ELECTRONIC RESEARCH ARCHIVE
Volume 28 Number 1, March 2020 Pages 221–261
eISSN: 2688-1594 AIMS (2020)
doi:10.3934/era.2020015

EXISTENCE AND UNIFORM DECAY ESTIMATES FOR THE FOURTH ORDER WAVE EQUATION WITH NONLINEAR BOUNDARY DAMPING AND INTERIOR SOURCE

HUAFEI DI*, YADONG SHANG AND JIALI YU

ABSTRACT. In this paper, we consider the initial boundary value problem for the fourth order wave equation with nonlinear boundary velocity feedbacks $f_1(u_{\nu t})$, $f_2(u_t)$ and internal source $|u|^\rho u$. Under some geometrical conditions, the existence and uniform decay rates of the solutions are proved even if the nonlinear boundary velocity feedbacks $f_1(u_{\nu t})$, $f_2(u_t)$ have not polynomial growth near the origin respectively. By the combination of the Galerkin approximation, potential well method and a special basis constructed, we first obtain the global existence and uniqueness of regular solutions and weak solutions. In addition, we also investigate the explicit decay rate estimates of the energy, the ideas of which are based on the construction of a special weight function $\phi(t)$ (that depends on the behaviors of the functions $f_1(u_{\nu t})$, $f_2(u_t)$ near the origin), nonlinear integral inequality and the Multiplier method.

1. Introduction

This paper is concerned with the existence and uniform decay rate estimates for the following initial boundary value problem:

$$\begin{cases} u_{tt} = -\triangle^2 u + |u|^\rho u, \ (x,t) \in \Omega \times (0,\infty), \\ u = u_\nu = 0, \ (x,t) \in \Gamma_0 \times (0,\infty), \\ u_{\nu\nu} = -f_1(u_{\nu t}), \ u_{\nu\nu\nu} = f_2(u_t), \ (x,t) \in \Gamma_1 \times (0,\infty), \\ u(x,0) = u^0, \ u_t(x,0) = u^1, \ x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n with C^4 boundary Γ. Let $\{\Gamma_0, \Gamma_1\}$ be a partition of its boundary Γ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and Γ_0, Γ_1 are positive measurable, endowed with the $(n-1)-$dimentional Lebesgue measure. Here, ν represents the unit outward normal to Γ, and f_i $(i = 1,2)$ are given functions satisfying certain conditions to be specified later.

For the linear second order wave equations with nonlinear boundary feedback, there is an abounding literature about its initial boundary value problem. In [43],

Received by the editors October, 2019 and, in revised form, January, 2020.
2010 *Mathematics Subject Classification.* Primary: 35L35; Secondary: 35D30, 35D35, 35B40.
Key words and phrases. Fourth order wave equation, boundary velocity feedbacks, interior source, potential well, multiplier method.
Huafei Di is supported by the NSF of China (11801108, 11701116), the Scientific Program (2016A030310262) of Guangdong Province, and the College Scientific Research Project (1201630180) of Guangzhou City.
* Corresponding author: Huafei Di, dihuafei@gzhu.edu.cn.

Zuazua studied the following second order wave equation

$$\begin{cases} u_{tt} - \triangle u = 0, \ (x,t) \in \Omega \times (0,\infty), \\ u = 0, \ (x,t) \in \Gamma_0 \times (0,\infty), \\ u_\nu = -\{m(x)\cdot\nu(x)\} f(u_t), \ (x,t) \in \Gamma_1 \times (0,\infty), \\ u(x,0) = u^0, \ u_t(x,0) = u^1, \ x \in \Omega, \end{cases} \tag{1.2}$$

where x_0 is a fixed point in R^n, and $m(x) = x - x_0$. When $f(y) = |y|^p$ on [0,1] for some $p \geq 1$, he proved that the energy decays exponentially if $p = 1$ and polynomially if $p > 1$. In the later case, he gave that there exists a positive constant C such that

$$\forall\ t \geq 0, \ E(t) \leq \frac{C}{(1+t)^{2/(p+1)}}. \tag{1.3}$$

When the nonlinear boundary velocity feedback $f(y)$ is weaker than any polynomial near the origin, for instance, $\forall\ y \in (0,1), \ f(y) = e^{-1/y}$. Lasiecka and Tataru [20] showed that the energy of solutions decays with the following rate:

$$\forall\ t \geq 0, \ E(t) \leq S\left(\frac{t}{T_0} - 1\right) E(0), \tag{1.4}$$

where $S(t)$ is the solutions (contraction semigroup) of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \qquad S(0) = E(0), \tag{1.5}$$

and q is closely related to the behavior of the feedback $f(y)$ near the origin. They were the first to consider that the energy decay rate estimates associated to the solutions of some differential equation and without assuming that the feedback has a polynomial behavior near the origin. Martinez [26] complemented Lasiecka and Tataru's work in [20] concerning the linear wave equation subject to nonlinear boundary feedback. He proved that the energy of problem (1.5) decays to zero with an explicit decay rate estimates. The process of the proof relies on the construction of some special weight functions and some nonlinear integral inequalities. The method presented in [26], gives us a variety of explicit decay rate estimates, although in some simple cases a direct application of the above method doesn't give us optimal decay rates. For instance, when $f(y) = y^p$, $p > 1$, by the method of [26], the energy decay is given by $E(t) \leq C(1+t)^{-2/p}$, which is less good estimate than the estimate of (1.3). In spite of this, it is possible to obtain optimal decay rate estimates by this method for some other example, see [26] for details.

The linear second order wave equations subject to nonlinear boundary feedback and source terms have also been widely studied. For instance, Vitillaro [33] studied the following problem

$$\begin{cases} u_{tt} - \triangle u = 0, \ (x,t) \in \Omega \times (0,\infty), \\ u = 0, \ (x,t) \in \Gamma_0 \times (0,\infty), \\ u_\nu = -|u_t|^{m-2}u_t + |u|^{p-2}u, \ (x,t) \in \Gamma_1 \times (0,\infty), \\ u(x,0) = u^0, \ u_t(x,0) = u^1, \ x \in \Omega. \end{cases} \tag{1.6}$$

He showed that the presence of the superlinear damping term $-|u_t|^{m-2}u_t$, when $2 \leq p \leq m$, implies the global existence of solutions for arbitrary initial data, in opposition with the nonexistence phenomenon occurring when $m = 2 < p$. Zhang and Hu [42] proved the asymptotic behavior of the solutions of problem (1.6), where the initial data is inside a stable set. The blow up phenomenon of the solutions occurs when the initial data is inside an unstable set. More results on the second order wave equations with nonlinear boundary source and damping terms, the reader can see [2, 3, 25] and papers cited therein.

It is worth mentioning that the potential well theory (stable or unstable sets) is a very important and popular way to study the qualitative properties of nonlinear evolution equations. This method was first introduced by Sattinger [30] to investigate the global existence of solutions for nonlinear hyperbolic equations. Hence, it has been widely used and extended by many authors to study different kinds of evolution equations, we refer the reader to see [6, 7, 30, 34–36, 38–40] and references therein.

Let us mention some known results about the second order wave equations with nonlinear internal damping and source terms

$$u_{tt} - \triangle u + g(u_t) = f(u),\ (x,t) \in \Omega \times (0,\infty). \tag{1.7}$$

Geogev and Todorova [13] investigated the initial boundary value problem of equation (1.7), where $g(u_t) = |u_t|^{m-1}u_t,\ f(u) = |u|^{p-1}u$. They proved the existence of global solutions under the condition $1 < p \le m$. When $p \ge m > 1$, they also obtained the finite time blow up of solutions for sufficient large initial data. Ikehata [15] studied the initial boundary value problem of equation (1.7), where $g(u_t) = \delta|u_t|^{m-1}u_t$ and $f(u) = |u|^{p-1}u$. He proved that $1 \le m < p < \infty$ if $n = 1, 2$, and $1 \le m < p \le \frac{n}{n-2}$ if $n \ge 3$, the problem has a global solution for sufficiently small initial data. When $g(u_t) = au_t(1 + |u_t|^{m-2}),\ f(u) = b|u|^{p-2}u$, Messaoudi [1, 27] investigated the global existence and exponential decay behavior of solutions respectively.

For the second order wave equations with nonlinear internal source and boundary velocity feedback, Cavalcanti et al. [4] studied the following initial boundary value problem

$$\begin{cases} u_{tt} - \triangle u = |u|^p u,\ (x,t) \in \Omega \times (0,\infty), \\ u = 0,\ (x,t) \in \Gamma_0 \times (0,\infty), \\ u_\nu = -f(u_t),\ (x,t) \in \Gamma_1 \times (0,\infty), \\ u(x,0) = u^0,\ u_t(x,0) = u^1,\ x \in \Omega. \end{cases} \tag{1.8}$$

They proved the existence of global solutions and uniform decay rate estimates of the energy provided that the nonlinear boundary feedback $f(u_t)$ has not a polynomial growth near the origin by using the potential well method and the Galerkin approximation. When $f(u_t) = -\alpha(x)|u_t|^{m-2}u_t$ or $f(u_t) = -\alpha(x)(|u_t|^{m-2}u_t + |u_t|^{\mu-2}u_t)$, $1 \le \mu \le m$, and $\alpha(x) \in L^\infty(\Gamma_1), \alpha(x) \ge 0$, Vitillaro [31] extended the potential well theory. He obtained the local existence, blow up and global existence results of solutions. More results on the initial boundary value problem for the wave equations with nonlinear internal source and boundary velocity feedback, we refer readers to see (Di and Shang [9], Feng and Li [11, 12], Liu, Sun and Li [24]) and the papers cited therein.

There are some literature on the initial boundary value problem or Cauchy problem for the fourth order wave equations with source and damping terms in the interior of Ω

$$u_{tt} + \triangle^2 u + g(u_t) = f(u),\ (x,t) \in \Omega \times (0,\infty). \tag{1.9}$$

For example, when $g(u_t) = a|u_t|^{m-2}u_t$, $f(u) = -q(x)u(x,t)$ with $q(x) > 0$, Guesmia [14] investigated the initial boundary value problem of equation (1.9). He obtained a global existence and a regularity result and proved that the solutions decay exponentially if $g(y)$ behaves like a linear functions. For more results on the qualitative problem of the fourth order wave equations with interior source and damping terms, the reader is referred to see [8, 37, 41] and references therein.

When people studied the small transversal vibrations of a thin plate (Lagnese and Lions [17], Lagnese [18]) and the strong or uniform stabilization of different plate and beam models (Lasiecka [19], Puel and Tucsnak [28]), some nonlinear evolution equations with the main part $u_{tt}+\triangle^2 u=0$ and different nonlinear boundary feedbacks were obtained. For example, Komornik [16] studied the following evolutionary problem:

$$\begin{cases} u_{tt}+\triangle^2 u=0,\ (x,t)\in\Omega\times(0,\infty), \\ u=u_\nu=0,\ (x,t)\in\Gamma_0\times(0,\infty), \\ u_{\nu\nu}+u_{\tau\tau}=0,\ on\ \Gamma_1\times(0,\infty), \\ u_{\nu\nu\nu}+(2-\mu)u_{\tau\tau\nu}=lf(u_t),\ (x,t)\in\Gamma_1\times(0,\infty), \\ u(x,0)=u^0,\ u_t(x,0)=u^1,\ x\in\Omega, \end{cases} \tag{1.10}$$

where $\mu\in(0,1)$, $l\in C^1(\Gamma_1)$, and $f:\mathrm{R}\to\mathrm{R}$ is a non-decreasing, continuous function. The subscripts ν and τ stand for the normal and tangential derivatives to Γ_0 and Γ_1. He proved the global existence, regularity results and gave some stabilization properties for problem (1.10) by using the Multiplier method. It is worth mentioning that the Multiplier method has already been used by many authors for different reasons, we also refer to the related papers [5, 16, 21] about the Multiplier method.

Motivated by the above results, in the present work we study the initial boundary value problem of the fourth order wave equation with an internal nonlinear source $|u|^\rho u$, and nonlinear boundary velocity feedbacks $f_1(u_{\nu t})$, $f_2(u_t)$. As far as we know, there is little information on the well-posedness and energy decay estimates for problem (1.1). Naturally, our attention of this paper is paid to the study of the related qualitative properties to problem (1.1). Here, when the boundary velocity feedbacks $f_1(u_{\nu t})$, $f_2(u_t)$ have not the polynomial behaviour near the origin for wave equation supplemented with an interior source $|u|^\rho u$ acting in the domain, we first investigate the global existence, uniqueness of regular solutions and weak solutions by the combination of Galerkin approximation, potential well method and a special basis constructed. In addition, we also prove that the energy of problem (1.1) decays uniformly to zero, which is based on a weight function $\phi(t)$ constructed, Multiplier method and nonlinear integral inequality.

Our paper is organized as follows. In Section 2, we introduce some potential wells, basic definitions, important lemmas, and main results of this paper. In Section 3-4, we show the global existence and uniqueness of the regular solutions and weak solutions respectively. In the last Section, we investigate the explicit decay rate estimates of the energy.

2. Preliminaries and main results

In order to state our results precisely, we first introduce some notations, basic definitions, important lemmas and some functional spaces.

Let Ω be a bounded domain of R^n with C^4 boundary Γ and x^0 be a fix point in R^n. We shall define

$$m(x)=x-x^0,\qquad R=\max_{x\in\overline{\Omega}}|x-x^0|,$$

and introduce a partition of the boundary Γ such that

$$\Gamma_0 = \{x \in \Gamma : \ m(x) \cdot \nu(x) \le 0\}, \quad \Gamma_1 = \{x \in \Gamma : \ m(x) \cdot \nu(x) > 0\}.$$

Throughout this paper, the following inner products and norms are used for precise statement:

$$(u,v) = \int_\Omega u(x)v(x)dx, \ (u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)d\Gamma,$$

$$\|u\|_p^p = \int_\Omega |u(x)|^p dx, \ \|u\|_{\Gamma_1,p}^p = \int_{\Gamma_1} |u(x)|^p d\Gamma, \quad \|u\|_\infty = \operatorname{ess}\sup_{t\ge 0} |u(x)|,$$

and the Hilbert space

$$V = \left\{u \in H^2(\Omega); \ u = u_\nu = 0 \text{ on } \Gamma_0\right\}.$$

Since Γ_0 has positive $(n-1)$ dimensional Lebesgue measure, by Poincaré inequality, we can endow V with the equivalent norm $\|u\|_V = \|\triangle u\|_2$ (see [22]).

To obtain the results of this paper, let us consider the potential energy

$$J(u) = \frac{1}{2}\|\triangle u\|_2^2 - \frac{1}{\rho+2}\|u\|_{\rho+2}^{\rho+2}, \tag{2.1}$$

and total energy

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\triangle u\|_2^2 - \frac{1}{\rho+2}\|u\|_{\rho+2}^{\rho+2} = \frac{1}{2}\|u_t\|_2^2 + J(u), \tag{2.2}$$

associated to the solutions of problem (1.1). We may define the (positive) number

$$d = \inf_{u \in V\setminus\{0\}} \left\{\sup_{\lambda>0} J(\lambda u)\right\}, \tag{2.3}$$

which is also called the depth of the potential well. Moreover, the value d is shown to be the Mountain pass level associated to the elliptic problem

$$\begin{cases} -\triangle^2 u = |u|^\rho u, \ x \in \Omega, \\ u = u_\nu = 0, \ x \in \Gamma_0, \\ u_{\nu\nu} = u_{\nu\nu\nu} = 0, \ x \in \Gamma_1. \end{cases} \tag{2.4}$$

Here, let $B_1 > 0$ be the optimal constant of Sobolev imbedding from V into $L^{\rho+2}(\Omega)$, which satisfies the inequality $\|u\|_{\rho+2} \le B_1\|\triangle u\|_2$, $\forall \ u \in V$. From this inequality, we discover that

$$\frac{\frac{1}{\rho+2}\|u\|_{\rho+2}^{\rho+2}}{\|\triangle u\|_2^{\rho+2}} \le \frac{B_1^{\rho+2}}{\rho+2}, \ \forall \ u \in V \setminus \{0\}. \tag{2.5}$$

Furthermore, setting

$$K_0 = \sup_{u \in V\setminus\{0\}} \left(\frac{\frac{1}{\rho+2}\|u\|_{\rho+2}^{\rho+2}}{\|\triangle u\|_2^{\rho+2}}\right) \le \frac{B_1^{\rho+2}}{\rho+2}, \tag{2.6}$$

and the function

$$f(\lambda) = \frac{1}{2}\lambda^2 - K_0\lambda^{\rho+2}, \ \lambda > 0. \tag{2.7}$$

We can easily see (the simple proof can be founded in [32]) that

$$\lambda_1 = \left(\frac{1}{K_0(\rho+2)}\right)^{\frac{1}{\rho}}, \ d = f(\lambda_1) = \lambda_1^2\left(\frac{1}{2} - \frac{1}{\rho+2}\right), \tag{2.8}$$

where λ_1 is the absolute maximum point of function f.

Now, we will give some basic hypotheses to establish the main results of this paper.

(A1) Suppose that $0 < \rho < \frac{4}{n-4}$, if $n \geq 5$ and $\rho > 0$, if $n = 1, 2, 3, 4$. Then, we have the following Sobolev imbedding

$$V \hookrightarrow L^{2(\rho+1)}(\Omega) \hookrightarrow L^{\rho+2}(\Omega). \tag{2.9}$$

(A2) Assumptions on the functions f_i $(i = 1, 2)$: f_i: R → R are nondecreasing C^1 functions such that $f_i(0) = 0$. In addition, there exist some strictly increasing and odd functions g_i of C^1 class on $[-1, 1]$ satisfy

$$\forall\ s \in [-1, 1],\ |g_i(s)| \leq |f_i(s)| \leq |g_i^{-1}(s)|, \tag{2.10}$$

$$\forall\ |s| > 1,\ C_{i1}|s| \leq |f_i(s)| \leq C_{i2}|s|, \tag{2.11}$$

where $g_i^{-1}(s)$ denote the inverse functions of $g_i(s)$ and C_{i1}, C_{i2} are positive constants.

In order to obtain the global existence of regular solutions, we shall need the following additional hypotheses.

(A3) Assumptions on the initial data: let us consider

$$\{u^0, u^1\} \in V \cap H^4(\Omega) \times V, \tag{2.12}$$

satisfying the compatibility conditions

$$u^0_{\nu\nu} + f_1(u^1_\nu) = 0,\ u^0_{\nu\nu\nu} - f_2(u^1) = 0,\ \text{on } \Gamma_1. \tag{2.13}$$

Moreover, assume that

(A4) $E(0) < d$ and $\|\triangle u^0\|_2 < \lambda_1$.

The next lemma will play an essential role for proving the global existence of regular (weak) solutions of problem (1.1).

Lemma 2.1. *Suppose that* (A1), (A2) *and* (A4) *hold. Let* u *be a solution of problem* (1.1)*, then for all* $t \geq 0$, $\|\triangle u(t)\|_2 < \lambda_1$.

Proof. In view of (2.2), (2.6) and (2.7), we deduce that

$$\begin{aligned} E(t) \geq J(u(t)) &= \frac{1}{2}\|\triangle u(t)\|_2^2 - \frac{1}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} \\ &= \frac{1}{2}\|\triangle u(t)\|_2^2 - \frac{\frac{1}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2}}{\|\triangle u(t)\|_2^{\rho+2}}\|\triangle u(t)\|_2^{\rho+2} \\ &\geq \frac{1}{2}\|\triangle u(t)\|_2^2 - K_0\|\triangle u(t)\|_2^{\rho+2} = f(\|\triangle u(t)\|_2), \end{aligned} \tag{2.14}$$

where $f(\lambda) = \frac{1}{2}\lambda^2 - K_0\lambda^{\rho+2}$, $\lambda > 0$, which is defined as (2.7). Of course, f is increasing for $0 < \lambda < \lambda_1$, decreasing for $\lambda > \lambda_1$, and $f(\lambda_1) = d$. From the definition of f, we also note that $f(\lambda) \to +\infty$ as $\lambda \to \infty$. Since $E(0) < d$, there exists $\lambda_2' < \lambda_1 < \lambda_2$ such that $f(\lambda_2') = f(\lambda_2) = E(0)$.

Multiplying the equation in (1.1) by $u_t(t)$, a direct computation gives that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_t(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\triangle u(t)\|_2^2 - \frac{1}{\rho+2}\frac{d}{dt}\|u(t)\|_{\rho+2}^{\rho+2} \\ &= -\int_{\Gamma_1} f_2(u_t(t))u_t(t)d\Gamma - \int_{\Gamma_1} f_1(u_{\nu t}(t))u_{\nu t}(t)d\Gamma. \end{aligned} \tag{2.15}$$

By the hypotheses that f_i are nondecreasing C^1 functions such that $f_i(0) = 0$, we know that $f_i(s)s > 0$ for $s \neq 0$. Hence, from the definition of $E(t)$, it follows that

$$E'(t) = -\int_{\Gamma_1} f_2(u_t(t))u_t(t)d\Gamma - \int_{\Gamma_1} f_1(u_{\nu t}(t))u_{\nu t}(t)d\Gamma \leq 0. \tag{2.16}$$

So we have $E(t) \leq E(0)$ for all $t \geq 0$. Denote $\lambda_0 = \|\triangle u^0\|_2$, from the hypotheses $(A4)$ we have $\lambda_0 < \lambda_1$. Furthermore, by (2.14), we have $f(\lambda_0) \leq E(0)$, which together with f is increasing in $[0, \lambda_1)$ and $f(\lambda_2') = E(0)$, it is easy to see that $\lambda_0 = \|\triangle u^0\|_2 < \lambda_2'$.

Next, we prove that $\|\triangle u(t)\|_2 \leq \lambda_2'$ for all $t \geq 0$. In deed, by contradiction, suppose that $\|\triangle u(t_0)\|_2 > \lambda_2'$ for some $t_0 \geq 0$. Using the continuity of $\|\triangle u(t)\|_2$, we also may suppose that $\|\triangle u(t_0)\|_2 < \lambda_1$. Thus, by (2.14) again, we see that

$$E(t_0) \geq f(\|\triangle u(t_0)\|_2) > f(\lambda_2') = E(0), \tag{2.17}$$

which contradicts (2.16). This completes the proof of Lemma 2.1. □

The following two technical lemmas are very crucial to derive the asymptotic behavior of the energy to problem (1.1).

Lemma 2.2. *Let $E : \mathrm{R}_+ \to \mathrm{R}_+$ be a non-increasing function and $\phi : \mathrm{R}_+ \to \mathrm{R}_+$ a strictly increasing function of C^1 class such that*

$$\phi(0) = 0 \text{ and } \phi(t) \to +\infty \text{ as } t \to +\infty. \tag{2.18}$$

Suppose that there exist $\sigma > 0$, $\sigma' \geq 0$ and $C > 0$ such that

$$\int_S^{+\infty} E(t)^{1+\sigma}\phi'(t)dt \leq CE(S)^{1+\sigma} + \frac{C}{(1+\phi(S))^{\sigma'}}E(0)^\sigma E(S), \ \forall\ S \geq 0. \tag{2.19}$$

Then, there exists $C > 0$ such that

$$E(t) \leq E(0)\frac{C}{(1+\phi(t))^{(1+\sigma')/\sigma}}, \ \forall\ t > 0. \tag{2.20}$$

Remark 2.1. Note that the above integral inequality was first introduced in Martinez [26], was used in Cavalcanti et al. [4] to prove the decay rate estimates of energy.

Lemma 2.3. *There exists a strictly increasing function $\phi : \mathrm{R}_+ \to \mathrm{R}_+$ of C^2 class on $(0, +\infty)$, and such that the following conditions hold*

$$\phi(t) \text{ is concave and } \phi(t) \to +\infty \text{ as } t \to +\infty, \tag{2.21}$$

$$\phi'(t) \to 0 \text{ as } t \to +\infty, \tag{2.22}$$

$$\int_1^{+\infty} \phi'(t)\big(g_1^{-1}(\phi'(t))\big)^2 dt < +\infty \text{ and } \int_1^{+\infty} \phi'(t)\big(g_2^{-1}(\phi'(t))\big)^2 dt < \infty, \tag{2.23}$$

where the functions $g_i^{-1}(s)$ $(i = 1, 2)$ were introduced in assumption $(A2)$.

Proof. These properties of the function ϕ are closely related to the behaviors of f_i $(i = 1, 2)$ near 0. We will present the construction method of a special weight function ϕ in Section 5.

Now, we are ready to state the main results of this paper.

Theorem 2.1 (Existence and uniqueness of regular solutions)**.** *Let the assumptions* $(A1)-(A4)$ *hold, then the problem* (1.1) *possesses a unique regular strong solution* u *satisfying*

$$u \in L^{\infty}(0,\infty;V),\ u_t \in L^{\infty}(0,\infty;V),$$

$$u_{tt} \in L^{\infty}(0,\infty;L^2(\Omega)),\ \triangle^2 u \in L^{\infty}(0,\infty;L^2(\Omega)),\ \|\triangle u\|_2 < \lambda_1,$$

for all $t \geq 0$*. Further, the following energy identity holds*

$$E(t) + \int_0^t \int_{\Gamma_1} f_2(u_t(s))u_t(s)d\Gamma ds + \int_0^t \int_{\Gamma_1} f_1(u_{\nu t}(s))u_{\nu t}(s)d\Gamma ds = E(0), \tag{2.24}$$

where the total energy $E(t)$ *has been defined by* (2.2)*.*

Theorem 2.2 (Existence and uniqueness of weak solutions)**.** *Given* $\{u^0, u^1\} \in V \cap L^2(\Omega)$*. Assume that the hypotheses* $(A1)$*,* $(A2)$ *and* $(A4)$ *hold, then the problem* (1.1) *possesses a unique weak solution satisfying*

$$u \in C(0,\infty;V) \cap C^1(0,\infty;L^2(\Omega)),\ \|\triangle u\|_2 < \lambda_1,$$

for all $t > 0$*. Besides, the weak solution has the same energy identity given as* (2.24)*.*

Theorem 2.3 (Uniform decay rates of energy)**.** *Assume that the hypotheses* $(A1)-(A4)$ *hold. Let* u *be a solution to problem* (1.1) *with the properties listed in Theorem 2.1. Then, the energy of problem* (1.1) *has the following decay rate*

$$\forall\ t \geq 1,\ E(t) \leq C\left(G^{-1}\left(\frac{1}{t}\right)\right)^2,$$

where the function $G(y) = y\frac{g_1(y)g_2(y)}{g_1(y)+g_2(y)}$ *and the constant* C *only depending on the initial data* $E(1)$ *in a continuous way.*

Remark 2.2. By a direct calculation, we can show that the $G(y) = y\frac{g_1(y)g_2(y)}{g_1(y)+g_2(y)}$ is an increasing function.

Remark 2.3. we also extend the decay rate estimate of regular solutions to the weak solutions of problem (1.1) by using the standard arguments of density.

3. Existence, uniqueness of regular solutions

In this section, we study the global existence and uniqueness of regular solutions of problem (1.1) by using the combination of the Galerkin approximation, potential well method and a special basis constructed.

The proof of Theorem 2.1 is divided into five steps.

Proof. **Step 1. Galerkin approximation**.

The main idea is to use the Galerkin's method. To do this, let us take a basis $\{w_j^*\}$ to V. We construct a special basis $\{w_j\}$ from basis $\{w_j^*\}$ which are associated with problem (1.1).

If u^0, u^1 are linearly independent, we take $w_1 = u^0$, $w_2 = u^1$, and w_i, $i \geq 3$ of $\{w_j^*\}$, which are chosen to be linearly independent with u^0, u^1. If u^0, u^1 are linearly dependent, we define $w_1 = u^0$, and w_i, $i \geq 2$ of $\{w_j^*\}$, which are chosen to be linearly independent with u^0. Thus, we represent by V_m a subspace of $\{w_j\}$ generated by $[w_1, \cdots, w_m]$.

Next, we construct an approximate solution of problem (1.1) by

$$u^m(t) = \sum_{j=1}^{m} d_m^j w_j(x),\ m = 1, 2, \cdots\cdots . \tag{3.1}$$

According to Galerkin's method, these coefficients $d_m^j(t)$ need to satisfy the following initial value problem of the nonlinear ordinary differential equation

$$\begin{cases} (u_{tt}^m(t), w_j) + (\triangle u^m(t), \triangle w_j) + (f_1(u_{\nu t}^m(t)), w_{j\nu})_{\Gamma_1} \\ \qquad +(f_2(u_t^m(t)), w_j)_{\Gamma_1} = (|u^m(t)|^\rho u^m(t), w_j), \\ u^m(x,0) = u^0,\ u_t^m(x,0) = u^1. \end{cases} \tag{3.2}$$

Note that we can solve system (3.2) by Picard's iteration method. In fact, the ordinary differential equation (3.2) has a local solution on the interval $[0, T_m)$. The extension of these solutions to the whole interval $[0, +\infty)$ is a consequence of a priori estimate which we are going to prove below.

Step 2. The first estimate.

Replacing w_j by u_t^m in (3.2), a direct computation gives that

$$E_m'(t) = -\int_{\Gamma_1} f_2(u_t^m)u_t^m d\Gamma - \int_{\Gamma_1} f_1(u_{\nu t}^m)u_{\nu t}^m d\Gamma \le 0, \tag{3.3}$$

which implies that $E_m(t)$ is a decreasing function.

Combining problem (3.2) and assumption $(A4)$, we obtain that

$$\|\triangle u^m(0)\|_2 = \|\triangle u^0\|_2 < \lambda_1.$$

Taking Lemma 2.1 into account, we conclude that $\|\triangle u^m(t)\|_2 < \lambda_1$, for all $t \ge 0$. Returning to the approximate problem, we deduce

$$\begin{aligned} &\frac{1}{2}\|u_t^m(t)\|_2^2 + \frac{1}{2}\|\triangle u^m(t)\|_2^2 - \frac{1}{\rho+2}\|u^m(t)\|_{\rho+2}^{\rho+2} \\ &\le \frac{1}{2}\|u^1\|_2^2 + \frac{1}{2}\|\triangle u^0\|_2^2 - \frac{1}{\rho+2}\|u^0\|_{\rho+2}^{\rho+2}. \end{aligned} \tag{3.4}$$

Considering assumption $(A1)$, we have Sobolev inequality $\|u^m(t)\|_{\rho+2} \le B_1 \|\triangle u^m(t)\|_2$, which together with above inequality, a simple calculation reveals that

$$\|u_t^m(t)\|_2^2 \le \|u^1\|_2^2 + 2\lambda_1^2 + \frac{4}{\rho+2}(B_1\lambda_1)^{\rho+2}. \tag{3.5}$$

Step 3. The second estimate.

Multiplying (3.2) by ${d''}_m^j(0)$, summing for $j = 1, 2, \cdots\cdots$, and considering $t = 0$, then we have

$$\begin{aligned} \|u_{tt}^m(0)\|_2^2 = &-(\triangle u^m(0), \triangle u_{tt}^m(0)) - (f_1(u_{\nu t}^m(0)), u_{\nu tt}^m(0))_{\Gamma_1} \\ &- (f_2(u_t^m(0), u_{tt}^m(0))_{\Gamma_1} + (|u^m(0)|^\rho u^m(0), u_{tt}^m(0)). \end{aligned} \tag{3.6}$$

Using the generalized Green Theorem, it follows that

$$\begin{aligned} \|u_{tt}^m(0)\|_2^2 = &-(\triangle^2 u^0, u_{tt}^m(0)) - (u_{\nu\nu}^0 + f_1(u_\nu^1), u_{\nu tt}^m(0))_{\Gamma_1} \\ &+ (u_{\nu\nu\nu}^0 - f_2(u^1), u_{tt}^m(0))_{\Gamma_1} + (|u^0|^\rho u^0, u_{tt}^m(0)). \end{aligned} \tag{3.7}$$

By Hölder inequality and the compatibility condition $(A3)$, we discover that

$$\|u_{tt}^m(0)\|_2 \le \|\triangle^2 u^0\|_2 + \|u^0\|_{2(\rho+1)}^{\rho+1}. \tag{3.8}$$

Differentiating equation in (3.2) with respect to t, and substituting w_j by u_{tt}^m, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|u_{tt}^m(t)\|_2^2+\frac{1}{2}\frac{d}{dt}\|\triangle u_t^m(t)\|_2^2+\int_{\Gamma_1} f_{1t}(u_{\nu t}^m(t))(u_{\nu tt}^m(t))^2 d\Gamma$$
$$+\int_{\Gamma_1} f_{2t}(u_t^m(t))(u_{tt}^m(t))^2 d\Gamma \le (\rho+1)\int_{\Omega}|u^m|^{\rho}|u_t^m||u_{tt}^m|dx. \tag{3.9}$$

We will give the estimate of $K_1=(\rho+1)\int_{\Omega}|u^m|^{\rho}|u_t^m||u_{tt}^m|dx$. From now on, we will denote by C various positive constants which may be different at different occurrences.

In view of the generalized Hölder inequality ($\frac{\rho}{2(\rho+1)}+\frac{1}{2(\rho+1)}+\frac{1}{2}=1$), Sobolev imbedding $V\hookrightarrow L^{2(\rho+1)}(\Omega)$ and Lemma 2.1, we conclude that

$$\begin{aligned}|K_1| &\le (\rho+1)\|u^m(t)\|_{2(\rho+1)}^{\rho}\|u_t^m(t)\|_{2(\rho+1)}\|u_{tt}^m(t)\|_2\\ &\le C\|\triangle u^m(t)\|_2^{\rho}\|\triangle u_t^m(t)\|_2\|u_{tt}^m(t)\|_2\\ &\le C[\|\triangle u_t^m(t)\|_2^2+\|u_{tt}^m(t)\|_2^2],\end{aligned} \tag{3.10}$$

where the constant C are positive constants independent of m and t. By (3.9) and (3.10), it is inferred that

$$\frac{1}{2}\frac{d}{dt}\|u_{tt}^m(t)\|_2^2+\frac{1}{2}\frac{d}{dt}\|\triangle u_t^m(t)\|_2^2+\int_{\Gamma_1} f_{1t}(u_{\nu t}^m(t))(u_{\nu tt}^m(t))^2 d\Gamma$$
$$+\int_{\Gamma_1} f_{2t}(u_t^m(t))(u_{tt}^m(t))^2 d\Gamma \le C[\|\triangle u_t^m(t)\|_2^2+\|u_{tt}^m(t)\|_2^2]. \tag{3.11}$$

Integrating the above inequality over $(0,t)$, and taking (3.8) into account, we get that

$$\|u_{tt}^m(t)\|_2^2+\|\triangle u_t^m(t)\|_2^2+2\int_0^t\int_{\Gamma_1} f_{1t}(u_{\nu t}^m(s))(u_{\nu tt}^m(s))^2 d\Gamma ds$$
$$+2\int_0^t\int_{\Gamma_1} f_{2t}(u_t^m(s))(u_{tt}^m(s))^2 d\Gamma ds$$
$$\le \|\triangle^2 u^0\|_2^2+\|u^0\|_{2(\rho+1)}^{2(\rho+1)}+\|\triangle^2 u^1\|_2^2+2C\int_0^t[\|u_{tt}^m\|_2^2+\|\triangle u_t^m\|_2^2]ds$$
$$+2C\int_0^t\int_0^s\int_{\Gamma_1}[f_{1t}(u_{\nu t}^m(\eta))(u_{\nu tt}^m(\eta))^2+f_{2t}(u_t^m(\eta))(u_{tt}^m(\eta))^2]d\Gamma d\eta ds. \tag{3.12}$$

The Gronwall Lemma guarantees that

$$\|u_{tt}^m(t)\|_2^2+\|\triangle u_t^m(t)\|_2^2+2\int_0^t\int_{\Gamma_1} f_{1t}(u_{\nu t}^m(s))(u_{\nu tt}^m(s))^2 d\Gamma ds$$
$$+2\int_0^t\int_{\Gamma_1} f_{2t}(u_t^m(s))(u_{tt}^m(s))^2 d\Gamma ds \le C. \tag{3.13}$$

From the inequality (3.13) and Trace Theorem [10], we also obtain the following estimate

$$\|\nabla u_t^m(t)\|_{\Gamma_1,2}^2 \le C\|\triangle u_t^m(t)\|_2^2 \le C, \tag{3.14}$$

where the constant $C>0$ is independent of m and t. Furthermore, taking assumption $(A2)$ into account, we know that if $|u_t^m(t)|>1$, then $|f_2(u_t^m(t))|\le C_{22}|u_t^m(t)|$.

If $|u_t^m(t)| \leq 1$, we obtain from the continuity of the function f_2 that $|f_2(u_t^m(t))| \leq C$. Thereby, we obtain that

$$\begin{aligned} &\|f_2(u_t^m(t))\|_{\Gamma_1,2}^2 \\ &= \int_{|u_t^m(t)|\leq 1} |f_2(u_t^m(t))|^2 d\Gamma + \int_{|u_t^m(t)|>1} |f_2(u_t^m(t))|^2 d\Gamma \\ &\leq C + C_{22}^2 \int_{\Gamma_1} |u_t^m(t)|^2 d\Gamma \leq C. \end{aligned} \tag{3.15}$$

Using analogous arguments, from the assumption $(A2)$ and (3.14), we also obtain that

$$\|f_1(u_{\nu t}^m(t))\|_{\Gamma_1,2}^2 \leq C. \tag{3.16}$$

Step 4. Global existence.

From the above estimates, we can show that there exists a subsequences of $\{u^m\}$ which from now on will be also denoted by $\{u^m\}$ and function $u: \ \Omega \times [0,T]$ such that

$$u^m \longrightarrow u \text{ in } L^\infty(0,T;V) \text{ weakly star}, \ m \longrightarrow \infty, \tag{3.17}$$

$$u_t^m \longrightarrow u_t \text{ in } L^\infty(0,T;V) \text{ weakly star}, \ m \longrightarrow \infty, \tag{3.18}$$

$$u_{tt}^m \longrightarrow u_{tt} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weakly star}, \ m \longrightarrow \infty. \tag{3.19}$$

Since $V \hookrightarrow L^{2(\rho+1)}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, thanks to Aubin-Lions Theorem [38, Chapter 1], we have that

$$u^m \longrightarrow u \text{ in } L^2(0,T;L^2(\Omega)) \text{ strongly}, \ m \longrightarrow \infty, \tag{3.20}$$

$$u^m \longrightarrow u \ a.e. \text{ in } Q_T = \Omega \times (0,T), \ m \longrightarrow \infty, \tag{3.21}$$

$$u_t^m \longrightarrow u_t \text{ in } L^2(0,T;L^2(\Omega)) \text{ strongly}, \ m \longrightarrow \infty, \tag{3.22}$$

$$u_t^m \longrightarrow u_t \ a.e. \text{ in } Q_T = \Omega \times (0,T), \ m \longrightarrow \infty. \tag{3.23}$$

Consequently, making use of Lion's Lemma [38, Lemma 1.3, Chapter 1], it follows that

$$|u^m|^\rho u^m \longrightarrow |u|^\rho u \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weakly star}, \ m \longrightarrow \infty. \tag{3.24}$$

In addition, we also obtain

$$u_t^m \longrightarrow u_t \text{ in } L^\infty(0,T;H^1(\Gamma_1)) \text{ weakly star}, \ m \longrightarrow \infty, \tag{3.25}$$

$$f_1(u_{\nu t}^m) \longrightarrow \chi_1 \text{ in } L^\infty(0,T;L^2(\Gamma_1)) \text{ weakly star}, \ m \longrightarrow \infty, \tag{3.26}$$

$$f_2(u_t^m) \longrightarrow \chi_2 \text{ in } L^\infty(0,T;L^2(\Gamma_1)) \text{ weakly star}, \ m \longrightarrow \infty. \tag{3.27}$$

Therefore, (3.19)-(3.27) permit us to pass to the limit in equation (3.2). Since $\{w_j\}$ is a basis of V, then for all $T > 0$, for all $d(t) \in D(0,T)$ and for all $w \in V$, we have

$$\begin{aligned} &\int_0^T (u_{tt}(t), w) d(t) dt + \int_0^T (\triangle u(t), \triangle w) d(t) dt + \int_0^T \int_{\Gamma_1} \chi_1 w_\nu d\Gamma d(t) dt \\ &\quad + \int_0^T \int_{\Gamma_1} \chi_2 w d\Gamma d(t) dt = \int_0^T (|u(t)|^\rho u(t), w) d(t) dt. \end{aligned} \tag{3.28}$$

Taking into account $w \in D(\Omega)$ and (3.28), we deduce that

$$u_{tt} + \triangle^2 u = |u|^\rho u, \text{ in } D'(\Omega \times (0,T)).$$

Utilizing the convergences of (3.19) and (3.24), there appear the relations that $u_{tt} \in L^\infty(0,T;L^2(\Omega))$ and $|u|^\rho u \in L^\infty(0,T;L^2(\Omega))$. Hence, we deduce that $\triangle^2 u \in L^\infty(0,T;L^2(\Omega))$ and

$$u_{tt} + \triangle^2 u = |u|^\rho u, \text{ in } L^\infty(0,T;L^2(\Omega)). \tag{3.29}$$

Combining (3.19) and (3.26), it is easy to see that the approximate solutions $\{u^m\}$ possess the following property

$$0 = \int_0^T (u^m_{\nu\nu} + f_1(u^m_{\nu t}), w)dt \to \int_0^T (u_{\nu\nu} + \chi_1, w)dt \text{ as } m \to \infty,$$

for all $w \in V$, which implies that

$$u_{\nu\nu} + \chi_1 = 0 \text{ in } D'(0,T;H^{\frac{3}{2}}(\Gamma_1)). \tag{3.30}$$

Taking (3.28)-(3.30) into account, and making use of generalized Green formula, we discover that

$$u_{\nu\nu\nu} - \chi_2 = 0 \text{ in } D'(0,T;H^{\frac{1}{2}}(\Gamma_1)). \tag{3.31}$$

Since $\chi_1, \chi_2 \in L^\infty(0,T;L^2(\Gamma_1))$, we deduce that

$$u_{\nu\nu} + \chi_1 = 0 \text{ and } u_{\nu\nu\nu} - \chi_2 = 0 \text{ in } L^\infty(0,T;L^2(\Gamma_1)). \tag{3.32}$$

Next, we need to prove that

$$\chi_1 = f_1(u_{\nu t}) \quad \text{ and } \quad \chi_2 = f_2(u_t). \tag{3.33}$$

In deed, replacing w_j by u^m in equation (3.2), and integrating the obtained expression over $(0,T)$, it is inferred that

$$\int_0^T (u^m_{tt}(t), u^m(t))dt + \int_0^T \|\triangle u^m\|_2^2 dt + \int_0^T (f_1(u^m_{\nu t}(t)), u^m_\nu(t))_{\Gamma_1} dt$$
$$+ \int_0^T (f_2(u^m_t(t)), u^m(t))_{\Gamma_1} dt = \int_0^T (|u^m(t)|^\rho u^m(t), u^m(t))dt. \tag{3.34}$$

In view of the first and second estimates, Sobolev imbedding, Poincaré inequality, and Trace Theorem [10], it follows that

$$V \hookrightarrow H^{\frac{3}{2}}(\Gamma_1) \hookrightarrow H^1(\Gamma_1) \hookrightarrow L^2(\Gamma_1),$$

which implies that

$$\|u^m(t)\|_{\Gamma_1,2} \le C\|\nabla u^m(t)\|_{\Gamma_1,2} \le C\|u^m(t)\|_{H^{\frac{3}{2}}(\Gamma_1)} \le C\|\triangle u^m(t)\|_2, \tag{3.35}$$

$$\|u^m_t(t)\|_{\Gamma_1,2} \le C\|\nabla u^m_t(t)\|_{\Gamma_1,2} \le C\|u^m_t(t)\|_{H^{\frac{3}{2}}(\Gamma_1)} \le C\|\triangle u^m_t(t)\|_2. \tag{3.36}$$

Making use of the Aubin-Lions Theorem [23, Chapter 1] again, we have that

$$u^m \longrightarrow u \text{ in } L^2(0,T;H^1(\Gamma_1)) \text{ strongly}, \ m \longrightarrow \infty, \tag{3.37}$$

$$u^m_t \longrightarrow u_t \text{ in } L^2(0,T;H^1(\Gamma_1)) \text{ strongly}, \ m \longrightarrow \infty. \tag{3.38}$$

Then, from the convergences (3.19), (3.24), (3.26), (3.27) and (3.37), we can pass to the limit in equation (3.34) to obtain

$$\lim_{m\to\infty} \int_0^T \|\triangle u^m\|_2^2 dt = -\int_0^T (u_{tt}(t), u(t))dt - \int_0^T (\chi_1, u_\nu(t))_{\Gamma_1} dt$$

$$-\int_0^T (\chi_2, u(t))_{\Gamma_1} dt + \int_0^T (|u(t)|^\rho u(t), u(t)) dt. \tag{3.39}$$

Combining (3.29), (3.32), (3.39) and the generalized Green formula, it is found that

$$\lim_{m\to\infty} \int_0^T \|\triangle u^m\|_2^2 dt = \int_0^T \|\triangle u\|_2^2 dt,$$

which implies that

$$\triangle u^m \longrightarrow \triangle u \text{ in } L^2(0,T;L^2(\Omega)) \text{ strongly, } m \longrightarrow \infty. \tag{3.40}$$

Now, in view of (3.26), (3.27), (3.38), and using the standard Lebesgue control-convergent Theorem, we obtain that

$$\lim_{m\to\infty} \int_0^T (f_1(u_{\nu t}^m(t)), u_{\nu t}^m(t))_{\Gamma_1} dt = \int_0^T (\chi_1, u_{\nu t}(t))_{\Gamma_1} dt, \tag{3.41}$$

$$\lim_{m\to\infty} \int_0^T (f_2(u_t^m(t)), u_t^m(t))_{\Gamma_1} dt = \int_0^T (\chi_2, u_t(t))_{\Gamma_1} dt. \tag{3.42}$$

Utilizing the non-decreasing monotonicity of functions f_i $(i = 1, 2)$, it follows that

$$\int_0^T (f_1(u_{\nu t}^m(t)) - f_1(\psi), u_{\nu t}^m(t) - \psi)_{\Gamma_1} dt \geq 0, \tag{3.43}$$

$$\int_0^T (f_2(u_t^m(t)) - f_2(\psi), u_t^m(t) - \psi)_{\Gamma_1} dt \geq 0, \tag{3.44}$$

for all $\psi \in L^2(\Gamma_1)$. Then, from the inequalities (3.43), (3.44), we discover that

$$\int_0^T (f_1(u_{\nu t}^m(t)), \psi)_{\Gamma_1} dt + \int_0^T (f_1(\psi), u_{\nu t}^m(t) - \psi)_{\Gamma_1} dt \leq \int_0^T (f_1(u_{\nu t}^m(t)), u_{\nu t}^m(t))_{\Gamma_1} dt, \tag{3.45}$$

$$\int_0^T (f_2(u_t^m(t)), \psi)_{\Gamma_1} dt + \int_0^T (f_2(\psi), u_t^m(t) - \psi)_{\Gamma_1} dt \leq \int_0^T (f_2(u_t^m(t)), u_t^m(t))_{\Gamma_1} dt, \tag{3.46}$$

and then passing to the limit as $m \to \infty$,

$$\int_0^T (\chi_1 - f_1(\psi), u_{\nu t}(t) - \psi)_{\Gamma_1} dt \geq 0, \tag{3.47}$$

$$\int_0^T (\chi_2 - f_2(\psi), u_t(t) - \psi)_{\Gamma_1} dt \geq 0. \tag{3.48}$$

In order to prove (3.33) from (3.47) and (3.48), we use the semi-continuous [23, Chapter 2]. Let $\psi = u_{\nu t} - \lambda\varphi$, $\forall\, \varphi \in L^2(\Gamma_1)$ and $\lambda \geq 0$, then we have

$$\lambda \int_0^T (\chi_1 - f_1(u_{\nu t} - \lambda\varphi), \varphi)_{\Gamma_1} dt \geq 0,$$

and

$$\int_0^T (\chi_1 - f_1(u_{\nu t} - \lambda\varphi), \varphi)_{\Gamma_1} dt \geq 0. \tag{3.49}$$

Pass to the limit as $\lambda \to 0$ gives that

$$\int_0^T (\chi_1 - f_1(u_{\nu t}), \varphi)_{\Gamma_1} dt \geq 0, \ \forall \varphi \in L^2(\Gamma_1). \tag{3.50}$$

In a similar way, let $\psi = u_{\nu t} - \lambda\varphi$, $\lambda \leq 0$ and $\forall \varphi \in L^2(\Gamma_1)$, we obtain

$$\int_0^T (\chi_1 - f_1(u_{\nu t}), \varphi)_{\Gamma_1} dt \leq 0, \ \forall \varphi \in L^2(\Gamma_1). \tag{3.51}$$

From (3.50) and (3.51), we see that

$$\chi_1 = f_1(u_{\nu t}).$$

Using the analogous arguments, taking $\psi = u_{\nu t} - \lambda\varphi$, and $\forall \varphi \in L^2(\Gamma_1)$, we also get from (3.48) that

$$\int_0^T (\chi_2 - f_2(u_t), \varphi)_{\Gamma_1} dt \leq 0 \text{ and } \int_0^T (\chi_2 - f_2(u_t), \varphi)_{\Gamma_1} dt \geq 0, \tag{3.52}$$

which implies that

$$\chi_2 = f_2(u_t).$$

Thus, we obtain that u is a global regular solutions of problem (1.1).

Step 5. Uniqueness.

Let u, $\widetilde{u}$ be two solutions of problem (1.1). Then, $y = u - \widetilde{u}$ satisfies

$$\begin{aligned}(y_{tt}(t), w) + (\triangle y(t), \triangle w) + (f_1(u_{\nu t}(t)) - f_1(\widetilde{u}_{\nu t}(t)), w_\nu)_{\Gamma_1} \\ + (f_2(u_t(t)) - f_2(\widetilde{u}_t(t)), w)_{\Gamma_1} = (|u(t)|^\rho u(t) - |\widetilde{u}(t)|^\rho \widetilde{u}(t), w),\end{aligned} \tag{3.53}$$

for all $w \in V$. Replacing w by y_t in the above identity, and noting that f_i $(i = 1, 2)$ are monotone functions, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|y_t(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\triangle y(t)\|_2^2 \\ &\quad \leq \int_\Omega \big(|u(t)|^\rho u(t) - |\widetilde{u}(t)|^\rho \widetilde{u}(t)\big) y_t dx \\ &\quad \leq (\rho + 1)\int_\Omega \sup\{|u(t)|^\rho, |\widetilde{u}(t)|^\rho\}|y(t)||y_t(t)|dx \\ &\quad \leq (\rho + 1)\int_\Omega (|u(t)|^\rho + |\widetilde{u}(t)|^\rho)|y(t)||y_t(t)|dx.\end{aligned}$$

Using the Hölder inequality, Sobolev imbedding $V \hookrightarrow L^{2(\rho+1)}(\Omega)$ and taking the first estimate into account, we thereby deduce that

$$\begin{aligned}&\frac{d}{dt}\left\{\|y_t(t)\|_2^2 + \|\triangle y(t)\|_2^2\right\} \\ &\quad \leq C\left(\|u(t)\|_{2(\rho+1)}^\rho + \|\widetilde{u}(t)\|_{2(\rho+1)}^\rho\right)\|y(t)\|_{2(\rho+1)}\|y_t(t)\|_2 \\ &\quad \leq C\left(\|\triangle y(t)\|_2^2 + \|y_t(t)\|_2^2\right).\end{aligned} \tag{3.54}$$

Then, apply the Gronwall Lemma yields that $\|y_t(t)\|_2^2 = \|\triangle y(t)\|_2^2 = 0$. This completes the proof of Theorem 2.1. □

4. Existence, uniqueness of weak solutions

Our attention in this section is turned to the existence, uniqueness of weak solutions for problem (1.1). Applying the standard density argument, we extend the existence, uniqueness results of regular solutions to the weak solutions.

Proof. The main idea of this proof is the density method. We will divided it into four steps.

Step 1. Galerkin approximation.
We start to approximate the initial data u^0 and u^1 with more regular data u^0_μ and u^1_μ, respectively. Indeed, let us assume that

$$\{u^0, u^1\} \in V \cap L^2(\Omega), \tag{4.1}$$

such that

$$\|\triangle u^0\|_2 < \lambda_1 \text{ and } E(0) < d.$$

Hence, we choose

$$\{u^0_\mu, u^1_\mu\} \in D(\triangle^2) \cap V, \tag{4.2}$$

where $D(\triangle^2) = \{u \in V \cap H^4(\Omega);\ u_{\nu\nu\nu} = u_{\nu\nu} = 0 \text{ on } \Gamma_1\}$ such that

$$u^0_\mu \to u^0, \text{ in } V \text{ and } u^1_\mu \to u^1, \text{ in } L^2(\Omega), \text{ as } \mu \to \infty. \tag{4.3}$$

Thus, it is easy to see that $\{u^0_\mu, u^1_\mu\}$ satisfies the compatibility conditions

$$u^0_{\mu\nu\nu} + f_1(u^1_{\mu\nu}) = 0,\ u^0_{\mu\nu\nu\nu} - f_2(u^1_\mu) = 0, \text{ on } \Gamma_1. \tag{4.4}$$

Moreover, using the continuity of functionals $\|\triangle u\|_2$, $E(u)$, we have

$$\lim_{\mu\to\infty} \|\triangle u^0_\mu\|_2 = \|\triangle u^0\|_2 < \lambda_1 \text{ and } \lim_{\mu\to\infty} E_\mu(0) = E(0) < d,$$

where $E_\mu(0) = E(u^0_\mu)$. Therefore, for sufficiently large $\mu \geq \mu_0$, we get

$$\|\triangle u^0_\mu\|_2 < \lambda_1 \text{ and } E_\mu(0) < d. \tag{4.5}$$

Thus, for each $\mu \geq \mu_0$, let u^μ be the solutions of problem (1.1) with the initial date $\{u^0_\mu, u^1_\mu\}$, which satisfies all the conditions of Theorem 2.1, so we obtain

$$\begin{gathered} u^\mu \in L^\infty(0,\infty;V),\ u^\mu_t \in L^\infty(0,\infty;V), \\ u^\mu_{tt} \in L^\infty(0,\infty;L^2(\Omega)),\ \triangle^2 u^\mu \in L^\infty(0,\infty;L^2(\Omega)),\ \|\triangle u^\mu\|_2 < \lambda_1, \end{gathered} \tag{4.6}$$

and verifies

$$\begin{cases} u^\mu_{tt} = -\triangle^2 u^\mu + |u^\mu|^\rho u^\mu,\ (x,t) \in \Omega \times (0,\infty), \\ u^\mu = u^\mu_\nu = 0,\ (x,t) \in \Gamma_0 \times (0,\infty), \\ u^\mu_{\nu\nu} = -f_1(u^\mu_{\nu t}),\ u^\mu_{\nu\nu\nu} = f_2(u^\mu_t),\ (x,t) \in \Gamma_1 \times (0,\infty), \\ u^\mu(x,0) = u^0_\mu,\ u^\mu_t(x,0) = u^1_\mu,\ x \in \Omega. \end{cases} \tag{4.7}$$

Step 2. Energy estimates and global existence.
Applying the analogous arguments used to prove the first estimate of the above section, we deduce that there exist constants C (various positive constants C may be different at different occurrences) which are independent of μ and $t \in [0,T]$, such that

$$\begin{gathered} \|u^\mu_t(t)\|_2^2 \leq C,\ \|\triangle u^\mu(t)\|_2^2 \leq C, \\ \|u^\mu_{\nu t}\|_{\Gamma_1,2} \leq C,\ \|f_1(u^\mu_{\nu t}(t))\|_2 \leq C,\ \|f_2(u^\mu_t(t))\|_2 \leq C. \end{gathered} \tag{4.8}$$

Let us define $y^{\mu,\sigma}(t) = u^{\mu}(t) - u^{\sigma}(t)$, $\mu, \sigma \in N$. From the monotonicity of functions f_i, $(i = 1, 2)$, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|y^{\mu,\sigma}(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\triangle y^{\mu,\sigma}(t)\|_2^2\\ &\leq (\rho+1)\int_{\Omega}(|u^{\mu}(t)|^{\rho} + |u^{\sigma}(t)|^{\rho})|y^{\mu,\sigma}(t)||y_t^{\mu,\sigma}(t)|dx,\end{aligned} \tag{4.9}$$

which together with the Hölder inequality, Sobelev imbedding from $V \hookrightarrow L^{2(\rho+1)}(\Omega)$ and (4.8) gives that

$$\begin{aligned}&\frac{d}{dt}\{\|y^{\mu,\sigma}(t)\|_2^2 + \|\triangle y^{\mu,\sigma}(t)\|_2^2\}\\ &\leq C\left(\|u^{\mu}(t)\|_{2(\rho+1)}^{\rho} + \|u^{\sigma}(t)\|_{2(\rho+1)}^{\rho}\right)\|y^{\mu,\sigma}(t)\|_{2(\rho+1)}\|y_t^{\mu,\sigma}(t)\|_2\\ &\leq C(\|y^{\mu,\sigma}(t)\|_2^2 + \|\triangle y^{\mu,\sigma}(t)\|_2^2).\end{aligned} \tag{4.10}$$

Then, the Gronwall Lemma reveals that

$$\begin{aligned}&\|u_t^{\mu}(t) - u_t^{\sigma}(t)\|_2^2 + \|\triangle u^{\mu}(t) - \triangle u^{\sigma}(t)\|_2^2\\ &\leq C\left[\|u_{\mu}^1 - u_{\sigma}^1\|_2^2 + \|\triangle u_{\mu}^0 - \triangle u_{\sigma}^0\|_2^2\right],\end{aligned} \tag{4.11}$$

where the constant $C > 0$ is independent of $\mu, \sigma \in N$.

Consequently, the estimates (4.11) and (4.3) permit us to obtain a subsequences of u^{μ} which from now on will be also denoted by u^{μ} and function u such that for all $T > 0$,

$$u^{\mu} \longrightarrow u \text{ in } C(0,T;V) \text{ strongly},\ \mu \longrightarrow \infty, \tag{4.12}$$

$$u_t^{\mu} \longrightarrow u_t \text{ in } C(0,T;L^2(\Omega)) \text{ strongly},\ \mu \longrightarrow \infty. \tag{4.13}$$

On the other hand, from (4.8) and (4.12), we also obtain

$$u_t^{\mu} \longrightarrow u_t \text{ in } L^{\infty}(0,T;H^1(\Gamma_1)) \text{ weakly star},\ \mu \longrightarrow \infty, \tag{4.14}$$

$$f_1(u_{\nu t}^{\mu}) \longrightarrow \chi_1 \text{ in } L^{\infty}(0,T;L^2(\Gamma_1)) \text{ weakly star},\ \mu \longrightarrow \infty, \tag{4.15}$$

$$f_2(u_t^{\mu}) \longrightarrow \chi_2 \text{ in } L^{\infty}(0,T;L^2(\Gamma_1)) \text{ weakly star},\ \mu \longrightarrow \infty, \tag{4.16}$$

$$|u^{\mu}|^{\rho}u^{\mu} \longrightarrow |u|^{\rho}u \text{ in } L^{\infty}(0,T;L^2(\Omega)) \text{ weakly star},\ \mu \longrightarrow \infty. \tag{4.17}$$

Considering the above convergences, making use of the arguments of compactness and generalized Green formula, we deduce that

$$u_{tt} + \triangle^2 u = |u|^{\rho}u, \text{ in } D'(\Omega \times (0,T)).$$

Combining (4.3), (4.12), (4.13) and (4.17), it follows that $\triangle^2 u \in C(0,T;H^{-2}(\Omega))$, $|u|^{\rho}u \in C(0,T;L^2(\Omega))$, and

$$u_{tt} + \triangle^2 u = |u|^{\rho}u, \text{ in } C(0,T;H^{-2}(\Omega)). \tag{4.18}$$

From the identity (4.18), making use of the Bochner's integral in $H^{-2}(\Omega)$, it follows that

$$u_t(t) - u_t(0) = \int_0^t \triangle^2 u(s)ds + \int_0^t |u(s)|^{\rho}u(s)ds. \tag{4.19}$$

Defining $Z(t) = \int_0^t u(s)ds$, so we obtain from (4.19) that

$$u_t(t) - u_t(0) = \triangle^2 Z(t) + \int_0^t |u(s)|^\rho u(s)ds. \tag{4.20}$$

Furthermore, thanks to (4.12), (4.13) and $(A1)$, we discover that

$$\int_0^t |u(s)|^\rho u(s)ds \in C(0,T;L^2(\Omega)) \quad \text{and} \quad u_t(t) \in C(0,T;L^2(\Omega)). \tag{4.21}$$

By the first equation of problem (1.1), we note that $\triangle^2 Z(t) \in C(0,T;L^2(\Omega))$, which implies that

$$Z(t) \in C(0,T;\mathcal{H}(\Omega)), \tag{4.22}$$

where $\mathcal{H}(\Omega) = \{u \in H^2(\Omega);\ \triangle^2 u \in L^2(\Omega)\}$. Together with the definition of $Z(t)$, we have that

$$\begin{gathered} Z'(t) = u(t) \in H^{-1}(0,T;\mathcal{H}(\Omega)), \\ u_{\nu\nu} \in H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma_1)),\ u_{\nu\nu\nu} \in H^{-1}(0,T;H^{-\frac{3}{2}}(\Gamma_1)). \end{gathered} \tag{4.23}$$

Similarly, if we define $Z^\mu(t) = \int_0^t u^\mu(s)ds$, using the same arguments as (4.12), (4.13) and (4.17), we obtain that

$$\begin{gathered} Z^\mu(t) \in C(0,T;\mathcal{H}(\Omega)),\ \triangle^2 Z^\mu(t) \in C(0,T;L^2(\Omega)), \\ u^\mu_{\nu\nu} \in H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma_1)),\ u^\mu_{\nu\nu\nu} \in H^{-1}(0,T;H^{-\frac{3}{2}}(\Gamma_1)). \end{gathered} \tag{4.24}$$

In view of (4.22)-(4.24), making use of Lion's Lemma [38, Lemma 1.3, Chapter 1] yields that

$$Z^\mu(t) \longrightarrow Z(t) \text{ in } C(0,T;\mathcal{H}(\Omega)) \text{ weakly star},\ \mu \longrightarrow \infty, \tag{4.25}$$

$$\triangle^2 Z^\mu(t) \longrightarrow \triangle^2 Z(t) \text{ in } C(0,T;L^2(\Omega)) \text{ weakly star},\ \mu \longrightarrow \infty, \tag{4.26}$$

$$Z^\mu_t(t) \longrightarrow Z_t(t) \text{ in } H^{-1}(0,T;\mathcal{H}(\Omega)) \text{ weakly},\ \mu \longrightarrow \infty, \tag{4.27}$$

$$f_1(u^\mu_{\nu t}) = -u^\mu_{\nu\nu} \longrightarrow -u_{\nu\nu} \text{ in } H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma_1)) \text{ weakly},\ \mu \longrightarrow \infty, \tag{4.28}$$

$$f_2(u^\mu_t) = -u^\mu_{\nu\nu\nu} \longrightarrow -u_{\nu\nu\nu} \text{ in } H^{-1}(0,T;H^{-\frac{3}{2}}(\Gamma_1)) \text{ weakly},\ \mu \longrightarrow \infty. \tag{4.29}$$

Combining (4.15), (4.16) and the above convergences, it is inferred that

$$u_{\nu\nu} = -\chi_1,\ u_{\nu\nu\nu} = \chi_2,\ \text{ in } L^\infty(0,T;L^2(\Gamma_1)). \tag{4.30}$$

On the other hand, from the convergences of (4.13) and (4.17), we know that $u_t(t) \in C(0,T;L^2(\Omega))$ and $|u|^\rho u \in L^\infty(0,T;L^2(\Omega))$. By the Sobolev embedding relations $C(0,T;L^2(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$ and $L^\infty(0,T;L^2(\Omega)) \hookrightarrow H^{-1}(0,T;L^2(\Omega))$, if follows that $u_t(t) \in L^2(0,T;L^2(\Omega))$ and $|u|^\rho u \in H^{-1}(0,T;L^2(\Omega))$. Hence, it is easy to see that $u_{tt}(t) \in H^{-1}(0,T;L^2(\Omega))$ and

$$u_{tt} + \triangle^2 u = |u|^\rho u,\ \text{ in } H^{-1}(0,T;L^2(\Omega)). \tag{4.31}$$

Utilizing the above identity, the generalized Green formula and (4.25), it is found that

$$\begin{aligned} \langle \triangle^2 u, v\rangle_{H^{-1}(0,T;L^2(\Omega))\times H^1_0(0,T;L^2(\Omega))} &= (\triangle u, \triangle v)_{L^2(0,T;L^2(\Omega))} \\ &+ (u_{\nu\nu\nu}, v)_{L^2(0,T;L^2(\Gamma_1))} - (u_{\nu\nu}, v_\nu)_{L^2(0,T;L^2(\Gamma_1))}, \end{aligned} \tag{4.32}$$

which along with Trace Theorem, Sobolev imbedding $L^2(0,T;V) \hookrightarrow L^2(0,T;H^{\frac{3}{2}}(\Gamma_1)) \hookrightarrow L^2(0,T;L^{\frac{1}{2}}(\Gamma_1))$ and Hölder inequality leads to

$$|\langle \triangle^2 u, v\rangle_{H^{-1}(0,T;L^2(\Omega))\times H_0^1(0,T;L^2(\Omega))}| \le C\|v\|_{L^2(0,T;V)}, \tag{4.33}$$

for all $v \in H_0^1(0,T;V)$. Thus, the term $\triangle^2 u$ possess a continuous extension to the space $L^2(0,T;V')$ such that

$$u_{tt} + \triangle^2 u = |u|^\rho u, \text{ in } L^2(0,T;V'). \tag{4.34}$$

Next, our goal is to show that

$$\chi_1 = f_1(u_{\nu t}) \quad \text{and} \quad \chi_2 = f_2(u_t).$$

In deed, multiplying the first equation in (4.7) by u_t^μ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_t^\mu(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\triangle u^\mu(t)\|_2^2 + \int_{\Gamma_1} f_1(u_{\nu t}^\mu(t))u_{\nu t}^\mu(t)d\Gamma\\ + \int_{\Gamma_1} f_2(u_t^\mu(t))u_t^\mu(t)d\Gamma = \frac{1}{(\rho+2)}\frac{d}{dt}\|u^\mu(t)\|_{\rho+2}^{\rho+2}.\end{aligned} \tag{4.35}$$

Integrate (4.35) over $(0,t)$ leads to

$$\begin{aligned}&\frac{1}{2}\|u_t^\mu(t)\|_2^2 + \frac{1}{2}\|\triangle u^\mu(t)\|_2^2 - \frac{1}{(\rho+2)}\frac{d}{dt}\|u^\mu(t)\|_{\rho+2}^{\rho+2}\\ &\quad + \int_0^t\int_{\Gamma_1} f_1(u_{\nu t}^\mu(s))u_{\nu t}^\mu(s)d\Gamma ds + \int_0^t\int_{\Gamma_1} f_2(u_t^\mu(s))u_t^\mu(s)d\Gamma ds\\ &= \frac{1}{2}\|u_\mu^1\|_2^2 + \frac{1}{2}\|\triangle u_\mu^0\|_2^2 - \frac{1}{(\rho+2)}\|u_\mu^0\|_{\rho+2}^{\rho+2}.\end{aligned} \tag{4.36}$$

Considering the convergences (4.3), (4.12) and (4.13), we deduce that

$$\begin{aligned}&\lim_{\mu\to\infty}\int_0^t\int_{\Gamma_1} f_1(u_{\nu t}^\mu(s))u_{\nu t}^\mu(s)d\Gamma ds + \lim_{\mu\to\infty}\int_0^t\int_{\Gamma_1} f_2(u_t^\mu(s))u_t^\mu(s)d\Gamma ds\\ &\quad = -\frac{1}{2}\|u_t(t)\|_2^2 - \frac{1}{2}\|\triangle u(t)\|_2^2 + \frac{1}{(\rho+2)}\|u(t)\|_{\rho+2}^{\rho+2}\\ &\qquad + \frac{1}{2}\|u^1\|_2^2 + \frac{1}{2}\|\triangle u^0\|_2^2 - \frac{1}{(\rho+2)}\|u^0\|_{\rho+2}^{\rho+2}.\end{aligned} \tag{4.37}$$

On the other hand, we assume that u is a weak solution to the problem

$$\begin{cases} u_{tt} = -\triangle^2 u + |u|^\rho u, \text{ in } L^2(0,\infty;V'),\\ u = u_\nu = 0, \text{ on } \Gamma_0\times(0,\infty),\\ u_{\nu\nu} = -\chi_1,\ u_{\nu\nu\nu} = \chi_2, \text{ in } L^\infty(0,\infty;L^2(\Gamma_1)),\\ u(x,0) = u^0,\ u_t(x,0) = u^1,\ x\in\Omega.\end{cases} \tag{4.38}$$

Adapting the ideas of Lasiecka and Tataru [2, Proposition 2.1], Komornik [33, Theorem 7.9] or Lions [38, Lemma 6.1], we obtain that the weak solutions u satisfy energy identity

$$\begin{aligned}&\int_0^t\int_{\Gamma_1}\chi_1 u_{\nu t}(s)d\Gamma ds + \int_0^t\int_{\Gamma_1}\chi_2 u_t(s)d\Gamma ds = -\frac{1}{2}\|u_t(t)\|_2^2 - \frac{1}{2}\|\triangle u(t)\|_2^2\\ &\quad + \frac{1}{(\rho+2)}\|u(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|u^1\|_2^2 + \frac{1}{2}\|\triangle u^0\|_2^2 - \frac{1}{(\rho+2)}\|u^0\|_{\rho+2}^{\rho+2},\end{aligned} \tag{4.39}$$

which along with (4.37) yields to

$$\lim_{\mu\to\infty}\int_0^t (f_1(u^\mu_{\nu t}(s)), u^\mu_{\nu t}(s))_{\Gamma_1} ds + \lim_{\mu\to\infty}\int_0^t (f_2(u^\mu_t(s)), u^\mu_t(s))_{\Gamma_1} ds$$

$$= \int_0^t (\chi_1, u_{\nu t}(s))ds + \int_0^t (\chi_2, u_t(s))ds. \tag{4.40}$$

Taking (4.14)-(4.16) into account, we get that

$$\lim_{\mu\to\infty}\int_0^t (f_1(u^\mu_{\nu t}(s)), u^\mu_{\nu t}(s))_{\Gamma_1} ds = \int_0^t (\chi_1, u_{\nu t}(s))ds, \tag{4.41}$$

and

$$\lim_{\mu\to\infty}\int_0^t (f_2(u^\mu_t(s)), u^\mu_t(s))_{\Gamma_1} ds = \int_0^t (\chi_2, u_t(s))ds. \tag{4.42}$$

By the analogous arguments which have been used in the proof's process of regular solutions. we also obtain from (4.36) and (4.37) that $\chi_1 = f_1(u_{\nu t})$, and $\chi_2 = f_2(u_t)$. Thus, we prove that there exists the global weak solutions u satisfying

$$\begin{cases} u_{tt} = -\triangle^2 u + |u|^\rho u, \ \text{ in } L^2(0,\infty; V'), \\ u = u_\nu = 0, \ \text{ on } \Gamma_0 \times (0,\infty), \\ u_{\nu\nu} = -f_1(u_{\nu t}), \ u_{\nu\nu\nu} = f_2(u_t), \ \text{ in } L^\infty(0,\infty; L^2(\Gamma_1)), \\ u(x,0) = u^0 \in V, \ u_t(x,0) = u^1 \in L^2(\Omega), \end{cases} \tag{4.43}$$

with $\|\triangle u(t)\|_2 < \lambda_1$ for all $t \geq 0$.

Step 3. Uniqueness.

Finally, we will use the standard energy estimate to get the uniqueness of weak solutions. Let u and $\widetilde{u}$ be the solutions of problem (4.43), then $y = u - \widetilde{u}$ satisfies

$$\begin{cases} y_{tt} = -\triangle^2 y + |u|^\rho u - |\widetilde{u}|^\rho \widetilde{u}, \ \text{ in } L^2(0,\infty; V'), \\ y = y_\nu = 0, \ \text{ on } \Gamma_0 \times (0,\infty), \\ y_{\nu\nu} = -f_1(u_{\nu t}) + f_1(\widetilde{u}_{\nu t}), \ y_{\nu\nu\nu} = f_2(u_t) - f_2(\widetilde{u}_t), \ \text{ in } L^\infty(0,\infty; L^2(\Gamma_1)), \\ y(x,0) = 0, \ y_t(x,0) = 0. \end{cases} \tag{4.44}$$

Making use of the same procedure to prove (4.39), we have the energy identity

$$\int_0^t (f_1(u_{\nu t}(s)) - f_1(\widetilde{u}_{\nu t}(s)), y_{\nu t}(s))_{\Gamma_1} ds + \int_0^t (f_2(u_t(s)) - f_2(\widetilde{u}_t(s)), y_t(s))_{\Gamma_1} ds$$
$$= -\frac{1}{2}\|y_t(t)\|_2^2 - \frac{1}{2}\|\triangle y(t)\|_2^2 + \int_0^t (|u(s)|^\rho u(s) - |\widetilde{u(s)}|^\rho \widetilde{u}(s), y_t(s))ds,$$

which together with the Hölder inequality, assumptions $(A2)$ and (4.8) leads to

$$\begin{aligned} &\|y_t(t)\|_2^2 + \|\triangle y(t)\|_2^2 \\ &\leq 2(\rho+1)\int_0^t \int_\Omega (|u(s)|^\rho + |\widetilde{u}(s)|^\rho)|y(s)||y_t(s)|dxds \\ &\quad - 2\int_0^t (f_1(u_{\nu t}(s)) - f_1(\widetilde{u}_{\nu t}(s)), y_{\nu t}(s))_{\Gamma_1} ds \\ &\quad - 2\int_0^t (f_2(u_t(s)) - f_2(\widetilde{u}_t(s)), y_t(s))_{\Gamma_1} ds \\ &\leq C\int_0^t \left(\|u(s)\|^\rho_{2(\rho+1)} + \|\widetilde{u}(s)\|^\rho_{2(\rho+1)}\right)\|y(s)\|_{2(\rho+1)}\|y_t(s)\|_2 ds \end{aligned}$$

$$\leq C\int_0^t (\|y_t(s)\|_2^2+\|\triangle y(s)\|_2^2)ds. \tag{4.45}$$

Employing the Gronwall Lemma, we get that $\|y_t(t)\|_2^2+\|\triangle y(t)\|_2^2=0$, which implies the uniqueness of weak solutions. This completes the proof of Theorem 2.2. □

5. Uniform decay rates of solutions

The focus of the development in this section is the decay rate estimates of the energy to problem (1.1). The proofs are based on the construction of a special weight function ϕ, nonlinear integral inequality and the Multiplier method.

First, by the virtue of Theorem 2.1, it is known that the solution u of problem (1.1) possesses the some properties listed in Theorem 2.1 and Theorem 2.2. Thus, we can apply the following energy identity

$$E'(t)=-\int_{\Gamma_1} f_2(u_t(t))u_t(t)d\Gamma-\int_{\Gamma_1} f_1(u_{\nu t}(t))u_{\nu t}(t)d\Gamma. \tag{5.1}$$

Taking into account that $f_i(s)s>0$ if $s\neq 0$, we see that $E(t)$ is a non-increasing function. Moreover, the weight function ϕ appeared in Lemma 2.3 (construction method of ϕ will be presented in the sequel) will play key role in the proof of energy decay rate estimates.

Now, let us multiply the equation in (1.1) by $E\phi' Mu$, where the function Mu is defined by

$$Mu=2(m\cdot\nabla u)+(n-1)u. \tag{5.2}$$

Then, considering $0\leq S<T<+\infty$ and applying the generalized Green formula, we deduce that

$$\begin{aligned}
0&=\int_S^T E\phi'\int_\Omega (u_{tt}+\triangle^2 u-|u|^\rho u)Mudxdt\\
&=\int_S^T E\phi'\int_\Omega (u_{tt}+\triangle^2 u-|u|^\rho u)(2m\cdot\nabla u+(n-1)u)dxdt\\
&=2\int_S^T E\phi'\int_\Omega u_{tt}(m\cdot\nabla u)dxdt+2\int_S^T E\phi'\int_\Omega \triangle u\triangle(m\cdot\nabla u)dxdt\\
&\quad+2\int_S^T E\phi'\int_\Gamma u_{\nu\nu\nu}(m\cdot u_\nu)d\Gamma dt-2\int_S^T E\phi'\int_\Gamma u_{\nu\nu}(m\cdot u_\nu)_\nu d\Gamma dt\\
&\quad-2\int_S^T E\phi'\int_\Omega |u|^\rho u(m\cdot\nabla u)dxdt+(n-1)\int_S^T E\phi'\int_\Omega u_{tt}udxdt\\
&\quad+(n-1)\int_S^T E\phi'\int_\Omega |\triangle u|^2dxdt+(n-1)\int_S^T E\phi'\int_{\Gamma_1} u_{\nu\nu\nu}ud\Gamma dt\\
&\quad-(n-1)\int_S^T E\phi'\int_{\Gamma_1} u_{\nu\nu}u_\nu d\Gamma dt-(n-1)\int_S^T E\phi'\int_\Omega |u|^{\rho+2}dxdt.
\end{aligned} \tag{5.3}$$

Estimate of $I_1=2\int_S^T E\phi'\int_\Omega u_{tt}(m\cdot\nabla u)dxdt$.

Applying integration by parts and Gauss Theorem, it follows that

$$\begin{aligned}
I_1&=2\left[E\phi'\int_\Omega u_t(m\cdot\nabla u)dx\right]_S^T-2\int_S^T(E'\phi'+E\phi'')\int_\Omega u_t(m\cdot\nabla u)dxdt\\
&\quad-2\int_S^T E\phi'\int_\Omega u_t(m\cdot\nabla u_t)dxdt
\end{aligned}$$

$$= 2\left[E\phi'\int_\Omega u_t(m\cdot\nabla u)dx\right]_S^T - 2\int_S^T (E'\phi' + E\phi'')\int_\Omega u_t(m\cdot\nabla u)dxdt$$

$$- \int_S^T E\phi'\int_{\Gamma_1} |u_t|^2(m\cdot\nu)d\Gamma dt + n\int_S^T E\phi'\int_\Omega |u_t|^2 dxdt. \tag{5.4}$$

Estimate of $I_2 = 2\int_S^T E\phi'\int_\Omega \triangle u\cdot\triangle(m\cdot\nabla u)dxdt$.
The application of Gauss Theorem gives that

$$\begin{aligned}
I_2 &= 2\int_S^T E\phi'\int_\Omega \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i\partial x_i}\cdot\sum_{j=1}^n \frac{\partial^2(m\cdot\nabla u)}{\partial x_j\partial x_j} \\
&= 2\int_S^T E\phi'\int_\Omega \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i\partial x_i}\cdot\sum_{j=1}^n\sum_{k=1}^n \frac{\partial^2(m_k\frac{\partial u}{\partial x_k})}{\partial x_j\partial x_j} \\
&= 4\int_S^T E\phi'\int_\Omega \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i\partial x_i}\cdot\sum_{j=1}^n \frac{\partial^2 u}{\partial x_j\partial x_j} \\
&\quad + 2\int_S^T E\phi'\int_\Omega \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i\partial x_i}\cdot\sum_{j=1}^n\sum_{k=1}^n m_k\frac{\partial^2 u}{\partial x_j\partial x_j\partial x_k} \\
&= 4\int_S^T E\phi'\int_\Omega |\triangle u|^2 dxdt + \int_S^T E\phi'\int_\Omega m\cdot\nabla(|\triangle u|^2)dxdt \\
&= (4-n)\int_S^T E\phi'\int_\Omega |\triangle u|^2 dxdt + \int_S^T E\phi'\int_\Gamma (m\cdot\nu)|u_{\nu\nu}|^2 d\Gamma dt. \qquad (5.5)
\end{aligned}$$

Estimate of $I_3 = (n-1)\int_S^T E\phi'\int_\Omega u_{tt}u dxdt$.
By the integration by parts again, we also obtain that

$$I_3 = (n-1)\left[E\phi'\int_\Omega u_t u dx\right]_S^T - (n-1)\int_S^T E\phi'\int_\Omega |u_t|^2 dxdt$$

$$- (n-1)\int_S^T (E'\phi' + E\phi'')\int_\Omega u_t u dxdt. \tag{5.6}$$

Inserting (5.4)-(5.6) into (5.3), noting that $u_{\nu\nu} = -f_1(u_{\nu t}),\ u_{\nu\nu\nu} = f_2(u_t)$ on Γ_1 and $\nabla u = u_\nu\cdot\nu$ on Γ_0, it follows that

$$\begin{aligned}
0 &= \left[E\phi'\int_\Omega u_t M u dx\right]_S^T - \int_S^T (E'\phi' + E\phi'')\int_\Omega u_t M u dxdt \\
&\quad + \int_S^T E\phi'\int_\Omega |u_t|^2 dxdt + 3\int_S^T E\phi'\int_\Omega |\triangle u|^2 dxdt \\
&\quad + \int_S^T E\phi'\int_{\Gamma_1} f_2(u_t)Mud\Gamma dt + \int_S^T E\phi'\int_{\Gamma_1} f_1(u_{\nu t})(Mu)_\nu d\Gamma dt \\
&\quad - \int_S^T E\phi'\int_{\Gamma_1} |u_t|^2(m\cdot\nu)d\Gamma dt + \int_S^T E\phi'\int_{\Gamma_0} (m\cdot\nu)|u_{\nu\nu}|^2 d\Gamma dt \\
&\quad + \int_S^T E\phi'\int_{\Gamma_1} (m\cdot\nu)|u_{\nu\nu}|^2 d\Gamma dt + 2\int_S^T E\phi'\int_{\Gamma_0} u_{\nu\nu\nu}u_\nu(m\cdot\nu)d\Gamma dt \\
&\quad - 2\int_S^T E\phi'\int_{\Gamma_0} (m\cdot\nu)|u_{\nu\nu}|^2 d\Gamma dt - 2\int_S^T E\phi'\int_\Omega |u|^\rho u(m\cdot\nabla u)dxdt
\end{aligned}$$

$$(5.7)\qquad -(n-1)\int_S^T E\phi'\int_\Omega |u|^{\rho+2}dxdt.$$

Using the definition of energy $E(t)$ and the identity (5.7), we obtain that

$$\begin{aligned}
&2\int_S^T E^2(t)\phi'(t)dt + 2\int_S^T E\phi'\int_\Omega |\triangle u|^2 dxdt\\
&= -\left[E\phi'\int_\Omega u_t Mudx\right]_S^T + \int_S^T (E'\phi' + E\phi'')\int_\Omega u_t Mudxdt\\
&\quad -\int_S^T E\phi'\int_{\Gamma_1} f_2(u_t)Mud\Gamma dt - \int_S^T E\phi'\int_{\Gamma_1} f_1(u_{\nu t})(Mu)_\nu d\Gamma dt\\
&\quad +\int_S^T E\phi'\int_{\Gamma_1}(m\cdot\nu)(|u_t|^2 - |u_{\nu\nu}|^2)d\Gamma dt + \int_S^T E\phi'\int_{\Gamma_0}(m\cdot\nu)|u_{\nu\nu}|^2 d\Gamma dt\\
(5.8)\quad &\quad +\left[n-1+\frac{2}{\rho+2}\right]\int_S^T E\phi'\int_\Omega |u|^{\rho+2}dxdt + 2\int_S^T E\phi'\int_\Omega |u|^\rho u(m\cdot\nabla u)dxdt.
\end{aligned}$$

Next, we shall estimate the last two terms of the right hand side of the above identity (5.8).

Estimate of $D_1 = \left[n-1+\frac{2}{\rho+2}\right]\int_S^T E\phi'\int_\Omega |u|^{\rho+2}dxdt$.

Taking into account that $\frac{1}{p} = \frac{\alpha}{2} + \frac{1-\alpha}{q}$, $\alpha\in[0,1]$, then by the interpolation inequality of $L^p(\Omega)$ spaces, $\|s\|_p \le \|s\|_2^\alpha \|s\|_q^{1-\alpha}$ with $p=\rho+2$, $q=2(\rho+1)$ and $\alpha = \frac{1}{\rho+2}$, we deduce that

$$(5.9)\qquad \|u\|_{\rho+2} \le \|u\|_2^{\frac{1}{\rho+2}}\|u\|_{2(\rho+1)}^{\frac{\rho+1}{\rho+2}}.$$

Setting $h = n-1+\frac{2}{\rho+2}$, by Poincaré inequality, Sobolev embedding from V into $L^{2(\rho+1)}(\Omega)$ and Young inequality, we obtain that

$$\begin{aligned}
h\|u\|_{\rho+2}^{\rho+2} &\le n\|u\|_2\|u\|_{2(\rho+1)}^{\rho+1} \le C\|\nabla u\|_2\|\triangle u\|_2^{\rho+1}\\
(5.10)\qquad &\le C(\varepsilon)\|\nabla u\|_2^2 + \frac{\varepsilon}{B_2}\|\triangle u\|_2^{2(\rho+1)},
\end{aligned}$$

for all $\varepsilon > 0$ and $B_2 = \left(\frac{2(\rho+2)}{\rho}\right)^{\rho+1}E(0)^\rho$. Combining (2.8) and (2.14), a direct computation gives that

$$\begin{aligned}
E(t) \ge J(u) &= \frac{1}{2}\|\triangle u\|_2^2 - \frac{1}{\rho+2}\|u\|_{\rho+2}^{\rho+2}\\
&\ge \frac{1}{2}\|\triangle u\|_2^2 - K_0\|\triangle u\|_2^{\rho+2} > \|\triangle u\|_2^2\left[\frac{1}{2} - \lambda_1^\rho K_0\right]\\
(5.11)\qquad &= \|\triangle u\|_2^2\left[\frac{1}{2} - \frac{1}{K_0(\rho+2)}K_0\right] = \|\triangle u\|_2^2\left[\frac{1}{2} - \frac{1}{\rho+2}\right],
\end{aligned}$$

which implies that

$$(5.12)\qquad \|\triangle u\|_2^2 \le \frac{2(\rho+2)}{\rho}E(t) \le \frac{2(\rho+2)}{\rho}E(0).$$

Furthermore, replace (5.12) in (5.10) gives that

$$(5.13)\qquad h\|u\|_{\rho+2}^{\rho+2} \le C(\varepsilon)\|\nabla u\|_2^2 + \frac{\varepsilon}{B_2}\|\triangle u\|_2^2\|\triangle u\|_2^{2\rho} \le C(\varepsilon)\|\nabla u\|_2^2 + \varepsilon E(t).$$

From (5.13), we obtain that

$$D_1 \le \varepsilon \int_S^T E^2(t)\phi'(t)dt + C(\varepsilon)\int_S^T E\phi' \int_\Omega |\nabla u|^2 dxdt. \tag{5.14}$$

Estimate of $D_2 = 2\int_S^T E\phi' \int_\Omega |u|^\rho u(m\cdot\nabla u)dxdt$.

By the Hölder inequality and Poincaré inequality, we have that

$$D_2 \le 2R\int_S^T E\phi' \int_\Omega |u|^{\rho+1}|\nabla u|dxdt \le 2CR\int_S^T E\phi' \|\triangle u\|_2 \|u\|_{2(\rho+1)}^{\rho+1}dt. \tag{5.15}$$

Taking into account that $0<\rho<\frac{4}{n-4}$, if $n>4$, and $0<s<\frac{2n}{n-4}-2(\rho+1)$, and considering the interpolation inequality $\|s\|_p \le \|s\|_2^\alpha \|s\|_q^{1-\alpha}$ with $p=2(\rho+1)$, $q=2(\rho+1)+s$, we discover that

$$\|u\|_{2(\rho+1)} \le \|u\|_2^{1-\alpha}\|u\|_{2(\rho+1)+s}^\alpha, \tag{5.16}$$

where $\alpha\in(0,1)$ is given by $\alpha = 1+\frac{s}{(\rho+1)[2-2(\rho+1)-s]}$, which implies that

$$\|u\|_{2(\rho+1)}^{\rho+1} \le \|u\|_2^{(1-\alpha)(\rho+1)}\|u\|_{2(\rho+1)+s}^{\alpha(\rho+1)}. \tag{5.17}$$

Applying Poincaré inequality and Sobolev embedding from $V \hookrightarrow L^{2(\rho+1)+s}(\Omega)$ $\left(2(\rho+1)+s<\frac{2n}{n-4}\right)$, then we have

$$\|u\|_{2(\rho+1)}^{\rho+1} \le C\|\nabla u\|_2^{(1-\alpha)(\rho+1)}\|\triangle u\|_2^{\alpha(\rho+1)}. \tag{5.18}$$

Combining (5.15) and (5.18), we conclude that

$$D_2 \le CR\int_S^T E\phi' \|\nabla u\|_2^{(1-\alpha)(\rho+1)}\|\triangle u\|_2^{\alpha(\rho+1)+1}dt. \tag{5.19}$$

From the Young inequality,

$$ab \le \frac{1}{\varepsilon^{p/p'}p}a^p + \frac{\varepsilon}{p'}b^{p'},\ \frac{1}{p}+\frac{1}{p'}=1, \tag{5.20}$$

for all $\varepsilon>0$. Let us take $p=\frac{2}{(1-\alpha)(\rho+1)}$ and $p'=\frac{2}{2-(1-\alpha)(\rho+1)}$, then we have

$$\begin{aligned} &CR\|\nabla u\|_2^{(1-\alpha)(\rho+1)}\|\triangle u\|_2^{\alpha(\rho+1)+1} \\ &\le \frac{(CR)^{2(1-\alpha)^{-1}(\rho+1)^{-1}}}{p\varepsilon^{\frac{2-(1-\alpha)(\rho+1)}{1-\alpha)(\rho+1)}}}\|\nabla u\|_2^2 + \frac{\varepsilon}{p'}\|\triangle u\|_2^{\frac{2[\alpha(\rho+1)+1]}{2-(1-\alpha)(\rho+1)}} \\ &= C(\varepsilon)\|\nabla u\|_2^2 + K\varepsilon E(t), \end{aligned} \tag{5.21}$$

where

$$C(\varepsilon) = \frac{(CR)^{2(1-\alpha)^{-1}(\rho+1)^{-1}}}{p\varepsilon^{\frac{2-(1-\alpha)(\rho+1)}{1-\alpha)(\rho+1)}}},\quad K=\frac{2(\rho+2)}{p'\rho}\left[\frac{2(\rho+2)}{\rho}E(0)\right]^{\frac{\rho}{2-(1-\alpha)(\rho+1)}}.$$

Combining (5.19) and (5.21), we have that

$$D_2 \le K\varepsilon\int_S^T E^2(t)\phi'(t)dt + C(\varepsilon)\int_S^T E\phi' \int_\Omega |\nabla u|^2 dxdt. \tag{5.22}$$

Therefore, in view of (5.8), (5.14) and (5.22), when $m\cdot\nu\le 0$ on Γ_0 and ε small enough, we can conclude that there exists $\delta_1,\delta_2>0$ such that

$$\delta_1\int_S^T E^2(t)\phi'(t)dt$$

$$\begin{aligned}&\leq -\left[E\phi'\int_\Omega u_t Mudx\right]_S^T + \int_S^T (E'\phi' + E\phi'')\int_\Omega u_t Mudxdt\\&\quad - \int_S^T E\phi'\int_{\Gamma_1} f_2(u_t)Mud\Gamma dt - \int_S^T E\phi'\int_{\Gamma_1} f_1(u_{\nu t})(Mu)_\nu d\Gamma dt\\&\quad + \int_S^T E\phi'\int_{\Gamma_1}(m\cdot\nu)(|u_t|^2 - |u_{\nu\nu}|^2)d\Gamma dt + \delta_2\int_S^T E\phi'\int_\Omega |\nabla u|^2 dxdt. \end{aligned} \tag{5.23}$$

In order to estimate the last term of (5.23), let us give the following lemma.

Lemma 5.1. *Under the hypotheses of Theorem 2.1. Let u be a solution to problem (1.1). Then for $T > T_0$, where T_0 is sufficiently large, we have*

$$\begin{aligned}\int_S^T \phi'\int_\Omega |\nabla u|^2 dxdt \leq C(T_0)\Big\{&\int_S^T \phi'\int_{\Gamma_1}(f_1(u_{\nu t}))^2 d\Gamma dt\\&+\int_S^T \phi'\int_{\Gamma_1}(f_2(u_t))^2 d\Gamma dt + \int_S^T \phi'\int_{\Gamma_1}|u_{\nu t}|^2 d\Gamma dt\Big\},\end{aligned} \tag{5.24}$$

for all $0 \leq S < T < +\infty$.

Proof. We shall argue by contradiction. Suppose that (5.24) is not verified. Let u_k be a sequence of solutions to problem (1.1) such that

$$\lim_{k\to\infty} \frac{\int_S^T \phi'\int_\Omega |\nabla u^k|^2 dxdt}{\int_S^T \phi'\int_{\Gamma_1}(f_1(u^k_{\nu t}))^2 d\Gamma dt + \int_S^T \phi'\int_{\Gamma_1}(f_2(u^k_t))^2 d\Gamma dt + \int_S^T \phi'\int_{\Gamma_1}|u^k_{\nu t}|^2 d\Gamma dt} = +\infty, \tag{5.25}$$

while the total energy $E_k(0)$ with initial data $\{u^k(0), u^k_t(0)\}$ remains uniformly bounded in k, that is , there exists $\widetilde{M} > 0$ such that $E_k(0) < \widetilde{M}$.

Since $E_k(0) < \widetilde{M}$, by the non-increasing property of $E_k(t)$, we have $E_k(t) < \widetilde{M}$. Hence, there exists a subsequence of the sequence $\{u_k\}$, still denoted by $\{u_k\}$, which satisfies

$$u^k \longrightarrow u \text{ in } H^1(0,T;H^2(\Omega)) \text{ weakly},\ k\longrightarrow\infty, \tag{5.26}$$

$$u^k \longrightarrow u \text{ in } L^\infty(0,T;V) \text{ weakly star},\ k\longrightarrow\infty. \tag{5.27}$$

Applying the similar methods used to prove (3.18) and (3.25), we have that

$$u^k_t \longrightarrow u_t \text{ in } L^\infty(0,T;V) \text{ weakly star},\ k\longrightarrow\infty, \tag{5.28}$$

$$u^k_t \longrightarrow u_t \text{ in } L^\infty(0,T;H^1(\Gamma_1)) \text{ weakly star},\ k\longrightarrow\infty. \tag{5.29}$$

Notice that the Aubin-Lions type compactness gives us

$$u^k \longrightarrow u \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ strongly},\ k\longrightarrow\infty, \tag{5.30}$$

$$u^k \longrightarrow u \text{ in } L^\infty(0,T;H^1(\Gamma)) \text{ strongly},\ k\longrightarrow\infty. \tag{5.31}$$

In what follows, we will apply the ideas contained in Lasiecka and Tataru [20] or Cavalcanti et al [4] to our context.

Case (i). Let us consider that $u \neq 0$. By (5.30), it follows that

$$|u^k|^\rho u^k \longrightarrow |u|^\rho u \text{ , a.e. in } Q_T = \Omega\times(0,T),\ k\longrightarrow\infty. \tag{5.32}$$

Since the sequence $\{|u^k|^\rho u^k\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, together with (5.32) and Lion's Lemma [23, Lemma 1.3, Chapter 1], we have

$$|u^k|^\rho u^k \longrightarrow |u|^\rho u \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weakly star},\ k \longrightarrow \infty. \tag{5.33}$$

Taking into account that the Poincaré inequality and the boundedness of $E_k(t)$, it is found that

$$\|\nabla u^k\|_2^2 \le C\|\triangle u^k\|_2^2 \le CE_k(t), \tag{5.34}$$

where C is a positive constant independent of k and t. Thus, we can deduce that the term $\int_S^T \phi' \int_\Omega |\nabla u^k|^2 dxdt$ is bounded. Therefore, we have from (5.25) that

$$\int_S^T \phi' \int_{\Gamma_1} (f_1(u^k_{\nu t}))^2 d\Gamma dt + \int_S^T \phi' \int_{\Gamma_1} (f_2(u^k_t))^2 d\Gamma dt + \int_S^T \phi' \int_{\Gamma_1} |u^k_{\nu t}|^2 d\Gamma dt \to 0, \tag{5.35}$$

as $k \to \infty$. Especially, (5.35) implies that

$$\int_S^T \phi' \int_{\Gamma_1} (f_1(u^k_{\nu t}))^2 d\Gamma dt \to 0, \text{ as } k \to +\infty. \tag{5.36}$$

Since $\phi(t)$ is concave, it follows that $\phi'(t) \ge \phi'(T)$, $t \in [S,T]$, for any $T > 0$, we also get

$$0 \le \phi'(T) \int_S^T \int_{\Gamma_1} (f_1(u^k_{\nu t}))^2 d\Gamma dt \le \int_S^T \phi' \int_{\Gamma_1} (f_1(u^k_{\nu t}))^2 d\Gamma dt. \tag{5.37}$$

Thus, combining (5.36) and (5.37), it follows that

$$\lim_{k\to+\infty} \int_S^T \int_{\Gamma_1} (f_1(u^k_{\nu t}))^2 d\Gamma dt = 0. \tag{5.38}$$

Considering that S is chosen in the interval $[0,T]$, so we write

$$\lim_{k\to+\infty} \int_0^T \int_{\Gamma_1} (f_1(u^k_{\nu t}))^2 d\Gamma dt = 0.$$

Therefore, we conclude

$$f_1(u^k_{\nu t}) \longrightarrow 0 \text{ in } L^2(0,T;L^2(\Gamma_1)) \text{ strongly},\ k \longrightarrow \infty. \tag{5.39}$$

In a similar way, we also conclude that

$$f_2(u^k_t) \longrightarrow 0 \text{ in } L^2(0,T;L^2(\Gamma_1)) \text{ strongly},\ k \longrightarrow \infty, \tag{5.40}$$

$$u^k_t \longrightarrow 0 \text{ in } L^2(0,T;H^1(\Gamma_1)) \text{ strongly},\ k \longrightarrow \infty. \tag{5.41}$$

Taking $k \to +\infty$ in the equation, we get for u

$$\begin{cases} u_{tt} = -\triangle^2 u + |u|^\rho u,\ (x,t) \in \Omega \times (0,\infty), \\ u = u_\nu = 0,\ (x,t) \in \Gamma_0 \times (0,\infty), \\ u_{\nu\nu} = u_{\nu\nu\nu} = 0,\ u_t = u_{\nu t} = 0,\ (x,t) \in \Gamma_1 \times (0,\infty), \end{cases} \tag{5.42}$$

and for $u_t = v$,

$$\begin{cases} v_{tt} = -\triangle^2 v + (\rho+1)|u|^\rho v,\ (x,t) \in \Omega \times (0,\infty), \\ v = v_\nu = 0,\ (x,t) \in \Gamma_0 \times (0,\infty), \\ v_{\nu\nu} = v_{\nu\nu\nu} = 0,\ v = v_\nu = 0,\ (x,t) \in \Gamma_1 \times (0,\infty). \end{cases} \tag{5.43}$$

Note that $u \in L^\infty(0,T;V)$ implies $u \in L^{\frac{4}{n-4}}(\Omega)$ and $(\rho+1)|u|^\rho \in L^\infty(0,T;L^n(\Omega))$. Applying the standard uniqueness results of [16, see Chapter 6] or the uniqueness results of [29] to our context, we conclude that $v = 0$, which means $u_t = 0$, for T suitably large.

Hence, the equation (5.42) reduce to the elliptic equation

$$\begin{cases} \triangle^2 u = |u|^\rho u, \;\; x \in \Omega, \\ u = u_\nu = 0, \;\; x \in \Gamma_0, \\ u_{\nu\nu} = u_{\nu\nu\nu} = 0, \;\; x \in \Gamma_1. \end{cases} \tag{5.44}$$

Multiplying the above elliptic equation by u, we have

$$\int_\Omega |\triangle u|^2 dx - \int_\Omega |u|^{\rho+2} dx = 0, \tag{5.45}$$

which implies that $J(u) = \frac{\rho}{2(\rho+2)}\|\triangle u\|_2^2$. But according to (5.11), it follows that

$$E(t) \geq J(u) \geq \frac{\rho}{2(\rho+2)}\|\triangle u\|_2^2, \tag{5.46}$$

for all $u \neq 0$. This is a contradiction.

Case (ii). Let us assume that $u \equiv 0$. Setting

$$c_k = \left[\int_S^T \phi' \int_\Omega |\nabla u^k|^2 dxdt\right]^{\frac{1}{2}} \text{ and } \widetilde{u}^k = \frac{u_k}{c_k}, \tag{5.47}$$

which implies

$$\int_S^T \phi' \int_\Omega |\nabla \widetilde{u}^k|^2 dxdt = \int_S^T \phi' \int_\Omega \frac{|\nabla u^k|^2}{c_k^2} dxdt = 1. \tag{5.48}$$

Besides,

$$\widetilde{E}_k(t) = \frac{1}{2}\|\widetilde{u}_t^k\|_2^2 + \frac{1}{2}\|\triangle \widetilde{u}^k\|_2^2 - \frac{1}{\rho+2}\|\widetilde{u}^k\|_{\rho+2}^{\rho+2} \leq \frac{1}{2c_k^2}\left(\|u_t^k\|_2^2 + \|\triangle u^k\|_2^2\right). \tag{5.49}$$

By the similar argument as (5.46), we deduce that

$$\frac{1}{2}\|\triangle u^k\|_2^2 \leq \frac{\rho+2}{\rho} J(u^k) = \frac{\rho+2}{\rho}\left(\frac{1}{2}\|\triangle u^k\|_2^2 - \frac{1}{\rho+2}\|u^k\|_{\rho+2}^{\rho+2}\right), \tag{5.50}$$

which along with (5.49) yields that

$$\begin{aligned} \widetilde{E}_k(t) &\leq \frac{1}{c_k^2}\left(\frac{1}{2}\|u_t^k\|_2^2 + \frac{\rho+2}{2\rho}\|\triangle u^k\|_2^2 - \frac{1}{\rho}\|u^k\|_{\rho+2}^{\rho+2}\right) \\ &= \frac{\rho+2}{\rho c_k^2}\left(\frac{\rho}{2(\rho+2)}\|u_t^k\|_2^2 + \frac{1}{2}\|\triangle u^k\|_2^2 - \frac{1}{\rho+2}\|u^k\|_{\rho+2}^{\rho+2}\right) \\ &\leq \frac{\rho+2}{\rho c_k^2} E_k(t). \end{aligned} \tag{5.51}$$

Also,

$$\begin{aligned} \widetilde{E}_k(t) &= \frac{1}{2}\|\widetilde{u}_t^k\|_2^2 + \frac{1}{2}\|\triangle \widetilde{u}^k\|_2^2 - \frac{1}{\rho+2}\|\widetilde{u}^k\|_{\rho+2}^{\rho+2} \\ &\geq \frac{1}{2}\|\widetilde{u}_t^k\|_2^2 + \frac{\rho}{2(\rho+2)}\|\triangle \widetilde{u}^k\|_2^2 \geq \frac{\rho}{(\rho+2)c_k^2} E_k(t). \end{aligned} \tag{5.52}$$

Furthermore, when $u \equiv 0$, we deduce that $c_k \to 0$ as $k \to +\infty$.

On the other hand, considering the energy identity,

$$E_k'(t) = -\int_{\Gamma_1} f_2(u_t^k(t))u_t^k(t)d\Gamma - \int_{\Gamma_1} f_1(u_{\nu t}^k(t))u_{\nu t}^k(t)d\Gamma, \tag{5.53}$$

and multiplying this identity by $E_k(t)$, then we obtain

$$\frac{1}{2}\frac{d}{dt}[E_k(t)]^2 = -E_k(t)\int_{\Gamma_1} f_2(u_t^k(t))u_t^k(t)d\Gamma - E_k(t)\int_{\Gamma_1} f_1(u_{\nu t}^k(t))u_{\nu t}^k(t)d\Gamma. \tag{5.54}$$

Integrating (5.54) with respect to t from S to T, we discover that

$$\begin{aligned} E_k^2(T) - E_k^2(S) = &-2\int_S^T E_k(t)\int_{\Gamma_1} f_2(u_t^k(t))u_t^k(t)d\Gamma dt \\ &- 2\int_S^T E_k(t)\int_{\Gamma_1} f_1(u_{\nu t}^k(t))u_{\nu t}^k(t)d\Gamma dt. \end{aligned} \tag{5.55}$$

In view of (5.54) and (5.55), we deduce that

$$\begin{aligned} &\int_S^T E_k^2(t)\phi'(t)dt \\ &\geq \int_S^T E_k^2(T)\phi'(t)dt = [\phi(T) - \phi(S)]E_k^2(S) \\ &\quad - 2[\phi(T) - \phi(S)]\int_S^T E_k(t)\int_{\Gamma_1} f_2(u_t^k(t))u_t^k(t)d\Gamma dt \\ &\quad - 2[\phi(T) - \phi(S)]\int_S^T E_k(t)\int_{\Gamma_1} f_1(u_{\nu t}^k(t))u_{\nu t}^k(t)d\Gamma dt. \end{aligned} \tag{5.56}$$

Replacing $Mu^k = 2(m\cdot\nabla u^k) + (n-1)u^k$ in inequality (5.23), we obtain that

$$\begin{aligned} &\delta_1\int_S^T E_k^2(t)\phi'(t)dt \\ &\leq -2\left[E_k\phi'\int_\Omega u_t^k(m\cdot\nabla u^k)dx\right]_S^T - (n-1)\left[E_k\phi'\int_\Omega u_t^k u^k dx\right]_S^T \\ &\quad + \int_S^T (E_k'\phi' + E_k\phi'')\int_\Omega u_t^k Mu^k dxdt - 2\int_S^T E_k\phi'\int_{\Gamma_1} f_2(u_t)(m\cdot u_\nu^k)d\Gamma dt \\ &\quad - (n-1)\int_S^T E_k\phi'\int_{\Gamma_1} f_2(u_t^k)u^k d\Gamma dt - 2\int_S^T E_k\phi'\int_{\Gamma_1} f_1(u_{\nu t}^k)(m\cdot u_\nu^k)_\nu d\Gamma dt \\ &\quad - (n-1)\int_S^T E_k\phi'\int_{\Gamma_1} f_1(u_{\nu t}^k)u_\nu^k d\Gamma dt + \int_S^T E_k\phi'\int_{\Gamma_1} |u_t^k|^2(m\cdot\nu)d\Gamma dt \\ &\quad - \int_S^T E_k\phi'\int_{\Gamma_1} |u_{\nu\nu}^k|^2(m\cdot\nu)d\Gamma dt + \delta_2\int_S^T E_k\phi'\int_\Omega |u^k|^2 dxdt. \end{aligned} \tag{5.57}$$

Estimate of $G_1 = -2\int_S^T E_k\phi'\int_{\Gamma_1} f_2(u_t^k)(m\cdot u_\nu^k)d\Gamma dt$.
Using Young inequality and a direct calculation gives that

$$\begin{aligned} G_1 &\leq 2R\int_S^T E_k\phi'\int_{\Gamma_1} |f_2(u_t^k)||u_\nu^k|d\Gamma dt \\ &\leq \eta\int_S^T E_k\phi'\int_{\Gamma_1} |u_\nu^k|^2 d\Gamma dt + \frac{R^2}{\eta}\int_S^T E_k\phi'\int_{\Gamma_1} |f_2(u_t^k)|^2 d\Gamma dt. \end{aligned} \tag{5.58}$$

for all $\eta > 0$.

Estimate of $G_2 = -2\int_S^T E_k\phi' \int_{\Gamma_1} f_1(u^k_{\nu t})(m\cdot u^k_\nu)_\nu d\Gamma dt$.

Applying integration by parts and the Young inequality, it follows that

$$
\begin{aligned}
G_2 = &-2\int_S^T E_k\phi' \int_{\Gamma_1} \sum_{j=1}^n \frac{\partial\left(\sum_{i=1}^n m_i \frac{\partial u^k}{\partial x_i}\right)}{\partial x_j}\nu_j f_1(u^k_{\nu t})d\Gamma dt \\
&-2\int_S^T E_k\phi' \int_{\Gamma_1}\left(\sum_{j=1}^n \frac{\partial u^k}{\partial x_j} + \sum_{j=1}^n\sum_{i=1}^n m_i \frac{\partial^2 u^k}{\partial x_i \partial x_j}\right)\nu_j f_1(u^k_{\nu t})d\Gamma dt \\
&-2\int_S^T E_k\phi' \int_{\Gamma_1} u^k_\nu f_1(u^k_{\nu t})d\Gamma dt - 2\int_S^T E_k\phi' \int_{\Gamma_1} m\cdot u^k_{\nu\nu} f_1(u^k_{\nu t})d\Gamma dt \\
\leq &\ \eta\int_S^T E_k\phi' \int_{\Gamma_1} |u^k_\nu|^2 d\Gamma dt + \frac{1}{\eta}\int_S^T E_k\phi' \int_{\Gamma_1} |f_1(u^k_{\nu t})|^2 d\Gamma dt \\
&+\eta\int_S^T E_k\phi' \int_{\Gamma_1} |u^k_{\nu\nu}|^2 d\Gamma dt + \frac{R^2}{\eta}\int_S^T E_k\phi' \int_{\Gamma_1} |f_1(u^k_{\nu t})|^2 d\Gamma dt. \qquad (5.59)
\end{aligned}
$$

Estimate of $G_3 = -2E_k\phi' \int_\Omega u^k_t(m\cdot\nabla u^k)dx$.

Considering Young inequality and Poincaré inequality, we obtain from the definition of $E_k(t)$ that

$$
\begin{aligned}
G_3 &\leq |-2E_k\phi' \int_\Omega u^k_t(m\cdot\nabla u^k)dx| \leq 2E_k(t)LR\int_\Omega |u^k_t||\nabla u^k|dx \\
&\leq E_k(t)LR\int_\Omega [|u^k_t|^2 + |\nabla u^k|^2]dx \leq CE_k^2(t), \qquad (5.60)
\end{aligned}
$$

where L is a positive constant which verifies $|\phi'(t)| \leq |\phi'(0)| = L, \forall\, t \geq 0$. Therefore, we have

$$
-2\left[E_k\phi' \int_\Omega u^k_t(m\cdot\nabla u^k)dx\right]_S^T \leq CE_k^2(T) + CE_k^2(S) \leq CE_k^2(S). \qquad (5.61)
$$

Estimate of $G_4 = -(n-1)\left[E_k\phi' \int_\Omega u^k_t u^k dx\right]_S^T$.

Analogously, considering the same procedure used to prove (5.61), we also get that

$$
-(n-1)\left[E_k\phi' \int_\Omega u^k_t u^k dx\right]_S^T \leq CE_k^2(S). \qquad (5.62)
$$

Estimate of $G_5 = \int_S^T (E'_k\phi' + E_k\phi'')\int_\Omega u^k_t M u^k dxdt$.

By Young inequality and Poincaré inequality, a simple computation reveals that

$$
\begin{aligned}
\int_\Omega u^k_t M u^k dx &= 2\int_\Omega (m\cdot\nabla u^k)u^k dx + (n-1)\int_\Omega u^k_t u^k dx \\
&\leq 2R\int_\Omega |\nabla u^k||u^k|dx + (n-1)\int_\Omega |u^k_t||u^k|dx \leq CE_k(t). \qquad (5.63)
\end{aligned}
$$

We thereby conclude that

$$
\begin{aligned}
G_5 &= \int_S^T (E'_k\phi' + E_k\phi'')\int_\Omega u^k_t M u^k dxdt \\
&\leq C\int_S^T |E'_k\phi' + E_k\phi''|E_k(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq LC\int_S^T -E_k'E_k dt + CE_k{}^2(S)\int_S^T -\phi'' dt\\
&= \frac{LC}{2}\int_S^T -\frac{d}{dt}E_k^2(t)dt + CE_k{}^2(S)\int_S^T -\phi'' dt\\
&= \frac{LC}{2}\left[E_k{}^2(S)-E_k{}^2(T)\right] + CE_k{}^2(S)\left[\phi'(S)-\phi'(T)\right]\\
&\leq \frac{LC}{2}E_k{}^2(S) + CE_k{}^2(S)\phi'(S). \qquad (5.64)
\end{aligned}
$$

Estimate of $G_6 = -(n-1)\int_S^T E_k\phi'\int_{\Gamma_1} f_2(u_t^k)u^k d\Gamma dt$.

Using Young inequality, there appears the relation

$$
G_6 \leq \gamma\int_S^T E_k\phi'\int_{\Gamma_1}|u^k|^2 d\Gamma dt + \frac{(n-1)^2}{4\gamma}\int_S^T E_k\phi'\int_{\Gamma_1}|f_2(u_t^k)|^2 d\Gamma dt, \qquad (5.65)
$$

for any $\gamma > 0$. Taking into account that the continuity of the linear trace operator $\mathfrak{B}$: $V \hookrightarrow H^1(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$, there exist two positive constants ξ_1, ξ_2 such that

$$
\|u\|_{L^2(\Gamma_1)} \leq \xi_1\|\triangle u\|_2,\ \|\nabla u\|_{L^2(\Gamma_1)} \leq \xi_2\|\triangle u\|_2, \qquad (5.66)
$$

for all $u \in V$. Hence, we deduce that

$$
G_6 \leq C\gamma\int_S^T E_k^2\phi' dt + \frac{(n-1)^2}{4\gamma}\int_S^T E_k\phi'\int_{\Gamma_1}|f_2(u_t^k)|^2 d\Gamma dt. \qquad (5.67)
$$

Estimate of $G_7 = -(n-1)\int_S^T E_k\phi'\int_{\Gamma_1} f_1(u_{\nu t}^k)u_\nu^k d\Gamma dt$.

Analogously, we obtain that

$$
G_7 \leq C\gamma\int_S^T E_k^2\phi' dt + \frac{(n-1)^2}{4\gamma}\int_S^T E_k\phi'\int_{\Gamma_1}|f_1(u_{\nu t}^k)|^2 d\Gamma dt. \qquad (5.68)
$$

Since $m\cdot\nu$ are sufficiently smooth and Γ_1 is compact, there exists $\delta > 0$ such that $m\cdot\nu \geq \delta > 0$ for all $x \in \Gamma_1$. Consequently, inserting the estimates $(G_1)-(G_7)$ into (5.57), we conclude that

$$
\begin{aligned}
&\delta_1\int_S^T E_k^2(t)\phi'(t)dt\\
&\leq \frac{C\eta}{\delta}\int_S^T E_k\phi'\int_{\Gamma_1}(m\cdot\nu)|u_{\nu\nu}^k|^2 d\Gamma dt + \frac{R^2}{\eta}\int_S^T E_k\phi'\int_{\Gamma_1}|f_2(u_t^k)|^2 d\Gamma dt\\
&\quad + \frac{C\eta}{\delta}\int_S^T E_k\phi'\int_{\Gamma_1}(m\cdot\nu)|u_{\nu\nu}^k|^2 d\Gamma dt + \frac{1}{\eta}\int_S^T E_k\phi'\int_{\Gamma_1}|f_1(u_{\nu t}^k)|^2 d\Gamma dt\\
&\quad + \frac{\eta}{\delta}\int_S^T E_k\phi'\int_{\Gamma_1}(m\cdot\nu)|u_{\nu\nu}^k|^2 d\Gamma dt + \frac{R^2}{\eta}\int_S^T E_k\phi'\int_{\Gamma_1}|f_1(u_{\nu t}^k)|^2 d\Gamma dt\\
&\quad + C\gamma\int_S^T E_k^2\phi' dt + \frac{(n-1)^2}{4\gamma}\int_S^T E_k\phi'\int_{\Gamma_1}|f_2(u_t^k)|^2 d\Gamma dt\\
&\quad + C\gamma\int_S^T E_k^2\phi' dt + \frac{(n-1)^2}{4\gamma}\int_S^T E_k\phi'\int_{\Gamma_1}|f_1(u_{\nu t}^k)|^2 d\Gamma dt\\
&\quad + (\frac{LC}{2}+C)E_k{}^2(S) + CE_k{}^2(S)\phi'(S) + \delta_2\int_S^T E_k\phi'\int_\Omega |u^k|^2 dx dt\\
&\quad + \int_S^T E_k\phi'\int_{\Gamma_1}|u_t^k|^2(m\cdot\nu)d\Gamma dt - \int_S^T E_k\phi'\int_{\Gamma_1}|u_{\nu\nu}^k|^2(m\cdot\nu)d\Gamma dt. \qquad (5.69)
\end{aligned}
$$

Taking η, γ small enough such that $\delta_1 - 2C\gamma > 0$ and $1 - \frac{\eta}{\delta} - \frac{2C\eta}{\delta} > 0$, then we have

$$
\begin{aligned}
&\int_S^T E_k^2(t)\phi'(t)dt \\
&\le C_1 \int_S^T E_k\phi' \int_{\Gamma_1} |f_2(u_t^k)|^2 d\Gamma dt + C_2 \int_S^T E_k\phi' \int_{\Gamma_1} |f_1(u_{\nu t}^k)|^2 d\Gamma dt \\
&\quad + C_3 {E_k}^2(S) + C_4 {E_k}^2(S)\phi'(S) + C_5 \int_S^T E_k\phi' \int_{\Gamma_1} |u_t^k|^2 (m\cdot\nu) d\Gamma dt \\
&\quad + C_6 \int_S^T E_k\phi' \int_\Omega |u^k|^2 dxdt,
\end{aligned} \tag{5.70}
$$

where C_i, $i = 1, \cdots, 6$ are positive constants. Combining (5.56) and (5.70), it is found that

$$
\begin{aligned}
&[\phi(T) - \phi(S)]E_k^2(S) \\
&\le C_1 \int_S^T E_k\phi' \int_{\Gamma_1} |f_2(u_t^k)|^2 d\Gamma dt \\
&\quad + C_2 \int_S^T E_k\phi' \int_{\Gamma_1} |f_1(u_{\nu t}^k)|^2 d\Gamma dt + C_3 {E_k}^2(S) + C_4 {E_k}^2(S)\phi'(S) \\
&\quad + C_5 \int_S^T E_k\phi' \int_{\Gamma_1} |u_t^k|^2 (m\cdot\nu) d\Gamma dt + C_6 \int_S^T E_k\phi' \int_\Omega |u^k|^2 dxdt \\
&\quad - 2[\phi(T) - \phi(S)]\frac{1}{\phi'(T)}\phi'(T) \int_S^T E_k(t) \int_{\Gamma_1} f_2(u_t^k(t))u_t^k(t) d\Gamma dt \\
&\quad - 2[\phi(T) - \phi(S)]\frac{1}{\phi'(T)}\phi'(T) \int_S^T E_k(t) \int_{\Gamma_1} f_1(u_{\nu t}^k(t))u_{\nu t}^k(t) d\Gamma dt.
\end{aligned} \tag{5.71}
$$

Furthermore, considering that $\phi'(t)$ is a non-increasing function, it is inferred that

$$
\begin{aligned}
&[\phi(T) - \phi(S) - C_3 - C_4\phi'(S)]\, E_k^2(S) \\
&\le \left(C_1 + \frac{\phi(T)}{\phi'(T)}\right) \int_S^T E_k\phi' \int_{\Gamma_1} |f_2(u_t^k(t))|^2 d\Gamma dt \\
&\quad + \left(C_1 + \frac{\phi(T)}{\phi'(T)}\right) \int_S^T E_k\phi' \int_{\Gamma_1} |f_1(u_{\nu t}^k(t))|^2 d\Gamma dt \\
&\quad + \left(C_5 + \frac{\phi(T)}{\phi'(T)}\right) \int_S^T E_k\phi' \int_{\Gamma_1} |u_t^k(t)|^2 d\Gamma dt \\
&\quad + \frac{\phi(T)}{\phi'(T)} \int_S^T E_k\phi' \int_{\Gamma_1} |u_{\nu t}^k(t)|^2 d\Gamma dt + C_6 \int_S^T E_k\phi' \int_\Omega |u^k|^2 dxdt.
\end{aligned} \tag{5.72}
$$

Since $\phi(t) \to +\infty$ as $t \to +\infty$, for a large T, it is noted that $\phi(T) - \phi(S) - C_3 - C_4\phi'(S) > 0$. Thus, using Poincaré inequality again, we deduce that

$$
\begin{aligned}
E_k(S) \le C(S,T,\phi,\phi') \Bigg\{ &\int_S^T \phi' \int_{\Gamma_1} |f_2(u_t^k)|^2 d\Gamma dt + \int_S^T \phi' \int_{\Gamma_1} |f_1(u_{\nu t}^k)|^2 d\Gamma dt \\
&+ \int_S^T \phi' \int_{\Gamma_1} |u_{\nu t}^k|^2 d\Gamma dt + \int_S^T \phi' \int_\Omega |u_\nu^k|^2 dxdt \Bigg\}.
\end{aligned}
$$

Dividing both sides of the last inequality by $\int_S^T \phi' \int_\Omega |\nabla u^k|^2 dxdt$, then for very $t \in [S,T]$, where $0 \le S < T < +\infty$, we have that

(5.73)
$$\frac{E_k(t)}{\int_S^T \phi' \int_\Omega |\nabla u^k|^2 dxdt} \le C(S,T,\phi,\phi') \times \left\{ \frac{\int_S^T \phi' \int_{\Gamma_1} |f_2(u_t^k)|^2 d\Gamma dt + \int_S^T \phi' \int_{\Gamma_1} |f_1(u_{\nu t}^k)|^2 d\Gamma dt + \int_S^T \phi' \int_{\Gamma_1} |u_{\nu t}^k|^2 d\Gamma dt}{\int_S^T \phi' \int_\Omega |\nabla u^k|^2 dxdt} + 1 \right\}.$$

By (5.25), we know that

(5.74)
$$\lim_{k\to\infty} \frac{\int_S^T \phi' \int_{\Gamma_1} (f_1(u_{\nu t}^k))^2 d\Gamma dt + \int_S^T \phi' \int_{\Gamma_1} (f_2(u_t^k))^2 d\Gamma dt + \int_S^T \phi' \int_{\Gamma_1} |u_{\nu t}^k|^2 d\Gamma dt}{\int_S^T \phi' \int_\Omega |\nabla u^k|^2 dxdt} = 0,$$

therefore, there exists $\widetilde{N} > 0$ such that

$$\frac{E_k(t)}{c_k^2} \le C(S,T,\phi,\phi')(\widetilde{N}+1), \tag{5.75}$$

for all $t \in [S,T]$, $0 \le S < T < +\infty$. Combining (5.51) and (5.75), we have that

$$\widetilde{E}_k(t) \le \frac{\rho+2}{\rho}\frac{1}{c_k^2} E_k(t) \le \frac{\rho+2}{\rho} C(S,T,\phi,\phi')(N+1), \tag{5.76}$$

which implies

$$\|\widetilde{u}_t^k\|_2^2 + \|\triangle \widetilde{u}^k\|_2^2 \le \frac{2(\rho+2)}{\rho} C(S,T,\phi,\phi')(N+1), \tag{5.77}$$

for all $t \in [S,T]$, $0 \le S < T < +\infty$.

Hence, there exists a subsequence of the sequence $\{\widetilde{u}_k\}$, still denoted by $\{\widetilde{u}_k\}$, which satisfies

$$\widetilde{u}^k \longrightarrow \widetilde{u} \text{ in } L^\infty(0,T;V) \text{ weakly star},\ k \longrightarrow \infty, \tag{5.78}$$

$$\widetilde{u}_t^k \longrightarrow \widetilde{u}_t \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weakly star},\ k \longrightarrow \infty, \tag{5.79}$$

$$\widetilde{u}^k \longrightarrow \widetilde{u} \text{ in } L^2(0,T;L^2(\Omega)) \text{ strongly},\ k \longrightarrow \infty. \tag{5.80}$$

In addition, $\widetilde{u}^k$ also satisfies

$$\begin{cases} \widetilde{u}_{tt}^k = -\triangle^2 \widetilde{u}^k + |u^k|^\rho \widetilde{u}^k,\ (x,t) \in \Omega \times (0,\infty), \\ \widetilde{u}^k = \widetilde{u}_\nu^k = 0,\ (x,t) \in \Gamma_0 \times (0,\infty), \\ \widetilde{u}_{\nu\nu}^k = -f_1(u_{\nu t}^k)\frac{1}{c_k},\ \widetilde{u}_{\nu\nu\nu}^k = f_2(u_t^k)\frac{1}{c_k},\ (x,t) \in \Gamma_1 \times (0,\infty). \end{cases} \tag{5.81}$$

From (5.74), we see that

$$\lim_{k\to\infty} \frac{\int_S^T \phi' \int_{\Gamma_1} (f_2(u_t^k))^2 d\Gamma dt}{c_k^2} = 0. \tag{5.82}$$

Since

$$0 \le \phi'(T) \int_S^T \int_{\Gamma_1} \left| \frac{f_2(u_t^k)}{c_k} \right|^2 d\Gamma dt \le \frac{\int_S^T \phi' \int_{\Gamma_1} |f_2(u_t^k)|^2 d\Gamma dt}{c_k^2}, \tag{5.83}$$

we thereby have

$$\lim_{k\to\infty}\int_S^T\int_{\Gamma_1}\left|\frac{f_2(u_t^k)}{c_k}\right|^2 d\Gamma dt=0, \tag{5.84}$$

which implies

$$\frac{f_2(u_t^k)}{c_k}\to 0 \text{ in } L^2(0,T;L^2(\Gamma_1)) \text{ as } k\to+\infty. \tag{5.85}$$

Making use of the same procedure used to prove (5.85), we deduce that

$$\frac{f_1(u_{\nu t}^k)}{c_k}\to 0 \text{ in } L^2(0,T;L^2(\Gamma_1)) \text{ as } k\to+\infty. \tag{5.86}$$

Further, there appear the relation

$$\begin{aligned}\int_0^T\int_\Omega||u^k|^\rho\widetilde{u}^k|^2dxdt&=\int_{Q_T}|u^k|^{2\rho}|\widetilde{u}^k|^2dxdt\\ &=\int_{|u^k|\le\varepsilon}|u^k|^{2\rho}|\widetilde{u}^k|^2dxdt+\int_{|u^k|>\varepsilon}|u^k|^{2\rho}|\widetilde{u}^k|^2dxdt.\end{aligned} \tag{5.87}$$

Considering that $|y|^\rho$ is a continuous in R, so we define $\widetilde{M_\varepsilon}=\sup_{|y|\le\varepsilon}|y|^\rho$. Therefore, we obtain

$$\int_0^T\int_\Omega||u^k|^\rho\widetilde{u}^k|^2dxdt\le\widetilde{M}_\varepsilon^2\|\widetilde{u}^k\|^2_{L^2(Q)}+c_k^{2\rho}\|\widetilde{u}^k\|^{2\rho+2}_{L^{2\rho+2}(Q)}. \tag{5.88}$$

Combining (5.77) and hypotheses $(A1)$, we deduce from the (5.88) that

$$\int_0^T\int_\Omega||u^k|^\rho\widetilde{u}^k|^2dxdt\le C[M_\varepsilon^2+c_k^{2\rho}]. \tag{5.89}$$

Then, taking $\varepsilon\to 0$ and $k\to+\infty$, we get that

$$|u^k|^\rho\widetilde{u}^k\to 0 \text{ in } L^2(0,T;L^2(\Omega)). \tag{5.90}$$

From what has been discussed above, passing to the limit in (5.81) as $k\to+\infty$, we have

$$\begin{cases}\widetilde{u}_{tt}+\triangle^2\widetilde{u}=0,\ (x,t)\in\Omega\times(0,\infty),\\ \widetilde{u}=\widetilde{u}_\nu=0,\ (x,t)\in\Gamma_0\times(0,\infty),\\ \widetilde{u}_{\nu\nu}=0,\ \widetilde{u}_{\nu\nu\nu}=0,\ (x,t)\in\Gamma_1\times(0,\infty).\end{cases} \tag{5.91}$$

Differentiating (5.91) with respect to t and taking $v=\widetilde{u}_t$, we conclude that

$$\begin{cases}v_{tt}+\triangle^2 v=0,\ (x,t)\in\Omega\times(0,\infty),\\ v=v_\nu=0,\ (x,t)\in\Gamma_0\times(0,\infty),\\ v_{\nu\nu}=0,\ v_{\nu\nu\nu}=0,\ (x,t)\in\Gamma_1\times(0,\infty).\end{cases} \tag{5.92}$$

Applying the standard uniqueness results of [16](see Chapter 6) or the uniqueness results of [29] to our context again, it comes that $v=0$, that is $\widetilde{u}_t=0$. Returning to the equation (5.91), we obtain

$$\begin{cases}\triangle^2\widetilde{u}=0,\ x\in\Omega,\\ \widetilde{u}=\widetilde{u}_\nu=0,\ x\in\Gamma_0,\\ \widetilde{u}_{\nu\nu}=0,\ \widetilde{u}_{\nu\nu\nu}=0,\ x\in\Gamma_1.\end{cases} \tag{5.93}$$

Multiplying the above problem by $\widetilde{u}$, we see that

$$0 = -\int_\Omega (\triangle^2 \widetilde{u})\widetilde{u}dx = -\int_\Omega |\triangle \widetilde{u}|^2 dx = -\|\widetilde{u}\|_V^2, \tag{5.94}$$

which implies that $\widetilde{u} = 0$. But from the (5.48) and (5.80), we conclude that $\widetilde{u} \neq 0$. This is a contraction. This completes the proof of Lemma 5.1. □

On the basis of Lemma 5.1, we are now in positive to give the straightforward proof of Theorem 2.3.

Proof of Theorem 2.3. Inserting the results of Lemma 5.1 into (5.23) and then using the similar calculation as (5.57) to (5.70), we have that

$$\begin{aligned}
&\int_S^T E^2(t)\phi'(t)dt \\
&\leq \widetilde{C}_1 E(S)\int_S^T \phi' \int_{\Gamma_1} |f_2(u_t)|^2 d\Gamma dt + \widetilde{C}_2 E(S)\int_S^T \phi' \int_{\Gamma_1} |f_1(u_{\nu t})|^2 d\Gamma dt \\
&\quad + \widetilde{C}_3 E^2(S) + \widetilde{C}_4 E^2(S)\phi'(S) + \widetilde{C}_5 E(S)\int_S^T \phi' \int_{\Gamma_1} |u_t|^2 (m\cdot\nu) d\Gamma dt \\
&\quad + \widetilde{C}_6 E(S)\int_S^T \phi' \int_{\Gamma_1} |u_{\nu t}|^2 d\Gamma dt.
\end{aligned} \tag{5.95}$$

Analysis of $J_1 = \int_S^T \phi' \int_{\Gamma_1} |u_{\nu t}|^2 d\Gamma dt$.

For every $t \geq 1$, let us define the following partition of Γ_1:

$$\begin{cases}
\Gamma_{1,1} = \{x \in \Gamma_1 : \ |u_{\nu t}| \leq h_1(t)\}, \\
\Gamma_{1,2} = \{x \in \Gamma_1 : \ h_1(t) < |u_{\nu t}| \leq h_1(1)\}, \\
\Gamma_{1,3} = \{x \in \Gamma_1 : \ |u_{\nu t}| > h_1(1)\},
\end{cases} \tag{5.96}$$

where $\Gamma_{1,1}, \Gamma_{1,2}, \Gamma_{1,3}$ depend on $t \in \mathrm{R}^+$ and $h_1(t) = g_1^{-1}(\phi'(t))$ is a decreasing positive function and satisfies $h_1(t) \to 0$, as $t \to +\infty$.

Estimate of $\Gamma_{1,3}:$ we note that $h_1(1) = 0 \Leftrightarrow g_1^{-1}(\phi'(1)) = 0 \Leftrightarrow \phi'(1) = g_1(0) = 0$. But, if $\phi'(1) = 0$, we have for $t \geq 0$ that $\phi'(t) \leq \phi'(1) = 0$. Consequently, $\phi'(t) = 0$, $\forall\ t \geq 1$, which contradicts the fact that ϕ is a strictly increasing function. Thus, we have $h_1(1) > 0$.

If $h_1(1) > 1$, we obtain from the hypotheses $(A2)$ that $|f_1(u_{\nu t})| \geq C_{11}|u_{\nu t}|$.

If $h_1(1) \leq 1$, we observe that the function $F:\ y \to \frac{f_1(y)}{y}$ is a positive and continuous on $[-1, -h_1(1)] \cup\ [h_1(1), 1]$, which implies that there exists a constant $\beta_1 > 0$ such that $\frac{f_1(y)}{y} \geq \beta_1$, that means $|f_1(u_{\nu t})| \geq \beta_1 |u_{\nu t}|$.

So we conclude that $|u_{\nu t}| \leq \frac{1}{d_0}|f_1(u_{\nu t})|$, where $d_0 = \min\{C_{11}, \beta_1\}$. Therefore, we have

$$\begin{aligned}
\int_S^T \phi' \int_{\Gamma_{1,3}} |u_{\nu t}|^2 d\Gamma dt &\leq \frac{1}{d_0}\int_S^T \phi' \int_{\Gamma_{1,3}} |u_{\nu t}||f_1(u_{\nu t})| d\Gamma dt \\
&\leq \frac{1}{d_0}\phi'(S)\int_S^T \int_{\Gamma_{1,3}} u_{\nu t} f_1(u_{\nu t}) d\Gamma dt \\
&\leq \frac{1}{d_0}\phi'(S)\int_S^T -E'(t)dt \leq \frac{\phi'(S)}{d_0}E(S).
\end{aligned} \tag{5.97}$$

Estimate of $\Gamma_{1,2}$: considering that g_1 is an increasing function, it follows that $\phi'(t) = g_1(h_1(t)) \leq g_1(|u_{\nu t}|) \leq |g_1(u_{\nu t})|$.

If $h_1(1) < 1$, we deduce that $|u_{\nu t}| < 1$. By the hypotheses $(A2)$, we get that $|g_1(u_{\nu t})| \leq |f_1(u_{\nu t})| \leq |g_1^{-1}(u_{\nu t})|$. Thus, it follows that $|u_{\nu t}|^2|g_1(u_{\nu t})| \leq |u_{\nu t}|^2|f_1(u_{\nu t})| \leq |u_{\nu t}||f_1(u_{\nu t})|$.

If $h_1(1) \geq 1$ and $|u_{\nu t}| \in [1, h_1(1)]$, we have that $-h_1(1) \leq u_{\nu t} \leq h_1(1)$. Since g_1 is an increasing and odd function, it is found that $|g_1(u_{\nu t})| \leq |g_1(h_1(1))|$. Thus, taking into account that $C_{11}|u_{\nu t}| \leq |f_1(u_{\nu t})|$, we have

$$\frac{1}{g_1(h_1(1))} \leq \frac{1}{|g_1(u_{\nu t})|} \leq \frac{|f_1(u_{\nu t})|}{C_{11}|u_{\nu t}||g_1(u_{\nu t})|} = \frac{|f_1(u_{\nu t})||u_{\nu t}|}{C_{11}|u_{\nu t}|^2|g_1(u_{\nu t})|}, \tag{5.98}$$

which implies that

$$|u_{\nu t}|^2|g_1(u_{\nu t})| \leq \frac{g_1(h_1(1))}{C_{11}} u_{\nu t} f_1(u_{\nu t}).$$

Hence, we discover that $|u_{\nu t}|^2|g_1(u_{\nu t})| \leq d_1 u_{\nu t} f_1(u_{\nu t})$, where $d_1 = \min\{1, \frac{g_1(h_1(1))}{C_{11}}\}$. Furthermore, taking into account that $|\phi'(t)| = |g_1(h_1(t))| \leq |g_1(u_{\nu t})|$, we obtain that

$$\begin{aligned}\int_S^T \phi' \int_{\Gamma_{1,2}} |u_{\nu t}|^2 d\Gamma dt &\leq \int_S^T \int_{\Gamma_{1,2}} |u_{\nu t}|^2 |g_1(u_{\nu t})| d\Gamma dt \\ &\leq d_1 \int_S^T \int_{\Gamma_{1,2}} |u_{\nu t}||f_1(u_{\nu t})| d\Gamma dt \\ &\leq d_1 \int_S^T \int_{\Gamma_1} |u_{\nu t}||f_1(u_{\nu t})| d\Gamma dt \\ &\leq d_1 \int_S^T -E'(t) dt \leq d_1 E(S). \end{aligned} \tag{5.99}$$

Estimate of $\Gamma_{1,1}$: thanks to the definition of this part of the boundary, we have that

$$\begin{aligned}\int_S^T \phi' \int_{\Gamma_{1,1}} |u_{\nu t}|^2 d\Gamma dt &\leq \int_S^T \int_{\Gamma_{1,1}} |h_1(t)|^2 d\Gamma dt \\ &\leq meas(\Gamma) \int_S^T \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt. \end{aligned} \tag{5.100}$$

Therefore, in view of (5.97)-(5.100), there appears the relation

$$\int_S^T \phi' \int_{\Gamma_1} |u_{\nu t}|^2 d\Gamma dt \leq L_1 E(S) + L_2 \int_S^T \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt, \tag{5.101}$$

where L_1, L_2 are positive constants.

Analysis of $J_2 = \int_S^T \phi' \int_{\Gamma_1} |u_t|^2 d\Gamma dt$.

For every $t \geq 1$, let us define the following partition of Γ_1:

$$\begin{cases} \Gamma_{1,4} = \{x \in \Gamma_1 : \ |u_{\nu t}| \leq h_1(t)\}, \\ \Gamma_{1,5} = \{x \in \Gamma_1 : \ h_1(t) < |u_{\nu t}| \leq h_1(1)\}, \\ \Gamma_{1,6} = \{x \in \Gamma_1 : \ |u_{\nu t}| > h_1(1)\}, \end{cases} \tag{5.102}$$

where $\Gamma_{1,4}, \Gamma_{1,5}, \Gamma_{1,6}$ depend on $t \in \mathrm{R}^+$ and $h_2(t) = g_2^{-1}(\phi'(t))$ is a decreasing positive function which satisfies $h_2(t) \to 0$, as $t \to +\infty$.

By a straightforward adaptation of the above result (5.101), we also obtain that

$$\int_S^T \phi' \int_{\Gamma_1} |u_t|^2 d\Gamma dt \le L_3 E(S) + L_4 \int_S^T \phi'(t)(g_2^{-1}(\phi'(t)))^2 dt, \tag{5.103}$$

where L_3, L_4 are positive constants.

Analysis of $J_3 = \int_S^T \phi' \int_{\Gamma_1} |f_1(u_{\nu t})|^2 d\Gamma dt$.

For every $t \ge 1$, let us define the following partition of Γ_1:

$$\begin{cases} \Gamma_{1,7} = \{x \in \Gamma_1 : \ |u_{\nu t}| \le \phi'(t)\}, \\ \Gamma_{1,8} = \{x \in \Gamma_1 : \ \phi'(t) < |u_{\nu t}| \le \phi'(1)\}, \\ \Gamma_{1,9} = \{x \in \Gamma_1 : \ |u_{\nu t}| > \phi'(1)\}, \end{cases} \tag{5.104}$$

where $\Gamma_{1,7}, \Gamma_{1,8}, \Gamma_{1,9}$ depend on $t \in \mathrm{R}^+$.

Estimate of $\Gamma_{1,9}$: if $\phi'(1) = 0$, we have that for all $t \ge 1$, $\phi'(t) \le \phi'(1) = 0$. Consequently, $\phi'(t) = 0$, $\forall\ t \ge 1$, which contradicts the fact that $\phi(t)$ is a strictly increasing function. Then, $\phi'(1) > 0$.

If $\phi'(1) > 1$, we obtain from the hypotheses $(A2)$ that $|f_1(u_{\nu t})| \le C_{12}|u_{\nu t}|$.

If $\phi'(1) \le 1$, we observe that the function $F : \ y \to \frac{f_1(y)}{y}$ is a positive and continuous on $[-1, -\phi'(1)] \cup [\phi'(1), 1]$ which implies that there exists a constant $\beta_2 > 0$ such that $\frac{f_1(y)}{y} \le \beta_2$, that means $|f_1(u_{\nu t})| \le \beta_2 |u_{\nu t}|$.

We conclude that $|f_1(u_{\nu t})| \le d_3 |u_{\nu t}|$, where $d_3 = \max\{C_{12}, \beta_2\}$. Therefore, we have that

$$\begin{aligned} \int_S^T \phi' \int_{\Gamma_{1,9}} |f_1(u_{\nu t})|^2 d\Gamma dt &\le d_3 \int_S^T \phi' \int_{\Gamma_{1,9}} |u_{\nu t}||f_1(u_{\nu t})| d\Gamma dt \\ &\le d_3 \phi'(S) \int_S^T \int_{\Gamma_{1,9}} u_{\nu t} f_1(u_{\nu t}) d\Gamma dt \\ &\le d_3 \phi'(S) \int_S^T -E'(t) dt \le d_3 \phi'(S) E(S). \end{aligned} \tag{5.105}$$

Estimate of $\Gamma_{1,8}$: considering the monotonicity of f_1, $f_1(\phi'(t)) < f_1(|u_{\nu t}|) \le f_1(\phi'(1))$, and the boundary conditions of this part, we discover that

$$\begin{aligned} \int_S^T \phi' \int_{\Gamma_{1,8}} |f_1(u_{\nu t})|^2 d\Gamma dt &\le f_1(\phi'(1)) \int_S^T \int_{\Gamma_{1,8}} |u_{\nu t}||f_1(u_{\nu t})| d\Gamma dt \\ &\le C \int_S^T \int_{\Gamma_{1,8}} u_{\nu t} f_1(u_{\nu t}) d\Gamma dt \le C E(S). \end{aligned} \tag{5.106}$$

Estimate of $\Gamma_{1,7}$: if $\phi'(1) \le 1$, we have from the hypotheses $(A2)$ that $|f_1(u_{\nu t})| \le |g_1^{-1}(u_{\nu t})| \le |g_1^{-1}(\phi'(t))|$. Then,

$$\begin{aligned} \int_S^T \phi' \int_{\Gamma_{1,7}} |f_1(u_{\nu t})|^2 d\Gamma dt &\le \int_S^T \phi' \int_{\Gamma_{1,7}} |g_1^{-1}(u_{\nu t})|^2 d\Gamma dt \\ &\le \int_S^T \phi' \int_{\Gamma_{1,7}} |g_1^{-1}(\phi'(t))|^2 d\Gamma dt \\ &\le meas(\Gamma) \int_S^T \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt. \end{aligned} \tag{5.107}$$

If $\phi'(1) > 1$, then $|u_{\nu t}| \in [1, \phi'(t)]$. From the hypotheses $(A2)$, we obtain $|f_1(u_{\nu t})| \leq C_{12}|u_{\nu t}|$, and

$$\begin{aligned}\int_S^T \phi' \int_{\Gamma_{1,7}} |f_1(u_{\nu t})|^2 d\Gamma dt &\leq C_{12} \int_S^T \phi' \int_{\Gamma_{1,7}} |u_{\nu t}||f_1(u_{\nu t})| d\Gamma dt \\ &\leq C_{12}\phi'(S) \int_S^T \int_{\Gamma_{1,7}} u_{\nu t} f_1(u_{\nu t}) d\Gamma dt \\ &\leq C_{12}\phi'(S) \int_S^T -E'(t)dt \leq \phi'(S) C_{12} E(S). \end{aligned} \tag{5.108}$$

Therefore, combining (5.105)-(5.108), we have

$$\int_S^T \phi' \int_{\Gamma_1} |f_1(u_{\nu t})|^2 d\Gamma dt \leq L_5 E(S) + L_6 \int_S^T \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt, \tag{5.109}$$

where L_5, L_6 are positive constants.

Analysis of $J_4 = \int_S^T \phi' \int_{\Gamma_1} |f_2(u_t)|^2 d\Gamma dt$.

For every $t \geq 1$, let us define the following partition of Γ_1:

$$\begin{cases} \Gamma_{1,10} = \{x \in \Gamma_1 : \ |u_t| \leq \phi'(t)\}, \\ \Gamma_{1,11} = \{x \in \Gamma_1 : \ \phi'(t) < |u_t| \leq \phi'(1)\}, \\ \Gamma_{1,12} = \{x \in \Gamma_1 : \ |u_t| > \phi'(1)\}, \end{cases} \tag{5.110}$$

where $\Gamma_{1,10}, \Gamma_{1,11}, \Gamma_{1,12}$ depend on $t \in \mathrm{R}^+$.

Using the analogous arguments as (5.109), we obtain

$$\int_S^T \phi' \int_{\Gamma_1} |f_2(u_t)|^2 d\Gamma dt \leq L_7 E(S) + L_8 \int_S^T \phi'(t)(g_2^{-1}(\phi'(t)))^2 dt, \tag{5.111}$$

where L_7, L_8 are positive constants.

Inserting (5.101), (5.103), (5.109) and (5.111) into inequality (5.95), it follows that

$$\begin{aligned}\int_S^T E^2(t)\phi'(t)dt &\leq \widetilde{C}_1' E^2(S) + \widetilde{C}_2' E^2(S)\phi'(S) \\ &+ \widetilde{C}_3' E(S) \int_S^T \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt \\ &+ \widetilde{C}_4' E(S) \int_S^T \phi'(t)(g_2^{-1}(\phi'(t)))^2 dt. \end{aligned} \tag{5.112}$$

Now assume that ϕ satisfies the following additional properties:

$$\int_1^\infty \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt \ \text{ and } \ \int_1^\infty \phi'(t)(g_2^{-1}(\phi'(t)))^2 dt \text{ are convergence.} \tag{5.113}$$

These properties are closely related to the behavior of g_i $(i = 1, 2)$ near 0 and the decay rate of ϕ' at infinity. Thus, we deduce from (5.112) and (5.113) that there exists positive constants C such that

$$\begin{aligned}\forall\ S \geq 1, \ \int_S^\infty E^2(t)\phi'(t)dt &\leq CE^2(S) + CE(S) \int_S^\infty \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt \\ &+ CE(S) \int_S^\infty \phi'(t)(g_2^{-1}(\phi'(t)))^2 dt. \end{aligned} \tag{5.114}$$

Next, the main problem is to find a strictly increasing function, which satisfies the following conditions:

$$\phi(t) \text{ is concave and } \phi(t) \to +\infty \text{ as } t \to +\infty,$$

$$\phi'(t) \to 0 \text{ as } t \to +\infty,$$

$$\int_1^\infty \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt \text{ and } \int_1^\infty \phi'(t)(g_2^{-1}(\phi'(t)))^2 dt \text{ are convergence.}$$

We consider, without loss of generality, that $\phi(1) = 1$. From this fact, observing that $\phi^{-1}(t) \to +\infty$ as $t \to +\infty$ and taking into account the change of variable $\tau = \phi(t)$, one have that

(5.115)
$$\int_1^\infty \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt = \int_1^\infty (g_1^{-1}(\phi'(\phi^{-1}(\tau))))^2 d\tau = \int_1^\infty (g_1^{-1}(\frac{1}{(\phi^{-1})'(\tau)}))^2 d\tau,$$

(5.116)
$$\int_1^\infty \phi'(t)(g_2^{-1}(\phi'(t)))^2 dt = \int_1^\infty (g_2^{-1}(\phi'(\phi^{-1}(\tau))))^2 d\tau = \int_1^\infty (g_2^{-1}(\frac{1}{(\phi^{-1})'(\tau)}))^2 d\tau.$$

Let us define the auxiliary functions ψ_1, ψ_2 on $[1, \infty)$ by

$$\forall\ t \geq 1,\ \psi_1(t) = \frac{1}{2} + \int_1^t \frac{1}{g_1(\frac{1}{\tau})} d\tau, \tag{5.117}$$

$$\forall\ t \geq 1,\ \psi_2(t) = \frac{1}{2} + \int_1^t \frac{1}{g_2(\frac{1}{\tau})} d\tau. \tag{5.118}$$

Then ψ_1, ψ_2 are strictly increasing functions of C^2 on $[1, \infty)$ and satisfy

$$\forall\ t \geq 1,\ \psi_1'(t) = \frac{1}{g_1(\frac{1}{t})} \to +\infty,\ \text{as } t \to +\infty, \tag{5.119}$$

$$\forall\ t \geq 1,\ \psi_2'(t) = \frac{1}{g_2(\frac{1}{t})} \to +\infty,\ \text{as } t \to +\infty, \tag{5.120}$$

and

(5.121)
$$\int_1^\infty (g_1^{-1}(\frac{1}{\psi_1'(\tau) + \psi_2'(\tau)}))^2 d\tau \leq \int_1^\infty (g_1^{-1}(\frac{1}{\psi_1'(\tau)}))^2 d\tau = \int_1^\infty \frac{1}{\tau^2} d\tau < +\infty,$$

(5.122)
$$\int_1^\infty (g_2^{-1}(\frac{1}{\psi_1'(\tau) + \psi_2'(\tau)}))^2 d\tau \leq \int_1^\infty (g_2^{-1}(\frac{1}{\psi_2'(\tau)}))^2 d\tau = \int_1^\infty \frac{1}{\tau^2} d\tau < +\infty.$$

By a direct computation, we can show that $\psi_1''(t), \psi_2''(t) \geq 0$ which imply that ψ_1', ψ_2' are non-increasing functions and ψ_1, ψ_2 are convex. Furthermore, let $\psi = \psi_1 + \psi_2$, then it is easy to verify that $\psi^{-1}(t)$ is concave on $[1, \infty)$. Take two derivatives of the expression $\psi(\psi^{-1}(t)) = t$, we deduce that

$$(\psi^{-1})''(t) = -\frac{\psi''(\psi^{-1}(t))((\psi^{-1})'(t))^2}{\psi'(\psi^{-1}(t))} = -\frac{\psi''(\psi^{-1}(t))}{(\psi'(\psi^{-1}(t)))^3} \leq 0. \tag{5.123}$$

That is why we define ϕ on $[1, \infty)$ by

$$\phi(t) = (\psi_1 + \psi_2)^{-1}(t) = (\psi)^{-1}(t). \tag{5.124}$$

Thus ϕ is a strictly increasing concave function of class C^2 on $[1,\infty)$, which satisfies all the assumptions we made in our computation. In addition, we deduce that

$$\begin{aligned}\phi'(t) &= \frac{1}{\psi'(\phi(t))} = \frac{1}{\psi_1'(\phi(t)) + \psi_2'(\phi(t))}\\ &= \frac{1}{\frac{1}{g_1(\frac{1}{\phi(t)})} + \frac{1}{g_2(\frac{1}{\phi(t)})}} = \frac{g_1(\frac{1}{\phi(t)})g_2(\frac{1}{\phi(t)})}{g_1(\frac{1}{\phi(t)}) + g_2(\frac{1}{\phi(t)})} \to 0. \end{aligned} \tag{5.125}$$

Note that $\phi(1) = 1$, because $\psi(1) = \psi_1(1) + \psi_2(1) = 1$, and $\phi'(1) = \frac{g_1(1)g_2(1)}{g_1(1)+g_2(1)} \leq \frac{1}{2}$, so it is easy to extend ϕ on $[0,\infty)$ such that it remains a concave and strictly increasing nonnegative function on $[0,\infty)$. Thus, we have explicitly constructed a function ϕ that satisfies the all the conditions of Lemma 2.3.

Furthermore, we deduce from the (5.114) that

$$\begin{aligned}\int_S^\infty E^2(t)\phi'(t)dt &\leq CE^2(S) + CE(S)\int_{\phi(S)}^\infty \phi'(t)(g_1^{-1}(\phi'(t)))^2 dt\\ &\quad + CE(S)\int_S^\infty \phi'(t)(g_2^{-1}(\phi'(t)))^2 dt\\ &= CE^2(S) + CE(S)\int_{\phi(S)}^\infty \left(g_1^{-1}\left(\phi'(\phi^{-1}(t)\right)\right)^2 dt\\ &\quad + CE(S)\int_{\phi(S)}^\infty \left(g_2^{-1}\left(\phi'(\phi^{-1}(t)\right)\right)^2 dt\\ &= CE^2(S) + CE(S)\int_{\phi(S)}^\infty \left(g_1^{-1}\left(\frac{1}{(\phi^{-1})'(\tau)}\right)\right)^2 dt\\ &\quad + CE(S)\int_{\phi(S)}^\infty \left(g_2^{-1}\left(\frac{1}{(\phi^{-1})'(\tau)}\right)\right)^2 dt\\ &\leq CE^2(S) + CE(S)\int_{\phi(S)}^\infty \frac{1}{t^2}dt\\ &= CE^2(S) + \frac{CE(S)}{\phi(S)}, \qquad \forall\ S \geq 1. \end{aligned} \tag{5.126}$$

We define $F(t) = E(t+1)$ and $\Phi(t) = \phi(t+1) - 1$ on $[0,+\infty)$, then (5.126) implies that

$$\int_S^\infty F^2(t)\Phi'(t)dt \leq CF^2(S) + \frac{CF(S)}{1+\Phi(S)}, \qquad \forall\ S \geq 0. \tag{5.127}$$

Noting that the function $F(t)$ and $\Phi(t)$ satisfy all the assumptions of Lemma 2.2 with $\sigma = \sigma' = 1$, so we obtain a decay rate estimate:

$$\forall\ t \geq 0,\ \ F(t) \leq \frac{C}{(1+\Phi(t))^2}. \tag{5.128}$$

By the method of variable substitution, we also obtain

$$\forall\ t \geq 1,\ \ E(t) \leq \frac{C}{\phi^2(t)}, \tag{5.129}$$

where C is depending on $E(1)$ in a continuous way.

Finally, it remains to estimate the growth of ϕ. Setting τ_0 such that

$$g_1\left(\frac{1}{\tau_0}\right) \leq 2 \quad \text{and} \quad g_2\left(\frac{1}{\tau_0}\right) \leq 2. \tag{5.130}$$

Using the monotonicity of g_i $i = 1, 2$, we have

$$\begin{aligned}
\forall\ \tau \geq \tau_0,\ \psi_1(\tau) &= \frac{1}{2} + \int_1^\tau \frac{1}{g_1(\frac{1}{s})} ds \leq \frac{1}{2} + (\tau - 1)\frac{1}{g_1(\frac{1}{\tau})} \\
&= \frac{1}{2} + \tau\frac{1}{g_1(\frac{1}{\tau})} - \frac{1}{g_1(\frac{1}{\tau})} \leq \tau\frac{1}{g_1(\frac{1}{\tau})},
\end{aligned}$$

$$\begin{aligned}
\forall\ \tau \geq \tau_0,\ \psi_2(\tau) &= \frac{1}{2} + \int_1^\tau \frac{1}{g_2(\frac{1}{s})} ds \leq \frac{1}{2} + (\tau - 1)\frac{1}{g_2(\frac{1}{\tau})} \\
&= \frac{1}{2} + \tau\frac{1}{g_2(\frac{1}{\tau})} - \frac{1}{g_2(\frac{1}{\tau})} \leq \tau\frac{1}{g_2(\frac{1}{\tau})}.
\end{aligned}$$

Hence, we have that

$$\psi_1(\tau) + \psi_2(\tau) = \psi(\tau) \leq \tau\frac{1}{g_1(\frac{1}{\tau})} + \tau\frac{1}{g_2(\frac{1}{\tau})}, \tag{5.131}$$

which implies that

$$\tau \leq \psi^{-1}\left(\tau\frac{1}{g_1(\frac{1}{\tau})} + \tau\frac{1}{g_2(\frac{1}{\tau})}\right) = \phi\left(\tau\frac{1}{g_1(\frac{1}{\tau})} + \tau\frac{1}{g_2(\frac{1}{\tau})}\right). \tag{5.132}$$

Taking into account that

$$t = \tau\frac{1}{g_1(\frac{1}{\tau})} + \tau\frac{1}{g_2(\frac{1}{\tau})} = \tau\frac{g_1(\frac{1}{\tau}) + g_2(\frac{1}{\tau})}{g_1(\frac{1}{\tau})g_2(\frac{1}{\tau})} = \frac{1}{G(\frac{1}{\tau})}, \tag{5.133}$$

so we have

$$\frac{1}{\phi(t)} \leq \frac{1}{\tau} = G^{-1}(\frac{1}{t}), \tag{5.134}$$

where the function $G(y) = y\frac{g_1(y)g_2(y)}{g_1(y)+g_2(y)}$. Thus, we thereby conclude that

$$\forall\ t \geq 1,\ E(t) \leq C\left(G^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{5.135}$$

which completes the proof of Theorem 2.3. □

Acknowledgments

The authors wish to express their gratitude to the Editors and Referees for giving a number of valuable comments and helpful suggestions, which led us to improve the presentation of original manuscript significantly. This work was initiated while Di was visiting Department of Mathematics, University of Texas at Arlington as a Postdoctoral Researcher during the year 2018-2020, who would like to thank the department for its warm hospitality and support.

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Huafei Di, School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
Email address: dihuafei@gzhu.edu.cn

Yadong Shang, School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
Email address: gzydshang@126.com

Jiali Yu, School of Science, Dalian Jiaotong University, Dalian 116028, China
Email address: jialiyu97@yahoo.com