

THE ANISOTROPIC INTEGRABILITY LOGARITHMIC REGULARITY CRITERION FOR THE 3D MHD EQUATIONS

AHMAD MOHAMMAD ALGHAMDI, SADEK GALA*
 CHENYIN QIAN AND MARIA ALESSANDRA RAGUSA

(Communicated by Alain Miranville)

ABSTRACT. This study is devoted to investigating the regularity criterion of the 3D MHD equations in terms of pressure in the framework of anisotropic Lebesgue spaces. The result shows that if a weak solution (u, b) satisfies

$$(1) \quad \int_0^T \frac{\left\| \partial_3 \pi(\cdot, t) \right\|_{L_{x_3}^\gamma}^q}{1 + \ln \left(e + \left\| \pi(\cdot, t) \right\|_{L^2}^2 \right)} dt < \infty,$$

where

$$\frac{1}{\gamma} + \frac{2}{q} + \frac{2}{\alpha} = \lambda \in [2, 3) \quad \text{and} \quad \frac{3}{\lambda} \leq \gamma \leq \alpha < \frac{1}{\lambda - 2},$$

then (u, b) is regular at $t = T$, which improve the previous results on the MHD equations

1. INTRODUCTION

Let us consider the following Cauchy problem of the incompressible magneto-hydrodynamic (MHD) equations in three-spatial dimensions :

$$(2) \quad \begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \pi - (b \cdot \nabla) b = 0, \\ \partial_t b - \Delta b + (u \cdot \nabla) b - (b \cdot \nabla) u = 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), \end{cases}$$

where u is the velocity field, b the magnetic field and π the pressure, while u_0, b_0 are given initial data with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the sense of distributions.

Due to the importance of both physics and mathematics, there is large literature on the well-posedness for weak solutions of magnetohydrodynamic equations. However, similar to the Navier-Stokes equations ($b = 0$), the question of global regularity of the weak solutions to (2) still one of the most challenging problems in the theory of PDE's (see for example [3, 4, 5, 6, 9, 10, 11, 12, 13] and the references therein). It is interesting to study the regularity of the weak solutions of (2) by imposing some growth sufficient conditions on the velocity or the pressure. In particular, the

Received by the editors October, 2019 and, in revised form, January, 2020.

2010 *Mathematics Subject Classification*. Primary: 35Q35, 35B65; Secondary: 76D05.

Key words and phrases. MHD equations, regularity criterion, anisotropic Lebesgue spaces, a priori estimates.

This research is partially supported by GNAMPA 2019. The fourth author wishes to thank the support of "RUDN University Program 5-100".

* Corresponding author: Sadek Gala, sgala793@gmail.com.

condition via only one directional derivative of the pressure was established in [2] and showed that the weak solution becomes regular if the pressure π satisfies

$$(3) \quad \partial_3 \pi \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \text{for} \quad \frac{2}{\alpha} + \frac{3}{\beta} = \frac{7}{4}, \quad \frac{12}{7} \leq \beta \leq 4.$$

Later on, Jia and Zhou [7] improve (3) as

$$(4) \quad \partial_3 \pi \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \text{for} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2, \quad 3 \leq \beta < \infty.$$

Recently, some interesting logarithmical pressure regularity criteria of MHD equations are studied. In particular, Benbernou et al. [1] refined (4) by imposing the following regularity criterion

$$\int_0^T \frac{\|\partial_3 \pi(\cdot, t)\|_{L^\lambda}^{\frac{2\lambda}{3-2\lambda}}}{1 + \ln(e + \|b(\cdot, t)\|_{L^3})} dt < \infty \quad \text{with} \quad \frac{3}{2} < \lambda \leq \infty.$$

The aim of this paper is to establish the logarithmical regularity criterion in terms of the partial derivative of pressure in the framework of anisotropic Lebesgue space.

Throughout the rest of this paper, we endow the usual Lebesgue space $L^p(\mathbb{R}^3)$ with the norm $\|\cdot\|_{L^p}$. We denote by $\partial_i = \frac{\partial}{\partial x_i}$ the partial derivative in the x_i -direction. Recall that the anisotropic Lebesgue space consists on all the total measurable real valued functions $h = h(x_1, x_2, x_3)$ with finite norm

$$\|h\|_{L_{x_i}^p} \|h\|_{L_{x_j x_k}^q} = \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |h(x)|^p dx_i \right)^{\frac{q}{p}} dx_j dx_k \right)^{\frac{1}{q}},$$

where (i, j, k) belongs to the permutation group $S = \text{span}\{1, 2, 3\}$.

Before giving the main result, let us first recall the definition of weak solutions for MHD equations (2).

Definition 1.1 (weak solutions). Let $(u_0, b_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the sense of distribution and $T > 0$. A pair vector field $(u(x, t), b(x, t))$ is called a weak solution of (2) on $(0, T)$ if (u, b) satisfies the following properties :

- (i): $(u, b) \in L^\infty((0, T); L^2(\mathbb{R}^3)) \cap L^2((0, T); H^1(\mathbb{R}^3))$;
- (ii): $\nabla \cdot u = \nabla \cdot b = 0$ in the sense of distribution;
- (iii): (u, b) verifies (2) in the sense of distribution.
- (iv): (u, b) satisfies the energy inequality, that is,

$$\|u(\cdot, t)\|_{L^2}^2 + \|b(\cdot, t)\|_{L^2}^2 + 2 \int_0^t (\|\nabla u(\cdot, \tau)\|_{L^2}^2 + \|\nabla b(\cdot, \tau)\|_{L^2}^2) d\tau \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2,$$

for all $t \in [0, T]$.

Now, our result read as follows.

Theorem 1.2. Let $(u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in \mathbb{R}^3 . Suppose that (u, b) is a weak solution of (2) in $(0, T)$. If the pressure π satisfies the condition

$$(5) \quad \int_0^T \frac{\left\| \|\partial_3 \pi(\cdot, t)\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^q}{1 + \ln(e + \|\pi(\cdot, t)\|_{L^2}^2)} dt < \infty,$$

where

$$\frac{1}{\gamma} + \frac{2}{q} + \frac{2}{\alpha} = \lambda \in [2, 3) \quad \text{and} \quad \frac{3}{\lambda} \leq \gamma \leq \alpha < \frac{1}{\lambda - 2},$$

then the weak solution (u, b) becomes a regular solution on $(0, T]$.

This allows us to obtain the regularity criterion of weak solutions via only one directional derivative of the pressure. This extends and improve some known regularity criterion of weak solutions in term of one directional derivative, including the notable works of Jia and Zhou [7].

As an application of Theorem 1.2, we also obtain the following regularity criterion of weak solutions.

Corollary 1.3. *Let $(u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the sense of distributions. Assume that (u, b) is a weak solution of (2) in $(0, T)$. If the pressure satisfies the condition*

$$(6) \quad \int_0^T \frac{\|\partial_3 \pi(\cdot, t)\|_{L^\alpha}^q}{1 + \ln(e + \|\pi(\cdot, t)\|_{L^2}^2)} dt < \infty,$$

where

$$\frac{2}{q} + \frac{3}{\alpha} = \lambda \in [2, 3) \quad \text{and} \quad \frac{3}{\lambda} \leq \alpha < \frac{1}{\lambda - 2},$$

then the weak solution (u, b) becomes a regular solution on $(0, T]$.

In order to prove the main result, we need to recall the following lemma, which is proved in [9] (see also [8]).

Lemma 1.4. *Let us assume that $r > 1$ and $1 < \gamma \leq \alpha < \infty$. Then for $f, g, \varphi \in C_0^\infty(\mathbb{R}^3)$, we have the following estimate*

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} f g \varphi dx_1 dx_2 dx_3 \right| \\ & \leq C \left\| \|\partial_3 \varphi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{1}{r}} \left\| \|\partial_3 \varphi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^{a\theta(r-1)}}^{\frac{\theta(r-1)}{r}} \left\| \|\varphi\|_{L_{x_3}^\beta} \right\|_{L_{x_1 x_2}^{b(1-\theta)(r-1)}}^{\frac{(1-\theta)(r-1)}{r}} \\ & \quad \times \|f\|_{L^2}^{\frac{r-1}{r}} \|\partial_1 f\|_{L^2}^{\frac{1}{2r}} \|\partial_2 f\|_{L^2}^{\frac{1}{2r}} \|g\|_{L^2}^{\frac{r-1}{r}} \|\partial_1 g\|_{L^2}^{\frac{1}{2r}} \|\partial_2 g\|_{L^2}^{\frac{1}{2r}}. \end{aligned}$$

where $0 \leq \theta \leq 1$ satisfying

$$(7) \quad \frac{1}{a} + \frac{1}{b} = \frac{\alpha - 1}{\alpha},$$

and

$$(8) \quad \frac{1}{\gamma(r-1)} + \frac{\theta}{\gamma} = \frac{1-\theta}{\beta(\gamma-1)}.$$

and C is a constant independent of f, g, φ .

2. PROOF OF THEOREM 1.2

We are now ready to give the proof of Theorem 1.2. Clearly, in order to prove Theorem 1.2, it suffices to show that the assumption (5) ensures the following a priori estimate :

$$\lim_{t \rightarrow T^-} (\|u(\cdot, t)\|_{L^4} + \|b(\cdot, t)\|_{L^4}) < \infty.$$

Proof. We first convert the MHD system equations into a symmetric form. Adding and subtracting (2)₁ and (2)₂, we get w^+ and w^- satisfy

$$(9) \quad \begin{cases} \partial_t w^+ + (w^- \cdot \nabla) w^+ = \Delta w^+ - \nabla \pi, \\ \partial_t w^- + (w^+ \cdot \nabla) w^- = \Delta w^- - \nabla \pi, \\ \nabla \cdot w^+ = \nabla \cdot w^- = 0, \\ w^+(x, 0) = u_0 + b_0, \quad w^-(x, 0) = u_0 - b_0, \end{cases}$$

with $w^\pm = u \pm b$.

Next, we establish some fundamental estimates between the pressure π and w^\pm . Taking the divergence operator $\nabla \cdot$ on both sides of the first and second equations of (2.1) gives

$$\begin{cases} \nabla \cdot (\partial_t w^+ + w^- \cdot \nabla w^+) = \nabla \cdot \Delta w^+ - \nabla \cdot (\nabla \pi), \\ \nabla \cdot (\partial_t w^- + w^+ \cdot \nabla w^-) = \nabla \cdot \Delta w^- - \nabla \cdot (\nabla \pi), \end{cases}$$

and hence

$$\Delta \pi = \operatorname{div}(w^- \cdot \nabla w^+) = \operatorname{div}(w^+ \cdot \nabla w^-) = \sum_{i,j=1}^3 \partial_j w_i^- \cdot \partial_i w_j^+,$$

where we have used the divergence free condition $\nabla \cdot w^+ = \nabla \cdot w^- = 0$. Due to the boundedness of Riesz transform in Lebesgue space L^p ($1 < p < \infty$), we have

$$\|\pi\|_{L^p} \leq C \|w^+\|_{L^{2p}} \|w^-\|_{L^{2p}},$$

Similarly, acting the operator $\nabla \operatorname{div}$ on both sides of the first and second equations of (9), one shows that

$$-\Delta \nabla \pi = \nabla \operatorname{div}(w^- \cdot \nabla w^+) = \nabla \operatorname{div}(w^+ \cdot \nabla w^-),$$

together with Calderón-Zygmund inequality, implies that for any $1 < p < \infty$,

$$(10) \quad \|\nabla \pi\|_{L^p} \leq C \|w^- \cdot \nabla w^+\|_{L^p}$$

or

$$(11) \quad \|\nabla \pi\|_{L^p} \leq C \|w^+ \cdot \nabla w^-\|_{L^p}.$$

By means of the local existence result, (2) with $(u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ admit a unique L^4 -strong solution (u, b) on a maximal time interval. For the notation simplicity, we may suppose that the maximal time interval is $[0, T)$. It is obvious that to prove regularity for u and b , it is sufficient to prove it for w^+ and w^- . We shall show

$$\lim_{t \rightarrow T^-} (\|w^+(\cdot, t)\|_{L^4} + \|w^-(\cdot, t)\|_{L^4}) < \infty,$$

and thus $\|u\|_{L^4}^2 + \|b\|_{L^4}^2$ are uniformly bounded on $[0, T]$. This will lead to a contradiction to the estimates to be derived below. We now begin to follow this argument.

We multiply (9)₁ by $w^+ |w^+|^2$, (9)₂ by $w^- |w^-|^2$ in $L^2(\mathbb{R}^3)$, respectively and integrating by parts, we obtain

$$\begin{aligned} & \frac{1}{4} \frac{d}{dt} (\|w^+\|_{L^4}^4 + \|w^-\|_{L^4}^4) + \int_{\mathbb{R}^3} (|w^+|^2 |\nabla w^+|^2 + |w^-|^2 |\nabla w^-|^2) dx \\ & + \frac{1}{2} \left(\left\| \nabla |w^+|^2 \right\|_{L^2}^2 + \left\| \nabla |w^-|^2 \right\|_{L^2}^2 \right) \\ & = \int_{\mathbb{R}^3} \pi w^+ \cdot \nabla |w^+|^2 dx + \int_{\mathbb{R}^3} \pi w^- \cdot \nabla |w^-|^2 dx = I + J \end{aligned}$$

By Hölder's and Young's inequalities,

$$(12) \quad I \leq C \int_{\mathbb{R}^3} |\pi|^2 |w^+|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |w^+|^2|^2 dx = A_1 + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |w^+|^2|^2 dx.$$

In the following, we establish the bound of the integral

$$A_1 = \int_{\mathbb{R}^3} |\pi|^2 |w^+|^2 dx$$

on the right-hand side of (12).

We select that

$$a = \frac{\alpha(\gamma + \alpha\gamma - \alpha)}{\alpha - \gamma} \quad \text{and} \quad b = \frac{\gamma + \alpha\gamma - \alpha}{\alpha(\gamma - 1)}$$

in Lemma 1.4, then the selected a and b satisfy (7). To bound A_1 and A_2 , we recall the

$$\begin{aligned} \|\pi\|_{L^p} &\leq C \|w^+\|_{L^{2p}} \|w^-\|_{L^{2p}}, \\ \|\nabla \pi\|_{L^p} &\leq C \|w^- \cdot \nabla w^+\|_{L^p}, \\ \|\nabla \pi\|_{L^p} &\leq C \|w^+ \cdot \nabla w^-\|_{L^p}, \end{aligned}$$

we can estimate A_1 as follows

$$\begin{aligned} A_1 &= \int_{\mathbb{R}^3} |\pi| |\pi| |w^+|^2 dx_1 dx_2 dx_3 \\ &\leq C \left\| \|\partial_3 \pi\|_{L^{\frac{\gamma}{\gamma+\beta(\gamma-1)}}_{x_3}} \right\|_{L^{\frac{\alpha}{\alpha(\gamma+\beta(\gamma-1))}}_{x_1 x_2}} \left\| \|\pi\|_{L^{\frac{\beta(\gamma-1)}{\gamma+\beta(\gamma-1)}}} \right\|_{L^2} \left\| \|\pi\|_{L^2}^{\frac{\alpha-\gamma+\alpha\beta(\gamma-1)}{\alpha(\gamma+\beta(\gamma-1))}} \right\|_{L^2} \left\| \|\partial_1 \pi\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{2\alpha(\gamma+\beta(\gamma-1))}} \right\|_{L^2} \left\| \|\partial_2 \pi\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{2\alpha(\gamma+\beta(\gamma-1))}} \right\|_{L^2} \\ &\quad \times \left\| \|w^+\|^2 \right\|_{L^2}^{\frac{\alpha-\gamma+\alpha\beta(\gamma-1)}{\alpha(\gamma+\beta(\gamma-1))}} \left\| \|\partial_1 |w^+|^2\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{2\alpha(\gamma+\beta(\gamma-1))}} \right\|_{L^2} \left\| \|\partial_2 |w^+|^2\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{2\alpha(\gamma+\beta(\gamma-1))}} \right\|_{L^2} \\ &\leq C \left\| \|\partial_3 \pi\|_{L^{\frac{\gamma}{\gamma+\beta(\gamma-1)}}_{x_3}} \right\|_{L^{\frac{\alpha}{\alpha(\gamma+\beta(\gamma-1))}}_{x_1 x_2}} \left\| \|\pi\|_{L^{\frac{\beta(\gamma-1)}{\gamma+\beta(\gamma-1)}}} \right\|_{L^{\frac{\alpha}{\alpha(\gamma+\beta(\gamma-1))}}} \|w^-\|_{L^4}^{\frac{\alpha-\gamma+\alpha\beta(\gamma-1)}{\alpha(\gamma+\beta(\gamma-1))}} \|w^+\|_{L^4}^{\frac{3(\alpha-\gamma+\alpha\beta(\gamma-1))}{\alpha(\gamma+\beta(\gamma-1))}} \\ &\quad \times \left\| \|\nabla \pi\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{\alpha(\gamma+\beta(\gamma-1))}} \right\|_{L^2} \left\| \|w^+\|^2 \right\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{\alpha(\gamma+\beta(\gamma-1))}} \\ &\leq C \left\| \|\partial_3 \pi\|_{L^{\frac{\gamma}{\gamma+\beta(\gamma-1)}}_{x_3}} \right\|_{L^{\frac{\alpha}{\alpha(\gamma+\beta(\gamma-1))}}_{x_1 x_2}} \left\| \|\pi\|_{L^{\frac{\beta(\gamma-1)}{\gamma+\beta(\gamma-1)}}} \right\|_{L^{\frac{\alpha}{\alpha(\gamma+\beta(\gamma-1))}}} \|w^-\|_{L^4}^{\frac{\alpha-\gamma+\alpha\beta(\gamma-1)}{\alpha(\gamma+\beta(\gamma-1))}} \|w^+\|_{L^4}^{\frac{3(\alpha-\gamma+\alpha\beta(\gamma-1))}{\alpha(\gamma+\beta(\gamma-1))}} \\ &\quad \times \left\| \|w^+| \cdot \nabla w^-\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{\alpha(\gamma+\beta(\gamma-1))}} \right\|_{L^2} \left\| \|\nabla |w^+|^2\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{\alpha(\gamma+\beta(\gamma-1))}} \right\|_{L^2} \\ &\leq C \left\| \|\partial_3 \pi\|_{L^{\frac{\gamma}{\gamma+\beta(\gamma-1)}}_{x_3}} \right\|_{L^{\frac{\alpha}{\alpha(\gamma+\beta(\gamma-1))}}_{x_1 x_2}} \left\| \|\pi\|_{L^{\frac{\beta(\gamma-1)}{\gamma+\beta(\gamma-1)}}} \right\|_{L^{\frac{\alpha}{\alpha(\gamma+\beta(\gamma-1))}}} \left(\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4 \right) \\ &\quad + \frac{1}{4} \left(\| |w^+| \cdot \nabla w^-\|_{L^2}^2 + \|\nabla |w^+|^2\|_{L^2}^2 \right), \end{aligned}$$

where α, β, r and θ satisfy the following identities

$$(13) \quad \begin{cases} \alpha = \theta(r-1)a, \\ \beta = (1-\theta)(r-1)b, \\ r = \frac{\alpha\gamma+\alpha\beta(\gamma-1)}{\gamma+\alpha\gamma-\alpha}, \\ \theta = \frac{\alpha-\gamma}{\alpha\beta(\gamma-1)+\alpha-\gamma}. \end{cases}$$

According to the fact that $2 \leq \lambda < 3$, we choose $r = \frac{(4-\lambda)\alpha\gamma}{\gamma+\alpha\gamma-\alpha}$, then it follows from (13)₃ that $\beta = \frac{(3-\lambda)\gamma}{\gamma-1}$. Now, on the one hand, observe that

$$\gamma < \frac{1}{\lambda - 2} \Leftrightarrow \frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} < 1.$$

On the other hand, since

$$\gamma \geq \frac{3}{\lambda} \Leftrightarrow \lambda\alpha\gamma - \alpha - 2\gamma \geq 2\alpha - 2\gamma$$

and since $\alpha \geq \gamma$, we get

$$\lambda\alpha\gamma - \alpha - 2\gamma \geq 0.$$

But you know, λ must be less than 3, hence

$$\begin{cases} (3-\lambda)\alpha\gamma > 0 \\ \alpha - \gamma \geq 0 \end{cases}$$

which implies that $(3-\lambda)\alpha\gamma + (\alpha - \gamma) > 0$. Gathering these estimates together, we obtain

$$0 \leq \frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} < 1,$$

and it is clear that

$$\frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} + \frac{2(3-\lambda)\alpha\gamma - \alpha(\lambda\gamma - 3)}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} = 1,$$

Now using Hölder inequality with exponents $\frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]}$ and $\frac{2(3-\lambda)\alpha\gamma - \alpha(\lambda\gamma - 3)}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]}$, A_1 can be further estimated as

$$\begin{aligned} A_1 &\leq \frac{1}{4}(\| |w^+| |\nabla w^-| \|_{L^2}^2 \\ &\quad + \| \nabla |w^+|^2 \|_{L^2}^2) + C \left\| \partial_3 \pi \right\|_{L_{x_3}^\gamma}^{\frac{1}{3-\lambda}} \left\| \pi \right\|_{L^3} (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4) \\ &\leq C \left\| \partial_3 \pi \right\|_{L_{x_3}^\gamma}^{\frac{1}{3-\lambda}} \left\| w^+ \right\|_{L^6} \|w^-\|_{L^6} (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4) \\ &\quad + \frac{1}{4}(\| |w^+| |\nabla w^-| \|_{L^2}^2 + \| \nabla |w^+|^2 \|_{L^2}^2) \\ &\leq C \left\| \partial_3 \pi \right\|_{L_{x_3}^\gamma}^{\frac{1}{3-\lambda}} \left\| \nabla w^+ \right\|_{L^2} \left\| \nabla w^- \right\|_{L^2} (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4) \\ &\quad + \frac{1}{4}(\| |w^+| |\nabla w^-| \|_{L^2}^2 + \| \nabla |w^+|^2 \|_{L^2}^2) + \\ &\leq \frac{1}{4}(\| |w^+| |\nabla w^-| \|_{L^2}^2 + \| \nabla |w^+|^2 \|_{L^2}^2) \\ &\quad + C \left\| \partial_3 \pi \right\|_{L_{x_3}^\gamma}^{\frac{1}{3-\lambda}} (\| \nabla w^+ \|_{L^2}^2 + \| \nabla w^- \|_{L^2}^2) (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4), \end{aligned}$$

when $\frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} = 0$ (i.e. $\alpha = \gamma = \frac{3}{\lambda}$) or

$$\begin{aligned} A_1 &\leq \frac{1}{4}(\| |w^+| |\nabla w^-| \|_{L^2}^2 + \| \nabla |w^+|^2 \|_{L^2}^2) \\ &\quad + C \left(\left\| \partial_3 \pi \right\|_{L_{x_3}^\gamma}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma - \alpha - 2\gamma}} + \left\| \pi \right\|_{L^\beta}^{\frac{2\gamma(3-\lambda)}{2\gamma(3-\lambda) - 3(\gamma-1)}} \right) (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4}(\| |w^+| |\nabla w^-| \|_{L^2}^2 + \| |\nabla w^+|^2 \|_{L^2}^2) \\
&\quad + C \left(\left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma - \alpha - 2\gamma}} + \|w^+\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda) - 3(\gamma-1)}} \|w^-\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda) - 3(\gamma-1)}} \right) \\
&\quad \times (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4),
\end{aligned}$$

when $0 < \frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} < 1$ (i.e. $\frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}$) and $\beta = \frac{(3-\lambda)\gamma}{\gamma-1}$.

A_2 can be bounded exactly as A_1 . Following the steps as in the bound of A_1 , we have

$$\begin{aligned}
A_2 &\leq \frac{1}{4}(\| |w^-| |\nabla w^+| \|_{L^2}^2 \\
&\quad + \| |\nabla w^-|^2 \|_{L^2}^2) + C \left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{1}{3-\lambda}} \|\pi\|_{L^3} (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4) \\
&\leq C \left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{1}{3-\lambda}} \|w^+\|_{L^6} \|w^-\|_{L^6} (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4) \\
&\quad + \frac{1}{4}(\| |w^-| |\nabla w^+| \|_{L^2}^2 + \| |\nabla w^-|^2 \|_{L^2}^2) \\
&\leq C \left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{1}{3-\lambda}} \|\nabla w^+\|_{L^2} \|\nabla w^-\|_{L^2} (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4) \\
&\quad + \frac{1}{4}(\| |w^-| |\nabla w^+| \|_{L^2}^2 + \| |\nabla w^-|^2 \|_{L^2}^2) \\
&\leq \frac{1}{4}(\| |w^-| |\nabla w^+| \|_{L^2}^2 + \| |\nabla w^-|^2 \|_{L^2}^2) \\
&\quad + C \left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{1}{3-\lambda}} (\|\nabla w^+\|_{L^2}^2 + \|\nabla w^-\|_{L^2}^2) (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4),
\end{aligned}$$

when $\frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} = 0$ (i.e. $\alpha = \gamma = \frac{3}{\lambda}$) or

$$\begin{aligned}
A_2 &\leq \frac{1}{4}(\| |w^-| |\nabla w^+| \|_{L^2}^2 + \| |\nabla w^-|^2 \|_{L^2}^2) \\
&\quad + C \left(\left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma - \alpha - 2\gamma}} + \|\pi\|_{L^\beta}^{\frac{2\gamma(3-\lambda)}{2\gamma(3-\lambda) - 3(\gamma-1)}} \right) (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4) \\
&\leq \frac{1}{4}(\| |w^-| |\nabla w^+| \|_{L^2}^2 + \| |\nabla w^-|^2 \|_{L^2}^2) \\
&\quad + C \left(\left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma - \alpha - 2\gamma}} + \|w^+\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda) - 3(\gamma-1)}} \|w^-\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda) - 3(\gamma-1)}} \right) \\
&\quad \times (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4),
\end{aligned}$$

when $0 < \frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} < 1$ (i.e. $\frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}$) and $\beta = \frac{(3-\lambda)\gamma}{\gamma-1}$.

Combining all the estimates from above, we find

$$(14) \quad \frac{d}{dt} (\|w^+(\cdot, t)\|_{L^4}^4 + \|w^-(\cdot, t)\|_{L^4}^4)$$

$$\leq \begin{cases} C \left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{1}{3-\lambda}} (\|\nabla w^+\|_{L^2}^2 + \|\nabla w^-\|_{L^2}^2) (\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4) \\ \quad + C(\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4), \\ \quad \text{if } \gamma = \alpha = \frac{3}{\lambda}, \\ \left(\left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma-\alpha-2\gamma}} + \|w^+\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda)-3(\gamma-1)}} \|w^-\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda)-3(\gamma-1)}} \right) \|u\|_{L^4}^4 \\ \quad + C(\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4), \\ \quad \text{if } \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}. \end{cases}$$

Defining

$$H(t) = e + \|w^+(\cdot, t)\|_{L^4}^4 + \|w^-(\cdot, t)\|_{L^4}^4,$$

and thanks to

$$\begin{aligned} 1 + \ln(1 + \|\pi\|_{L^2}^2) &\leq 1 + \ln(e + \|w^+\|_{L^4}^4 + \|w^-\|_{L^4}^4) \\ &\leq 1 + \ln(e + \|w^+\|_{L^4}^4 + \|w^-\|_{L^4}^4), \end{aligned}$$

inequality (14) implies that

$$\frac{d}{dt} H(t) \leq \begin{cases} C \frac{\left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{1}{3-\lambda}}}{1 + \ln(1 + \|\pi\|_{L^2}^2)} (\|\nabla w^+\|_{L^2}^2 + \|\nabla w^-\|_{L^2}^2) H(t) (1 + \ln H(t)), \\ \quad \text{if } \gamma = \alpha = \frac{3}{\lambda}, \\ \left(\frac{\left\| \|\partial_3 \pi\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma-\alpha-2\gamma}}}{1 + \ln(1 + \|\pi\|_{L^2}^2)} + \|w^+\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda)-3(\gamma-1)}} \|w^-\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda)-3(\gamma-1)}} \right) H(t) (1 + \ln H(t)), \\ \quad \text{if } \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}, \end{cases}$$

and hence

$$\begin{aligned} &\frac{d}{dt} (1 + \ln H(t)) \\ &\leq \begin{cases} C(\|\nabla w^+\|_{L^2}^2 + \|\nabla w^-\|_{L^2}^2) F(t) (1 + \ln H(t)), & \text{if } \gamma = \alpha = \frac{3}{\lambda}, \\ \left(F(t) + \|w^+\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda)-3(\gamma-1)}} \|w^-\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda)-3(\gamma-1)}} \right) (1 + \ln H(t)), \\ & \text{if } \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}, \end{cases} \end{aligned}$$

here

$$F(t) = \begin{cases} \frac{\left\| \|\partial_3 \pi(\cdot, t)\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{1}{3-\lambda}}}{1 + \ln(1 + \|\pi(\cdot, t)\|_{L^2}^2)}, & \text{if } \gamma = \alpha = \frac{3}{\lambda}, \\ \frac{\left\| \|\partial_3 \pi(\cdot, t)\|_{L_{x_3}^\gamma} \right\|_{L_{x_1 x_2}^\alpha}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma-\alpha-2\gamma}}}{1 + \ln(1 + \|\pi(\cdot, t)\|_{L^2}^2)}, & \text{if } \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}, \end{cases}$$

Thanks to (w^+, w^-) is a weak solution of the 3D MHD equations (2), that is

$$(15) \quad (w^+, w^-) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)),$$

together with the interpolation inequality yields that

$$(w^+, w^-) \in L^s(0, T; L^r(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{s} + \frac{3}{r} = \frac{3}{2} \quad \text{and} \quad 2 \leq r \leq 6.$$

On the other hand, since

$$\gamma < \frac{1}{\lambda - 2} \Leftrightarrow \frac{(3 - \lambda)\gamma}{\gamma - 1} > 1$$

and

$$\gamma > \frac{3}{\lambda} \Leftrightarrow \frac{(3 - \lambda)\gamma}{\gamma - 1} < 3,$$

it is easy to see that

$$2 < \frac{2(3 - \lambda)\gamma}{\gamma - 1} < 6 \quad \text{if} \quad \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda - 2}$$

and consequently

$$\frac{2}{\frac{4\gamma(3-\lambda)}{2(3-\lambda)\gamma-3(\gamma-1)}} + \frac{3}{\frac{2\gamma(3-\lambda)}{\gamma-1}} = \frac{2(3-\lambda)\gamma-3(\gamma-1)}{2\gamma(3-\lambda)} + \frac{3(\gamma-1)}{2\gamma(3-\lambda)} = \frac{3}{2}.$$

Hence, one has

$$(16) \quad (w^+, w^-) \in L^{\frac{4\gamma(3-\lambda)}{2(3-\lambda)\gamma-3(\gamma-1)}}(0, T; L^{\frac{2\gamma(3-\lambda)}{\gamma-1}}(\mathbb{R}^3)), \quad \text{if} \quad \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda - 2}.$$

Applying the Gronwall inequality yields that

(17)

$$\begin{aligned} & \ln(H(t)) \\ \leq & C(T, w_0^+, w_0^-) \left\{ \begin{array}{l} \exp \left\{ C \sup_{0 \leq t \leq T} F(t) \int_0^t (\|\nabla w^+(\cdot, \tau)\|_{L^2}^2 + \|\nabla w^-(\cdot, \tau)\|_{L^2}^2) d\tau \right\}, \\ \exp \left\{ \int_0^t \left((F(\tau) + \|w^+(\cdot, \tau)\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda)-3(\gamma-1)}} \|w^-(\cdot, \tau)\|_{L^{2\beta}}^{\frac{2\gamma(3-\beta)}{2\gamma(3-\lambda)-3(\gamma-1)}}) d\tau \right) \right\}, \\ \text{if } \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}, \end{array} \right. \end{aligned}$$

which together with (5) implies that

$$(18) \quad \sup_{0 \leq t \leq T} (\|w^+(\cdot, t)\|_{L^4} + \|w^-(\cdot, t)\|_{L^4}) < \infty.$$

Hence, it follows from the triangle inequality and (18) that

$$\begin{aligned} \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^4} &= \frac{1}{2} \sup_{0 \leq t \leq T} \|(u+b)(\cdot, t) + (u-b)(\cdot, t)\|_{L^4} \\ &\leq \frac{1}{2} \sup_{0 \leq t \leq T} (\|(u+b)(\cdot, t)\|_{L^4} + \|(u-b)(\cdot, t)\|_{L^4}) \\ &\leq \frac{1}{2} \sup_{0 \leq t \leq T} (\|w^+(\cdot, t)\|_{L^4} + \|w^-(\cdot, t)\|_{L^4}) < \infty, \end{aligned}$$

and

$$\begin{aligned} \sup_{0 \leq t \leq T} \|b(\cdot, t)\|_{L^4} &= \frac{1}{2} \sup_{0 \leq t \leq T} \|(u+b)(\cdot, t) - (u-b)(\cdot, t)\|_{L^4} \\ &\leq \frac{1}{2} \sup_{0 \leq t \leq T} (\|(u+b)(\cdot, t)\|_{L^4} + \|(u-b)(\cdot, t)\|_{L^4}) \\ &\leq \frac{1}{2} \sup_{0 \leq t \leq T} (\|w^+(\cdot, t)\|_{L^4} + \|w^-(\cdot, t)\|_{L^4}) < \infty. \end{aligned}$$

Thus,

$$\sup_{0 \leq t \leq T} (\|u(\cdot, t)\|_{L^4} + \|b(\cdot, t)\|_{L^4}) < \infty.$$

This completes the proof of Theorem 1.2. \square

ACKNOWLEDGMENTS

This work was done while the second author was visiting the Catania University in Italy. He would like to thank the hospitality and support of the University, where this work was completed. This research is partially supported by Piano della Ricerca 2016-2018 - Linea di intervento 2: “Metodi variazionali ed equazioni differenziali”. The fourth author wishes to thank the support of “RUDN University Program 5-100”. The authors wish to express their thanks to the referee for his/her very careful reading of the paper, giving valuable comments and helpful suggestions.

REFERENCES

- [1] S. Benbernou, M. A. Ragusa and M. Terbeche, [A logarithmically improved regularity criterion for the MHD equations in terms of one directional derivative of the pressure](#), *Appl. Anal.*, **96** (2017), 2140–2148. [MR 3667851](#)
- [2] C. Cao and J. Wu, [Two regularity criteria for the 3D MHD equations](#), *J. Differential Equations*, **248** (2010), 2263–2274. [MR 2595721](#)
- [3] J. Fan, S. Jiang, G. Nakamura and Y. Zhou, [Logarithmically improved regularity criteria for the Navier-Stokes and MHD equations](#), *J. Math. Fluid Mech.*, **13** (2011), 557–571. [MR 2847281](#)
- [4] S. Gala, [Extension criterion on regularity for weak solutions to the 3D MHD equations](#), *Math. Methods Appl. Sci.*, **33** (2010), 1496–1503. [MR 2680689](#)
- [5] X. Jia and Y. Zhou, [Ladyzhenskaya-Prodi-Serrin type regularity criteria for the 3D incompressible MHD equations in terms of \$3 \times 3\$ mixture matrices](#), *Nonlinearity*, **28** (2015), 3289–3307. [MR 3403399](#)
- [6] X. Jia and Y. Zhou, [Regularity criteria for the 3D MHD equations via partial derivatives. II](#), *Kinet. Relat. Models*, **7** (2014), 291–304. [MR 3195076](#)
- [7] X. Jia and Y. Zhou, [A new regularity criterion for the 3D incompressible MHD equations in terms of one component of the gradient of pressure](#), *J. Math. Anal. Appl.*, **396** (2012), 345–350. [MR 2956967](#)
- [8] Q. Liu, [The 3D Boussinesq equations with regularity in one directional derivative of the pressure](#), *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 3005–3019. [MR 3999076](#)
- [9] C. Qian, [A generalized regularity criterion for the 3D Navier-Stokes equations in terms of one velocity component](#), *J. Differential Equations*, **260** (2016), 3477–3494. [MR 3434406](#)
- [10] C. Qian, [The anisotropic integrability regularity criterion to 3D magnetohydrodynamic equations](#), *Math. Methods Appl. Sci.*, **40** (2017), 5461–5469. [MR 3702472](#)
- [11] Y. Zhou, [Regularity criteria for the 3D MHD equations in terms of the pressure](#), *Internat. J. Non-Linear Mech.*, **41** (2006), 1174–1180. [MR 2308660](#)
- [12] Y. Zhou and J. Fan, [Logarithmically improved regularity criteria for the 3D viscous MHD equations](#), *Forum Math.*, **24** (2012), 691–708. [MR 2949119](#)
- [13] Y. Zhou and M. Pokorný, [On the regularity of the solutions of the Navier-Stokes equations via one velocity component](#), *Nonlinearity*, **23** (2010), 1097–1107. [MR 2630092](#)

AHMAD MOHAMMAD ALGHAMDI, DEPARTMENT OF MATHEMATICAL SCIENCE, FACULTY OF APPLIED SCIENCE, UMM ALQURA UNIVERSITY, P. O. BOX 14035, MAKKAH 21955, SAUDI ARABIA
Email address: amghamdi@uqu.edu.sa

SADEK GALA, DEPARTMENT OF MATHEMATICS, ENS OF MOSTAGANEM, BOX 227, MOSTAGANEM 27000, ALGERIA; DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI CATANIA, VIALE A. DORIA, 6 - 95125 CATANIA, ITALY
Email address: sgala793@gmail.com

CHENYIN QIAN, DEPARTMENT OF MATHEMATICS, ZHEJIANG NORMAL UNIVERSITY, JINHUA, CHINA

Email address: `qcyjcsx@163.com`

MARIA ALESSANDRA RAGUSA, DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI CATANIA, VIALE A. DORIA, 6 - 95125 CATANIA, ITALY; RUDN UNIVERSITY, 6 MIKLUKHO - MAKLAY ST, MOSCOW, 117198, RUSSIA

Email address: `mariaalessandra.ragusa@unict.it`, `maragusa@dmf.unict.it`