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GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS FOR A WAVE EQUATION WITH NON-CONSTANT DELAY AND NONLINEAR WEIGHTS

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Abstract. We consider the wave equation with a weak internal damping with non-constant delay and nonlinear weights given by

$$u_{tt}(x,t) - u_{xx}(x,t) + \mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau(t)) = 0$$

in a bounded domain. Under proper conditions on nonlinear weights $\mu_1(t), \mu_2(t)$ and non-constant delay $\tau(t)$, we prove global existence and estimative the decay rate for the energy.

1. Introduction

This paper is concerned with the initial boundary value problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau(t)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(0,t) = u(L,t) = 0 & \text{on }]0,+\infty[, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } \Omega, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in } \Omega\times]0,\tau(0)[, \end{cases} \tag{1}$$

where $\Omega =]0, L[$, $0 < \tau(t)$ are a non-constant time delay, $\mu_1(t), \mu_2(t)$ are non-constant weights and the initial data (u_0, u_1, f_0) belong to a suitable function space.

This problem has been first proposed and studied in Nicaise and Pignotti [22] in case of constant coefficients μ_1, μ_2 and constant time delay. Under suitable assumptions, the authors proved the exponential stability of the solution by introducing suitable energies and by using some observability inequalities. Some instability results are also given for the case of the some assumptions is not satisfied.

With a weight depending on time, $\mu_1(t), \mu_2(t)$ and constant time delay, this problem was studied in [2], where the existence of solution was made by Faedo-Galerkin method and a decay rate estimate for the energy was given by using the multiplier method.

W. Liu in [19] studied the weak viscoelastic equation with an internal time varying delay term. By introducing suitable energy and Lyapunov functionals, he establishes a general decay rate estimate for the energy under suitable assumptions.

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F. Tahamtani and A. Peyravi [29] investigated the nonlinear viscoelastic wave equation with source term. Using the Potential well theory they showed that the solutions blow up in finite time under some restrictions on initial data and for arbitrary initial energy.

Global existence and asymptotic behavior of solutions to the viscoelastic wave equation with a constant delay term was considered by M. Remil and A. Hakem in [28].

Global existence and asymptotic stability for a coupled viscoelastic wave equation with time-varying delay was studied in [3] by combining the energy method with the Faedo-Galerkin's procedure.

The stabilization problem by interior damping of the wave equation with boundary or internal time-varying delay was studied in [23] by introducing suitable Lyapunov functionals.

Energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks was considered in [11].

For problems with delay in different contexts we cite [9, 10, 30, 32] with references therein. In absence of delay ($\mu_2(t) = 0$), the problem (1) is exponentially stable provided that $\mu_1(t)$ is constant, see, for instance [5, 6, 16, 17, 21] and reference therein.

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect, and the central question is that delays source can destabilize a system that is asymptotically stable in the absence of delays, see [7]. In fact, an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used, see for example [8, 12, 31]

By energy method in [24] was studied the stabilization of the wave equation with boundary or internal distributed delay. By semigroup approach in [27] was proved the well-posedness and exponential stability for a wave equation with frictional damping and nonlocal time-delayed condition. Transmission problem with distributed delay was studied in [18] where was established the exponential stability of the solution by introducing a suitable Lyapunov functional.

Here we consider a wave equation with non-constant delay and nonlinear weights, thus, the present paper is a generalization of the previous ones. The remaining part of this paper is organized as follows. In the section 2 we introduce some notations and prove the dissipative property of the full energy of the system. In the section 3, for an approach combining semigroup theory (see [21] and [4]) with the energy estimate method we prove the existence and uniqueness of solution. In section 4 we present the result of exponential stability.

2. Notation and preliminaries

We will need the following hypotheses:

(H1) $\mu_1 : \mathbb{R}_+ \to]0, +\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\left|\frac{\mu_1'(t)}{\mu_1(t)}\right| \leq M_1, \quad 0 < \alpha_0 \leq \mu_1(t), \quad \forall t \geq 0, \tag{2}$$

where α_0 and M_1 are constants such that $M_1 > 0$.

(H2) $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$,which is not necessarily positives or monotones, such that

$$|\mu_2(t)| \leq \beta\mu_1(t), \tag{3}$$

$$|\mu_2'(t)| \le M_2\mu_1(t), \tag{4}$$

for some $0 < \beta < \sqrt{1-d}$ and $M_2 > 0$.

We now state a lemma needed later.

Lemma 2.1 (Sobolev-Poincare's inequality)**.** *Let q be a number with $2 \le q \le +\infty$. Then there is a constant $c_* = c_*(]0,L[,q)$ such that*

$$\|\Psi\|_q \le c_* \|\Psi_x\|_2\,, \quad \textit{for } \Psi \in H_0^1(]0,L[).$$

Lemma 2.2 ([13][16])**.** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and assume that there are two constants $\sigma > -1$ and $\omega > 0$ such that*

$$\int_S^{+\infty} E^{1+\sigma}(t)\,dt \le \frac{1}{\omega}E^{\sigma}(0)E(S),\ \ 0 \le S < +\infty.$$

Then

$$\begin{aligned} &E(t) = 0\ \forall t \ge \frac{E^{\sigma}(0)}{\omega|\sigma|}, \quad \textit{if } -1 < \sigma < 0,\\ &E(t) \le E(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}} \quad \forall t \ge 0, \quad \textit{if } \sigma > 0,\\ &E(t) \le E(0)e^{1-\omega t}\ \forall t \ge 0, \quad \textit{if } \sigma = 0. \end{aligned}$$

As in [23], we assume that

$$\tau(t) \in W^{2,+\infty}([0,T]), \quad \text{for } T > 0 \tag{5}$$

and there exist positive constants τ_0, τ_1 and d satisfying

$$0 < \tau_0 \le \tau(t) \le \tau_1,\ \forall t > 0 \tag{6}$$

and

$$\tau'(t) \le d < 1,\ \forall t > 0. \tag{7}$$

We introduce the new variable

$$z(x,\rho,t) = u_t(x, t - \tau(t)\rho),\ \ x \in \Omega, \rho \in]0,1[, t > 0. \tag{8}$$

Then

$$\tau(t)z_t(x,\rho,t) + (1 - \tau'(t)\rho)z_\rho(x,\rho,t) = 0,\ \ x \in \Omega,\ \ \rho \in]0,1[,\ \ t > 0$$

and problem (1) takes the form

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \mu_1(t)u_t(x,t) + \mu_2(t)z(x,1,t) = 0 & \text{in } \Omega\times]0,+\infty[,\\ \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0 & \text{in } \Omega\times]0,1[\times]0,+\infty[,\\ u(0,t) = u(L,t) = 0 & \text{on }]0,+\infty[,\\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } \Omega,\\ z(x,\rho,0) = u_t(x,-\tau(0)\rho) = f_0(x,-\tau(0)\rho) & \text{in } \Omega\times]0,1[. \end{cases} \tag{9}$$

We define the energy of the solution of problem (9) by

$$E(t) = \frac{1}{2}\|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_x\|^2_{L^2(\Omega)} + \frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx, \tag{10}$$

where

$$\xi(t) = \bar{\xi}\mu_1(t) \tag{11}$$

is a non-increasing function of class $C^1(\mathbb{R}_+)$ and $\bar{\xi}$ be a positive constant such that

$$\frac{\beta}{\sqrt{1-d}} < \bar{\xi} < 2 - \frac{\beta}{\sqrt{1-d}}. \tag{12}$$

Our first result states that the energy is a non-increasing function.

Lemma 2.3. *Let (u, z) be a solution to the problem (9). Then, the energy functional defined by (10) satisfies*

$$\begin{aligned} E'(t) &\leq -\mu_1(t)\left(1 - \frac{\bar{\xi}}{2} - \frac{\beta}{2\sqrt{1-d}}\right)\|u_t\|^2_{L^2(\Omega)} \\ &\quad - \mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2} - \frac{\beta\sqrt{1-d}}{2}\right)\|z(x,1,t)\|^2_{L^2(\Omega)} \\ &\leq 0. \end{aligned} \tag{13}$$

Proof. Multiplying the first equation (9) by $u_t(x,t)$, integrating on Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2_{L^2(\Omega)} + \|u_x\|^2_{L^2(\Omega)}\right) + \mu_1(t)\|u_t\|^2_{L^2(\Omega)} + \mu_2(t)\int_\Omega z(x,1,t)u_t\,dx. \tag{14}$$

Now multiplying the second equation (9) by $\xi(t)z(x,\rho,t)$ and integrate on $\Omega\times]0,1[$, to obtain

$$\tau(t)\xi(t)\int_\Omega\int_0^1 z_t(x,\rho,t)z(x,\rho,t)\,d\rho\,dx = -\frac{\xi(t)}{2}\int_\Omega\int_0^1 (1-\tau'(t)\rho)\frac{\partial}{\partial\rho}(z(x,\rho,t))^2\,d\rho\,dx.$$

Consequently,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right) &= -\frac{\xi(t)}{2}\int_\Omega\int_0^1 (1-\tau'(t)\rho)\frac{\partial}{\partial\rho}(z(x,\rho,t))^2\,d\rho\,dx \\ &\quad + \frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx \\ &= \frac{\xi(t)}{2}\int_\Omega (z^2(x,0,t) - z^2(x,1,t))\,dx \\ &\quad + \frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,1,t)\,d\rho\,dx \\ &\quad + \frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx. \end{aligned} \tag{15}$$

From (10), (14) and (15) we obtain

$$\begin{aligned} E'(t) &= \frac{\xi(t)}{2}\|u_t\|^2_{L^2(\Omega)} - \frac{\xi(t)}{2}\|z(x,1,t)\|^2_{L^2(\Omega)} \\ &\quad + \frac{\xi(t)\tau'(t)}{2}\|z(x,1,t)\|^2_{L^2(\Omega)} + \frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx \\ &\quad - \mu_1(t)\|u_t\|^2_{L^2(\Omega)} - \mu_2(t)\int_\Omega z(x,1,t)u_t\,dx. \end{aligned} \tag{16}$$

Due to Young's inequality, we have

$$\mu_2(t)\int_\Omega z(x,1,t)u_t\,dx \leq \frac{|\mu_2(t)|}{2\sqrt{1-d}}\|u_t\|^2_{L^2(\Omega)} + \frac{|\mu_2(t)|\sqrt{1-d}}{2}\|z(x,1,t)\|^2_{L^2(\Omega)}. \tag{17}$$

Inserting (17) into (16), we obtain

$$
\begin{aligned}
E'(t) \le & - \left(\mu_1(t) - \frac{\xi(t)}{2} - \frac{|\mu_2(t)|}{2\sqrt{1-d}} \right) \|u_t\|^2_{L^2(\Omega)} \\
& - \left(\frac{\xi(t)}{2} - \frac{\xi(t)\tau'(t)}{2} - \frac{|\mu_2(t)|\sqrt{1-d}}{2} \right) \|z(x,1,t)\|^2_{L^2(\Omega)} \\
& + \frac{\xi'(t)\tau(t)}{2} \int_\Omega \int_0^1 z^2(x,\rho,t)\, d\rho\, dx \\
\le & -\mu_1(t) \left(1 - \frac{\bar{\xi}}{2} - \frac{\beta}{2\sqrt{1-d}} \right) \|u_t\|^2_{L^2(\Omega)} \\
& - \mu_1(t) \left(\frac{\bar{\xi}(1-\tau'(t))}{2} - \frac{\beta\sqrt{1-d}}{2} \right) \|z(x,1,t)\|^2_{L^2(\Omega)} \\
\le & \ 0.
\end{aligned}
$$

□

Lemma 2.4. *Let (u,z) be a solution to the problem (9). Then the energy functional defined by (10) satisfies*

$$
\|u_t(x,t)\|^2_{L^2(\Omega)} < -\frac{1}{\sigma} E'(t),
$$

where $\sigma = a_0 \left(1 - \frac{\bar{\xi}}{2} - \frac{\beta}{2\sqrt{1-d}} \right)$.

Proof. From Lemma 2.3, we have that

$$
\begin{aligned}
-E'(t) \ge & \ \mu_1(t) \left(1 - \frac{\bar{\xi}}{2} + \frac{\beta}{2\sqrt{1-d}} \right) \|u_t\|^2_{L^2(\Omega)} \\
& + \mu_1(t) \left(\frac{\bar{\xi}(1-\tau'(t))}{2} + \frac{\beta\sqrt{1-d}}{2} \right) \|z(x,1,t)\|^2_{L^2(\Omega)} \\
\ge & \ 0
\end{aligned}
$$

and from (H1), we obtain

$$
\begin{aligned}
0 \le & \ a_0 \left(1 - \frac{\bar{\xi}}{2} + \frac{\beta}{2\sqrt{1-d}} \right) \|u_t\|^2_{L^2(\Omega)} \\
\le & \ \mu_1(t) \left(1 - \frac{\bar{\xi}}{2} + \frac{\beta}{2\sqrt{1-d}} \right) \|u_t\|^2_{L^2(\Omega)} \\
\le & -E'(t)
\end{aligned}
$$

and the lemma is proved. □

3. Global solution

For the semigroup setup we $U = (u, u_t, z)^T$ and rewrite (9) as

$$
\left\{
\begin{array}{l}
U_t = \mathcal{A}(t)U, \\
U(0) = U_0 = (u_0, u_1, f_0(\cdot, -, \tau(0)))^T,
\end{array}
\right. \tag{18}
$$

where the operator $\mathcal{A}(t)$ is defined by

$$
\mathcal{A}U = \left(v, u_{xx} - \mu_1(t)v - \mu_2(t)z(x,1,t), -\frac{1-\tau'(t)\rho}{\tau(t)} z_\rho(x,\rho,t) \right)^T. \tag{19}
$$

We introduce the phase space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times]0,1[)$$

and the domain of $\mathcal{A}$ is defined by

$$D(\mathcal{A}(t)) = \left\{(u,v,z)^T \in H / v = z(\cdot,0) \text{ in } \Omega\right\}, \tag{20}$$

where

$$H = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega; H_0^1(]0,1[)).$$

Notice that the domain of the operator $\mathcal{A}(t)$ is independent of the time t, i.e.,

$$D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \quad \forall t > 0. \tag{21}$$

$\mathcal{H}$ is a Hilbert space provided with the inner product

$$\langle U, \bar{U}\rangle_{\mathcal{H}} = \int_\Omega u_x \bar{u}_x \, dx + \int_\Omega v\bar{v}\, dx + \xi(t)\tau(t) \int_\Omega \int_0^1 z\bar{z}\, d\rho\, dx, \tag{22}$$

for $U = (u,v,z)^T$ and $\bar{U} = (\bar{u},\bar{v},\bar{z})^T$.

Using this time-dependent inner product and the next theorem we will get a result of existence and uniqueness.

Theorem 3.1. *Assume that*

(i) $Y = D(\mathcal{A}(0))$ *is dense subset of* $\mathcal{H}$,
(ii) (21) *holds,*
(iii) for all $t \in [0,T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and the family* $\mathcal{A}(t) = \{\mathcal{A}(t)/t \in [0,T]\}$ *is stable with stability constants* C *and* m *independent of* t *(i.e., the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $\mathcal{A}(t)$ *satisfies* $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$, *for all* $u \in \mathcal{H}$ *and* $s \geq 0$*),*
(iv) $\partial_t \mathcal{A}(t)$ *belongs to* $L_*^\infty([0,T], B(Y,\mathcal{H}))$, *which is the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0,T]$ *into the set* $B(Y,\mathcal{H})$ *of bounded operators from* Y *into* $\mathcal{H}$.

Then, problem (18) *has a solution* $U \in C([0,T],Y) \cap C^1([0,T],\mathcal{H})$ *for any initial datum in* Y.

Our goal is then to check the above assumptions for problem (18).

First, we prove $D(\mathcal{A}(0))$ is dense in $\mathcal{H}$.

The proof is the same as the one Lemma 2.2 of [25], we give it for the sake of completeness.

Let $(f,g,h)^T$ be orthogonal to all elements of $D\mathcal{A}(0)$, namely

$$0 = \langle (u,v,z)^T, (f,g,h)^T\rangle_{\mathcal{H}} = \int_\Omega u_x f_x\, dx + \int_\Omega vg\, dx + \xi(t)\tau(t)\int_\Omega\int_0^1 zh\, d\rho\, dx,$$

for all $(u,v,z)^T \in D(\mathcal{A}(0))$.

We first take $u = v = 0$ and $z \in D(\Omega\times]0,1[)$. As $(0,0,z)^T \in D(\mathcal{A}(0))$, we get

$$\int_\Omega \int_0^1 zh\, d\rho\, dx = 0.$$

Since $D(\Omega\times]0,1[)$ is dense in $L^2(\Omega\times]0,1[)$, we deduce that $h = 0$. In the same manner, by taking $u = z = 0$ e $v \in D(\Omega)$ we see that $g = 0$.

The above orthogonality condition is then reduced to

$$0 = \int_\Omega u_x f_x\, dx, \quad \forall (u,v,z)^T \in D(\mathcal{A}(0)).$$

By restricting ourselves to $v = 0$ and $z = 0$, we obtain

$$0 = \int_\Omega u_x f_x \, dx, \quad \forall (u, 0, 0)^T \in D(\mathcal{A}(0)).$$

Since $D(\Omega)$ is dense in $H_0^1(\Omega)$ (equipped with the inner product $\langle \cdot, \cdot \rangle_{H_0^1(\Omega)}$), we deduce that $f = 0$.

We consequently

$$D(\mathcal{A}(0) \text{ is dense in } \mathcal{H}. \tag{23}$$

Secondly, we notice that

$$\frac{\|\Phi\|_t}{\|\Phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \quad \forall t, s \in [0, T], \tag{24}$$

where $\Phi = (u, v, z)^T$ and c is a positive constant and $\| \cdot \|$ is the norm associated the inner product (22). For all $t, s \in [0, T]$, we have

$$\begin{aligned}\|\Phi\|_t^2 - \|\Phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} = &\left(1 - e^{\frac{c}{2\tau_0}|t-s|}\right) \left(\|u_x\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2\right) \\ &+ \left(\xi(t)\tau(t) - \xi(s)\tau(s) e^{\frac{c}{\tau_0}|t-s|}\right) \int_\Omega \int_0^1 z^2(x, \rho, t) \, d\rho \, dx.\end{aligned}$$

It is clear that $1 - e^{\frac{c}{\tau_0}|t-s|} \leq 0$. Now we will prove $\xi(t)\tau(t) - \xi(s)\tau(s)e^{\frac{c}{\tau_0}|t-s|} \leq 0$ for some $c > 0$. To do this , we have

$$\tau(t) = \tau(s) + \tau'(r)(t - s),$$

where $r \in]s, t[$.

Hence ξ is a non increasing function and $\xi > 0$, we get

$$\xi(t)\tau(t) \leq \xi(s)\tau(s) + \xi(s)\tau'(r)(t - s),$$

which implies

$$\frac{\xi(t)\tau(t)}{\xi(s)\tau(s)} \leq 1 + \frac{|\tau'(r)|}{\tau(s)}|t - s|.$$

Using (5) and τ' is bounded, we deduce that

$$\frac{\xi(t)\tau(t)}{\xi(s)\tau(s)} \leq 1 + \frac{c}{\tau_0}|t - s| \leq e^{\frac{c}{\tau_0}|t-s|},$$

which proves (24) and therefore (iii) follows.

Now we calculate $\langle \mathcal{A}(t)U, U \rangle_t$ for a fixed t. Take $U = (u, v, z)^T \in D(\mathcal{A}(t))$. Then

$$\begin{aligned}\langle \mathcal{A}(t)U, U \rangle_t = &\int_\Omega v_x u_x \, dx + \int_\Omega (u_{xx} - \mu_1(t)v - \mu_2(t)z(\cdot, 1)) \, v \, dx \\ &- \xi(t) \int_\Omega \int_0^1 (1 - \tau'(t)\rho) \, z_\rho(x, \rho) z(x, \rho) \, d\rho \, dx.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\langle \mathcal{A}(t)U, U \rangle_t = &- \mu_1(t)\|v\|_{L^2(\Omega)}^2 - \mu_2(t) \int_\Omega z(\cdot, 1)v \, dx \\ &- \int_\Omega \int_0^1 (1 - \tau'(t)\rho) \, \frac{\partial}{\partial \rho} z^2(x, \rho) \, d\rho \, dx.\end{aligned}$$

Since

$$(1 - \tau'(t)\rho) \, \frac{\partial}{\partial \rho} z^2(x, \rho) = \frac{\partial}{\partial \rho} \left((1 - \tau'(t)\rho) \, z^2(x, \rho)\right) + \tau'(t) z^2(x, \rho),$$

we have

$$\int_0^1 (1-\tau'(t)\rho)\,\frac{\partial}{\partial\rho}z^2(x,\rho)\,d\rho = (1-\tau'(t))\,z^2(x,1) - z^2(x,0) + \tau'(t)\int_0^1 z^2(x,\rho)\,d\rho.$$

So we get

$$\begin{aligned}\langle \mathcal{A}(t)U,U\rangle_t = &-\mu_1(t)\|v\|^2_{L^2(\Omega)} - \mu_2(t)\int_\Omega z(x,1)v\,dx + \frac{\xi(t)}{2}\|z(x,0)\|^2_{L^2(\Omega)}\\ &-\frac{\xi(t)\,(1-\tau'(t))}{2}\|z(x,1)\|^2_{L^2(\Omega)} - \frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho)\,d\rho\,dx.\end{aligned}$$

Therefore, from (16) and (17), we deduce

$$\begin{aligned}\langle \mathcal{A}(t)U,U\rangle_t \le &-\mu_1(t)\left(1-\frac{\bar{\xi}}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\|v\|^2_{L^2(\Omega)}\\ &-\mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\|z(x,1,t)\|^2_{L^2(\Omega)}\\ &+\frac{\xi(t)|\tau'(t)|}{2\tau(t)}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho)\,d\rho\,dx.\end{aligned}$$

Then, we have

$$\begin{aligned}\langle \mathcal{A}(t)U,U\rangle_t \le &-\mu_1(t)\left(1-\frac{\bar{\xi}}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\|v\|^2_{L^2(\Omega)}\\ &-\mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\|z(x,1,t)\|^2_{L^2(\Omega)}\\ &+\kappa(t)\langle U,U\rangle_t,\end{aligned}$$

where

$$\kappa(t) = \frac{\sqrt{1+\tau'(t)^2}}{2\tau(t)}.$$

From the (13), we obtain

$$\langle \mathcal{A}(t)U,U\rangle_t - \kappa(t)\langle U,U\rangle_t \le 0, \tag{25}$$

which means that the operator $\tilde{\mathcal{A}} = \mathcal{A}(t) - \kappa(t)I$ is dissipative.

Moreover, $\kappa'(t) = \frac{\tau'(t)\tau''(t)}{2\tau(t)\sqrt{1+\tau'(t)^2}} - \frac{\tau'(t)\sqrt{1+\tau'(t)^2}}{2\tau(t)^2}$ is bounded on $[0,T]$ for all $T>0$ (by (5) and (12)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \left(0, 0, \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2}z_\rho\right)^T,$$

with $\frac{\tau''(t)\tau(t)\rho-\tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2}$ bounded on $[0,T]$ by (5) and (12). Thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L^\infty_*([0,T], B(D(\mathcal{A}(0)),\mathcal{H})), \tag{26}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0,T]$ into $B(D(\mathcal{A}(0)),\mathcal{H})$.

Now, we will show that $\lambda I - \mathcal{A}(t)$ is surjective for fixed $t>0$ and $\lambda>0$. For this purpose, let $F=(f_1,f_2,f_3)^T \in \mathcal{H}$, we seek $U=(u,v,z)^T \in D(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t))\,U = F,$$

that is verifying following system of equations

$$\left\{\begin{array}{l}\lambda u - v = f_1,\\ \lambda v - u_{xx} + \mu_1(t)v - \mu_2(t)z(\cdot,1) = f_2,\\ \lambda z + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho = f_3.\end{array}\right. \tag{27}$$

Suppose that we have found u with the appropriated regularity. Then

$$v = \lambda u - f_1. \tag{28}$$

It is clear that $v \in H_0^1(\Omega)$. Furthermore, by (27) we can find z. From (20), we have

$$z(x,0) = v(x), \quad \text{for } x \in \Omega. \tag{29}$$

Following the same approach as in [22], we obtain, by using equation for z in (27),

$$z(x,\rho) = v(x)e^{-\vartheta(\rho,t)} + \tau(t)e^{-\vartheta(\rho,t)} \int_0^\rho f_3(x,s)e^{\vartheta(s,t)}\, ds,$$

if $\tau'(t) = 0$, where $\vartheta(\ell,t) = \lambda\ell\tau(t)$, and

$$z(x,\rho) = v(x)e^{\zeta(\rho,t)} + e^{\zeta(\rho,t)} \int_0^\rho \frac{\tau(t) f_3(x,s)}{1 - s\tau'(s)} e^{-\zeta(s,t)}\, ds,$$

otherwise, where $\zeta(\ell,t) = \lambda\frac{\tau(t)}{\tau'(t)} \ln(1 - \ell\tau'(t))$.

From (28), we obtain

$$z(x,\rho) = \lambda u(x)e^{-\vartheta(\rho,t)} - f_1(x,\rho)e^{-\vartheta(\rho,t)} + \tau(t)e^{-\vartheta(\rho,t)} \int_0^\rho f_3(x,s)e^{\vartheta(s,t)}\, ds, \tag{30}$$

if $\tau'(t) = 0$, and

$$z(x,\rho) = \lambda u(x)e^{\zeta(\rho,t)} - f_1(x,\rho)e^{\zeta(\rho,t)} + e^{\zeta(\rho,t)} \int_0^\rho \frac{\tau(t) f_3(x,s)}{1 - s\tau'(s)} e^{-\zeta(s,t)}\, ds, \tag{31}$$

otherwise.

In particular, if $\tau'(t) = 0$ and from (30), we have

$$z(x,1) = \lambda u(x)e^{-\vartheta(1,t)} - f_1(x,1)e^{-\vartheta(1,t)} + \tau(t)e^{-\vartheta(1,t)} \int_0^1 f_3(x,s)e^{\vartheta(s,t)}\, ds, \tag{32}$$

and if $\tau'(t) \neq 0$ and from (31), we have

$$z(x,1) = \lambda u(x)e^{\zeta(1,t)} - f_1(x,1)e^{\zeta(1,t)} + e^{\zeta(1,t)} \int_0^1 \frac{\tau(t) f_3(x,s)}{1 - s\tau'(s)} e^{-\zeta(s,t)}\, ds. \tag{33}$$

By using (27) and (28), the function u satisfies

$$\lambda^2 u - u_{xx} + \mu_1(t)v + \mu_2(t)z(\cdot,1) = f_2 + \lambda f_1. \tag{34}$$

Solving the equation (34) is equivalent to finding $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\int_\Omega \left(\lambda^2 u\eta + u_x\eta_x + \mu_1(t)v\eta + \mu_2(t)z(\cdot,1)\eta\right)\, dx = \int_\Omega (f_2 + \lambda f_1)\eta\, dx, \tag{35}$$

for all $\eta \in H_0^1(\Omega)$.

Consequently, the equation (35) is equivalent to the problem

$$\Upsilon(u,\eta) = L(\eta), \tag{36}$$

where the bilinear form

$$\Upsilon : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

and the linear form

$$L : H_0^1(\Omega) \to \mathbb{R}$$

are defined by

$$\Upsilon(u,\eta) = \int_\Omega \left(\lambda^2 u\eta + u_x\eta_x\right)\, dx + \int_\Omega \lambda u \left(\mu_1(t) + \mu_2(t)N_1\right)\eta\, dx$$

and

$$L(\eta) = \int_\Omega \left(\mu_1(t) f_1 \eta + \mu_2(t) N_2\right)\eta\, dx + \int_\Omega (f_2 + \lambda f_1)\eta\, dx,$$

where

$$N_1 = \begin{cases} e^{-\vartheta(1,t)}, & \text{if} \quad \tau'(t) = 0, \\ e^{\zeta(1,t)}, & \text{if} \quad \tau'(t) \neq 0 \end{cases}$$

and

$$N_2 = \begin{cases} -f_1(x,1)e^{-\vartheta(1,t)} + \tau(t)e^{-\vartheta(1,t)}\int_0^1 f_3(x,s)e^{\vartheta(s,t)}\, ds, & \text{if} \quad \tau'(t) = 0, \\ -f_1(x,1)e^{zeta(1,t)} + e^{zeta(1,t)}\int_0^1 \frac{\tau(t) f_3(x,s)}{1-s\tau'(t)} e^{-\zeta(s,t)}\, ds, & \text{if} \quad \tau'(t) \neq 0. \end{cases}$$

It is easy to verify that Υ is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $\eta \in H_0^1(\Omega)$ the problem (36) admits a unique solution

$$u \in H_0^1(\Omega).$$

Applying the classical elliptic regularity, it follows from (35) that

$$u \in H^2(\Omega).$$

Therefore, the operator $\lambda I - \mathcal{A}(t)$ is surjective for any $\lambda > 0$ and $t > 0$. Again as $\kappa(t) > 0$, this prove that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))\, I - \mathcal{A}(t) \text{ is surjective,} \tag{37}$$

for any $\lambda > 0$ and $t > 0$.

Then, (24), (25) and (37) imply that the family $\tilde{\mathcal{A}} = \left\{\tilde{\mathcal{A}}(t)/t \in [0,T]\right\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of t, by Proposition 1.1 from [14]. Therefore, the assumptions $(i) - (iv)$ of Theorem 3.1 are verified by (21), (24), (25), (26), (37) and (23), and thus, the problem

$$\begin{cases} \tilde{U}_t = \tilde{\mathcal{A}}(t)\tilde{U}, \\ \tilde{U}(0) = U_0 = (u_0, u_1, f_0(\cdot, -, \tau(0)))^T \end{cases} \tag{38}$$

has a unique solution $\tilde{U} \in C\left([0,+\infty[, D(\mathcal{A}(0))\right) \cap C^1\left([0,+\infty[, \mathcal{H}\right)$ for $U_0 \in D(\mathcal{A}(0))$. The requested solution of (18) is then given by

$$U(t) = e^{\int_0^t \kappa(s)\, ds}\tilde{U}(t)$$

because

$$\begin{aligned} U_t(t) &= \kappa(t)e^{\int_0^t \kappa(s)\, ds}\tilde{U}(t) + e^{\int_0^t \kappa(s)\, ds}\tilde{U}_t(t) \\ &= e^{\int_0^t \kappa(s)\, ds}\left(\kappa(t) + \tilde{\mathcal{A}}(t)\right)\tilde{U}(t) \\ &= \mathcal{A}(t)e^{\int_0^t \kappa(s)\, ds}\tilde{U}(t) \\ &= \mathcal{A}(t)U(t), \end{aligned}$$

which concludes the proof.

The existence and uniqueness are obtained by the following result.

Theorem 3.2 (Global solution)**.** *For any initial datum $U_0 \in \mathcal{H}$ there exists a unique solution U satisfying*

$$U \in C([0, +\infty[, \mathcal{H})$$

for problem (18)*.*

Moreover, if $U_0 \in D(\mathcal{A}(0))$, then

$$U \in C([0, +\infty[, D(\mathcal{A}(0))) \cap C^1([0, +\infty[, \mathcal{H}).$$

Proof. A general theory for equations of type (18) has been developed using semigroup theory [14], [15] and [26]. The simplest way to prove existence and uniqueness results in to show that the triplet $\{(\mathcal{A}, \mathcal{H}, Y)\}$, with $\mathcal{A} = \{\mathcal{A}(t)/t \in [0, T]\}$, for some fixed $T > 0$ and $Y = \mathcal{A}(0)$, forms a CD-systems (or constant domain system, see [14] and [15]). More precisely, the following theorem gives the existence and uniqueness results and is proved in Theorem 1.9 of [14] (see also Theorem 2.13 of [15] or [1]). □

4. Asymptotic behavior

In this section we shall investigate the asymptotic behavior of problem (1). The stability result will be obtained using Lemma 2.2.

Theorem 4.1 (Stability Result)**.** *Let $(u_0, u_1, f_0(\cdot, -, \tau(0))) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega\times]0, 1[)$. Assume that the hypotheses (H1), (H2) and* (5)*-*(7) *hold. Then problem* (1) *admits a unique solution*

$$u \in C([0, +\infty[, H_0^1(\Omega)) \cap C^1([0, +\infty[, L^2(\Omega)),$$
$$z \in C([0, +\infty[, L^2(\Omega)\times]0, 1[).$$

Proof. From now on, we denote by c various positive constants which may be different at different occurrences.

Given $0 \leq S < T < \infty$ we start by multiplying the first equation of (9) by uE^q and then integrating over $(S, T) \times \Omega$, we obtain

$$\int_S^T E^q \int_\Omega u \left(u_{tt} - u_{xx} + \mu_1(t)u_t + \mu_2(t)z(x, 1, t)\right) dx\, dt = 0.$$

Notice that

$$u_{tt}u = \left(u_t u\right)_t - u_t^2,$$

using integration by parts and the boundary conditions we know that

$$\begin{aligned} 0 = {} & \left[E^q(t) \int_\Omega u u_t \, dx\right]_S^T - \int_S^T qE^{q-1}(t)E'(t) \int_\Omega u u_t \, dx\, dt \\ & - \int_S^T E^q(t) \|u_t\|^2_{L^2(\Omega)} \, dt + \int_S^T E^q(t) \|u_x\|^2_{L^2(\Omega)} \, dt \\ & + \int_S^T E^q(t) \int_\Omega \mu_1(t) u u_t \, dx\, dt + \int_S^T E^q(t) \int_\Omega \mu_2(t) u z(x, 1, t) \, dx\, dt. \end{aligned} \tag{39}$$

Similarly, we multiply the second equation of (9) by $E^q \xi(t) e^{-2\rho\tau(t)} z(x, \rho, t)$ and then integrate over $\Omega \times (0, 1) \times (S, T)$ to see that

$$0 = \int_S^T \int_\Omega \int_0^1 E^q(t)\xi(t)e^{-2\rho\tau(t)} z \left(\tau(t)z_t + (1 - \rho\tau'(t))\, z_\rho\right) d\rho\, dx\, dt$$

$$
\begin{aligned}
&= \frac{1}{2} \int_\Omega \int_0^1 \int_S^T E^q(t)\xi(t) e^{-2\rho\tau(t)} \frac{\partial}{\partial t} z^2 \, dt \, d\rho \, dx \\
&\quad + \frac{1}{2} \int_S^T E^q(t)\xi(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} \left(1 - \rho\tau'(t)\right) \frac{\partial}{\partial \rho} z^2 \, d\rho \, dx \, dt.
\end{aligned}
$$

Using integration by parts and the boundary conditions we know that

$$
\begin{aligned}
0 &= \left[\frac{\xi(t)\tau(t)}{2} E^q(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \right]_S^T \\
&\quad - \frac{1}{2} \int_S^T q E^{q-1}(t) E'(t) \xi(t) \tau(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \, dt \\
&\quad - \frac{1}{2} \int_S^T q E^q(t) \xi'(t) \tau(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \, dt \\
&\quad + \frac{1}{2} \int_S^T E^q(t)\xi(t) \int_\Omega \left[e^{-2\rho\tau(t)} \left(1 - \tau'(t)\right) z^2(x,1,t) - z^2(x,0,t) \right] \, dx \, dt \\
&\quad + \int_S^T E^q(t)\xi(t)\tau(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \, dt. \qquad (40)
\end{aligned}
$$

Since μ_1 is a non-increasing function of class $C_1(\mathbb{R})$, its derivatives is non-positive, which implies that $\xi'(t) \leq 0$. This result this

$$
\int_S^T q E^q(t) \xi'(t) \tau(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \, dt \leq 0. \qquad (41)
$$

Moreover, as

$$
-\frac{1}{2} \int_S^T E^q(t)\xi(t) \int_\Omega e^{-2\rho\tau(t)} \left(1 - \tau'(t)\right) z^2(x,1,t) \, dx \, dt \leq 0, \qquad (42)
$$

then, from (40), (41) and (42), we have that

$$
\begin{aligned}
&\int_S^T E^q(t)\xi(t)\tau(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \, dt \\
&\qquad \leq - \left[\frac{\xi(t)\tau(t)}{2} E^q(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \right]_S^T \\
&\qquad\quad + \frac{1}{2} \int_S^T q E^{q-1}(t) E'(t) \xi(t) \tau(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \, dt \\
&\qquad\quad - \frac{1}{2} \int_S^T E^q(t)\xi(t) \int_\Omega z^2(x,0,t) \, dx \, dt. \qquad (43)
\end{aligned}
$$

Using the definition of E, (39) and (43), we get

$$
\begin{aligned}
\gamma_0 \int_S^T E^{q+1} \, dt &\leq - \left[E^q(t) \int_\Omega u u_t \, dx \right]_S^T - \left[\frac{\xi(t)\tau(t)}{2} E^q(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \right]_S^T \\
&\quad + q \int_S^T E^{q-1}(t) E'(t) \int_\Omega u u_t \, dx \, dt \\
&\quad + q \int_S^T \frac{\xi(t)\tau(t)}{2} E^{q-1}(t) E'(t) \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2 \, d\rho \, dx \, dt
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\int_S^T E^q(t)\|u_t\|^2_{L^2(\Omega)}\,dt - \int_S^T E^q(t)\int_\Omega \mu_1(t)uu_t\,dx\,dt \\
&- \int_S^T E^q(t)\int_\Omega \mu_2(t)uz(x,1,t)\,dx\,dt \\
&+ \frac{1}{2}\int_S^T \xi(t)E^q(t)e^{-2\rho\tau(t)}\int_\Omega z^2(x,0,t)\,dx\,dt, \qquad (44)
\end{aligned}
$$

where $\gamma_0 = 2\min\{1, e^{-2\tau_1}\}$.

Using the Young and Sobolev-Poincaré inequalities and Lemma 2.3, we find that

$$
\begin{aligned}
-\left[E^q(t)\int_\Omega uu_t\,dx\right]_S^T &\le E^q(S)\int_\Omega u(x,S)u_t(x,S)\,dx \\
&\quad - E^q(T)\int_\Omega u(x,T)u_t(x,T)\,dx \\
&\le cE^{q+1}(S).
\end{aligned}
$$

Now, we known that

$$
\begin{aligned}
-&\left[\frac{\xi(t)\tau(t)}{2}E^q(t)\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2\,d\rho\,dx\right]_S^T \\
&\le \frac{\xi(S)\tau(S)}{2}E^q(S)\int_\Omega\int_0^1 e^{-2\rho\tau(S)}z^2(x,\rho,S)\,d\rho\,dx \\
&\le cE^q(S)\xi(S)\tau(S)\int_\Omega\int_0^1 z^2(x,\rho,S)\,d\rho\,dx \\
&\le cE^{q+1}(S).
\end{aligned}
$$

By (13), we have

$$
\int_S^T E^{q-1}(t)E'(t)\int_\Omega uu_t\,dx\,dt \le c\int_S^T (-E'(t))E^q(t)\,dt \le cE^{q+1}(S).
$$

Similarly,

$$
\int_S^T E^{q-1}(t)E'(t)\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2\,d\rho\,dx\,dt \le cE^{q+1}(S).
$$

From Lemma 2.4, we deduce that

$$
\int_S^T E^q(t)\|u_t\|^2_{L^2(\Omega)}\,dt \le -c\int_S^T E^q(t)E'(t)\,dt \le cE^{q+1}(S).
$$

Now, we get that

$$
\begin{aligned}
\left|\int_S^T E^q(t)\int_\Omega \mu_1(t)uu_t\,dx\,dt\right| &\le \mu_1(0)\left|\int_S^T E^q(t)\int_\Omega uu_t\,dx\,dt\right| \\
&\le c(\varepsilon_1)\int_S^T E^q(t)\int_\Omega u_t^2\,dx\,dt + \varepsilon_1\int_S^T E^q(t)\int_\Omega u_x^2\,dx\,dt \\
&\le c(\varepsilon_1)\int_S^T E^q(t)(-E'(t))\,dt + \varepsilon_1\int_S^T E^q(t)E(t)\,dt \\
&\le c(\varepsilon_1)E^{q+1}(S) + \varepsilon_1\int_S^T E^{q+1}(t)\,dt \qquad (45)
\end{aligned}
$$

and from (H2) we obtain that

$$\begin{aligned}\left|\int_S^T E^q(t)\int_\Omega \mu_2(t)uz(x,1,t)\,dx\,dt\right| &\le \beta\mu_1(0)\left|\int_S^T E^q(t)\int_\Omega \varphi z(x,1,t)\,dx\,dt\right|\\ &\le c(\varepsilon_2)E^{q+1}(S)+\varepsilon_2\int_S^T E^{q+1}(t)\,dt. \end{aligned}\tag{46}$$

Finally,

$$\begin{aligned}\frac{1}{2}\int_S^T E^q(t)\xi(t)\int_\Omega z^2(x,0,t)\,dx\,dt &\le \frac{\bar{\xi}\mu_1(0)}{2}\int_S^T E^q(t)\|u_t\|^2_{L^2(\Omega)}\,dt\\ &\le c\int_S^T E^q(t)(-E'(t))\,dt \le cE^{q+1}(S).\end{aligned}$$

Choosing ε_1 and ε_2 small enough, we deduce from (45) and (46) that

$$\int_S^T E^{q+1}\,dt \le \frac{1}{\gamma}E^{q+1}(S).$$

Since $E(S)\le E(0)$ for $S\ge 0$, we have that

$$\int_S^T E^{q+1}\,dt \le \frac{1}{\gamma}E(0)E^q(S).$$

We choose $q=0$, we conclude from Lemma 2.2 that

$$E(t)\le E(0)e^{1-\gamma t}.$$

This ends the proof of Theorem 4.1. □

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