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A NOTE ON SIGN-CHANGING SOLUTIONS FOR THE SCHRÖDINGER POISSON SYSTEM

HUI GUO* AND TAO WANG

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Abstract. We consider the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi(x)u = f(u) & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2,\ \lim_{|x|\to\infty}\phi(x) = 0 & x \in \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ and f is continuous. By combining delicate analysis and the method of invariant subsets of descending flow, we prove the existence and asymptotic behavior of infinitely many radial sign-changing solutions for odd f. The nonlinearity covers the case of pure power-type nonlinearity $f(u) = |u|^{p-2}u$ with the less studied situation $p \in (3, 4)$. This result extends and complements the ones in [Z. Liu, Z. Q. Wang, and J. Zhang, Ann. Mat. Pura Appl., 2016] from the coercive potential case to the constant potential case.

1. Introduction

In this paper, we are concerned with the existence of sign-changing solutions for the Schrödinger-Poisson system

$$(1)\qquad \begin{cases} -\Delta u + u + \lambda\phi(x)u = f(u) & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2,\ \lim\limits_{|x|\to\infty}\phi(x) = 0 & x \in \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is fixed and f satisfies

(f1): $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim\limits_{s\to 0} \frac{f(s)}{s} = 0$;

(f2): $\limsup\limits_{|s|\to\infty} \frac{|f(s)|}{|s|^{p-1}} < \infty$ for some $p \in (3, 6)$;

(f3): there exists $\mu > 3$ such that $sf(s) \geq \mu F(s) > 0$ for all $s \neq 0$.

This system arises from the study of quantum mechanics and describes the interaction of a charged particle with an electromagnetic field. For more details on the physical aspect of (1), one can refer to [3] and references therein.

System (1) has been studied extensively in the last twenty years, and there are fruitful results on the existence, nonexistence and multiplicity of radial positive solutions [1, 2, 9, 11]. In particular, when $f(u) = |u|^{p-2}u$, Ruiz [9] proved that if

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* Corresponding author: Hui Guo.

$\lambda > \frac{1}{4}$, there is no nontrivial solution when $p \in (2,3]$, and if $\lambda > 0$, there is one radial positive solution when $p \in (3,6)$. This result shows that $p = 3$ is a critical value for the existence of positive solutions. Later, Ambrosetti and Ruiz [1] proved that for any $\lambda > 0$, system (1) admits infinitely many solutions for $p \in (3,6)$. Seok [11] extended this result for general nonlinearity.

However, the signs of these solutions are not known in the above papers. When $f(u) = |u|^{p-2}u$ and $p \in (4,6)$, Kim and Seok [6] and Ianni [5] proved the existence of radial solutions of (1) with prescribed numbers of nodal domains by using Nehari type manifold and heat flow method, respectively. Wang and Zhou [13] obtained a least energy sign-changing solution of (1) in $H^1_r(\mathbb{R}^3)$, and Guo [4] proved the nonexistence of least energy nodal solution in $H^1(\mathbb{R}^3)$ and $H^1_{odd}(\mathbb{R}^3)$. Recall that a solution (u, ϕ) of (1) is called a sign-changing solution if u changes its sign. For more related results, please see [4, 5, 6, 11, 13] and references therein. However, as far as we know, when $p \in (3,4)$, there is few result on infinitely many sign-changing solutions in the literature except [8]. In [8], Liu, Wang and Zhang obtained infinitely many sign-changing solutions to the Schrodinger Poisson system

$$
(2) \qquad \begin{cases} -\Delta u + V(x)u + \phi(x)u = f(u) & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \ \lim\limits_{|x|\to\infty} \phi(x) = 0 & x \in \mathbb{R}^3, \end{cases}
$$

where f satisfies **(f1)-(f3)** and V is coercive, i.e. $\lim_{|x|\to\infty} V(x) = +\infty$ and $\inf_{x\in\mathbb{R}^3} V(x) > 0$, and satisfies some suitable conditions. A natural and interesting question arises whether system (2) admits a sign-changing solution or infinitely many sign-changing solutions for odd f when $V \equiv$ *constant*. To the best of our knowledge, this question is still unknown. In this paper, we shall give a positive answer. For simplicity, we assume that $V \equiv 1$ and our result is as follows.

Theorem 1.1. *Assume that* **(f1)-(f3)** *hold. Then for any* $\lambda > 0$, *problem* (1) *has one radial sign-changing solution. Furthermore, if* f *is odd, then problem* (1) *possesses infinitely many radial sign-changing solutions. Moreover, these solutions converge to the solutions of the limit problem*

$$
(3) \qquad -\Delta u + u = f(u) \quad \text{in } \mathbb{R}^3,
$$

as $\lambda \to 0^+$.

When $p \in (3,4)$, the main difficulty lies in whether or not a (P.S.) sequence of the action functional associated with (1) is bounded. Recall that Liu, Wang and Zhang [8] overcame this difficulty by introducing a family of auxiliary equations approximating (2). They can deduce that any (P.S.) sequence of these action functionals associated with the family of auxiliary equations is bounded, which relies essentially on the compactly embedding theorem $E \hookrightarrow L^2(\mathbb{R}^3)$, where $E := \{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 < \infty\}$. However, in view of (1), even if the radial Sobolev space $H^1_r(\mathbb{R}^3)$ is considered, the arguments in [8] can not be applied directly, because $H^1_r(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is not compact. This results in that we have to resort to new techniques to overcome the difficulties in establishing the (P.S.) condition and constructing the invariant subsets of the descending flow. So Strauss's radial lemma and some delicate analysis are needed to prove the existence and multiplicity results for sign-changing solutions. Besides, the asymptotic behaviors of these solutions will be also investigated.

The outline of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove the existence results for the auxiliary equation. Based on these existence results, Section 4 is devoted to the proof of Theorem 1.1.

2. Preliminaries

In this paper, we collect the following notations and assumptions.

- Let $H^1(\mathbb{R}^3)$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)$ be, respectively, endowed with the inner product $(u,v) = \int_{\mathbb{R}^3} \nabla u \nabla v + uv$ and $(u,v)_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^3} \nabla u \nabla v$. So their corresponding norms are $\|u\| := (\int_{\mathbb{R}^3} |\nabla u|^2 + u^2)^{1/2}$ and $\|u\|_{\mathcal{D}^{1,2}} = (\int_{\mathbb{R}^3} |\nabla u|^2)^{1/2}$, respectively. Let $H^{-1}(\mathbb{R}^3)$ be the dual space of $H^1(\mathbb{R}^3)$ and $\langle \cdot, \cdot \rangle$ denote its duality pairing.
- $\|u\|_{L^s} := (\int |u|^s)^{1/s}$ for $u \in L^s(\mathbb{R}^3)$ and we use $\int$ instead of $\int_{\mathbb{R}^3}$ for simplicity.
- C, C_j denote possibly different positive constants.

For any given $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem shows that there is a unique

$$\phi_u = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \in \mathcal{D}^{1,2}(\mathbb{R}^3)$$

such that $-\Delta \phi_u = u^2$. As is well known, by substituting $\phi = \phi_u$, the system (1) is equivalent to a single equation

$$-\Delta u + u + \lambda \phi_u u = f(u), \qquad u \in H^1(\mathbb{R}^3). \tag{4}$$

Its corresponding functional $I^\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ is defined by

$$I^\lambda(u) := \frac{1}{2}\|u\|^2 + \frac{\lambda}{4} \int \phi_u u^2 - \int F(u), \tag{5}$$

where $F(u) = \int_0^u f(s)ds$. It is easy to see that $(u, \phi_u) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a weak solution of (1) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of I^λ. By standard regularity argument, the weak solutions are also classical solutions of (1)(see [9]). We now list some properties of ϕ_u for whose proofs one can refer to [2, 9].

Lemma 2.1. *The following properties hold:*

(i): $\phi_u \geq 0$, *and* ϕ_u *is radial if* u *is radial;*
(ii): $\int \phi_u u^2 \leq C\|u\|^4$;
(iii): $\phi_{u_n} \to \phi_u$ *if* $u_n \to u$ *in* $L^{12/5}(\mathbb{R}^3)$.

Denote $D(u,v) = \frac{1}{4\pi} \int \int \frac{u(x)v(y)}{|x-y|} dxdy$. Then $D(u^2,u^2) = \int \phi_u u^2$ for $u \in H_r^1(\mathbb{R}^3)$. Now we give the following lemma.

Lemma 2.2. *The following statements are true:*

(i): $D^2(u,v) \leq D(u,u)D(v,v)$ *for any* $u, v \in L^{6/5}(\mathbb{R}^3)$;
(ii): $D^2(uv,uv) \leq D^2(u^2,u^2)D^2(v^2,v^2)$ *for any* $u, v \in L^{12/5}(\mathbb{R}^3)$.

One can see [7, p.250] and [10] for the proofs of (i) and (ii). In the sequel, a radial lemma is listed below, which is important for the proof of Theorem 1.1.

Lemma 2.3. *(Radial lemma[12]) Let* $N \geq 2$. *Then for all radial function* $u \in H_r^1(\mathbb{R}^N)$, *there holds*

$$|u(x)| \leq a_0 |x|^{(1-N)/2} \|u\| \quad \textit{for almost everywhere } |x| \geq 1,$$

where a_0 *depends only on* N.

3. The auxiliary equation and its results

In this section, we always assume $\lambda > 0$. Since $\mu > 3$, it is usually not easy to verify the P.S. condition. Motivated by [8], we first study an auxiliary equation. Let $r \in (\max\{4, p\}, 6)$ and $\theta \in (0, 1]$, and consider the following auxiliary equation

$$-\Delta u + u + \lambda\phi_u u = f(u) + \theta|u|^{r-2}u, \qquad u \in H_r^1(\mathbb{R}^3). \tag{6}$$

Clearly, the corresponding functional is

$$I_\theta^\lambda(u) = I^\lambda(u) - \frac{\theta}{r}\int |u|^r dx \in C^1(H_r^1(\mathbb{R}^3), \mathbb{R}),$$

where $I^\lambda(u)$ is defined as in (5). By the principle of symmetric criticality, a critical point of I_θ^λ in $H_r^1(\mathbb{R}^3)$ is also a critical point of I_θ^λ in $H^1(\mathbb{R}^3)$. So we consider it in the radial space $H_r^1(\mathbb{R}^3)$.

Note that for any $u \in H_r^1(\mathbb{R}^3)$, there exists a unique solution $v_\theta \in H_r^1(\mathbb{R}^3)$ to the following equation

$$-\Delta v + v + \lambda\phi_u v = f(u) + \theta|u|^{r-2}u, \qquad u \in H_r^1(\mathbb{R}^3).$$

We define an operator $A_\theta : H_r^1(\mathbb{R}^3) \to H_r^1(\mathbb{R}^3)$ by $v_\theta = A_\theta(u)$. Obviously, if f is odd, A_θ is odd. Moreover, the following three statements are equivalent: $u \in H_r^1(\mathbb{R}^3)$ is a solution of (6), $u \in H_r^1(\mathbb{R}^3)$ is a critical point of functional I_θ^λ, and u is a fixed point of A_θ.

Define the positive and negative cone

$$P^+ := \{u \in H_r^1(\mathbb{R}^3) : u \geq 0\} \quad \text{and} \quad P^- := \{u \in H_r^1(\mathbb{R}^3) : u \leq 0\}.$$

For any $\epsilon > 0$, set

$$P_\epsilon^+ := \{u \in H_r^1(\mathbb{R}^3) : dist(u, P^+) < \epsilon\} \text{and} P_\epsilon^- := \{u \in H_r^1(\mathbb{R}^3) : dist(u, P^-) < \epsilon\},$$

where $dist(u, P^\pm) := \inf_{v \in P^\pm} \|u - v\|$. Clearly, $P_\epsilon^- = -P_\epsilon^+$ and $W := P_\epsilon^+ \cup P_\epsilon^-$ is open, symmetric in $H_r^1(\mathbb{R}^3)$. As stated in [8, Lemmas 3.1, 4.1, 4.3], the operator A_θ is well defined and is continuous and compact; and there exists $\bar{\epsilon_0} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon_0})$, $A_\theta(\partial P_\epsilon^\pm) \subset P_\epsilon^\pm$, and there exists $C > 0$ independent of θ such that

$$\|(I_\theta^\lambda)'(u)\| \leq \|u - A_\theta(u)\|(1 + C\|u\|^2), \ \forall \ u \in H_r^1(\mathbb{R}^3). \tag{7}$$

Lemma 3.1. *For any $a < b$ and $\alpha > 0$, if $u \in H_r^1(\mathbb{R}^3)$ satisfies $I_\theta^\lambda(u) \in [a, b]$ and $\|(I_\theta^\lambda)'(u)\| \geq \alpha$, then there exists $\beta > 0$ depending on θ such that $\|u - A_\theta(u)\| \geq \beta$.*

Proof. Take $\gamma \in (4, r)$. Then for $u \in H_r^1(\mathbb{R}^3)$, we have

$$\begin{aligned} I_\theta^\lambda(u) - \frac{1}{\gamma}(u, u - A_\theta(u)) =& (\frac{1}{2} - \frac{1}{\gamma})\|u\|^2 + (\frac{\lambda}{4} - \frac{\lambda}{\gamma})\int \phi_u u^2 + \frac{\lambda}{\gamma}\int \phi_u u(u - A_\theta(u)) \\ &+ \int (\frac{1}{\gamma}f(u)u - F(u)) + (\frac{\theta}{\gamma} - \frac{\theta}{r})\int |u|^r. \end{aligned}$$

By **(f1)** and **(f2)**, it yields

$$\begin{aligned} &\|u\|^2 + \lambda\int \phi_u u^2 + \theta\int |u|^r \\ \leq& C_1\left[|I_\theta^\lambda(u)| + \|u\|\|u - A_\theta(u)\| + \|u\|_{L^p}^p + \left|\lambda\int \phi_u u(u - A_\theta(u))\right|\right]. \end{aligned}$$

Since Lemma 2.2 (i) and the Hardy-Littlewood-Sobolev inequality [7] imply that

$$\left|\int \phi_u u(u - A_\theta(u))\right| \leq C_2\|u\|\|u - A_\theta(u)\| \left(\int \phi_u u^2\right)^{1/2},$$

by the Young inequality, we get that

$$\begin{aligned} &\|u\|^2 + \frac{\lambda}{2}\int \phi_u u^2 + \theta \int |u|^r \\ \leq &C_3\left[|I_\theta^\lambda(u)| + \|u\|\|u - A_\theta(u)\| + \|u\|_{L^p}^p + \|u\|^2\|u - A_\theta(u)\|^2\right]. \end{aligned} \tag{8}$$

Then we shall prove the lemma by contradiction. Suppose on the contrary that there exists $\{u_n\}_n \subset H_r^1(\mathbb{R}^3)$ with $I_\theta^\lambda(u_n) \in [a, b]$ and $\|(I_\theta^\lambda)'(u_n)\| \geq \alpha$ such that $\|u_n - A_\theta(u_n)\| \to 0$ as $n \to \infty$. Then it follows from (8) that for large n,

$$\|u_n\|^2 + \frac{\lambda}{2}\int \phi_{u_n} u_n^2 + \theta \int |u_n|^r \leq C_4(1 + \|u_n\|_{L^p}^p), \tag{9}$$

where $C_4 > 0$ is independent of n.

Now, we claim that $\{u_n\}_n$ is bounded in $H_r^1(\mathbb{R}^3)$. Otherwise, suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Then it follows from (9) that for large n,

$$\frac{1}{4}\|u_n\|^2 + \frac{\lambda}{2}\int \phi_{u_n} u_n^2 + \int (\frac{1}{2}u_n^2 + \theta|u_n|^r - C_4|u_n|^p) \leq 0. \tag{10}$$

Define a function

$$h : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, \quad h(u) = \frac{1}{2}u^2 + \theta|u|^r - C_4|u|^p.$$

Clearly, since $p \in (3, r)$, h is positive for $u \to 0^+$ or $u \to +\infty$. So the value $m_0 := \min_{\mathbb{R}^+\cup\{0\}} h > -\infty$. If $m_0 = 0$, the claim follows immediately. Hence we assume $m_0 < 0$. Obviously, the set $\{u > 0 : h(u) < 0\}$ must be of the form (c, d) with $c, d > 0$. It follows from (10) that

$$\begin{aligned} 0 &\geq \frac{1}{4}\|u_n\|^2 + \frac{\lambda}{2}\int \phi_{u_n} u_n^2 + \int h(u_n) \\ &\geq \frac{1}{4}\|u_n\|^2 + \frac{\lambda}{2}\int \phi_{u_n} u_n^2 + \int_{u_n \in (c,d)} h(u_n) \\ &\geq \frac{1}{4}\|u_n\|^2 + \frac{\lambda}{2}\int \phi_{u_n} u_n^2 + m_0|A_n| \end{aligned}$$

where $A_n = \{x \in \mathbb{R}^3 : u_n(x) \in (c, d)\}$ and $|A_n|$ denotes its Lebesgue measure. Thus we have

$$|m_0||A_n| \geq \frac{1}{4}\|u_n\|^2 + \frac{\lambda}{2}\int \phi_{u_n} u_n^2 \to \infty \quad \text{as } n \to \infty. \tag{11}$$

Note that the set A_n is spherically symmetric. Let $\rho_n := \sup\{|x| : x \in A_n\}$ and take $x \in \mathbb{R}^3$ with $|x| = \rho_n$. According to real analysis, the functions are identified if they are equal almost everywhere. So $u_n(x) = c$ and by Lemma 2.3 and (11),

$$0 < c = u_n(x) \leq a_0|\rho_n|^{-1}\|u_n\| \leq a_0|\rho_n|^{-1}(2|m_0||A_n|)^{1/2} \Rightarrow C_5\rho_n \leq |A_n|^{1/2} \tag{12}$$

for some $C_5 > 0$ independent of n.

On the other hand, the inequality (11) yields $\frac{\lambda}{2}\int \phi_{u_n} u_n^2 \le |m_0||A_n|$ and then

$$|m_0||A_n| \ge \frac{\lambda}{2}\int \phi_{u_n} u_n^2 \ge \frac{\lambda}{8\pi}\int_{A_n}\int_{A_n} \frac{u_n^2(x)u_n^2(y)}{|x-y|}dxdy \ge \frac{\lambda c^4}{8\pi}\int_{A_n}\int_{A_n}\frac{1}{|x-y|}dxdy \ge \frac{\lambda c^4}{8\pi}\frac{|A_n|^2}{2\rho_n}.$$

Thus,

$$C_6\rho_n \ge |A_n|$$

for some $C_6 > 0$. Clearly, it yields a contradiction with (11) and (12). So the claim is verified.

According to (7), it follows that $\|(I_\theta^\lambda)'(u_n)\| \to 0$ as $n \to \infty$, which contradicts our assumptions. Hence the proof is completed. □

Lemma 3.2. *(P.S. condition) Let $c \in \mathbb{R}$ and $\{u_n\}_n \subset H_r^1(\mathbb{R}^3)$ be a P.S. sequence of (6) at level c, namely,*

$$I_\theta^\lambda(u_n) \to c \text{ and } (I_\theta^\lambda)'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Then $\{u_n\}_n$ has a convergent subsequence.

Proof. Let $\gamma \in (4, r)$. Then

$$\begin{aligned}&I_\theta^\lambda(u_n) - \frac{1}{\gamma}\langle (I_\theta^\lambda)'(u_n), u_n\rangle \\ =&(\frac{1}{2}-\frac{1}{\gamma})\|u_n\|^2 + (\frac{\lambda}{4}-\frac{\lambda}{\gamma})\int \phi_{u_n}u_n^2 + \int (\frac{1}{\gamma}f(u_n)u_n - F(u_n)) + (\frac{\theta}{\gamma}-\frac{\theta}{r})\|u_n\|_{L^r}^r,\end{aligned}$$

and by **(f1)** and **(f2)**, it follows

$$\|u_n\|^2 + \lambda\int \phi_{u_n}u_n^2 + \theta\|u_n\|_{L^r}^r \le C(|I_\theta^\lambda(u_n)| + \|u_n\|\|(I_\theta^\lambda)'(u_n)\| + \|u_n\|_{L^p}^p),$$

where $C > 0$ is independent of n. Furthermore, by the conditions and Young inequality, it follows that for n large enough,

$$\|u_n\|^2 + \lambda\int \phi_{u_n}u_n^2 + \theta\|u_n\|_{L^r}^r \le C(1 + \|u_n\|_{L^p}^p).$$

As in the proof of Lemma 3.1, by using a similar argument as (9), one can deduce that $\{u_n\}_n$ is bounded in $H_r^1(\mathbb{R}^3)$. Thus, without loss of generality, we assume $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$ up to a subsequence. Since the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 < s < 6)$ is compact, we deduce that $\int F(u_n) \to \int F(u)$ and then $u_n \to u$ in $H_r^1(\mathbb{R}^3)$. The proof is completed. □

With the aid of Lemmas 3.1 and 3.2, one can use similar arguments as [8, Corollary 3.1, Theorem 1.2] to prove that $\{P_\epsilon^+, P_\epsilon^-\}$ is an admissible family of invariant subsets for small $\epsilon > 0$ independent of λ and ν, and the following results hold true. The interested readers can refer to [8] for the details, here we omit the proof.

Proposition 1. *Suppose that f satisfies assumptions* **(f1)** − **(f3)**. *Let $\theta \in (0, 1]$ and $r \in (\max\{4, p\}, 6)$. Then*

- **(i):** *equation (6) has one sign-changing solution $u_\theta^\lambda \in H_r^1(\mathbb{R}^3)$ such that $I_\theta^\lambda(u_\theta^\lambda) = c_\theta$, where*

$$c^\lambda(\theta) = \inf_{\psi\in\Gamma}\sup_{u\in\psi(\Delta)\setminus W} I_\theta^\lambda(u) \ge \frac{\epsilon^2}{2} > 0 \tag{13}$$

with small $\epsilon > 0$, *where* $\Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 \leq 1\}$, $\partial_1\Delta = \{0\} \times [0, 1]$, $\partial_2\Delta = [0, 1] \times \{0\}$ *and* $\partial_0\Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 = 1\}$, $\Gamma := \{\psi \in C(\Delta, X) : \psi(\partial_1\Delta) \subset P_\epsilon^+, \psi(\partial_2\Delta) \subset P_\epsilon^-, \psi|_{\partial_0\Delta} = \psi_0|_{\partial_0\Delta}\}$ *and* $\psi_0(t, s)(\cdot) = R^2(tv_1(R\cdot) + sv_2(R\cdot))$ *with large* $R > 0$.

(ii): *if* f *is odd, then equation* (6) *has infinitely many sign-changing solutions* $\{u_{\theta,j}^\lambda\}_{j\geq 2} \subset H_r^1(\mathbb{R}^3)$ *such that* $I_\theta^\lambda(u_{\theta,j}^\lambda) = c_j(\theta)$, *where*

$$c_j^\lambda(\theta) = \inf_{B\in\Gamma_j} \sup_{u\in B\setminus W} I_\theta^\lambda(u) \geq \frac{\epsilon^2}{2} > 0, \tag{14}$$

where

$$\Gamma_j := \{B \in X : B = \psi(B_n\setminus Y) \text{ for some } \psi \in G_n, Y \subset B_n \\ \text{with } n \geq j, \text{ such that } Y = -Y \text{ and } \gamma(\overline{Y}) \leq n - j\}$$

with $B_n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ *and* γ *denotes the genus of closed symmetric subsets,*

$$G_n := \{\psi \in C(B_n, X) : \psi(-t) = G\psi(t) \text{ for } t \in B_n, \psi(0) \in M \text{ and} \\ \psi|_{\partial B_n} = \psi_n|_{\partial B_n}\},$$

the group $G = \{id, -id\}$ *and* $\psi_n(t)(\cdot) = R_n^2 \sum_{i=1}^n t_i v_i(R_n\cdot)$ *with large* $R_n > 0$ *and* $t = (t_1, \cdots, t_n) \in B_n$.

4. Proof of Theorem 1.1

We shall complete the proof by using Propositions 1 and passing to the limit as $\theta \to 0^+$.

(Existence part and asymptotic behaviors): According to Proposition 1, for given $\lambda > 0$ and any $\theta \in (0, 1]$, equation (6) admits one radial sign-changing solution u_θ^λ such that $I_\theta^\lambda(u_\theta^\lambda) = c^\lambda(\theta)$. By the definition of $c^\lambda(\theta)$ in (13), we see that

$$\frac{\epsilon^2}{2} \leq c^\lambda(\theta) \leq \sup_{u\in\psi_0(\Delta)} I_\theta^\lambda(u) \leq \sup_{u\in\psi_0(\Delta)} I^\lambda(u) < +\infty.$$

Observe that $c^\lambda(\theta)$ is non-increasing in θ. Then

$$c^\lambda = \lim_{\theta\to 0^+} c^\lambda(\theta) \in (\frac{\epsilon^2}{2}, \infty) \tag{15}$$

is well-defined. In addition, solutions $\{u_\theta^\lambda\}_{\theta\in(0,1]}$ satisfy

$$c^\lambda(\theta) = \frac{1}{2}\|u_\theta^\lambda\|^2 + \frac{\lambda}{4}\int \phi_{u_\theta^\lambda}|u_\theta^\lambda|^2 - \int (F(u_\theta^\lambda) + \frac{\theta}{r}|u_\theta^\lambda|^r), \tag{16}$$

$$0 = \|u_\theta^\lambda\|^2 + \lambda\int \phi_{u_\theta^\lambda}|u_\theta^\lambda|^2 - \int (u_\theta^\lambda f(u_\theta^\lambda) + \theta|u_\theta^\lambda|^r) \tag{17}$$

and Pohozaev identity

$$0 = \frac{1}{2}\|\nabla u_\theta^\lambda\|_{L^2}^2 + \frac{3}{2}\|u_\theta^\lambda\|_{L^2}^2 + \frac{5\lambda}{4}\int \phi_{u_\theta^\lambda}|u_\theta^\lambda|^2 - \int (3F(u_\theta^\lambda) + \frac{3\theta}{r}|u_\theta^\lambda|^r). \tag{18}$$

Since **(f2)** and **(f3)** imply $3 < \mu \le p \le r < 6$, by multiplying (16) and (17) by μ and -2 respectively, and adding them to (18), we get that

$$\begin{aligned}\mu c^\lambda(\theta) &= \frac{\mu-3}{2}\|\nabla u_\theta^\lambda\|_{L^2}^2 + \frac{\mu-1}{2}\|u_\theta^\lambda\|_{L^2}^2 + \frac{\lambda(\mu-3)}{4}\int \phi_{u_\theta^\lambda}|u_\theta^\lambda|^2 \\ &\quad + \int\left(2u_\theta^\lambda f(u_\theta^\lambda) - (\mu+3)F(u_\theta^\lambda) + \frac{(2r-\mu-3)\theta}{r}|u_\theta^\lambda|^r\right) \\ &\ge \frac{\mu-3}{2}\|\nabla u_\theta^\lambda\|_{L^2}^2 + \frac{\mu-1}{2}\|u_\theta^\lambda\|_{L^2}^2 + \frac{\lambda(\mu-3)}{4}\int \phi_{u_\theta^\lambda}|u_\theta^\lambda|^2.\end{aligned} \tag{19}$$

This implies that $\{u_\theta^\lambda\}_{\theta\in(0,1]}$ are bounded.

Without loss of generality, assume that up to a subsequence, $u_{\theta_n}^\lambda \rightharpoonup u^\lambda$ in $H_r^1(\mathbb{R}^3)$ as $\theta_n \to 0^+$. Then by (iii) of Lemma 2.1 and a standard argument, we have $(I^\lambda)'(u^\lambda) = 0, I^\lambda(u^\lambda) = c^\lambda$ and $u_{\theta_n}^\lambda \to u^\lambda$ in $H_r^1(\mathbb{R}^3)$ as $\theta_n \to 0^+$. Moreover, $u^\lambda \in H_r^1(\mathbb{R}^3)\backslash(P_\epsilon^+ \cup P_\epsilon^-)$, because $u_{\theta_n}^\lambda \in H_r^1(\mathbb{R}^3)\backslash(P_\epsilon^+ \cup P_\epsilon^-)$. Thus, u^λ is a radial sign-changing solution of (4) with positive energy c^λ.

Note that c^λ is non-decreasing with respect to $\lambda > 0$. Then it follows from (15) that the limit

$$c^0 := \lim_{\lambda\to 0^+} c^\lambda$$

exists and $c^0 \ge \frac{\epsilon^2}{2}$. Thus $\{c^\lambda\}_{\lambda\to 0^+}$ is bounded. Since $I^\lambda(u^\lambda) = c^\lambda$ and $(I^\lambda)'(u^\lambda) = 0$, we can argue similarly as (16)-(18) to derive that $\{u^\lambda\}_{\lambda\to 0^+}$ is bounded in $H_r^1(\mathbb{R}^3)$. Then there is a subsequence $\{\lambda_n\}$ with $\lambda_n \to 0_+$ such that $u^{\lambda_n} \rightharpoonup u^0$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$. It follows from $(I^{\lambda_n})'(u^{\lambda_n})$ that $\mathcal{I}'(u^0) = 0$, where $\mathcal{I}$ is the functional associated to (3). By the compactly embedding $H_r^1(\mathbb{R}^3) \to L^s(\mathbb{R}^3)$ with $s \in (2,6)$, it is standard to conclude that $u^{\lambda_n} \to u^0$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$. Then $\mathcal{I}(u^0) = c^0$ and $\mathcal{I}'(u^0) = 0$. So u^0 is a radial solution of (3).

(Multiplicity part and asymptotic behaviors): According to Proposition 1 (ii), for any $\theta \in (0,1]$, equation (6) admits infinitely many radial sign-changing solutions $\{u_{\theta,j}^\lambda\}_{j\ge 2}$ such that $I_\theta^\lambda(u_{\theta,j}^\lambda) = c_j^\lambda(\theta)$. In a similar way as (19), we can prove that for any fixed $j \ge 2$, the sequence $\{u_{\theta,j}^\lambda\}_{\theta\in(0,1]}$ is bounded in $H_r^1(\mathbb{R}^3)$. Without loss of generality, we assume that $u_{\theta,j}^\lambda \rightharpoonup u_j^\lambda$ for some $u_j^\lambda \in H_r^1(\mathbb{R}^3)$ as $\theta \to 0^+$. Note that $c_j^\lambda(\theta)$ is decreasing in θ and $c_j^\lambda(\theta) \le \sup_{u\in B\backslash W} I_\theta^\lambda(u) < +\infty$. Then by (14), $c_j^\lambda := \lim_{\theta\to 0^+} c_j^\lambda(\theta)$ is well defined and

$$\frac{\epsilon^2}{2} \le c_j^\lambda(\theta) \le c_j^\lambda \le \sup_{u\in B\backslash W} I^\lambda(u) < \infty \tag{20}$$

for all $\theta \in (0,1]$. By the compactly embedding theorem and standard arguments, it follows that $u_{\theta,j}^\lambda \to u_j^\lambda$ in $H_r^1(\mathbb{R}^3)$ as $\theta \to 0^+$ for some $u_j^\lambda \in H_r^1(\mathbb{R}^3)\backslash W$. Thus $I^\lambda(u_j^\lambda) = c_j^\lambda$ and $(I^\lambda)'(u_j^\lambda) = 0$. Since $c_j^\lambda(\theta) \to +\infty$ as $j \to \infty$ in (14), we see that $c_j^\lambda \to +\infty$ as $j \to +\infty$. Therefore, (4) (or (1)) has infinitely many radial sign-changing solutions $\{u_j^\lambda\}_{j\ge 2}$.

Since c_j^λ is non-decreasing in $\lambda > 0$. Then by (20), the limit

$$c_j := \lim_{\lambda\to 0^+} c_j^\lambda$$

exists and $\frac{\epsilon^2}{2} \le c_j < \infty$. Clearly, for fixed $j \ge 2$, $\{c_j^\lambda\}_{\lambda\to 0^+}$ is bounded. We can also argue similarly as (16)-(18) to deduce that $\{u_j^\lambda\}_{\lambda\to 0_+}$ is bounded in E, since $I^\lambda(u_j^\lambda) = c_j^\lambda$ and $(I^\lambda)'(u_j^\lambda) = 0$. Thus there exists a sequence $\{\lambda_n\}$ tending to 0 and some $u_j \in H_r^1(\mathbb{R}^3)$ such that $u_j^{\lambda_n} \rightharpoonup u_j$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$. By the weakly sequentially continuity, it follows immediately that $\mathcal{I}'(u_j) = 0$, where $\mathcal{I}$ is the functional for (3). By the compactly embedding $H_r^1(\mathbb{R}^3) \to L^s(\mathbb{R}^3)$ with $s \in (2,6)$, we can deduce from the standard arguments that $u_j^{\lambda_n} \to u_j$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$. So $\mathcal{I}(u_j) = c_j$ and $\mathcal{I}'(u_j) = 0$. Namely, u_j is a radial solution of (3). The proof is complete. □

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Hui Guo, College of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, China
Email address: huiguo_math@163.com

Tao Wang, College of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, China
Email address: wt_61003@163.com