

GLOBAL EXISTENCE AND PROPAGATION SPEED FOR A DEGASPERIS-PROCESI EQUATION WITH BOTH DISSIPATION AND DISPERSION

GUENBO HWANG AND BYUNGSOO MOON*

ABSTRACT. In this paper, we consider the dissipative Degasperis-Procesi equation with arbitrary dispersion coefficient and compactly supported initial data. We establish the simple condition on the initial data which lead to guarantee that the solution exists globally. We also investigate the propagation speed for the equation under the initial data is compactly supported.

1. INTRODUCTION

In the present paper, we study the Cauchy problem to the dissipative Degasperis-Procesi equation with arbitrary dispersion term and compactly supported initial data:

$$(1) \quad \begin{cases} u_t - u_{txx} + k(u - u_{xx})_x + 4uu_x + \lambda(u - u_{xx}) = 3u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $\text{supp } u_0 \subset [a, b]$ is a compactly supported initial data, $k \in \mathbb{R}$ is an arbitrary dispersion coefficient, and $\lambda > 0$ is a dissipative parameter.

When $k = 0$ and $\lambda = 0$, (1) becomes the Degasperis-Procesi equation [3]

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R},$$

which arises in the shallow-water medium-amplitude regime [2, 10], introduced to capture stronger nonlinear effects that will allow for breaking waves, since the latter are not modeled by the shallow-water small-amplitude regime characteristic for the KdV equation. In this regime only the Camass-Holm equation [1] and the Degasperis-Procesi equation [4] arise as integrable model equations, with the same accuracy of approximation to the governing equations for water waves. It has bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa-Holm peakons. After the Degasperis-Procesi equation was derived, many papers were devoted to its study, cf. [3, 8, 12, 14, 21] and the citations therein.

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* Corresponding author: Byungsoo Moon.

If $k = 0$ and $\lambda > 0$ in (1), then it becomes the weakly dissipative Degasperis-Procesi equation [5, 6, 18, 19]

$$(2) \quad v_t - v_{txx} + 4vv_x + \lambda(v - v_{xx}) = 3v_xv_{xx} + vv_{xxx}, \quad t > 0, \quad x \in \mathbb{R},$$

which has been analyzed in several papers: Local well-posedness of (2) in $H^s, s > 3/2$, was proved in [5]. Existence and uniqueness of global weak solutions of (2) was established in [6]. Further global existence, blow-up results, persistence properties, and propagation speed were derived in [18, 5]. Lenells and Wunsch [12] point out that the weakly dissipative the Degasperis-Procesi equation (2) are equivalent to their non-dissipative counterparts up to a simple change of variables. More precisely, if $u(t, x)$ and $v(t, x)$ are related by

$$v(t, x) = e^{-\lambda t} u \left(\frac{1 - e^{-\lambda t}}{\lambda}, x \right),$$

then $v(t, x)$ satisfies the weakly dissipative Degasperis-Procesi equation (2) if and only if $u(t, x)$ satisfies the respective non-dissipative equation. It was observed already in [18] that the properties of (2) are similar to the properties of non-dissipative Degasperis-Procesi equation restricted to a finite time interval. It was also noted that there are considerable differences between these equations with respect to their long time behavior. However, the work done above was only involved in dissipative terms. It is found [13, 15, 20] that such kinds of considerable differences were investigated for the different model (nonlinear wave equations). Recently, Novruzov and Hagverdiyev [17] analyzed the behavior of solutions to the dissipative Camassa-Holm equation with arbitrary dispersion coefficient.

In the present paper, we discuss the global existence and the propagation speed of strong solutions to the equation (1). Our result shows that in comparison between the Degasperis-Procesi equation ($k = 0$ and $\lambda = 0$) and (1) ($k \neq 0$ and $\lambda > 0$), some behaviors of solutions to the dissipative Degasperis-Procesi equation (1) with arbitrary dispersion are similar to the ones of Degasperis-Procesi equation ($k = 0$ and $\lambda = 0$), such as, the local well-posedness and the blow-up scenario. However, the dissipative term $\lambda(u - u_{xx})$ and the dispersive term $k(u_x - u_{xxx})$ in (1) do have impacts on the global existence and the propagation speed of its solutions, which are shown below in Theorem 3.3 and Theorem 4.1, respectively. In particular, the propagation speed is seriously affected by both the dissipative parameter λ and the dispersion coefficient k .

Our paper is organized as follows. In Section 2, we provide some preliminary materials which are crucial in the proof of our result. We gave the global existence result in Section 3. We study the propagation speed of strong solutions to the equation (1) under the condition that the initial data has compact support in Section 4.

2. PRELIMINARIES

Since we shall also use a priori estimates and further properties of solutions in $H^s(\mathbb{R})$, $s > 3/2$, we briefly collect the needed results from [21] in order to pursue our goal.

With $m := u - u_{xx}$, The equation (1) takes the following form of a quasi-linear evolution equation of hyperbolic type:

$$(3) \quad \begin{cases} m_t + (u + k)m_x + 3u_xm + \lambda m = 0, & t > 0, \quad x \in \mathbb{R}, \\ m(0, x) = u_0(x) - u_{0,xx}(x), & x \in \mathbb{R}. \end{cases}$$

Note that if $p(x) : \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1}f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * m = u$. Using this identity, we can rewrite equation (3) as follows:

$$(4) \quad \begin{cases} u_t + (u + k)u_x + \partial_x p * \left(\frac{3}{2}u^2\right) + \lambda u = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

The local well-posedness of the Cauchy problem (4) with initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, and $\lambda = 0$ can be obtained by applying the Kato theorem [11]. It is easy to see that the same result holds for the Cauchy problem (4). As a result, we have the following well-posedness result.

Lemma 2.1. [19] *Given $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, there exists a maximal time $T = T(u_0, k, \lambda) > 0$ and a unique solution u to initial-value problem (4), such that*

$$u = u(\cdot, u_0) \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \rightarrow u(\cdot, u_0) : H^s(\mathbb{R}) \rightarrow C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}))$ is continuous and the maximal time of existence $T > 0$ can be chosen to be independent of s .

Consider the following differential equation:

$$(5) \quad \begin{cases} \varphi_t = u(t, \varphi(t, x)) + k, & t \in [0, T], \\ \varphi(0, x) = x, & x \in \mathbb{R}, \end{cases}$$

where $u(t, x)$ is the corresponding strong solution to (1). Applying classical results in the theory of ordinary differential equations, one can obtain the following two results on φ which are crucial in the proof of global existence and blow-up solutions.

Lemma 2.2. [19] *Let $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, and let $T > 0$ be the maximal existence time of the corresponding solution u to equation (4). Then equation (5) has a unique solution $\varphi \in C^1([0, T] \times \mathbb{R}, \mathbb{R})$. Moreover, the map $\varphi(t, \cdot)$ is an increasing diffeomorphism of \mathbb{R} with*

$$\varphi_x(t, x) = \exp\left(\int_0^t u_x(\tau, \varphi(\tau, x))d\tau\right) > 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}.$$

Lemma 2.3. *Let $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, and let $T > 0$ be the maximal existence time of the corresponding solution u to equation (4). Then we have*

$$(6) \quad m(t, \varphi(t, x))\varphi_x^3(t, x) = m_0(x)e^{-\lambda t}.$$

Proof. Differentiating the left-hand side of equation (6) with respect to t , in view of the relations (5) and (3), we obtain

$$\begin{aligned} & \frac{d}{dt} \{m(t, \varphi(t, x))\varphi_x^3(t, x)\} \\ &= (m_t(t, \varphi) + m_x(t, \varphi)\varphi_t(t, x))\varphi_x^3(t, x) + 3m(t, \varphi)\varphi_x^2(t, x)\varphi_{xt}(t, x) \\ &= [m_t(t, \varphi) + (u(t, \varphi) + k)m_x(t, \varphi) + 3m(t, \varphi)u_x(t, \varphi)]\varphi_x^3(t, x) \\ &= -\lambda m(t, \varphi(t, x))\varphi_x^3(t, x), \end{aligned}$$

which completes the proof of the Lemma 2.3. \square

3. GLOBAL EXISTENCE

In this section, we will derive a conservation law for strong solutions to equation (4). we then establish a priori estimate for the L^∞ -norm of the strong solution by using conservation law. This enables us to guarantee that the solution exists globally.

Lemma 3.1. *If $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, then as long as the solution $u(t, x)$ given by Lemma 2.1 exists, we have*

$$(7) \quad \int_{-\infty}^{\infty} m(t, x)v(t, x)dx = e^{-2\lambda t} \int_{-\infty}^{\infty} m(0, x)v(0, x)dx,$$

where $m(t, x) = u(t, x) - u_{xx}(t, x)$ and $v(t, x) = (4 - \partial_x^2)^{-1}u$. Moreover, we have

$$(8) \quad \|u(t)\|_{L^2}^2 \leq 4e^{-2\lambda t} \|u_0\|_{L^2}^2.$$

Proof. Applying Lemma 2.1 and a simple density argument, we only need to show that this lemma with some $s > 3/2$. Thus we take $s = 3$ in the proof. Let $T > 0$ be the maximal time of existence of the solution u to equation (4) with initial data $u_0 \in H^3(\mathbb{R})$ such that $u \in C([0, T]; H^3(\mathbb{R})) \cap C^1([0, T]; H^2(\mathbb{R}))$, which is guaranteed by the local well-posedness Lemma 2.1. Applying the operator $(1 - \partial_x^2)$ on the both sides of equation (4), we have

$$m_t + (1 - \partial_x^2)\partial_x \left(\frac{1}{2}u^2 + ku \right) + \partial_x \left(\frac{3}{2}u^2 \right) + \lambda m = 0.$$

Multiplying the above equation by $v(t, x)$ and integrating by parts with respect to x , in view of $4v - v_{xx} = u$, we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} vm_t dx \\ &= - \int_{-\infty}^{\infty} v(1 - \partial_x^2)\partial_x \left(\frac{1}{2}u^2 + ku \right) dx - \frac{3}{2} \int_{-\infty}^{\infty} v\partial_x(u^2) dx - \lambda \int_{-\infty}^{\infty} vm dx \\ &= \int_{-\infty}^{\infty} v_x(1 - \partial_x^2) \left(\frac{1}{2}u^2 + ku \right) dx + \frac{3}{2} \int_{-\infty}^{\infty} v_x u^2 dx - \lambda \int_{-\infty}^{\infty} vm dx \\ &= 2 \int_{-\infty}^{\infty} v_x u^2 dx - \frac{1}{2} \int_{-\infty}^{\infty} v_{xxx} u^2 dx + k \int_{-\infty}^{\infty} v_x(1 - \partial_x^2)(4v - v_{xx}) dx - \lambda \int_{-\infty}^{\infty} vm dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} u_x u^2 dx + k \int_{-\infty}^{\infty} v_x(4v - 5v_{xx} + v_{xxxx}) dx - \lambda \int_{-\infty}^{\infty} vm dx \\ &= -\lambda \int_{-\infty}^{\infty} vm dx. \end{aligned}$$

Since

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} mv dx = \frac{1}{2} \int_{-\infty}^{\infty} m_t v dx + \frac{1}{2} \int_{-\infty}^{\infty} m v_t dx = \int_{-\infty}^{\infty} m_t v dx,$$

it follows that

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} mv dx = -\lambda \int_{-\infty}^{\infty} vm dx,$$

which implies the desired conserved quantity. In view of the above conservation law, it then follows that

$$\begin{aligned}\|u(t, \cdot)\|_{L^2}^2 &= \|\hat{u}(t, \cdot)\|_{L^2}^2 \leq 4 \int_{-\infty}^{\infty} \frac{1 + \xi^2}{4 + \xi^2} |\hat{u}(t, \xi)|^2 d\xi = 4(\hat{m}(t), \hat{v}(t)) \\ &= 4(m(t), v(t)) = 4(m_0, v_0)e^{-2\lambda t} = 4(\hat{m}_0, \hat{v}_0)e^{-2\lambda t} \\ &\leq 4e^{-2\lambda t} \int_{-\infty}^{\infty} \frac{1 + \xi^2}{4 + \xi^2} |\hat{u}_0(\xi)|^2 d\xi \leq 4e^{-2\lambda t} \|\hat{u}_0\|_{L^2}^2 = 4e^{-2\lambda t} \|u_0\|_{L^2}^2.\end{aligned}$$

This completes the proof of Lemma 3.1. \square

The following important estimate can be obtained by Lemma 3.1.

Lemma 3.2. *Assume $u_0 \in H^s(\mathbb{R})$, $s > 3/2$. Let $T > 0$ be the maximal existence time of the solution u to the equation (4) guaranteed by Lemma 2.1. Then we have*

$$\|u(t, x)\|_{L^\infty} \leq e^{-\lambda t} \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right).$$

Proof. Applying Lemma 2.1 and a simple density argument, it suffices to consider $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to (4) with the initial value $u_0 \in H^3(\mathbb{R})$ such that $u \in C([0, T]; H^3(\mathbb{R})) \cap C^1([0, T]; H^2(\mathbb{R}))$, which is guaranteed by Lemma 2.1. By (4), we obtain

$$(9) \quad u_t + (u + k)u_x = -3p * (uu_x) - \lambda u.$$

Note that

$$\begin{aligned}(10) \quad -3p * (uu_x) &= -\frac{3}{2} \int_{-\infty}^{\infty} e^{-|x-\eta|} uu_\eta d\eta \\ &= -\frac{3}{2} \int_{-\infty}^x e^{-x+\eta} uu_\eta d\eta - \frac{3}{2} \int_x^{\infty} e^{x-\eta} uu_\eta d\eta \\ &= \frac{3}{4} \int_{-\infty}^x e^{-|x-\eta|} u^2 d\eta - \frac{3}{4} \int_x^{\infty} e^{-|x-\eta|} u^2 d\eta.\end{aligned}$$

In view of (5), we have

$$\frac{d(u(t, \varphi(t, x)))}{dt} = u_t(t, \varphi(t, x)) + u_x(t, \varphi(t, x))\varphi_t(t, x) = (u_t + (u + k)u_x)(t, \varphi(t, x)),$$

where $\varphi = \varphi(t, x)$ is the C^1 solution to (5). It then follows from (9) and (10) that

$$-\frac{3}{4} \int_{\varphi(t, x)}^{\infty} e^{-|\varphi(t, x)-\eta|} u^2 d\eta \leq \frac{du(t, \varphi(t, x))}{dt} + \lambda u(t, \varphi(t, x)) \leq \frac{3}{4} \int_{-\infty}^{\varphi(t, x)} e^{-|\varphi(t, x)-\eta|} u^2 d\eta.$$

It thus transpires that

$$\left| \frac{du(t, \varphi(t, x))}{dt} + \lambda u(t, \varphi(t, x)) \right| \leq \frac{3}{4} \int_{-\infty}^{\infty} e^{-|\varphi(t, x)-\eta|} u^2 d\eta \leq \frac{3}{4} \int_{-\infty}^{\infty} u^2(t, \eta) d\eta.$$

In view of Lemma 3.1, we have

$$-3e^{-2\lambda t} \|u_0\|_{L^2}^2 \leq \frac{du(t, \varphi(t, x))}{dt} + \lambda u(t, \varphi(t, x)) \leq 3e^{-2\lambda t} \|u_0\|_{L^2}^2.$$

Integrating the above inequality with respect to $t < T$ on $[0, t]$ yields

$$-\frac{3}{\lambda} (1 - e^{-\lambda t}) \|u_0\|_{L^2}^2 \leq e^{\lambda t} u(t, \varphi(t, x)) - u_0(x) \leq \frac{3}{\lambda} (1 - e^{-\lambda t}) \|u_0\|_{L^2}^2.$$

Thus,

$$(11) \quad |u(t, \varphi(t, x))| \leq \|u(t, \varphi(t, x))\|_{L^\infty} \leq e^{-\lambda t} \left(\frac{3}{\lambda} (1 - e^{-\lambda t}) \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right).$$

Using the Sobolev embedding to ensure the uniform boundedness of $u_x(\tau, \eta)$ for $(\tau, \eta) \in [0, t] \times \mathbb{R}$ with $t \in [0, T]$, in view of Lemma 2.3, we obtain for every $t \in [0, T]$ a constant $\kappa(t) > 0$ such that

$$e^{-\kappa(t)} \leq \varphi_x(t, x) \leq e^{\kappa(t)}, \quad x \in \mathbb{R}.$$

We now deduce from the above equation that the function $\varphi(t, \cdot)$ is strictly increasing on \mathbb{R} with $\lim_{x \rightarrow \pm\infty} \varphi(t, x) = \pm\infty$ as long as $t \in [0, T]$. Thus, by (11) we can obtain

$$(12) \quad \|u(t, x)\|_{L^\infty} = \|u(t, \varphi(t, x))\|_{L^\infty} \leq e^{-\lambda t} \left(\frac{3}{\lambda} (1 - e^{-\lambda t}) \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right).$$

This completes the proof of Lemma 3.2. \square

We now present the global existence result.

Theorem 3.3. *Assume $u_0 \in H^s(\mathbb{R})$, $s > 3/2$. If $m_0 = u_0 - u_{0,xx}$ satisfying $\|m_0\|_{L^2} < \frac{4\lambda}{5}$, then the corresponding solution $u(t, x)$ to equation (4) exists globally.*

Proof. We only assume $s = 3$ to prove the above theorem. Let $T > 0$ be the maximal time of existence of the solution u to equation (4) with initial data $u_0 \in H^3(\mathbb{R})$. Multiplying equation (3) by m and using integration by parts with respect to x , we have

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \int_{-\infty}^{\infty} m^2 dx \\ &= -k \int_{-\infty}^{\infty} m m_x dx - \lambda \int_{-\infty}^{\infty} m^2 dx - 3 \int_{-\infty}^{\infty} m^2 u_x dx - \int_{-\infty}^{\infty} u m m_x dx \\ &= -\lambda \int_{-\infty}^{\infty} m^2 dx - \frac{5}{2} \int_{-\infty}^{\infty} m^2 u_x dx. \end{aligned}$$

Again, multiplying the above equality by $2e^{2\lambda t}$, we get

$$e^{2\lambda t} \frac{d}{dt} \int_{-\infty}^{\infty} m^2 dx + 2e^{2\lambda t} \lambda \int_{-\infty}^{\infty} m^2 dx = -5e^{2\lambda t} \int_{-\infty}^{\infty} m^2 u_x dx.$$

Therefore,

$$(13) \quad \frac{d}{dt} \left(e^{2\lambda t} \int_{-\infty}^{\infty} m^2 dx \right) = 5e^{2\lambda t} \int_{-\infty}^{\infty} m^2 u_x dx.$$

On the other hand, note that if $p(x) = \frac{1}{2}e^{-|x|}$, then $(1 - \partial_x^2)^{-1}f = p * f$ for all $f \in L^2(\mathbb{R})$ and $u = p * m$. From this relation, we find

$$(14) \quad \|u_x\|_{L^\infty} \leq \|p_x\|_{L^2} \|m\|_{L^2} \leq \frac{1}{2} \|m\|_{L^2}.$$

Using (14), we from (13) obtain

$$\frac{d}{dt} \left(e^{2\lambda t} \int_{-\infty}^{\infty} m^2 dx \right) \leq \frac{5}{2} e^{2\lambda t} \left(\int_{-\infty}^{\infty} m^2 dx \right)^{\frac{3}{2}} = \frac{5}{2} e^{-\lambda t} \left(e^{2\lambda t} \int_{-\infty}^{\infty} m^2 dx \right)^{\frac{3}{2}}.$$

From the above inequality, we easily derive that

$$\frac{d}{dt} \left(e^{2\lambda t} \int_{-\infty}^{\infty} m^2 dx \right)^{-\frac{1}{2}} \geq -\frac{5}{4} e^{-\lambda t}.$$

Integrating the above inequality with respect to t yields

$$\left(e^{2\lambda t} \int_{-\infty}^{\infty} m^2 dx\right)^{-\frac{1}{2}} - \left(\int_{-\infty}^{\infty} m_0^2 dx\right)^{-\frac{1}{2}} \geq \frac{5}{4\lambda}(e^{-\lambda t} - 1).$$

Thus,

$$\left(e^{2\lambda t} \int_{-\infty}^{\infty} m^2 dx\right)^{-\frac{1}{2}} \geq \left(\int_{-\infty}^{\infty} m_0^2 dx\right)^{-\frac{1}{2}} - \frac{5}{4\lambda}.$$

We deduce from the above inequality that

$$\frac{1}{\left(\int_{-\infty}^{\infty} m_0^2 dx\right)^{-\frac{1}{2}} - \frac{5}{4\lambda}} \geq \left(e^{2\lambda t} \int_{-\infty}^{\infty} m^2 dx\right)^{\frac{1}{2}},$$

which implies

$$(15) \quad \|m\|_{L^2} \leq \frac{1}{e^{\lambda t} \left(\|m_0\|_{L^2}^{-1} - \frac{5}{4\lambda}\right)}.$$

Using (14), (15), and the condition of Theorem, we have

$$\|u_x\|_{L^\infty} \leq \frac{1}{2}\|m\|_{L^2} < \|m\|_{L^2} \leq \frac{1}{e^{\lambda t} \left(\|m_0\|_{L^2}^{-1} - \frac{5}{4\lambda}\right)}.$$

The above inequality and Lemma 2.2 imply $T = \infty$. This proves that the solution u exists globally in time. This completes the proof of Theorem 3.3. \square

Remark 1. Theorem 3.3 demonstrate the difference of the previous global existence result for the Degasperis-Procesi equation. The obtained condition is simple and convenient since we do not use a condition of the sign of the potential m_0 in the point x_0 .

4. INFINITE PROPAGATION SPEED

Recently, the results of the infinite propagation speed for the Camassa-Holm equation and the Degasperis-Procesi equation was extensively established [7, 22]. Infinite propagation speed means that they loose instantly the property of having compact x -support. Motivated by recently work [9, 16], the purpose of this section is to give a more detailed description on the corresponding strong solution $u(t, x)$ to (4) in its lifespan with initial data $u_0(x)$ being compactly supported.

Theorem 4.1. *Assume that for some $T > 0$ and $s > 3/2$, $u \in C([0, T]; H^s(\mathbb{R}))$ is a strong solution of (1). If $u_0(x) = u(0, x)$ has compact support in $[a, b]$, then for any $t \in [0, T]$, we have*

$$(16) \quad u(t, x) = \begin{cases} E_+(t)e^{-x}, & x \geq \varphi(t, b), \\ E_-(t)e^x, & x \leq \varphi(t, a), \end{cases}$$

where E_+ and E_- are continuous non-vanishing functions with $E_+(0) = E_-(0) = 0$ and $E_+(t) > 0$ for $t \in (0, T)$ is a strictly increasing function, while $E_-(t) < 0$ for $t \in (0, T)$ is a strictly decreasing function. Furthermore, we get

$$E_+(t) \leq c_1(\lambda, \|u_0\|_{L^2}, \|u_0\|_{L^\infty}, b)e^{(k-\lambda)t} \text{ and } |E_-(t)| \leq c_2(\lambda, \|u_0\|_{L^2}, \|u_0\|_{L^\infty}, a)e^{-(k+\lambda)t}.$$

Proof. From Lemma 2.3, we see that

$$(17) \quad m(t, x) = (I - \partial_x^2)u(t, x) = \frac{(I - \partial_x^2)u_0(\varphi^{-1}(t, x))e^{-\lambda t}}{(\partial_x \varphi(t, \varphi^{-1}(t, x)))^3}.$$

In addition, u_0 has a compact support in x in the interval $[a, b]$ for any $t \in [0, T]$. Therefore so does $m(t, \cdot)$ in the interval $[\varphi(t, a), \varphi(t, b)]$. By the relation $u = p * m$ with $p(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, we have

$$(18) \quad u(t, x) = \frac{1}{2}e^{-x} \int_{-\infty}^x e^{\eta} m(t, \eta) d\eta + \frac{1}{2}e^x \int_x^{\infty} e^{-\eta} m(t, \eta) d\eta,$$

and

$$(19) \quad u_x(t, x) = -\frac{1}{2}e^{-x} \int_{-\infty}^x e^{\eta} m(t, \eta) d\eta + \frac{1}{2}e^x \int_x^{\infty} e^{-\eta} m(t, \eta) d\eta.$$

Moreover, defining

$$f_+(t) = \int_{\varphi(t, a)}^{\varphi(t, b)} e^{\eta} m(t, \eta) d\eta, \quad f_-(t) = \int_{\varphi(t, a)}^{\varphi(t, b)} e^{-\eta} m(t, \eta) d\eta,$$

one has from (17) that

$$(20) \quad \begin{aligned} u(t, x) &= \frac{1}{2}e^{-x} \left(\int_{-\infty}^{\varphi(t, a)} + \int_{\varphi(t, a)}^{\varphi(t, b)} + \int_{\varphi(t, b)}^{\infty} \right) e^{\eta} m(t, \eta) d\eta + \frac{1}{2}e^x \int_x^{\infty} e^{-\eta} m(t, \eta) d\eta \\ &= \frac{1}{2}e^{-x} f_+(t), \quad x \geq \varphi(t, b). \end{aligned}$$

In the same way, we obtain

$$u(t, x) = \frac{1}{2}e^x f_-(t), \quad x \leq \varphi(t, a).$$

It then from (19) and (20) follows that

$$(21) \quad u(t, x) = -u_x(t, x) = u_{xx}(t, x) = \frac{1}{2}e^{-x} f_+(t), \quad x \geq \varphi(t, b),$$

and similarly,

$$(22) \quad u(t, x) = u_x(t, x) = u_{xx}(t, x) = \frac{1}{2}e^x f_-(t), \quad x \leq \varphi(t, a).$$

By the definition of $f_+(t)$, we get

$$\begin{aligned} f_+(0) &= \int_{-\infty}^{\infty} e^{\eta} m_0(\eta) d\eta \\ &= \int_{-\infty}^{\infty} e^{\eta} (u_0 - u_{0,xx})(\eta) d\eta = \int_{-\infty}^{\infty} e^{\eta} u_0 d\eta + \int_{-\infty}^{\infty} e^{\eta} u_{0,xx}(\eta) d\eta = 0. \end{aligned}$$

Since $m(t, \cdot)$ is supported on compact interval $[\varphi(t, a), \varphi(t, b)]$, for fixed t we have

$$\frac{df_+}{dt} = \int_{\varphi(t, a)}^{\varphi(t, b)} e^{\eta} m_t(t, \eta) d\eta = \int_{-\infty}^{\infty} e^{\eta} m_t(t, \eta) d\eta.$$

Next, integration by parts, (18), (19) and equation using equation (3) yield the following identities

$$\begin{aligned}
\frac{df_+(t)}{dt} &= \int_{-\infty}^{\infty} e^{\eta} m_t(t, \eta) d\eta \\
&= - \int_{-\infty}^{\infty} e^{\eta} ((mu)_x + (u^2 - u_x^2)_x + \lambda m + km_x) d\eta \\
&= -mu e^{\eta}|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{\eta} mud\eta - (u^2 - u_x^2)e^{\eta}|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{\eta} (u^2 - u_x^2) d\eta \\
&\quad - \lambda \int_{-\infty}^{\infty} e^{\eta} m d\eta - k \int_{-\infty}^{\infty} e^{\eta} m_x d\eta \\
&= \int_{-\infty}^{\infty} e^{\eta} (u^2 + u_x^2) d\eta + \int_{-\infty}^{\infty} e^{\eta} (u^2 - u_x^2) d\eta - uu_x e^{\eta}|_{-\infty}^{\infty} + \frac{1}{2} u^2 e^{\eta}|_{-\infty}^{\infty} \\
&\quad - \frac{1}{2} \int_{-\infty}^{\infty} e^{\eta} u^2 d\eta - (\lambda - k) \int_{-\infty}^{\infty} e^{\eta} m d\eta \\
&= \frac{3}{2} \int_{-\infty}^{\infty} e^{\eta} u^2 d\eta - (\lambda - k) \int_{-\infty}^{\infty} e^{\eta} m d\eta.
\end{aligned}$$

Therefore,

$$\frac{df_+(t)}{dt} + (\lambda - k)f_+(t) \geq \frac{3}{2} \int_{-\infty}^{\infty} e^{\eta} u^2 d\eta.$$

Multiplying the above inequality by $e^{(\lambda-k)t}$, we obtain

$$\frac{d(f_+(t)e^{(\lambda-k)t})}{dt} > 0,$$

so that $f_+(t) > 0$ for any $t > 0$ is a strictly increasing function. Next, along the curve $\varphi(t, b)$, simple calculations yields the following estimation

$$\begin{aligned}
\varphi(t, b) &= \int_0^t u(\tau, \varphi(\tau, 0)) d\tau + kt + b \\
&\leq \int_0^t e^{-\lambda\tau} \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right) d\tau + kt + b \\
&= - \left(\frac{e^{-\lambda t}}{\lambda} - \frac{1}{\lambda} \right) \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right) + kt + b \\
&\leq \frac{1}{\lambda} \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right) + kt + b.
\end{aligned}$$

It then follows from (21) that

$$(23) \quad u(t, \varphi(t, b)) = \frac{1}{2} e^{-\varphi(t, b)} f_+(t) \geq \frac{1}{2} e^{-\frac{1}{\lambda} \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right) + |b| - kt} f_+(t).$$

On the other hand, by Lemma 3.2 we know that

$$(24) \quad u(t, \varphi(t, b)) \leq \|u(t, \varphi(t, b))\|_{L^\infty} \leq e^{-\lambda t} \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right).$$

Combining (23) with (24), we deduce that

$$\begin{aligned} f_+(t) &\leq 2 \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right) e^{\frac{1}{\lambda} \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right) - |b| + (k-\lambda)t} \\ &:= c_1(\lambda, \|u_0\|_{L^2}, \|u_0\|_{L^\infty}, b) e^{(k-\lambda)t}. \end{aligned}$$

In an analogous way, we can easily verify that

$$f_-(0) = 0$$

and

$$\frac{df_-(t)}{dt} + (\lambda + k)f_-(t) \leq -\frac{3}{2} \int_{-\infty}^{\infty} e^{-\eta} u^2 d\eta.$$

Following the similar argument of the function $f_+(t)$ and curve $\varphi(t, b)$, it is found that

$$\begin{aligned} f_-(t) e^{(\lambda+k)t} &\leq 0, \\ f_-(t) &< 0, \end{aligned}$$

and

$$(25) \quad \varphi(t, a) \geq -\frac{1}{\lambda} \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right) + kt + a.$$

Combining (22) with (25), in view of Lemma 3.2, we obtain

$$\begin{aligned} |f_-(t)| &\leq 2 \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right) e^{\frac{1}{\lambda} \left(\frac{3}{\lambda} \|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty} \right) - |a| - (k+\lambda)t} \\ &:= c_2(\lambda, \|u_0\|_{L^2}, \|u_0\|_{L^\infty}, a) e^{-(k+\lambda)t}. \end{aligned}$$

By taking $E_\pm(t) = \frac{1}{2} f_\pm(t)$, this completes the proof of Theorem 4.1. \square

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REFERENCES

- [1] R. Camassa and D. D. Holm, [An integrable shallow water equation with peaked solitons](#), *Phys. Rev. Lett.*, **71** (1993), 1661–1664. [MR 1234453](#)
- [2] A. Constantin and D. Lannes, [The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations](#), *Arch. Rational Mech. Anal.*, **192** (2009), 165–186. [MR 2481064](#)
- [3] A. Degasperis, D. Holm and A. Hone, [A new integrable equation with peakon solution](#), *Theoret. and Math. Phys.*, **133** (2002), 1463–1474. [MR 2001531](#)
- [4] H. R. Dullin, G. A. Gottwald and D. D. Holm, [Camassa-Holm, Korteweg-de Vries-5 and other asymptotically equivalent equations for shallow water waves](#), *Fluid Dynam. Res.*, **33** (2003), 73–95. [MR 1995028](#)
- [5] Z. Guo, [Some properties of solutions to the weakly dissipative Degasperis-Procesi equation](#), *J. Differential Equations*, **246** (2009), 4332–4344. [MR 2517774](#)
- [6] Z. Guo, S. Lai and Y. Wang, [Global weak solutions to the weakly dissipative Degasperis-Procesi equation](#), *Nonlinear Anal.*, **74** (2011), 4961–4973. [MR 2810679](#)
- [7] D. Henry, [Persistence properties for the Degasperis-Procesi equation](#), *J. Hyperbolic Differ. Equ.*, **5** (2008), 99–111. [MR 2405853](#)
- [8] D. Henry, [Infinite propagation speed for a two component Camassa-Holm equation](#), *Discrete Contin. Dyn. Syst. Ser. B*, **12** (2009), 597–606. [MR 2525158](#)
- [9] A. Himonas, G. Misiolek, G. Ponce and Y. Zhou, [Persistence properties and unique continuation of solutions of the Camassa-Holm equation](#), *Comm. Math. Phys.*, **271** (2007), 511–522. [MR 2287915](#)
- [10] R. S. Johnson, [Camassa-Holm, Korteweg-de Vries and related models for water waves](#), *J. Fluid Mech.*, **455** (2002), 63–82. [MR 1894796](#)

- [11] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, in: *Spectral Theory and Differential Equations, Lecture Notes in Math.*, Springer Verlag, Berlin, **448** (1975), 25–70. [MR 0407477](#)
- [12] J. Lenells and M. Wunsch, On the weakly dissipative Camassa-Holm, Degasperis-Procesi, and Novikov equations, *J. Differential Equations*, **255** (2013), 441–448. [MR 3053472](#)
- [13] W. Lian and R. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, *Adv. Nonlinear Anal.*, **9** (2020), 613–632. [MR 3980417](#)
- [14] Y. Liu and Z. Yin, Global existence and blow-up phenomena for the Degasperis-Procesi equation, *Comm. Math. Phys.*, **267** (2006), 801–820. [MR 2249792](#)
- [15] Y. Liu and R. Xu, Wave equations and reaction-diffusion equations with several nonlinear source terms of different sign, *Discrete Contin. Dyn. Syst. Ser. B*, **7** (2007), 171–189. [MR 2257457](#)
- [16] O. G. Mustafa, A note on the Degasperis-Procesi equation, *J. Nonlinear Math. Phys.*, **12** (2005), 10–14. [MR 2122861](#)
- [17] E. Novruzova and A. Hagverdiyevb, On the behavior of the solution of the dissipative Camassa-Holm equation with the arbitrary dispersion coefficient, *J. Differential Equations*, **257** (2014), 4525–4541. [MR 3268734](#)
- [18] S. Wu, J. Escher and Z. Yin, Global existence and blow-up phenomena for a weakly dissipative Degasperis-Procesi equation, *Discrete Contin. Dyn. Syst. Ser. B*, **12** (2009), 633–645. [MR 2525161](#)
- [19] S. Wu and Z. Yin, Blow-up and decay of the solution of the weakly dissipative Degasperis-Procesi equation, *SIAM J. Math. Anal.*, **40** (2008), 475–490. [MR 2438773](#)
- [20] R. Xu, M. Zhang, S. Chen, Y. Yang and J. Shen, The initial-boundary value problems for a class of sixth order nonlinear wave equation, *Discrete Contin. Dyn. Syst.*, **37** (2017), 5631–5649. [MR 3681953](#)
- [21] Z. Yin, On the Cauchy problem for an integrable equation with peakon solutions, *Illinois J. Math.*, **47** (2003), 649–666. [MR 2007229](#)
- [22] R. Zheng and Z. Yin, Symmetry analysis, persistence properties and unique continuation for the cross-coupled Camassa-Holm system, *Discrete Contin. Dyn. Syst.*, **38** (2018), 329–341. [MR 3708164](#)

GUENBO HWANG, DEPARTMENT OF MATHEMATICS, DAEGU UNIVERSITY, GYEONGSAN, GYEONG-BUK 38453, REPUBLIC OF KOREA

Email address: ghwang@daegu.ac.kr

BYUNGSOO MOON, DEPARTMENT OF MATHEMATICS, INCHEON NATIONAL UNIVERSITY, INCHEON 22012, REPUBLIC OF KOREA

Email address: bsmoon@inu.ac.kr