THE ANISOTROPIC INTEGRABILITY LOGARITHMIC
REGULARITY CRITERION FOR THE 3D MHD EQUATIONS

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Abstract. This study is devoted to investigating the regularity criterion of
the 3D MHD equations in terms of pressure in the framework of anisotropic
Lebesgue spaces. The result shows that if a weak solution \((u, b)\) satisfies
\[
\int_0^T \left\| \partial_3 \pi(\cdot, t) \right\|_{L_3^{\lambda}}^q \frac{1 + \ln \left( e + \|\pi(\cdot, t)\|_{L_2^2}^2 \right)}{dt < \infty,
\]
where \(\frac{1}{\gamma} + \frac{2}{q} + \frac{2}{\alpha} = \lambda \in [2, 3)\) and \(3 \leq \gamma \leq \alpha < \frac{1}{\lambda - 2}\),
then \((u, b)\) is regular at \(t = T\), which improve the previous results on the MHD
equations.

1. Introduction

Let us consider the following Cauchy problem of the incompressible magnetohydrodynamic (MHD) equations in three-spatial dimensions:
\[
\begin{cases}
\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \pi - (b \cdot \nabla) b = 0, \\
\partial_t b - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0, \\
\nabla \cdot u = \nabla \cdot b = 0, \\
u(x, 0) = u_0(x), \ b(x, 0) = b_0(x),
\end{cases}
\]
where \(u\) is the velocity field, \(b\) the magnetic field and \(\pi\) the pressure, while \(u_0, b_0\)
are given initial data with \(\nabla \cdot u_0 = \nabla \cdot b_0 = 0\) in the sense of distributions.

Due to the importance of both physics and mathematics, there is large literature
on the well-posedness for weak solutions of magnetohydrodynamic equations. How-
however, similar to the Navier-Stokes equations \((b = 0)\), the question of global regularity
of the weak solutions to (2) still one of the most challenging problems in the theory
of PDE’s (see for example [3, 4, 5, 6, 9, 10, 11, 12, 13] and the references therein).
It is interesting to study the regularity of the weak solutions of (2) by imposing
some growth sufficient conditions on the velocity or the pressure. In particular, the
condition via only one directional derivative of the pressure was established in [2] and showed that the weak solution becomes regular if the pressure \( \pi \) satisfies

\[
\frac{\partial_3 \pi}{\partial t} \in L^p(0, T; L^\beta(\mathbb{R}^3)), \quad \text{for} \quad \frac{2}{\alpha} + \frac{3}{\beta} = \frac{7}{4}, \quad \frac{12}{7} \leq \beta \leq 4.
\]

Later on, Jia and Zhou [7] improve (3) as

\[
\frac{\partial_3 \pi}{\partial t} \in L^p(0, T; L^\beta(\mathbb{R}^3)), \quad \text{for} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2, \quad 3 \leq \beta < \infty.
\]

Recently, some interesting logarithmical pressure regularity criteria of MHD equations are studied. In particular, Benbernou et al. [1] refined (4) by imposing the following regularity criterion

\[
\int_0^T \frac{\|\partial_3 \pi(\cdot, t)\|_{L^\alpha}}{1 + \ln (e + \|\pi(\cdot, t)\|_{L^3})} \, dt < \infty \quad \text{with} \quad \frac{3}{2} < \lambda \leq \infty.
\]

The aim of this paper is to establish the logarithmical regularity criterion in terms of the partial derivative of pressure in the framework of anisotropic Lebesgue space.

Throughout the rest of this paper, we endow the usual Lebesgue space \( L^p(\mathbb{R}^3) \) with the norm \( \|\cdot\|_{L^p} \). We denote by \( \partial_i = \frac{\partial}{\partial x_i} \) the partial derivative in the \( x_i \)-direction. Recall that the anisotropic Lebesgue space consists on all the total measurable real valued functions \( h = h(x_1, x_2, x_3) \) with finite norm

\[
\left\| h \right\|_{L^p_{x_1, x_2}} := \left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} |h(x)|^p \, dx_k \right)^{\frac{1}{p}} \, dx_j \right)^{\frac{1}{q}},
\]

where \( (i, j, k) \) belongs to the permutation group \( S = \text{span}\{1, 2, 3\} \).

Before giving the main result, let us first recall the definition of weak solutions for MHD equations (2).

**Definition 1.1** (weak solutions). Let \( (u_0, b_0) \in L^2(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = \nabla \cdot b_0 = 0 \) in the sense of distribution and \( T > 0 \). A pair vector field \((u(x, t), b(x, t))\) is called a weak solution of (2) on \((0, T)\) if \((u, b)\) satisfies the following properties:

(i): \((u, b) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))\);

(ii): \(\nabla \cdot u = \nabla \cdot b = 0\) in the sense of distribution;

(iii): \((u, b)\) verifies (2) in the sense of distribution.

(iv): \((u, b)\) satisfies the energy inequality, that is,

\[
\|u(\cdot, t)\|_{L^2}^2 + \|b(\cdot, t)\|_{L^2}^2 + 2 \int_0^t \left( \|\nabla u(\cdot, \tau)\|_{L^2}^2 + \|\nabla b(\cdot, \tau)\|_{L^2}^2 \right) \, d\tau \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2,
\]

for all \( t \in [0, T] \).

Now, our result read as follows.

**Theorem 1.2.** Let \((u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)\) with \( \nabla \cdot u_0 = \nabla \cdot b_0 = 0 \) in \( \mathbb{R}^3 \). Suppose that \((u, b)\) is a weak solution of (2) in \((0, T)\). If the pressure \( \pi \) satisfies the condition

\[
\int_0^T \frac{\|\partial_3 \pi(\cdot, t)\|_{L^{\alpha}}^q}{1 + \ln \left( e + \|\pi(\cdot, t)\|_{L^3}^2 \right)} \, dt < \infty,
\]

where

\[
\frac{1}{\gamma} + \frac{2}{q} + \frac{2}{\alpha} = \lambda \in [2, 3) \quad \text{and} \quad \frac{3}{\lambda} \leq \gamma \leq \alpha < \frac{1}{\lambda - 2},
\]

then...
then the weak solution \((u, b)\) becomes a regular solution on \((0, T]\).

This allows us to obtain the regularity criterion of weak solutions via only one directional derivative of the pressure. This extends and improve some known regularity criterion of weak solutions in term of one directional derivative, including the notable works of Jia and Zhou [7].

As an application of Theorem 1.2, we also obtain the following regularity criterion of weak solutions.

**Corollary 1.3.** Let \((u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)\) with \(\nabla \cdot u_0 = \nabla \cdot b_0 = 0\) in the sense of distributions. Assume that \((u, b)\) is a weak solution of (2) in \((0, T]\). If the pressure satisfies the condition

\[
\int_0^T \frac{\|\partial_3 \pi(\cdot,t)\|^q_{L^{\alpha}}}{1 + \ln \left(e + \|\pi(\cdot,t)\|_{L^2}^2\right)} dt < \infty,
\]

where

\[
\frac{2}{q} + \frac{3}{\alpha} = \lambda \in [2, 3) \quad \text{and} \quad \frac{3}{\lambda} \leq \alpha < \frac{1}{\lambda - 2},
\]

then the weak solution \((u, b)\) becomes a regular solution on \((0, T]\).

In order to prove the main result, we need to recall the following lemma, which is proved in [9] (see also [8]).

**Lemma 1.4.** Let us assume that \(r > 1\) and \(1 < \gamma \leq \alpha < \infty\). Then for \(f, g, \varphi \in C_0^\infty(\mathbb{R}^3)\), we have the following estimate

\[
\left| \int_{\mathbb{R}^3} fg \varphi dx_1 dx_2 dx_3 \right| \leq C \left( \left\| \partial_3 \varphi \right\|_{L^{1_2}} \left\| \partial_3 \varphi \right\|_{L^{3_1}} \left\| \partial_1 \varphi \right\|_{L^{\theta(r-1)}} \left\| \partial_2 \varphi \right\|_{L^{\beta(r-1)}} \right)^{\frac{1}{\gamma(r-1)}} \left( \left\| \partial_1 f \right\|_{L^2} \left\| \partial_2 f \right\|_{L^2} \left\| g \right\|_{L^2} \left\| \partial_1 g \right\|_{L^2} \left\| \partial_2 g \right\|_{L^2} \right)^{\frac{1}{\gamma}},
\]

where \(0 \leq \theta \leq 1\) satisfying

\[
\frac{1}{\theta} + \frac{1}{\beta} = \frac{\alpha - 1}{\alpha},
\]

and

\[
\frac{1}{\gamma(r-1)} + \frac{\theta}{\gamma} = \frac{1 - \theta}{\beta(r - 1)},
\]

and \(C\) is a constant independent of \(f, g, \varphi\).

2. **Proof of Theorem 1.2**

We are now ready to give the proof of Theorem 1.2. Clearly, in order to prove Theorem 1.2, it suffices to show that the assumption (5) ensures the following a priori estimate :

\[
\lim_{t \to T^-} (\|u(\cdot, t)\|_{L^4} + \|b(\cdot, t)\|_{L^4}) < \infty.
\]
Proof. We first convert the MHD system equations into a symmetric form. Adding and substracting \((2)_1\) and \((2)_2\), we get \(w^+\) and \(w^-\) satisfy

\[
\begin{cases}
\partial_t w^+ + (w^- \cdot \nabla) w^+ = \Delta w^+ - \nabla \pi, \\
\partial_t w^- + (w^+ \cdot \nabla) w^- = \Delta w^- - \nabla \pi,
\end{cases}
\]

(9)

with \(w^\pm = u \pm b\).

Next, we establish some fundamental estimates between the pressure \(\pi\) and \(w^\pm\). Taking the divergence operator \(\nabla \cdot\) on both sides of the first and second equations of (2.1) gives

\[
\begin{align*}
\nabla \cdot (\partial_t w^+ + w^- \cdot \nabla w^+) &= \nabla \cdot \Delta w^+ - \nabla \cdot (\nabla \pi), \\
\nabla \cdot (\partial_t w^- + w^+ \cdot \nabla w^-) &= \nabla \cdot \Delta w^- - \nabla \cdot (\nabla \pi),
\end{align*}
\]

and hence

\[
\Delta \pi = \text{div}(w^- \cdot \nabla w^+) = \text{div}(w^+ \cdot \nabla w^-) = \sum_{i,j=1}^3 \partial_j w^-_i \cdot \partial_i w^+_j,
\]

where we have used the divergence free condition \(\nabla \cdot w^+ = \nabla \cdot w^- = 0\). Due to the boundedness of Riesz transform in Lebesgue space \(L^p\) \((1 < p < \infty)\), we have

\[
\|\pi\|_{L^p} \leq C \|w^+\|_{L^{2p}} \|w^-\|_{L^{2p}},
\]

Similarly, acting the operator \(\nabla \text{div}\) on both sides of the first and second equations of (9), one shows that

\[
-\Delta \nabla \pi = \nabla \text{div}(w^- \cdot \nabla w^+) = \nabla \text{div}(w^+ \cdot \nabla w^-),
\]

together with Calderón-Zygmund inequality, implies that for any \(1 < p < \infty\),

\[
\|\nabla \pi\|_{L^p} \leq C \|w^- \cdot \nabla w^+\|_{L^p}
\]

or

\[
\|\nabla \pi\|_{L^p} \leq C \|w^+ \cdot \nabla w^-\|_{L^p}.
\]

By means of the local existence result, (2) with \((u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)\) admit a unique \(L^4\)-strong solution \((u, b)\) on a maximal time interval. For the notation simplicity, we may suppose that the maximal time interval is \([0, T]\). It is obvious that to prove regularity for \(u\) and \(b\), it is sufficient to prove it for \(w^+\) and \(w^-\). We shall show

\[
\lim_{t \to T^-} \left( \|w^+ (\cdot, t)\|_{L^4} + \|w^- (\cdot, t)\|_{L^4} \right) < \infty,
\]

and thus \(\|w^+\|_{L^4}^4 + \|b\|_{L^4}^4\) are uniformly bounded on \([0, T]\). This will lead to a contradiction to the estimates to be derived below. We now begin to follow this argument.

We multiply (9)_1 by \(w^+ |w^+|^2\), (9)_2 by \(w^- |w^-|^2\) in \(L^2(\mathbb{R}^3)\), respectively and integrating by parts, we obtain

\[
\frac{1}{4} \frac{d}{dt} (\|w^+\|_{L^4}^4 + \|w^-\|_{L^4}^4) + \int_{\mathbb{R}^3} (|w^+|^2 |\nabla w^+|^2 + |w^-|^2 |\nabla w^-|^2) \, dx \\
+ \frac{1}{2} \left( \|\nabla w^+\|_{L^2}^2 + \|\nabla w^-\|_{L^2}^2 \right) \\
= \int_{\mathbb{R}^3} \pi w^+ \cdot \nabla |w^+|^2 \, dx + \int_{\mathbb{R}^3} \pi w^- \cdot \nabla |w^-|^2 \, dx = I + J
\]
By Hölder’s and Young’s inequalities,
\begin{equation}
I \leq C \int_{\mathbb{R}^3} |\pi|^2 |w^+|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |w^+|^2|^2 \, dx = A_1 + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |w^+|^2|^2 \, dx.
\end{equation}

In the following, we establish the bound of the integral
\[A_1 = \int_{\mathbb{R}^3} |\pi|^2 |w^+|^2 \, dx\]
on the right-hand side of (12).

We select that
\[a = \frac{\alpha (\gamma + \alpha \gamma - \alpha)}{\alpha - \gamma}\] and \[b = \frac{\gamma + \alpha \gamma - \alpha}{\alpha (\gamma - 1)}\]
in Lemma 1.4, then the selected \(a\) and \(b\) satisfy (7). To bound \(A_1\) and \(A_2\), we recall the
\[
\begin{align*}
\|\pi\|_{L^p} \leq C \|w^+\|_{L^{2p}} \|w^-\|_{L^{2p}}, \\
\|\nabla \pi\|_{L^p} \leq C \|w^- \cdot \nabla w^+\|_{L^p}, \\
\|\nabla \pi\|_{L^p} \leq C \|w^+ \cdot \nabla w^-\|_{L^p},
\end{align*}
\]
we can estimate \(A_1\) as follows
\[
A_1 = \int_{\mathbb{R}^3} |\pi| |\pi| |w^+|^2 \, dx_1 dx_2 dx_3
\]
\[
\leq C \int_{\mathbb{R}^3} \left( \|\partial_3 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_1 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_2 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\times \left( \|\partial_1 |w^+|^2\|_{L^2} \right)^2 \left( \|\partial_2 |w^+|^2\|_{L^2} \right)^2 \left( \|\partial_3 |w^+|^2\|_{L^2} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\leq C \left( \|\partial_3 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_1 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_2 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\times \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\leq C \left( \|\partial_3 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_1 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_2 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\times \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\leq C \left( \|\partial_3 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_1 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_2 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\times \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\leq C \left( \|\partial_3 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_1 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_2 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\times \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\leq C \left( \|\partial_3 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_1 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\partial_2 \pi\|_{L^\frac{3}{2}} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
\times \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\nabla |w^+|^2\|_{L^2} \right)^2 \left( \|\pi\|_{L^\frac{3}{2}} \right)^2 \]
\[
+ \frac{1}{4} \left( \| |w^+| \| \nabla w^- \|^2 \right)^2_{L^2} + \left( \| \nabla |w^+|^2 \|^2 \right)^2_{L^2},
\]
where \(\alpha, \beta, r\) and \(\theta\) satisfy the following identities
\begin{align}
\alpha &= \theta (r - 1)a, \\
\beta &= (1 - \theta)(r - 1)b, \\
r &= \frac{\alpha \gamma + \beta (\gamma - 1)}{\gamma + \alpha \gamma - \alpha}, \\
\theta &= \frac{\alpha \gamma + \beta (\gamma - 1)}{\gamma + \alpha \gamma - \alpha}.
\end{align}

According to the fact that \(2 \leq \lambda < 3\), we choose \(r = \frac{(4 - \lambda) \alpha \gamma}{\gamma + \alpha \gamma - \alpha}\), then it follows from (13) that \(\beta = \frac{(3 - \lambda) \gamma}{\gamma + \alpha \gamma - \alpha}\). Now, on the one hand, observe that
$$\gamma < \frac{1}{\lambda - 2} \iff \frac{\lambda \alpha \gamma - \alpha - 2\gamma}{2[(3 - \lambda)\alpha \gamma - \gamma + \alpha]} < 1.$$  

On the other hand, since

$$\gamma \geq \frac{3}{\lambda} \iff \lambda \alpha \gamma - \alpha - 2\gamma \geq 2\alpha - 2\gamma$$  

and since $\alpha \geq \gamma$, we get

$$\lambda \alpha \gamma - \alpha - 2\gamma \geq 0.$$  

But you know, $\lambda$ must be less than 3, hence

$$\begin{cases} (3 - \lambda)\alpha \gamma > 0 \\ \alpha - \gamma \geq 0 \end{cases}$$

which implies that $(3 - \lambda)\alpha \gamma + (\alpha - \gamma) > 0$. Gathering these estimates together, we obtain

$$0 \leq \frac{\lambda \alpha \gamma - \alpha - 2\gamma}{2[(3 - \lambda)\alpha \gamma - \gamma + \alpha]} < 1,$$

and it is clear that

$$\frac{\lambda \alpha \gamma - \alpha - 2\gamma}{2[(3 - \lambda)\alpha \gamma - \gamma + \alpha]} + \frac{2(3 - \lambda)\alpha \gamma - \alpha(\lambda \gamma - 3)}{2[(3 - \lambda)\alpha \gamma - \gamma + \alpha]} = 1,$$

Now using Hölder inequality with exponents $\frac{\lambda \alpha \gamma - \alpha - 2\gamma}{2[(3 - \lambda)\alpha \gamma - \gamma + \alpha]}$ and $\frac{2(3 - \lambda)\alpha \gamma - \alpha(\lambda \gamma - 3)}{2[(3 - \lambda)\alpha \gamma - \gamma + \alpha]}$, $A_1$ can be further estimated as

$$A_1 \leq \frac{1}{4} (\|w^+\| \|\nabla w^-\|_{L^2}^2$$

$$+ \|\nabla |w^+|^2\|_{L^2}^2 + C \|\partial_3 \pi\|_{L_{3}^{2}} \|\pi\|_{L^3} \|\nabla w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4$$

$$\leq C \|\partial_3 \pi\|_{L_{3}^{2}} \|\nabla w^+\|_{L^6} \|w^-\|_{L^6} \left(\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4\right)$$

$$+ \frac{1}{4} (\|w^+\| \|\nabla w^-\|_{L^2}^2 + \|\nabla |w^+|^2\|_{L^2}^2)$$

$$\leq C \|\partial_3 \pi\|_{L_{3}^{2}} \|\nabla w^+\|_{L^2} \|\nabla w^-\|_{L^2} \left(\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4\right)$$

$$+ \frac{1}{4} (\|w^+\| \|\nabla w^-\|_{L^2}^2 + \|\nabla |w^+|^2\|_{L^2}^2)$$

$$\leq \frac{1}{4} \left(\|w^+\| \|\nabla w^-\|_{L^2}^2 + \|\nabla |w^+|^2\|_{L^2}^2\right)$$

$$+ C \|\partial_3 \pi\|_{L_{3}^{2}} \|\nabla w^+\|_{L^2} \|\nabla w^-\|_{L^2} \left(\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4\right),$$

when $\frac{\lambda \alpha \gamma - \alpha - 2\gamma}{2[(3 - \lambda)\alpha \gamma - \gamma + \alpha]} = 0$ (i.e. $\alpha = \gamma = \frac{3}{\lambda}$) or

$$A_1 \leq \frac{1}{4} (\|w^+\| \|\nabla w^-\|_{L^2}^2 + \|\nabla |w^+|^2\|_{L^2}^2)$$

$$+ C \left(\|\partial_3 \pi\|_{L_{3}^{2}} \|\nabla w^+\|_{L^2} \|\nabla w^-\|_{L^2} \right) \left(\|w^-\|_{L^4}^4 + \|w^+\|_{L^4}^4\right).$$
Combining all the estimates from above, we find

\[ \leq \frac{1}{4} \left( \| w^+ \| \| \nabla w^- \|_{L^2}^2 + \| \nabla \| w^+ \|_{L^2}^2 \right) 
+ C \left( \left\| \partial_3 \pi \right\|_{L^3} \left\| \frac{1}{\lambda^{\gamma}} \right\|_{L^6_{t^2}} + \left\| \partial_3 \pi \right\|_{L^3} \left\| \frac{1}{\lambda^{\gamma}} \right\|_{L^6_{t^2}} \left( \| w^- \|_{L^4}^4 + \| w^+ \|_{L^4}^4 \right) \right) 
\times (\| w^- \|_{L^4}^4 + \| w^+ \|_{L^4}^4), \]

when \( 0 < \frac{\lambda^{\alpha - \gamma} - 2\gamma}{2 + (3 - \lambda) \gamma + \alpha - \gamma} < 1 \) (i.e. \( \frac{3}{X} < \gamma \leq \alpha < \frac{1}{\lambda - 2} \)) and \( \beta = \frac{(3 - \lambda)\gamma}{\gamma - 1} \).

\( A_2 \) can be bounded exactly as \( A_1 \). Following the steps as in the bound of \( A_1 \), we have

\[ A_2 \leq \frac{1}{4} \left( \| w^- \| \| \nabla w^+ \|_{L^2}^2 \right) 
+ C \left( \left\| \partial_3 \pi \right\|_{L^3} \left\| \frac{1}{\lambda^{\gamma}} \right\|_{L^6_{t^2}} \left( \| w^+ \|_{L^4}^4 + \| w^- \|_{L^4}^4 \right) \right) 
\times \left( \left\| \partial_3 \pi \right\|_{L^3} \left\| \frac{1}{\lambda^{\gamma}} \right\|_{L^6_{t^2}} \left( \| w^- \|_{L^4}^4 + \| w^+ \|_{L^4}^4 \right) \right) 
\times (\| w^- \|_{L^4}^4 + \| w^+ \|_{L^4}^4), \]

when \( \frac{\lambda^{\alpha - \gamma} - 2\gamma}{2 + (3 - \lambda) \gamma + \alpha - \gamma} = 0 \) (i.e. \( \alpha = \gamma = \frac{3}{X} \)) or

\[ A_2 \leq \frac{1}{4} \left( \| w^- \| \| \nabla w^+ \|_{L^2}^2 + \| \nabla \| w^+ \|_{L^2}^2 \right) 
+ C \left( \left\| \partial_3 \pi \right\|_{L^3} \left\| \frac{1}{\lambda^{\gamma}} \right\|_{L^6_{t^2}} + \left\| \partial_3 \pi \right\|_{L^3} \left\| \frac{1}{\lambda^{\gamma}} \right\|_{L^6_{t^2}} \| w^+ \|_{L^4}^4 \right) 
\times (\| w^- \|_{L^4}^4 + \| w^+ \|_{L^4}^4), \]

when \( 0 < \frac{\lambda^{\alpha - \gamma} - 2\gamma}{2 + (3 - \lambda) \gamma + \alpha - \gamma} < 1 \) (i.e. \( \frac{3}{X} < \gamma \leq \alpha < \frac{1}{\lambda - 2} \)) and \( \beta = \frac{(3 - \lambda)\gamma}{\gamma - 1} \).

Combining all the estimates from above, we find

\[ \frac{d}{dt} \left( \| w^+ \|_{L^4}^4 + \| w^- \|_{L^4}^4 \right) \]

(14)
Defining
\[ H(t) = e + \left\| w^+ (\cdot, t) \right\|_{L^4}^4 + \left\| w^- (\cdot, t) \right\|_{L^4}^4, \]
and thanks to
\[ 1 + \ln(1 + \| \pi \|_{L^2}^2) \leq 1 + \ln(e + \left\| w^+ \right\|_{L^4}^4 + \left\| w^- \right\|_{L^4}^4) \leq 1 + \ln(e + \left\| w^+ \right\|_{L^4}^4 + \left\| w^- \right\|_{L^4}^4), \]
inequality (14) implies that
\[
\frac{d}{dt} H(t) \leq \left\{ \begin{array}{ll}
C \left\| \partial \pi \right\|_{L_{T^2}} \left\| \pi \right\|_{L_{T^2}}^{1/4} \left( \left\| \nabla w^+ \right\|_{L^2} + \left\| \nabla w^- \right\|_{L^2} \right) H(t)(1 + \ln H(t)), & \text{if } \gamma = \alpha = \frac{3}{\lambda}, \\
\frac{\left\| \partial \pi \right\|_{L_{T^2}}}{1 + \ln(1 + \| \pi \|_{L^2}^2) + \left\| w^+ \right\|_{L^2}^{2(3-\beta)} + \left\| w^- \right\|_{L^2}^{2(3-\beta)}} H(t)(1 + \ln H(t)), & \text{if } \frac{3}{\lambda} \leq \alpha < \frac{1}{1-\gamma}, \\
\end{array} \right.
\]
and hence
\[
\frac{d}{dt} \left( 1 + \ln H(t) \right) \leq \left\{ \begin{array}{ll}
C \left( \left\| \nabla w^+ \right\|_{L^2} + \left\| \nabla w^- \right\|_{L^2} \right) F(t)(1 + \ln H(t)), & \text{if } \gamma = \alpha = \frac{3}{\lambda}, \\
\left( F(t) + \left\| w^+ \right\|_{L^2}^{2(3-\beta)} + \left\| w^- \right\|_{L^2}^{2(3-\beta)} \right) \left( 1 + \ln H(t) \right), & \text{if } \frac{3}{\lambda} \leq \alpha < \frac{1}{1-\gamma}, \\
\end{array} \right.
\]
here
\[
F(t) = \left\{ \begin{array}{ll}
\left\| \partial \pi (\cdot, t) \right\|_{L_{T^2}} \left\| \pi \right\|_{L_{T^2}}^{1/4}, & \text{if } \gamma = \alpha = \frac{3}{\lambda}, \\
\left\| \partial \pi (\cdot, t) \right\|_{L_{T^2}} \left\| \pi \right\|_{L_{T^2}}^{2(3-\beta)}, & \text{if } \frac{3}{\lambda} \leq \alpha < \frac{1}{1-\gamma}. \\
\end{array} \right.
\]
Thanks to \((w^+, w^-)\) is a weak solution of the 3D MHD equations (2), that is
\[
(w^+, w^-) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)),
\]
for \(s > r \geq 2 \leq 6\)},
On the other hand, since
\[ \gamma < \frac{1}{\lambda - 2} \Leftrightarrow \frac{(3 - \lambda)\gamma}{\gamma - 1} > 1 \]
and
\[ \gamma > \frac{3}{\lambda} \Leftrightarrow \frac{(3 - \lambda)\gamma}{\gamma - 1} < 3, \]
it is easy to see that
\[ 2 < \frac{2(3 - \lambda)\gamma}{\gamma - 1} < 6 \text{ if } \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda - 2} \]
and consequently
\[ \frac{2}{2(3 - \lambda)\gamma - 3(\gamma - 1)} + \frac{3}{2(3 - \lambda)\gamma} = \frac{2(3 - \lambda)\gamma - 3(\gamma - 1)}{2(3 - \lambda)\gamma} + \frac{3(\gamma - 1)}{2(3 - \lambda)\gamma} = \frac{3}{2}. \]
Hence, one has
\[ (w^+, w^-) \in L^{\frac{2(3 - \lambda)}{2(3 - \lambda)\gamma - 3(\gamma - 1)}}(0, T; L^{\frac{2(3 - \lambda)}{2(3 - \lambda)\gamma - 3(\gamma - 1)}}(\mathbb{R}^3)), \]
if \( \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda - 2} \).

Applying the Gronwall inequality yields that
\[ \ln(H(t)) \leq C(T, w_0^+, u_0^-) \exp \left\{ C \sup_{0 \leq t \leq T} \int_0^t \left( (3 - \lambda)\gamma \right) \right\} \]
\[ \leq C(T, w_0^+, u_0^-) \exp \left\{ \int_0^t \left( (F(\tau) + \|w^+(\cdot, \tau)\|_{L^2}) + \|w^-(\cdot, \tau)\|_{L^2} \right) d\tau \right\}, \]
which together with (5) implies that
\[ \sup_{0 \leq t \leq T} \left( \|w^+(\cdot, t)\|_{L^4} + \|w^-(\cdot, t)\|_{L^4} \right) < \infty. \]

Hence, it follows from the triangle inequality and (18) that
\[ \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^4} = \frac{1}{2} \sup_{0 \leq t \leq T} \|u + b(\cdot, t) - (u - b)(\cdot, t)\|_{L^4} \]
\[ \leq \frac{1}{2} \sup_{0 \leq t \leq T} \left( \|u + b(\cdot, t)\|_{L^4} + \|(u - b)(\cdot, t)\|_{L^4} \right) \]
\[ \leq \frac{1}{2} \sup_{0 \leq t \leq T} \left( \|u^+(\cdot, t)\|_{L^4} + \|u^-(\cdot, t)\|_{L^4} \right) < \infty, \]
and
\[ \sup_{0 \leq t \leq T} \|b(\cdot, t)\|_{L^4} = \frac{1}{2} \sup_{0 \leq t \leq T} \|u + b(\cdot, t) - (u - b)(\cdot, t)\|_{L^4} \]
\[ \leq \frac{1}{2} \sup_{0 \leq t \leq T} \left( \|u + b(\cdot, t)\|_{L^4} + \|(u - b)(\cdot, t)\|_{L^4} \right) \]
\[ \leq \frac{1}{2} \sup_{0 \leq t \leq T} \left( \|u^+(\cdot, t)\|_{L^4} + \|u^-(\cdot, t)\|_{L^4} \right) < \infty. \]
Thus,
\[
\sup_{0 \leq t \leq T} (\|u(\cdot, t)\|_{L^4} + \|b(\cdot, t)\|_{L^4}) < \infty.
\]
This completes the proof of Theorem 1.2. \qed

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