In this section, we consider the global existence and energy decay of the solution for problem (1). First, we introduce the following lemmas which play an important role in studying the decay estimate of global solution for the problem (1).

Lemma 4.1. [33] *Let $\phi(t)$ be a nonincreasing and nonnegative function on $[0,T]$, $T > 1$, such that*

$$\phi(t)^{1+r} \leq \omega_0(\phi(t) - \phi(t+1)) \quad \text{on } [0,T],$$

where ω_0 is a positive constant and r is a nonnegative constant. Then we have

(i): *if $r > 0$, then*

$$\phi(t) \leq \left(\phi(0)^{-r} + \omega_0^{-1} r[t-1]^+\right)^{-\frac{1}{r}} \quad \text{on} \quad [0,T];$$

(ii): *if* $r = 0$, *then*

$$\phi(t) \leq \phi(0)e^{-\omega_1[t-1]^+} \quad on \quad [0,T],$$

where $\omega_1 = \log\left(\frac{w_0}{\omega_0 - 1}\right)$, *here* $\omega_0 > 1$.

Now, we establish the global existence and energy decay results.

Theorem 4.2. *Let u be the unique local solution to problem (1). Assume (2) and $2 \leq m < p$ hold. If $u_0 \in W$, $u_1 \in L^2(\Omega)$ and $E(0) < d$, then $u(t)$ is the global solution to the problem (1). Moreover it has the following decay property*

$$E(t) \leq Ke^{-\kappa t}, \text{ if } m = 2;$$

and

$$E(t) \leq \left(E(0)^{-\frac{m-2}{2}} + \frac{(m-2)\tau}{2}[t-1]^+\right)^{-\frac{2}{m-2}}, \text{ if } m > 2,$$

where K and κ are positive constants, τ is given by (47).

Proof. **Step 1**. Global existence. It suffices to show that $\|u_t\|^2 + \|\Delta u\|^2$ is uniformly bounded with respect to t. It follows from Lemma 2.5 (i) that $u(t) \in W$ on $[0,T]$. Using (13), we have the following estimate

$$\begin{aligned} d > E(0) \geq E(t) &= \frac{1}{2}\|u_t\|^2 + J(u) \\ &= \frac{1}{2}\|u_t\|^2 + \frac{1}{p}I(u) + \left(\frac{1}{2} - \frac{1}{p}\right)\|\Delta u\|^2 + \frac{k}{p^2}\|u\|_p^p \qquad (33) \\ &> \frac{1}{2}\|u_t\|^2 + \frac{p-2}{2p}\|\Delta u\|^2, \end{aligned}$$

which yields that

$$\|u_t\|^2 + \|\Delta u\|^2 \leq \frac{2p}{p-2}d < +\infty.$$

The above inequality and the continuation principle imply the global existence, i.e. $T = +\infty$.

Step 2. We claim that there exists constant $\theta \in (0,1)$ such that

$$I(u) \geq \theta\|\Delta u\|^2. \qquad (34)$$

In fact, it follows from $I(u(t)) > 0$ for all $t \geq 0$ and Lemma 2.1 that there exists a $\lambda_0 > 1$ such that $I(\lambda_0 u(t)) = 0$. Making use of (33), we have

$$\begin{aligned} d \leq J(\lambda_0 u(t)) &= \frac{1}{p}I(\lambda_0 u) + \left(\frac{1}{2} - \frac{1}{p}\right)\|\Delta(\lambda_0 u)\|^2 + \frac{k}{p^2}\|\lambda_0 u\|_p^p \\ &= \frac{p-2}{2p}\lambda_0^2\|\Delta u\|^2 + \frac{k}{p^2}\lambda_0^p\|u\|_p^p \\ &= \lambda_0^p\left(\frac{p-2}{2p}\lambda_0^{2-p}\|\Delta u\|^2 + \frac{k}{p^2}\|u\|_p^p\right) \\ &\leq \lambda_0^p\left(\frac{p-2}{2p}\|\Delta u\|^2 + \frac{k}{p^2}\|u\|_p^p\right) \\ &< \lambda_0^p E(0), \end{aligned}$$

which implies that

$$\lambda_0 > \left(\frac{d}{E(0)}\right)^{\frac{1}{p}} > 1. \tag{35}$$

It follows from (12) that

$$\begin{aligned} 0 = I(\lambda_0 u) &= \|\Delta(\lambda_0 u)\|^2 - \int_\Omega |\lambda_0 u|^p \log|\lambda_0 u|^k \mathrm{d}x \\ &= \lambda_0^2\|\Delta u\|^2 - \lambda_0^p k \int_\Omega |u|^p \log|u| \mathrm{d}x - (\lambda_0^p k \log\lambda_0)\|u\|_p^p \\ &= \lambda_0^p I(u) - \lambda_0^p\|\Delta u\|^2 + \lambda_0^2\|\Delta u\|^2 - (\lambda_0^p k \log\lambda_0)\,\|u\|_p^p \\ &= \lambda_0^p I(u) - (\lambda_0^p - \lambda_0^2)\|\Delta u\|^2 - (\lambda_0^p k \log\lambda_0)\,\|u\|_p^p. \end{aligned}$$

Combining this equality with (35), we have

$$\begin{aligned} \lambda_0^p I(u) &= (\lambda_0^p - \lambda_0^2)\|\Delta u\|^2 + (\lambda_0^p k \log\lambda_0)\|u\|_p^p \\ &\geq (\lambda_0^p - \lambda_0^2)\|\Delta u\|^2, \end{aligned}$$

which implies that

$$I(u) \geq (1 - \lambda_0^{2-p})\|\Delta u\|^2.$$

Hence, the inequality (34) holds with $\theta = 1 - \lambda_0^{2-p}$.

Step 3. Energy decay. By integrating (18) over $[t, t+1], t > 0$, we obtain

$$E(t) - E(t+1) \equiv D(t)^m, \tag{36}$$

where

$$D(t)^m = \int_t^{t+1} \|u_\tau\|_m^m \,\mathrm{d}\tau. \tag{37}$$

In view of (37) and the embedding $L^m(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$\int_t^{t+1} \int_\Omega |u_t|^2 \,\mathrm{d}x\mathrm{d}t \leq c(\Omega) D(t)^2. \tag{38}$$

Thus, from (38), there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$\|u_t(t_i)\|_2^2 \leq 4c(\Omega) D(t)^2, \quad i = 1, 2. \tag{39}$$

On the other hand, multiplying $(1.1)_1$ by u and integrating over $\Omega \times [t_1, t_2]$, we have

$$\begin{aligned} \int_{t_1}^{t_2} I(u)\mathrm{d}t = &\int_{t_1}^{t_2} \|u_t\|^2 \,\mathrm{d}t + (u_t(t_1), u(t_1)) - (u_t(t_2), u(t_2)) \\ &- \int_{t_1}^{t_2} \int_\Omega |u_t|^{m-2} u_t u \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{40}$$

It follows from (33) that

$$
\begin{aligned}
\left|\int_{t_1}^{t_2}\int_\Omega |u_t^{m-2}u_t u \mathrm{d}x\mathrm{d}t\right| &\leq \int_{t_1}^{t_2}\|u\|_m\|u_t\|_m^{m-1}\mathrm{d}t\\
&\leq C\int_{t_1}^{t_2}\|\Delta u\|\,\|u_t\|_m^{m-1}\mathrm{d}t\\
&\leq C\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\|u_t\|_m^{m-1}\mathrm{d}t\\
&\leq C\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}D(t)^{m-1}.
\end{aligned}\tag{41}
$$

By using (33) and (39), we also have

$$
\|u_t(t_i)\|_2\,\|u(t_i)\|_2\leq C_1D(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}},\quad i=1,2.\tag{42}
$$

Combining (38), (41) with (42), we have from (40) that

$$
\begin{aligned}
\int_{t_1}^{t_2}I(u)\mathrm{d}t \leq& c(\Omega)D(t)^2+2C_1D(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\\
&+C\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}D(t)^{m-1}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}.
\end{aligned}\tag{43}
$$

Moreover, using (33) and (34), it is easy to see that

$$
\|u\|_p^p\leq C_p^p\|\Delta u\|^p\leq C_p^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\frac{1}{\theta}I(u).
$$

Thus, we deduce that

$$
E(t)\leq\frac{1}{2}\|u_t\|^2+C_2I(u),\tag{44}
$$

where $C_2=\frac{1}{p}+\frac{p-2}{2p\theta}+\frac{kC_p^p}{p^2\theta}\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}$. By integrating (44) over (t_1,t_2), we have

$$
\int_{t_1}^{t_2}E(t)\mathrm{d}t\leq\frac{1}{2}\int_{t_1}^{t_2}\|u_t\|_2^2\mathrm{d}t+C_2\int_{t_1}^{t_2}I(u)\mathrm{d}t.\tag{45}
$$

By integrating (18) over $[t,t_2]$, we obtain

$$
E(t)=E(t_2)+\int_t^{t_2}\|u_t\|_m^m\mathrm{d}s
$$

Since $t_2-t_1\geq\frac{1}{2}$, it easy to see that

$$
E(t_2)\leq 2\int_{t_1}^{t_2}E(t)\mathrm{d}t.
$$

Then, in view of (36), we have

$$
E(t)=E(t+1)+D(t)^m\leq E(t_2)+D(t)^m\leq 2\int_{t_1}^{t_2}E(t)\mathrm{d}t+D(t)^m.
$$

Thus, combining (38) with (45), we get that

$$
\begin{aligned}
E(t) \leq & \left(c(\Omega)+2c(\Omega)C_2\right) D(t)^2 + D(t)^m 2 \\
& + C_2\left[2C_1 D(t) + C\left(\frac{2p}{(p-2)}\right)^{\frac{1}{2}} D(t)^{m-1}\right] \sup_{t_1\leq s\leq t_2} E(s)^{\frac{1}{2}} \\
\leq & \left(c(\Omega)+2c(\Omega)C_2\right) D(t)^2 + D(t)^m \\
& + 2C_2\left[2C_1 D(t) + C\left(\frac{2p}{(p-2)}\right)^{\frac{1}{2}} D(t)^{m-1}\right] E(t)^{\frac{1}{2}}.
\end{aligned}
$$

Hence, it follows from Young's inequality that

$$
E(t) \leq C_3\left[D(t)^2 + D(t)^m + D(t)^{2(m-1)}\right] \tag{46}
$$

holds with some positive constant $C_3 > 1$. It follows from (36) and (46), we deduce that

$$
\begin{aligned}
E(t) &\leq C_3\left[1 + D(t)^{m-2} + D(t)^{2m-4}\right] D(t)^2 \\
&\leq C_3\left[1 + E(0)^{\frac{m-2}{m}} + E(0)^{\frac{2m-4}{m}}\right] D(t)^2,
\end{aligned}
$$

which implies that

$$
E(t)^{\frac{m}{2}} \leq \left(C_4(E(0))\right)^{\frac{m}{2}} D(t)^m = \left(C_4(E(0))\right)^{\frac{m}{2}} \big(E(t) - E(t+1)\big),
$$

where $C_4(E(0)) = C_3\left[1 + E(0)^{\frac{m-2}{m}} + E(0)^{\frac{2m-4}{m}}\right]$. Notice that

$$
\lim_{E(0)\to 0} C_4(E(0)) = C_3
$$

Hence, the energy decay estimates hold with

$$
K = E(0)e^{\kappa}, \quad \kappa = \log\frac{3C_3}{3C_3 - 1} \text{ and } \tau = \left(C_4(E(0))\right)^{-\frac{m}{2}}. \tag{47}
$$

□

5. Blow up for negative energy

In this section, we will establish that the solution of problem (1) blows up in finite time provided $E(0) < 0$. For this purpose, we give some useful lemmas.

Lemma 5.1. *Assume that (2) holds. Then there exists a positive constant C such that*

$$
\left(\int_\Omega |u|^p \log|u|^k \mathrm{d}x\right)^{s/p} \leq C\left[\int_\Omega |u|^p \log|u|^k \mathrm{d}x + \|\Delta u\|_2^2\right],
$$

for any $u \in H_0^2(\Omega)$ and $2 \leq s \leq p$, provided that $\int_\Omega |u|^p \log|u|^k \mathrm{d}x \geq 0$.

Lemma 5.2. *Assume that (2) holds. Then there exists a positive constant C such that*

$$
\|u\|_p^p \leq C\left[\int_\Omega |u|^p \log|u|^k \mathrm{d}x + \|\Delta u\|^2\right],
$$

for any $u \in H_0^2(\Omega)$, provided that $\int_\Omega |u|^p \log|u|^k \mathrm{d}x \geq 0$.

Lemma 5.3. *Assume that (2) holds. Then there exists a positive constant $C > 1$ such that*

$$\|u\|_p^s \leq C\left[\|u\|_p^p + \|\nabla u\|_2^2\right],$$

for any $u \in H_0^2(\Omega)$ and $2 \leq s \leq p$.

The proof of lemma 5.1-5.3 is similar to the proof in [22], we omit the details.

Lemma 5.4. *Assume that (2) and $m < p$ hold. Then there exists a positive constant C such that*

$$\|u\|_m^m \leq C\left[\left(\int_\Omega |u|^p \log|u|^k \mathrm{d}x\right)^{\frac{m}{p}} + \|\Delta u\|^{\frac{2m}{p}}\right],$$

for any $u \in H_0^2(\Omega)$, provided that $\int_\Omega |u|^p \log|u|^k \mathrm{d}x \geq 0$.

Proof. Noting $m \leq p$ and using the fact that $\|u\|_m^m \leq C\left(\|u\|_p^p\right)^{\frac{m}{p}}$, we can obtain the result from Lemma 5.2. ☐

Now we are in the position to state and prove the blow up result for $E(0) < 0$.

Theorem 5.5. *Suppose that the conditions in Lemma 5.4 hold. Then the solution to the problem (1) blows up in finite time provided that $E(0) < 0$.*

Proof. We denote $H(t) = -E(t)$. It follows from (17) and (18) that

$$E(t) \leq E(0) < 0, \qquad H'(t) = -E'(t) = \|u_t\|_m^m.$$

and

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\int_\Omega |u|^p \log|u|^k \mathrm{d}x. \tag{48}$$

We define

$$L(t) = H^{1-\beta}(t) + \varepsilon\int_\Omega u u_t \mathrm{d}x, \quad t \geq 0,$$

where $\varepsilon > 0$ to be determined later and

$$\frac{2(p-m)}{(m-1)p^2} < \beta < \frac{p-m}{2(m-1)p} < 1. \tag{49}$$

By taking a derivation of $L(t)$, we get

$$\begin{aligned} L'(t) =&(1-\beta)H^{-\beta}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\Delta u\|^2 \\ &- \varepsilon\int_\Omega |u_t|^{m-2}u_t u \mathrm{d}x + \varepsilon\int_\Omega |u|^p \log|u|^k \mathrm{d}x. \end{aligned}$$

Adding and subtracting $\varepsilon p(1-a)H(t)$ for some $a \in (0,1)$ in the RHS of the above equation, then using the definition of $H(t)$, we obtain

$$\begin{aligned} L'(t) =&(1-\beta)H^{-\beta}(t)H'(t) + \varepsilon\frac{p(1-a)+2}{2}\|u_t\|^2 + \varepsilon\frac{p(1-a)-2}{2}\|\Delta u\|^2 \\ &+ \varepsilon p(1-a)H(t) - \varepsilon\int_\Omega |u_t|^{m-2}u_t u \mathrm{d}x + \varepsilon a\int_\Omega |u|^p \log|u|^k \mathrm{d}x \\ &+ \varepsilon\frac{(1-a)k}{p}\|u\|_p^p. \end{aligned} \tag{50}$$

In view of Young's inequality, we have

$$\int_\Omega |u_t|^{m-2}u_t u \mathrm{d}x \leq \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{-m/(m-1)}\|u_t\|_m^m$$

for any $\delta > 0$, which yields, by substitution in (50),

$$
\begin{aligned}
L'(t) \geq & \left[(1-\beta)H^{-\beta}(t) - \frac{m-1}{m}\varepsilon\delta^{-m/(m-1)}\right]\|u_t\|_m^m - \varepsilon\frac{\delta^m}{m}\|u\|_m^m \\
& + \varepsilon\frac{p(1-a)+2}{2}\|u_t\|^2 + \varepsilon\frac{p(1-a)-2}{2}\|\Delta u\|^2 + \varepsilon p(1-a)H(t) \\
& + \varepsilon a\int_\Omega |u|^p \log|u|^k \mathrm{d}x + \varepsilon\frac{(1-a)k}{p}\|u\|_p^p.
\end{aligned}
\tag{51}
$$

Since the integral is taken over the x variable, (50) holds even if δ is time dependent. Thus by choosing δ so that $\delta^{-m/(m-1)} = MH^{-\beta}(t)$, for large M to be determined later, substituting in (51), we obtain

$$
\begin{aligned}
L'(t) \geq & \left[(1-\beta) - \frac{m-1}{m}\varepsilon M\right]H^{-\beta}(t)\|u_t\|_m^m - \varepsilon\frac{M^{1-m}}{m}H^{\beta(m-1)}\|u\|_m^m \\
& + \varepsilon\frac{p(1-a)+2}{2}\|u_t\|^2 + \varepsilon\frac{p(1-a)-2}{2}\|\Delta u\|^2 + \varepsilon p(1-a)H(t) \\
& + \varepsilon a\int_\Omega |u|^p \log|u|^k \mathrm{d}x + \varepsilon\frac{(1-a)k}{p}\|u\|_p^p.
\end{aligned}
\tag{52}
$$

Making using of (48), Lemma 5.4 and Young's inequality, we find

$$
\begin{aligned}
& H^{\beta(m-1)}\|u\|_m^m \\
\leq & \left(\int_\Omega |u|^p \log|u|^k \mathrm{d}x\right)^{\beta(m-1)}\|u\|_m^m \\
\leq & C\left[\left(\int_\Omega |u|^p \log|u|^k \mathrm{d}x\right)^{\beta(m-1)+\frac{m}{p}} + \left(\int_\Omega |u|^p \log|u|^k \mathrm{d}x\right)^{\beta(m-1)}\|\Delta u\|^{\frac{2m}{p}}\right] \\
\leq & C\left[\left(\int_\Omega |u|^p \log|u|^k \mathrm{d}x\right)^{\beta(m-1)+\frac{m}{p}} + \left(\int_\Omega |u|^p \log|u|^k \mathrm{d}x\right)^{\beta(m-1)\cdot\frac{p}{p-m}} + \|\Delta u\|^2\right].
\end{aligned}
$$

Hence, it follows from Lemma 5.1 that

$$
2 < \beta(m-1)p + m \leq p \quad \text{and} \quad 2 < \frac{\beta(m-1)p^2}{p-m} \leq p.
$$

Thus, Lemma 5.1 implies

$$
H^{\beta(m-1)}\|u\|_m^m \leq C\left(\int_\Omega |u|^p \log|u|^k \mathrm{d}x + \|\Delta u\|^2\right).
\tag{53}
$$

Combining (52) and (53), we have

$$
\begin{aligned}
& L'(t) \\
\geq & \left[(1-\beta) - \frac{m-1}{m}\varepsilon M\right]H^{-\beta}(t)\|u_t\|_m^m + \varepsilon\frac{p(1-a)+2}{2}\|u_t\|^2
\end{aligned}
$$

$$\begin{aligned} &+\varepsilon\left[\frac{p(1-a)-2}{2}-\frac{M^{1-m}}{m}C\right]\|\Delta u\|^2+\varepsilon\left[a-\frac{M^{1-m}}{m}C\right]\int_\Omega |u|^p\log|u|^k\mathrm{d}x \\ &+\varepsilon p(1-a)H(t)+\varepsilon\frac{(1-a)k}{p}\|u\|_p^p. \end{aligned} \tag{54}$$

Now, we choose $a>0$ sufficiently small that

$$\frac{p(1-a)-2}{2}>0$$

and M sufficiently large that

$$\frac{p(1-a)-2}{2}-\frac{M^{1-m}}{m}C>0 \quad \text{and} \quad a-\frac{M^{1-m}}{m}C>0.$$

Once M and a are fixed, we choose ε sufficiently small that

$$(1-\beta)-\frac{m-1}{m}\varepsilon M>0 \quad \text{and} \quad L(0)=H^{1-\beta}(0)+\varepsilon\int_\Omega u_0u_1\mathrm{d}x>0.$$

Thus, for some constant $\gamma>0$, (54) has the form

$$L'(t)\geq\gamma\left[H(t)+\|u_t\|^2+\|\Delta u\|^2+\|u\|_p^p+\int_\Omega |u|^p\log|u|^k\mathrm{d}x\right]. \tag{55}$$

Consequently we have

$$L(t)\geq L(0), \text{ for all } t>0.$$

On the other hand, using Lemma 5.3, by the same method as in [32], we can deduce

$$L^{\frac{1}{1-\beta}}(t)\leq C\left[H(t)+\|u_t\|^2+\|\Delta u\|^2+\|u\|_p^p\right],\quad t\geq 0. \tag{56}$$

Combining (55) and (56), we obtain

$$L'(t)\geq\lambda L^{\frac{1}{1-\beta}}(t),\quad t\geq 0. \tag{57}$$

where $\lambda>0$ is constant depending only on γ and C. By a simple integration of (57) over $(0,t)$, we have

$$L^{\beta/(1-\beta)}(t)\geq\frac{1}{L^{-\beta/(1-\beta)}(0)-\lambda t\beta/(1-\beta)}.$$

which implies that $L(t)$ blow up in finite time

$$T\leq T^*=\frac{1-\beta}{\lambda\beta L^{\beta/(1-\beta)}(0)}.$$

This completes the proof of Theorem 5.1. □

6. Arbitrarily high initial energy for linear damping

In this section, we consider the problem (1) with the linear damping term, i.e. $m=2$. We will establish the finite time blow-up result by the method of the so called concavity method. For simplicity, we denote $\|u\|^2\leq B_0\|\Delta u\|^2$.

Lemma 6.1. [14] *Let $\delta\geq 0, T>0$ and h be a Lipschitzian function over $[0,T)$. Assume that $h(0)\geq 0$ and $h'(t)+\delta h(t)>0$ for a.e. $t\in[0,T)$. Then $h(t)>0$ for all $t\in(0,T)$.*

Lemma 6.2. *Suppose that $m = 2$. Let $I(u_0) < 0$ and $u_1 \in L^2(\Omega)$ such that*

$$\int_\Omega u_0 u_1 \mathrm{d}x \geq 0.$$

Let $u(t)$ be the solution with the initial data (u_0, u_1). Then the map $\{t \to \|u(t)\|^2\}$ is strictly increasing provided $I(u(t)) < 0$.

Proof. Let $F(t) = \|u(t)\|^2$ and $G(t) = F'(t) = 2\int_\Omega uu_t \mathrm{d}x$. A direct computation yields

$$\langle u_{tt}, u\rangle = \frac{d}{dt}(u_t, u) - \|u_t\|^2 \ for\ a.e.t \geq 0,$$

Moreover, by testing the equation with $u(t)$, we have

$$\langle u_{tt}, u\rangle + \|\Delta u\|^2 + (u_t, u) = \int_\Omega |u|^p \log|u|^k \mathrm{d}x,$$

which implies

$$\frac{d}{dt}\left((u_t, u) + \frac{1}{2}\|u\|^2\right) = \|u_t\|^2 - I(u).$$

Hence, if $I(u(t)) > 0$, we can deduce

$$G'(t) + G(t) = 2\|u_t\|^2 - 2I(u(t)) > 0 \ for\ a.e.t \in [0, T).$$

Therefore, it follows from Lemma 6.1 with $\delta = 1$ that $G(t) = F'(t) > 0$. Hence $F(t)$ is strictly increasing provided $I(u(t)) < 0$. □

Lemma 6.3. *Let $m = 2$. Assume that $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$ and (2) holds. Assume that the initial data satisfies*

$$\|u_1\|^2 - 2(u_1, u_0) + \Lambda E(0) < 0, \tag{58}$$

where $\Lambda = \frac{2B_0 p}{p-2}$. Then the solution $u(t)$ of the problem (1) with $E(0) > 0$ satisfies $I(u(t)) < 0$ provided $I(u_0) < 0$.

Proof. If this was not the case, by the continuity of $I(u(t))$ in t, then there would exist a first time $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$ and $I(u(t)) < 0$ for $t \in [0, t_0)$. It follows form the Cauchy-Schwarz inequality that

$$(u_1, u_0) \leq \|u_1\|\|u_0\| \leq \frac{1}{2}\big(\|u_1\|^2 + \|u_0\|^2\big). \tag{59}$$

By Lemma 6.2, (58) and (59), we deduce that

$$F(t) = \|u(t)\|^2 > \|u_0\|^2 \geq 2(u_1, u_0) - \|u_1\|^2 > \Lambda E(0) \text{ for } t \in (0, t_0), \tag{60}$$

which implies

$$F(t_0) = \|u(t_0)\|^2 > \Lambda E(0) \tag{61}$$

by the continuity of $u(t)$ in t. Moreover, it follows from (13) and (17) that

$$\begin{aligned} E(0) \geq E(t_0) &= \frac{1}{p}I(u(t_0)) + \left(\frac{1}{2} - \frac{1}{p}\right)\|\Delta u(t_0)\|^2 + \frac{k}{p^2}\|u(t_0)\|_p^p \\ &\geq \frac{p-2}{2p}\|\Delta u(t_0)\|^2 \end{aligned}$$

that is

$$\|\Delta u(t_0)\|^2 \leq \frac{2p}{p-2}E(0).$$

Hence, we have

$$F(t_0) = \|u(t_0)\|^2 \le B_0 \|\Delta u(t_0)\|^2 \le \frac{2B_0 p}{p-2} E(0) = \Lambda E(0),$$

which is a contradiction with (61). The proof is complete. □

We now present the main blow-up result for the weak solution of problem (1) with $m = 2$ with arbitrary positive initial energy.

Theorem 6.4. *Assume the conditions of Lemma 6.3 hold. Then the weak solution $u(t)$ of the problem (1) blows up in finite time provided that $E(0) > 0$ and $I(u_0) < 0$.*

Proof. It follows from Lemma 6.3 that $I(u(t)) < 0$ for $t \in [0, T)$. By contradiction, we assume now that $u(t)$ is global, namely $T = \infty$. Then, for any $T_0 > 0$, we may consider $\eta : [0, T_0] \to \mathbb{R}^+$ defined by

$$\eta(t) = \|u\|^2 + \int_0^t \|u(\tau)\|^2 \mathrm{d}\tau + (T_0 - t)\|u_0\|^2.$$

Notice $\eta(t) > 0$ for all $t \in [0, T_0]$; hence, since η is continuous, there exists $\varrho > 0$ (independent of the choice of T_0) such that

$$\eta(t) \ge \varrho \ \text{ for all } t \in [0, T_0]. \tag{62}$$

Moreover,

$$\eta'(t) = 2(u_t, u) + \|u\|^2 - \|u_0\|^2 = 2(u_t, u) + 2\int_0^t (u_\tau, u)\mathrm{d}\tau, \tag{63}$$

hence, we have

$$\begin{aligned} \eta''(t) &= 2\|u_t\|^2 + 2\langle u_{tt}, u\rangle + 2(u_t, u) \\ &= 2\left(\|u_t\|^2 - \|\Delta u\|^2 + \int_\Omega |u|^p \log|u|^k \mathrm{d}x\right) \qquad (64) \\ &= 2\|u_t\|^2 - 2I(u(t)). \end{aligned}$$

It follows from (63) that

$$(\eta'(t))^2 = 4\left((u_t, u)^2 + 2(u_t, u)\int_0^t (u_\tau, u)\mathrm{d}\tau + \Big(\int_0^t (u_\tau, u)\mathrm{d}\tau\Big)^2\right).$$

By the Cauchy-Schwarz inequality, we obtain

$$\|u_t\|^2 \, \|u\|^2 \ge (u_t, u)^2$$

$$\int_0^t \|u\|^2 \mathrm{d}\tau \int_0^t \|u_\tau\|^2 \,\mathrm{d}\tau \ge \left(\int_0^t (u_\tau, u)\,\mathrm{d}\tau\right)^2$$

and

$$\begin{aligned} &2(u_t, u)\int_0^t (u_\tau, u)\mathrm{d}\tau \\ \le& 2\|u_t\|\|u\|\Big(\int_0^t \|u_\tau\|^2 \mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_0^t \|u\|^2 \mathrm{d}\tau\Big)^{\frac{1}{2}} \\ \le& \|u_t\|^2 \int_0^t \|u\|^2 \mathrm{d}\tau + \|u\|^2 \int_0^t \|u_\tau\|^2 \mathrm{d}\tau. \end{aligned}$$

Combining the above inequalities, we have

(65)
$$\begin{aligned}&\eta'(t)^2\\ \leq&4\left(\|u_t\|^2\|u\|^2+\|u_t\|^2\int_0^t\|u\|^2\mathrm{d}\tau+\|u\|^2\int_0^t\|u_\tau\|^2\mathrm{d}\tau+\int_0^t\|u_\tau\|^2\mathrm{d}\tau\int_0^t\|u\|^2\mathrm{d}\tau\right)\\ =&4\left(\|u\|^2+\int_0^t\|u\|^2\mathrm{d}\tau\right)\left(\|u_t\|^2+\int_0^t\|u_\tau\|^2\mathrm{d}\tau\right)\\ \leq&4\eta(t)\left(\|u_t\|^2+\int_0^t\|u_\tau\|^2\mathrm{d}\tau\right).\end{aligned}$$

Hence, it follows from (64) and (65) that

$$\begin{aligned}&\eta''(t)\eta(t)-\frac{p+2}{4}\eta'(t)^2\\ \geq&\eta(t)\left(\eta''(t)-(p+2)\big(\|u_t\|^2+\int_0^t\|u_\tau\|^2\mathrm{d}\tau\big)\right)\\ =&\eta(t)\left(-p\|u_t\|^2-2\|\Delta u\|^2+2\int_\Omega|u|^p\log|u|^k\mathrm{d}x-(p+2)\int_0^t\|u_\tau\|^2\mathrm{d}\tau\right).\end{aligned} \tag{66}$$

Now, we define

$$\xi(t)=-p\|u_t\|^2-2\|\Delta u\|^2+2\int_\Omega|u|^p\log|u|^k\mathrm{d}x-(p+2)\int_0^t\|u_\tau\|^2\mathrm{d}\tau.$$

Noticing $m=2$ in this section, using (13) and (17), we obtain

$$\begin{aligned}\xi(t)&=(p-2)\|\Delta u\|^2-2pE(t)-(p+2)\int_0^t\|u_\tau\|^2\mathrm{d}\tau+\frac{2k}{p}\|u\|_p^p\\ &=(p-2)\|\Delta u\|^2-2pE(0)+(p-2)\int_0^t\|u_\tau\|^2\mathrm{d}\tau+\frac{2k}{p}\|u\|_p^p\\ &\geq(p-2)\|\Delta u\|^2-2pE(0).\end{aligned}$$

From (60) and Lemma 6.2, we deduce that

$$2pE(0)<\frac{p-2}{B_0}\|u_0\|^2<\frac{p-2}{B_0}\|u\|^2<(p-2)\|\Delta u\|^2.$$

which yields that $\xi(t)\geq\varsigma>0$. Then, (66) can be rewritten as

$$\eta''(t)-\frac{p+2}{4}\eta'(t)^2\geq\varrho\varsigma,\ \ t\in[0,T_0],$$

which implies that

$$\big(\eta^{-\frac{p-2}{4}}(t)\big)''\leq-\frac{p-2}{4}\varrho\varsigma(\eta(t))^{-\frac{p+6}{4}}<0.$$

Hence, it follows that there exists a $T^*>0$ such that

$$\lim_{t\to T^*}\eta^{-\frac{p-2}{4}}(t)=0,$$

that is

$$\lim_{t\to T^*}\eta(t)=+\infty.$$

In turn, this implies that

$$\lim_{t\to T^*}\|\Delta u(t)\|^2=+\infty. \tag{67}$$

In fact, if $\|u(t)\|\to+\infty$ as $t\to T^*$, then (67) immediately follows. On the contrary, if $\|u(t)\|$ remains bounded on $[0,T^*)$, then

$$\lim_{t\to T^*}\int_0^t\|u(\tau)\|^2\mathrm{d}\tau=+\infty$$

so that (67) is also satisfied. Hence (67) is a contraction with $T=+\infty$. The proof is complete. □

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