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NON-EXISTENCE OF SOLUTIONS TO SOME DEGENERATE COERCIVITY ELLIPTIC EQUATIONS INVOLVING MEASURES DATA

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Abstract. In this paper, we main consider the non-existence of solutions u by approximation to the following quasilinear elliptic problem with principal part having degenerate coercivity:

$$\begin{cases} -\operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{(p-1)\theta}}\right)+|u|^{q-1}u=\lambda, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

provided

$$q>\frac{r(p-1)[1+\theta(p-1)]}{r-p},$$

where Ω is a bounded smooth subset of $\mathbb{R}^N (N>2)$, $1<p<N$, $q>1$, $0\le\theta<1$, λ is a measure which is concentrated on a set with zero r capacity ($p<r\le N$).

1. Introduction

In this article, we prove the non-existence of solutions to the following quasilinear elliptic problem which has degenerate coercivity in their principal part by approximation,

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u))+|u|^{q-1}u=\lambda, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{1}$$

where $1<p<N, q>1$ and λ is a Radon measure. Ω is a bounded smooth subset of $\mathbb{R}^N (N>2)$. $a(x,t,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is the Carathéodory function (i.e: $a(x,t,\xi)$ is measure on Ω for every (t,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and $a(\cdot,t,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω), such that the following assumptions hold,

$$a(x,t,\xi)\cdot\xi\ge\frac{c|\xi|^p}{(1+|t|)^{\theta(p-1)}}, \tag{2}$$

$$|a(x,t,\xi)|\le c_0(|\xi|^{p-1}+b(x)), \tag{3}$$

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$$[a(x,t,\xi)-a(x,t,\xi')]\cdot[\xi-\xi']>0, \tag{4}$$

for almost every $x\in\Omega, t\in\mathbb{R}, \xi,\xi'\in\mathbb{R}^N$ with $\xi\neq\xi'$, where $0\leq\theta<1$, c and c_0 are two positive constants, $b\in L^{p'}(\Omega)$ is a non-negative function, p' is the conjugate Hölder exponent of p.

It is well-known that[3, 9], problem $-\Delta u+|u|^{q-1}u=\delta_0$ has no distributional solution if $q\geq\frac{N}{N-2}$. On the other hand, if $q<\frac{N}{N-2}$, then there exists a unique solution to

$$\begin{cases} -\Delta u+|u|^{q-1}u=\delta_0, & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases}$$

In the famous work [9], Brezis proved that if $\{u_n\}$ is sequence of solution to the nonlinear elliptic problem

$$\begin{cases} -\Delta u_n+|u_n|^{q-1}u_n=f_n, & x\in\Omega,\\ u_n=0, & x\in\partial\Omega,\end{cases} \tag{5}$$

with $q>\frac{N}{N-2}$, and $f_n\in L^\infty(\Omega)$ is a sequence functions such that, for any $\varrho>0$,

$$\lim_{n\to\infty}\int_{\Omega\setminus B_\varrho(0)}|f_n-f|=0.$$

Then u_n converges to the unique solution u to the following equation

$$\begin{cases} -\Delta u+|u|^{q-1}u=f, & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases}$$

This fact shows that $B_\varrho(0)$ is a removable singularity set of solution to equation (5) provided $q>\frac{N}{N-2}$. Orsina and Prignet[24] extended the result of [9] to more general operator $-\mathrm{div}(a(x,u,\nabla u))$, where $a(x,u,\nabla u)$ satisfies (2)-(4) with $\theta=0$. The main results of [24] shown that problem (1) with $\theta=0$ has a solution for every given bounded measure λ if $q<\frac{r(p-1)}{r-p}$. Some other related results see [12, 6, 10, 8, 14, 23, 26, 27, 21, 19, 16] and references therein.

The main goal of this paper is to study the non-existence of solutions to problem (1). More precisely, consider the limit of approximating equation (9)(see Theorem 1.2 below), our main task is to understand which is the limit of solutions to (9) and what equation it satisfies. A point worth emphasizing is that, even if $p=2$, the convergence of solutions is not true if the right hand side are distributions weakly converging in $W^{-1,2}(\Omega)$, see [5] for some counterexamples.

In order to state the main results of this paper, we need some definitions.

Let K be a compact subset of Ω, $r>1$ is a real number. The r capacity of K respect to Ω is defined as

$$\mathrm{cap}_r(K,\Omega)=\inf\{\|u\|^r_{W_0^{1,r}}: u\in C_c^\infty(\Omega), u\geq\chi_K\},$$

where χ_K is the characteristic function of K.

Let λ be a bounded measure on Ω, we say that λ is concentrated on a set E if $\lambda(B)=\lambda(B\cap E)$ for every Borel subset B of Ω. Thanks to the Hahn decomposition, λ can be decomposed as the difference of two nonnegative mutually singular measure, that is $\lambda=\lambda^+-\lambda^-$.

If λ is concentrated on a set E, as a consequence of the fact that λ^+ and λ^- are mutually singular, we have that λ^+ and λ^- concentrated a set E^+ and E^- respectively and $E^+\cap E^-=\emptyset$.

Let $\lambda = \lambda^+ - \lambda^-$ be a measure, $f_n = f_n^+ - f_n^-$ approximations of λ in the following way:

$$\lim_{n\to+\infty}\int_\Omega f_n^+\varphi dx = \int_\Omega \varphi d\lambda^+, \ \lim_{n\to+\infty}\int_\Omega f_n^-\varphi dx = \int_\Omega \varphi d\lambda^-, \tag{6}$$

for every function φ, which is continuous and bounded on Ω, where $\{f_n^+\}$ and $\{f_n^-\}$ are sequences of nonnegative $L^\infty(\Omega)$ functions. We not assume that f_n^+ and f_n^- are the positive and negative part of f_n. Observe that choosing $\varphi \equiv 1$ in (6), we obtain

$$\|f_n^+\|_{L^1(\Omega)} \le C, \ \|f_n^-\|_{L^1(\Omega)} \le C. \tag{7}$$

For all $k > 0, s \in \mathbb{R}$, define

$$T_k(s) = \max\{-k, \min\{k, s\}\}, \ G_k(s) = s - T_k(s).$$

Firstly we stale the existence result.

Theorem 1.1. *Let Ω be a bounded smooth subset of $\mathbb{R}^N (N > 2)$, $1 < p < N$, $g \in L^1(\Omega)$ and (2)-(4) hold. Then there exists a unique entropy solution $u \in W_0^{1,p}(\Omega)$ to problem*

$$\begin{cases} -div(a(x,u,\nabla u)) + |u|^{q-1}u = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{8}$$

if

$$q < \frac{N(1-\theta)}{N - (1 + \theta(p-1))}.$$

Moreover,

$$u \in M^{p_1}(\Omega), \ |\nabla u| \in M^{p_2}(\Omega),$$

where M^{p_1}, M^{p_2} represents the Marcinkiewicz space with exponent

$$p_1 = \frac{N(p-1)(1-\theta)}{N-p}, \ p_2 = \frac{N(p-1)(1-\theta)}{N - (1 + \theta(p-1))}.$$

Remark 1. The previous result gives existence and uniqueness of the entropy solution $u \in W_0^{1,p}(\Omega)$ to (8) for every $1 < p < N$ and $0 < \theta < 1$. If $\theta = 0$, the same result for (8) can be proved by the same techniques of [2].

Our main results are following:

Theorem 1.2. *Let $1 < p < r \le N$ and $\lambda = \lambda^+ - \lambda^-$ be a bounded Radon measure which is concentrated on a set E with zero r capacity. Let $f_n = f_n^+ - f_n^-$ be a sequence of $L^\infty(\Omega)$ functions which converge to λ in the sense of (6). $g \in L^1(\Omega)$ and let g_n is a sequence of $L^\infty(\Omega)$ functions which converge to g weakly in $L^1(\Omega)$. Suppose $u_n \in W_0^{1,p}(\Omega)$ is the solution to problem:*

$$\begin{cases} -div(a(x,u_n,\nabla u_n)) + |u_n|^{q-1}u_n = f_n + g_n, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

Then $|\nabla u_n|^{p-1}$ strong converges to $|\nabla u|^{p-1}$ in $L^\sigma(\Omega)$ as $n \to \infty$ for

$$\sigma < \frac{pq}{(q + 1 + \theta(p-1))(p-1)},$$

if

$$q > \frac{r(p-1)[1 + \theta(p-1)]}{r-p}, \tag{10}$$

where u is unique solution of (8). *Moreover,*

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{q-1}u_n\varphi dx=\int_\Omega |u|^{q-1}u\varphi dx+\int_\Omega \varphi d\lambda,\ \forall\varphi\in C(\Omega). \tag{11}$$

Remark 2. The above theorem shows that there is not a solution to problem (1) can be obtained by approximation, if q is large enough and the measure λ is concentrated on a set with zero r capacity.

Remark 3. Boccardo et.al [7] considered the non-existence result to the following problem

$$\begin{cases} -\mathrm{div}\left(\frac{a(x,\nabla u)}{(1+u)^\gamma}\right)+u=\mu, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{12}$$

where $\gamma>1$ and μ is a non-negative Radon measure, concentrated on a set E with zero harmonic capacity, $a(x,\xi)$ satisfies (2)-(4) with $\theta=0$, $p=2$ and $b(x)=0$. While in Theorem 1.2, λ is a bounded Radon measure concentrated on a set E with zero r capacity with $p<r\le N$, instead of p capacity. Therefore Theorem 1.2 is not a triviality extend the results of Theorem 4.1 of [7]. Furthermore, in Theorem 1.2, $\theta(p-1)\in(0,p-1)$ since $\theta\in(0,1)$. Note that, in problem (12), they required that $\gamma>1$. It is worth pointing out that different ranges of γ have an important impact on the behavior of solutions to problem (12), more details see [25, 18, 1, 17, 13, 4].

The structure of this paper is as follows: Section 2 mainly gives some lemmas which play a important role in the process of proof of the main theorem. The proof of theorem 1.1 and 1.2 are given in Section 3.

2. Useful tools and function setting

In the following, C is a constant and its value may changes from line to line.

In order to prove Theorem 1.1 and 1.2, the following basic lemmas and definitions are required.

Lemma 2.1. *(see Lemma 2.1 of [22]) Let K^+ and K^- be two disjoin compact subsets of Ω with zero r capacity, $\lambda=\lambda^+-\lambda^-$ be a measure which is concentrated on a set with zero r capacity with $1<r\le N$, Then there exist two functions ψ_δ^+ and ψ_δ^- in $C_c^\infty(\Omega)$, such that*

$$\begin{gathered} 0\le\psi_\delta^+\le1,\ 0\le\psi_\delta^-\le1,\ \int_\Omega|\nabla\psi_\delta^+|^r dx\le\delta,\ \int_\Omega|\nabla\psi_\delta^-|^r dx\le\delta,\\ 0\le\int_\Omega(1-\psi_\delta^+)d\lambda^+\le\delta,\ 0\le\int_\Omega(1-\psi_\delta^-)d\lambda^-\le\delta,\\ 0\le\int_\Omega\psi_\delta^- d\lambda^+\le\delta,\ 0\le\int_\Omega\psi_\delta^+)d\lambda^-\le\delta,\\ \psi_\delta^+\equiv1, x\in K^+,\ \psi_\delta^+\le1, x\in K^-, \end{gathered} \tag{13}$$

for every $\delta>0$.

Definition 2.2. Let u be an measurable function on Ω such that $T_k(u)\in W_0^{1,p}(\Omega)$ for every $k>0$. Then there exist a unique measurable function $v:\Omega\to\mathbb{R}^N$ such that

$$\nabla T_k(u)=v\chi_{\{|u|\le k\}},\ a.e\ in\ \Omega\ and\ for\ every\ k>0.$$

Define the gradient of u as the function v and denote it by $v=\nabla u$.

Definition 2.3. Let $f \in L^1(\Omega)$, $q > 0$ and (2)-(4) hold. A measurable function u is an entropy solution to problem (8), if $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$, $|u|^q \in L^1(\Omega)$ and

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla T_k(u-\varphi)dx + \int_\Omega |u|^{q-1}uT_k(u-\varphi)dx \le \int_\Omega gT_k(u-\varphi)dx,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^1(\Omega)$.

Definition 2.4. Marcinkiewicz space $M^s(\Omega)(s > 0)$ is the space composed of all the measurable functions υ that satisfy

$$|\{|\upsilon| \ge k\}| \le \frac{C}{k^s},$$

for any $k > 0$, where the constant $C > 0$.

If $|\Omega|$ is bounded and $0 < \varepsilon < s - 1$, then the following embedding relationship hold:

$$L^s(\Omega) \subset M^s(\Omega) \subset L^{s-\varepsilon}(\Omega).$$

Lemma 2.5. *Let $u \in M^s(\Omega)$ with $s > 0$. If there exist a constant $\rho > 0$, such that for any $k > 0$,*

$$\int_\Omega |\nabla T_k(u)|^p dx \le Ck^\rho,$$

for some positive constant C. Then

$$|\nabla u| \in M^{\frac{ps}{s+\rho}}(\Omega).$$

Proof. Let σ be a fixed positive real number, for every $k > 0$,

$$\begin{aligned} |\{|\nabla u| > \sigma\}| &= |\{|\nabla u| > \sigma, |u| \le k\}| + |\{|\nabla u| > \sigma, |u| > k\}| \\ &\le |\{|\nabla T_k(u)| > \sigma\}| + |\{|u| > k\}|. \end{aligned} \tag{14}$$

Moreover,

$$|\{|\nabla T_k(u)| > \sigma\}| \le \frac{1}{\sigma^p}\int_\Omega |\nabla T_k(u)|^p dx \le C\frac{k^\rho}{\sigma^p}. \tag{15}$$

Since $u \in M^s(\Omega)$, by Definition 2.4, there exist a constant C such that

$$|\{|u| > k\}| \le \frac{C}{k^s}. \tag{16}$$

Combining (14)-(16), we have

$$|\{|\nabla u| > \sigma\}| \le C\frac{k^\rho}{\sigma^p} + \frac{C}{k^s} \le \frac{C}{k^{\frac{ps}{s+\rho}}}.$$

Therefore, by Definition 2.4, we get $|\nabla u| \in M^{\frac{ps}{s+\rho}}$. □

Lemma 2.6. *Let $\{u_n\}$ be a sequence in $W_0^{1,p}(\Omega)$ and assume that there exist positive constants ρ and C with $p > \rho$, such that*

$$\int_\Omega |\nabla T_k(u_n)|^p dx \le Ck^\rho,$$

for any k and n. Then there exists a subsequence, still denoted by $\{u_n\}$, which converges to a measurable function v almost everywhere in Ω.

Lemma 2.7. *Let u be an entropy solution to (8), then*

$$\int_{\{k<|u|<k+h\}} |\nabla u|^p dx \le Ck^{\theta(p-1)}.$$

Proof. For any given h and $k>0, s\in\mathbb{R}$, define

$$T_{k,h}(s)=T_h(s-T_k(s))=\begin{cases} s-k\ sgn(s), & k\le |s|<k+h,\\ h, & |s|\ge k+h,\\ 0, & |s|\le k.\end{cases}$$

Take $T_{k,h}(u)$ as test function in (8), we have

$$\int_{\{k<|u|<k+h\}} (a(x,u,\nabla u)\cdot\nabla u)dx+\int_\Omega |u|^{q-1}uT_{k,h}(u)dx=\int_\Omega gT_{k,h}(u)dx. \tag{17}$$

Since $uT_{k,h}(u)\ge 0$, we find

$$\int_{\{k<|u|<k+h\}} (a(x,u,\nabla u)\cdot\nabla u)dx\le\int_\Omega gT_{k,h}(u)dx, \tag{18}$$

and

$$\int_\Omega gT_{k,h}(u)dx\le h\int_{\{|u|>k\}}|g|dx\le C. \tag{19}$$

According to the assumption (2) and (17)-(19), we get,

$$\int_{\{k<|u|<k+h\}} |\nabla u|^p dx \le Ck^{\theta(p-1)}.$$

□

Proposition 1. *Let $u\in W_0^{1,p}(\Omega)$ be an entropy solution to (8) and satisfy*

$$\int_{\{|u|<k\}} |\nabla u|^p dx\le Ck^\rho \tag{20}$$

for every $k>0$ and $p>\rho$. Then $u\in M^{p_1}(\Omega)$, where $p_1=N(p-\rho)/(N-p)$. More precisely, there exists $C=C(N,p,\theta)>0$ such that

$$|\{|u|>k\}|\le Ck^{-p_1}.$$

Proof. For every $k>0$, by the Sobolev embedding theorem and (20),

$$\|T_k(u)\|_{p^*}\le C(N,p,\theta)\|\nabla T_k(u)\|_p\le Ck^{\frac{\rho}{p}},$$

where $p^*=\frac{Np}{N-p}$. For $0<\eta\le k$, we have

$$\{|u|\ge\eta\}=\{|T_k(u)\ge\eta|\}.$$

Hence

$$|\{|u|>\eta\}|\le\frac{\|T_k(u)\|_{p^*}^{p^*}}{\eta^{p^*}}\le C(k^\rho)^{\frac{p*}{p}}\eta^{-p^*}.$$

Setting $\eta=k$, we obtain

$$|\{|u|>k\}|\le Ck^{-\frac{N(p-\rho)}{N-p}}.$$

This fact shows that $u\in M^{p_1}(\Omega)$ with $p_1=N(p-\rho)/(N-p)$. □

Proposition 2. *Assume that* $u \in W_0^{1,p}(\Omega)$ *is an entropy solution to* (8), *which satisfies* (20) *for every* k. *Then* $\nabla u \in M^{p_2}(\Omega)$, *where* $p_2 = N(p-\rho)/(N-\rho)$, *that is there exists* $C = C(N,p,\theta) > 0$ *such that*

$$|\{|\nabla u| > h\}| \leq Ch^{-p_2},$$

for every $h > 0$.

Proof. For $k, \lambda > 0$, set

$$\psi(k,\lambda) = |\{|\nabla u|^p > \lambda, |u| > k\}|.$$

Using the fact that the function $\lambda \mapsto \psi(k,\lambda)$ is nonincreasing, we get, for $k, \lambda > 0$,

$$\begin{aligned}\psi(0,\lambda) =& |\{|\nabla u|^p > \lambda\}| \leq \frac{1}{\lambda}\int_0^\lambda \psi(0,s)ds \\ \leq& \psi(k,0) + \frac{1}{\lambda}\int_0^\lambda \psi(0,s) - \psi(k,s)ds. \end{aligned} \tag{21}$$

By Proposition 1,

$$\psi(k,0) \leq Ck^{-p_1}, \tag{22}$$

where $p_1 = N(p-\rho)/(N-p)$. Since $\psi(0,s) - \psi(k,s) = |\{|\nabla u|^p > s, |u| < k\}|$, thanks to (20), we have

$$\int_0^\infty \psi(0,s) - \psi(k,s)ds = \int_{\{|u|<k\}} |\nabla u|^p dx \leq Ck^\rho. \tag{23}$$

Combining (21)-(23), we arrive at

$$\psi(0,\lambda) \leq \frac{Ck^\rho}{\lambda} + Ck^{-p_1}. \tag{24}$$

Let $\frac{Ck^\rho}{\lambda} = Ck^{-p_1}$ and $\lambda = h^p$, (24) implies that

$$|\{|\nabla u| > h\}| \leq Ch^{-\frac{N(p-\rho)}{N-\rho}}.$$

That is $\nabla u \in M^{p_2}(\Omega)$ with $p_2 = N(p-\rho)/(N-\rho)$. ☐

3. Proof of main theorem

In this section we prove Theorem 1.1 and 1.2 combining the results of Sections 2.

In the proofs of Theorem 1.1 and 1.2, $\omega(n,m,\delta)$ will denote any quantity (depending on n, m and δ) such that

$$\lim_{\delta\to 0^+}\lim_{m\to+\infty}\lim_{n\to+\infty}\omega(n,m,\delta) = 0.$$

If the quantity does not depend on one or more of the three parameters n, m and δ, we will omit the dependence from it in ω. For example, $\omega(n,\delta)$ is any quantity such that

$$\lim_{\delta\to 0^+}\lim_{n\to+\infty}\omega(n,\delta) = 0.$$

3.1. **Proof of Theorem 1.1.**

The proof of Theorem 1.1 will be divided in several steps.

Proof. (1)Uniqueness: Let u_1 and u_2 be two entropy solutions to equation (8). The proof of the fact that $u_1 = u_2$ will follow from the following four steps.

Step 1. Assume that $g_i \in L^1(\Omega), (i = 1, 2)$. Choosing $T_k(u_1 - T_h u_2)$ and $T_k(u_2 - T_h u_1)$ as test function in (8) respectively, we get

$$
\begin{aligned}
I :=& \int_\Omega a(x, u_1, \nabla u_1) \cdot \nabla T_k(u_1 - T_h u_2)dx + \int_\Omega a(x, u_2, \nabla u_2) \cdot \nabla T_k(u_2 - T_h u_1)dx \\
=& -\int_\Omega |u_1|^{q-1}u_1 T_k(u_1 - T_h u_2)dx - \int_\Omega |u_2|^{q-1}u_2 T_k(u_2 - T_h u_1)dx \\
&+ \int_\Omega g_1 T_k(u_1 - T_h u_2)dx + \int_\Omega g_2 T_k(u_2 - T_h u_1)dx. \qquad (25)
\end{aligned}
$$

Step 2. Denote

$$
\begin{aligned}
A_0 &= \{x \in \Omega : |u_1 - u_2| < k, |u_1| < h, |u_2| < h\}, \\
A_1 &= \{x \in \Omega : |u_1 - T_h u_2| < k, |u_2| \geq h\}, \\
A_2 &= \{x \in \Omega : |u_1 - T_h u_2| < k, |u_2| < h, |u_1| \geq h\}.
\end{aligned}
$$

For $x \in A_0$,

$$\nabla T_k(u_1 - T_h u_2) = \nabla(u_1 - u_2)$$

and

$$\nabla T_k(u_2 - T_h u_1) = \nabla T_k(u_2 - u_1).$$

Thus, for every $x \in A_0$,

$$
\begin{aligned}
&\int_\Omega a(x, u_1, \nabla u_1) \cdot \nabla T_k(u_1 - T_h u_2)dx + \int_\Omega a(x, u_2, \nabla u_2) \cdot \nabla T_k(u_2 - T_h u_1)dx \\
&= \int_{A_0} [a(x, u_1, \nabla u_1) - a(x, u_2, \nabla u_2)] \cdot \nabla(u_1 - u_2)dx := I_0. \qquad (26)
\end{aligned}
$$

For $x \in A_1$, $\nabla T_k(u_1 - T_h u_2) = \nabla(u_1 - h) = \nabla u_1$. By (2), we get

$$\int_\Omega a(x, u_1, \nabla u_1) \cdot \nabla T_k(u_1 - T_h u_2)dx = \int_{A_1} a(x, u_1, \nabla u_1) \cdot \nabla u_1 dx \geq 0. \qquad (27)$$

For $x \in A_2$, $\nabla T_k(u_1 - T_h u_2) = \nabla(u_1 - u_2)$. Thus

$$\int_\Omega a(x, u_1, \nabla u_1) \cdot \nabla T_k(u_1 - T_h u_2)dx \geq -\int_{A_2} a(x, u_1, \nabla u_1) \cdot \nabla u_2 dx. \qquad (28)$$

Similarly, denote

$$
\begin{aligned}
A_1^* &= \{x \in \Omega : |u_2 - T_h u_1| < k, |u_1| \geq h\}, \\
A_2^* &= \{x \in \Omega : |u_2 - T_h u_1| < k, |u_1| < h, |u_2| \geq h\}.
\end{aligned}
$$

Then for $x \in A_1^*$, $\nabla T_k(u_2 - T_h u_1) = \nabla(u_2 - h) = \nabla u_2$. By (2), we get

$$\int_\Omega a(x, u_2, \nabla u_2) \cdot \nabla T_k(u_2 - T_h u_1)dx = \int_{A_1^*} a(x, u_2, \nabla u_2) \cdot \nabla u_2 dx \geq 0. \qquad (29)$$

For $x \in A_2^*$, $\nabla T_k(u_2 - T_h u_1) = \nabla(u_2 - u_1)$. Thus

$$\int_\Omega a(x, u_2, \nabla u_2) \cdot \nabla T_k(u_2 - T_h u_1)dx \geq -\int_{A_2^*} a(x, u_2, \nabla u_2) \cdot \nabla u_1 dx. \tag{30}$$

Summing up (26)-(30) in the form $I \geq I_0 - I_1$, where

$$\begin{aligned} I_1 = &\int_{A_2} a(x, u_1, \nabla u_1) \cdot \nabla u_2 dx + \int_{A_2^*} a(x, u_2, \nabla u_2) \cdot \nabla u_1 dx \\ := &I_{11} + I_{12}. \end{aligned}$$

Now, we estimate I_{11}. By the Hölder inequality and (3), we have

$$\begin{aligned} I_{11} \leq &\|a(x, u_1, \nabla u_1)\|_{L^{p'}(\{h \leq |u_1| \leq h+k\})} \|\nabla u_2\|_{L^p(\{h-k \leq |u_2| \leq h\})} \\ \leq &c_0(\|\nabla u_1\|^{p-1}_{L^{p'}(\{h \leq |u_1| \leq h+k\})} + \|b(x)\|_{L^{p'}(\{|u_1| \geq h\})})\|\nabla u_2\|_{L^p(\{h-k \leq |u_2| \leq h\})}. \end{aligned}$$

Therefore, by Lemma 2.7 and Proposition 2, $I_{11} \to 0$ as $h \to \infty$ for every $k > 0$. $I_{12} \to 0$ as $h \to \infty$ for every $k > 0$ can be obtained in the same way.

Hence, we find

$$\begin{aligned} &\int_\Omega a(x, u_1, \nabla u_1) \cdot \nabla T_k(u_1 - T_h u_2)dx + \int_\Omega a(x, u_2, \nabla u_2) \cdot \nabla T_k(u_2 - T_h u_1)dx \\ &= \int_{A_0} [a(x, u_1, \nabla u_1) - a(x, u_2, \nabla u_2)] \cdot \nabla(u_1 - u_2)dx + \varepsilon(h). \end{aligned} \tag{31}$$

Step 3. Now estimate the terms on the right hand side of (25). Denote

$$\begin{aligned} B_0 &= \{x \in \Omega : |u_1| < h, |u_2| < h\}, \\ B_1 &= \{x \in \Omega : |u_1| \geq h\}, \\ B_2 &= \{x \in \Omega : |u_2| \geq h\}. \end{aligned}$$

For $x \in B_0$, since $T_k(u_1 - T_h u_2) = T_k(u_1 - u_2)$ and $T_k(u_2 - T_h u_1) = T_k(u_2 - u_1)$, we arrive at

$$\begin{aligned} &\int_\Omega |u_1|^{q-1} u_1 T_k(u_1 - T_h u_2)dx + \int_\Omega |u_2|^{q-1} u_2 T_k(u_2 - T_h u_1)dx \\ &= \int_{B_0} (|u_1|^{q-1} u_1 - |u_2|^{q-1} u_2) T_k(u_1 - u_2)dx \geq 0, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\int_\Omega g_1 T_k(u_1 - T_h u_2)dx + \int_\Omega g_2 T_k(u_2 - T_h u_1)dx \\ &= \int_{B_0} (g_1 - g_2) T_k(u_1 - u_2)dx \leq 0. \end{aligned} \tag{33}$$

For $x \in B_1$, since $T_k(u_2 - T_h u_1) = T_k(u_2 - h)$. Then

$$\begin{aligned} &\int_\Omega |u_1|^{q-1} u_1 T_k(u_1 - T_h u_2)dx + \int_\Omega |u_2|^{q-1} u_2 T_k(u_2 - T_h u_1)dx \\ \leq &k \int_{B_1} (|u_1|^{q-1} u_1 + |u_2|^{q-1} u_2)dx := J_1, \end{aligned}$$

and

$$\int_\Omega g_1 T_k(u_1 - T_h u_2)dx + \int_\Omega g_2 T_k(u_2 - T_h u_1)dx \leq k \int_{B_1} (|g_1| + |g_2|)dx := J_2.$$

For $x \in B_2$, since $T_k(u_1 - T_h u_2) = T_k(u_1 - h)$, we get

$$\begin{aligned}&\int_\Omega |u_1|^{q-1}u_1 T_k(u_1 - T_h u_2)dx + \int_\Omega |u_2|^{q-1}u_2 T_k(u_2 - T_h u_1)dx \\ \leq & k\int_{B_2} (|u_1|^{q-1}u_1 + |u_2|^{q-1}u_2)dx := J_1^*,\end{aligned}$$

and

$$\int_\Omega g_1 T_k(u_1 - T_h u_2)dx + \int_\Omega g_2 T_k(u_2 - T_h u_1)dx \leq k\int_{B_2} (|g_1| + |g_2|)dx := J_2^*.$$

According to $|B_1| \to 0, |B_2| \to 0$ as $h \to \infty$ and $|u|^q \in L^1(\Omega)$ for fixed $k > 0$, we get

$$J_1 + J_2 + J_1^* + J_2^* \to 0 \ as \ h \to \infty. \tag{34}$$

Step 4. Combining (25) and (31)-(34), we have

$$\int_{A_0} [a(x, u_1, \nabla u_1) - a(x, u_2, \nabla u_2)] \cdot \nabla(u_1 - u_2)dx \leq \varepsilon(h),$$

where $\varepsilon(h) \to 0$ as $h \to \infty$. Since A_0 converges to $\{x \in \Omega : |u_1 - u_2| < k\}$ by measure as $h \to \infty$ for fixed $k > 0$, we conclude that

$$\int_{\{|u_1-u_2|<k\}} [a(x, u_1, \nabla u_1) - a(x, u_2, \nabla u_2)] \cdot \nabla(u_1 - u_2)dx \leq 0,$$

for all $k > 0$. This fact, combine with (4), implies that $\nabla u_1 = \nabla u_2$ a.e in Ω. Then we get $u_1 = u_2$ a.e in Ω.

(2) Existence:

Step 1. Let

$$F(x, u) = g(x) - \beta(u),$$

where $\beta(u) = |u|^{q-1}u$, which is continuous with respect to u. Then $g(x) = F(x, 0) \in L^1(\mathbb{R}^N)$ and β is monotonous nondecreasing with respect to u with $\beta(0) = 0$ and $\beta(u)u \geq 0$.

Let $g_n \in C_0^\infty$, such that g_n converges to g in $L^1(\Omega)$, with $\|g_n\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega)}$ for every $n \geq 1$. Define $\beta_n(s) = T_n(\beta)$. In this way, $|\beta_n(s)| \leq |\beta(s)|$ for every $s \in \mathbb{R}$ and $x \in \Omega$. Finally we take

$$\gamma_n(s) = \beta_n(s) + \frac{1}{n}|s|^{p-2}s.$$

Then by [20], there exists $u_n \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} -\text{div}a(x, u_n, \nabla u_n) + \gamma_n(x, u_n) = g_n, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega, \end{cases} \tag{35}$$

holds in the sense of distributions in Ω.

By density arguments, we can take $T_h(u_n - T_k(u_n))$ and $T_k(u_n)$ as the test function in (35) respectively, we have

$$\begin{aligned}&\int_{\{k\leq|u_n|<k+h\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx + \int_{\{|u_n|>k\}} \gamma_n T_h(u_n - T_k(u_n))dx \\ =&\int_{\{|u_n|>k\}} g_n T_h(u_n - T_k(u_n))dx,\end{aligned} \tag{36}$$

and

$$\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n dx + \int_\Omega \gamma_n T_k(u_n)dx = \int_\Omega g_n T_k(u_n)dx. \tag{37}$$

Combine (36) with (2) (fix the ellipticity constant $c=1$) and $\gamma_n T_h(u_n - T_k(u_n)) \geq 0$, we get,

$$\begin{aligned}&\int_{\{k<|u_n|<k+h\}} |\nabla u_n|^p dx\\ \leq & hk^{\theta(p-1)}\int_{\{|u_n|>k\}} g_n dx \leq hk^{\theta(p-1)}\|g_n\|_{L^1(\Omega)} = Ck^{\theta(p-1)}.\end{aligned} \tag{38}$$

Since $a(x,u_n,\nabla u_n)\cdot\nabla u_n \geq 0$ by (2), we have

$$\int_{\{|u_n|>k\}} |\gamma_n(u_n)|dx \leq \int_{\{|u_n|>k\}} |g_n|dx \leq \|g_n\|_{L^1(\Omega)} \leq C. \tag{39}$$

Combine (37) with $\gamma_n T_k(u_n) \geq 0$, we have

$$\int_{\{|u_n|<k\}} |\nabla u_n|^p dx \leq Ck^{1+\theta(p-1)}. \tag{40}$$

Step 2. Convergence. Using (38) and Proposition 1, we have $|\{|u_n|>k\}|$ is bounded uniformly for every $k>0$. Thanks to (40), we see that $\{\nabla T_k(u_n)\}$ is bounded in $L^p_{\text{loc}}(\Omega)$ for every $k>0$.

Next we prove that $u_n \to u$ locally in measure.

For $t,\epsilon>0$, we have

$$\{|u_n-u_m|>t\} \subset \{|u_n|>k\}\cup\{|u_m|>k\}\cup\{|T_k(u_n)-T_k(u_m)|>t\}.$$

Thus

$$|\{|u_n-u_m|>t\}| \leq |\{|u_n|>k\}| + |\{|u_m|>k\}| + |\{|T_k(u_n)-T_k(u_m)|>t\}|.$$

Choosing k large enough such that $|\{|u_n|>k\}|<\epsilon$ and $|\{|u_m|>k\}|<\epsilon$. Since $\{\nabla T_k(u_n)\}_n$ is bounded in $L^p(\Omega)$ and $T_k(u_n)\in W_0^{1,p}(\Omega)$ for every $k>0$. Assume that $\{T_k(u_n)\}$ is a Cauchy sequence in $L^q(\Omega\cap B_R)$ for any $q<pN/(N-p)$ and any $R>0$,

$$T_k(u_n)\to T_k(u) \text{ in } L^p_{loc}(\Omega) \text{ and a.e in } \Omega.$$

Then

$$|\{|T_k(u_n)-T_k(u_m)|>t\}\cap B_R| \leq t^{-q}\int_{\Omega\cap B_R} |T_k(u_n)-T_k(u_m)|^q dx \leq \epsilon,$$

for all $n,m\geq n_0(k,t,R)$. This show that $\{u_n\}$ is a Cauchy sequence in B_R. Hence that $u_n\to u$ locally.

Now to prove that ∇u_n converges to some function v locally. We need to prove that $\{\nabla u_n\}$ is a Cauchy sequence in any ball B_R. Let $t,\epsilon>0$ again, then

$$\begin{aligned}\{|\nabla u_n-\nabla u_m|>t\}\cap B_R \subset&\{|u_n-u_m|\leq k, |\nabla u_n|\leq l, |\nabla u_m|\leq l, |\nabla u_n-\nabla u_m|>t\}\\ &\cup\{|\nabla u_n|>l\}\cup\{|\nabla u_m|>l\}\cup(\{|u_n-u_m|>k\}\cap B_R).\end{aligned}$$

Choose l large enough such that $|\{|\nabla u_n|>l\}|\leq\epsilon$ for all $n\in\mathbb{N}$. If a is a continuous function independent of x, then by (4), there exists a $\mu>0$, such that $|\xi|<l, |\xi'|<l$ and $|\xi-\xi'|>t$ means

$$[a(x,t,\xi)-a(x,t,\xi')]\cdot[\xi-\xi']\geq\mu.$$

This is a consequence of continuity and strict monotonicity of a. Set

$$d_n = g_n - \gamma_n(x, u_n). \tag{41}$$

Taking $T_k(u_n - u_m)$ as the test function of (35) and by (37), (41), we have

$$\begin{aligned}&\int_{\{|u_n-u_m|<k\}} [a(x,u_n,\nabla u_n) - a(x,u_m,\nabla u_m)]\cdot\nabla(u_n-u_m)dx\\ =&\int_\Omega (d_n-d_m)T_k(u_n-u_m)dx\\ \le& Ck^{1+\theta(p-1)}.\end{aligned}$$

Then

$$\begin{aligned}&\{|u_n-u_m|\le k, |\nabla u_n|\le l, |\nabla u_m|\le l, |\nabla u_n-\nabla u_m|>t\}\\ \le&\frac{1}{\mu}\int_{\{|u_n-u_m|<k\}} [a(x,u_n,\nabla u_n) - a(x,u_m,\nabla u_m)]\cdot\nabla(u_n-u_m)dx\\ \le&\frac{1}{\mu}Ck^{1+\theta(p-1)}\le\epsilon,\end{aligned}$$

if k is small enough such that $k^{1+\theta(p-1)} \le \mu\epsilon/C$.

Since l and k have been confirmed, if n_0 large enough, we have $|(\{|u_n - u_m| > k\} \cap B_R)| \le \epsilon$ for $n, m \ge n_0$. Then we get $|\{|\nabla u_n - \nabla u_m| > t\} \cap B_R| \le 4\epsilon$. This prove that ∇u_n converges to some function v locally.

Finally, since $\{\nabla T_k(u_n)\}_n \in L^p(\Omega)$ for every $k > 0$, it converges weakly to $\{\nabla T_k(u)\}$ in $L^p_{\text{loc}}(\Omega)$. We have $u \in W_0^{1,p}(\Omega)$ and $\nabla u = v$ a.e in Ω.

Step 3. In order to prove the existence of the solution completely, we still need to prove that sequence $\{a(x, u, \nabla u)\}_n$ is bounded in $L^q_{\text{loc}}(\Omega)$ for all

$$q \in \left(1, \frac{N(1-\theta)}{N-(1+\theta(p-1))}\right).$$

Indeed, by Proposition 2, $|\nabla u_n|^{p-1} \in M^{\frac{N(1-\theta)}{N-(1+\theta(p-1))}} \subset L^q_{\text{loc}}(\Omega)$. And by (3), we have $|a(x, u_n, \nabla u_n)| \in L^{p'}(\Omega) \subset L^q_{\text{loc}}(\Omega)$. According to the Nemitskii's theorem, $\nabla u_n \to \nabla u$ implies that

$$a(x, u_n, \nabla u_n) \to a(x, u, \nabla u).$$

It follows that

$$a(x, u, \nabla u) \in M^{\frac{N(1-\theta)}{N-(1+\theta(p-1))}} \subset L^q_{\text{loc}}(\Omega),$$

for all $q \in \left(1, \frac{N(1-\theta)}{N-(1+\theta(p-1))}\right)$. □

3.2. **Proof of Theorem 1.2.** In this subsection, we give the proof of Theorem 1.2 following some ideas in [11, 22].

Proof. **Step 1 (A priori estimates).** Firstly, choosing $T_k(u_n)(1 - \varphi_\delta)^s$ as test function in the weak formulation of (9), where $s = \frac{\eta}{\eta-p+1}$ and η will be given in

(48), we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)(1-\varphi_\delta)^s dx+\int_\Omega |u_n|^{q-1}u_nT_k(u_n)(1-\varphi_\delta)^s dx\\ =&s\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi_\delta T_k(u_n)(1-\varphi_\delta)^{s-1}dx+\int_\Omega g_nT_k(u_n)(1-\varphi_\delta)^s dx\\ &+\int_\Omega f_n^+T_k(u_n)(1-\varphi_\delta)^s dx+\int_\Omega f_n^-T_k(u_n)(1-\varphi_\delta)^s dx.\end{aligned}\tag{42}$$

By (2), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)d\mu\ge c\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{\theta(p-1)}}d\mu,\tag{43}$$

here $d\mu:=(1-\varphi_\delta)^s dx$.

Since $u_nT_k(u_n)\ge 0$,

$$\begin{aligned}\int_\Omega |u_n|^{q-1}u_nT_k(u_n)(1-\varphi_\delta)^s dx\ge&\int_{\{|u_n|\ge k\}}|u_n|^{q-1}u_nT_k(u_n)d\mu\\ \ge& k^{q+1}\mu(\{|u_n|\ge k\}).\end{aligned}\tag{44}$$

Using (3) and the Young inequality, we find

$$\begin{aligned}&\int_\Omega |a(x,u_n,\nabla u_n)\cdot\nabla\varphi_\delta T_k(u_n)(1-\varphi_\delta)^{s-1}|dx\\ \le& c_0k\int_\Omega(|\nabla u_n|^{p-1}+b(x))(|\nabla\varphi_\delta^+|+|\nabla\varphi_\delta^+|)(1-\varphi_\delta)^{s-1}dx\\ \le& Ck\int_\Omega(|\nabla u_n|^{(p-1)r'}+|b(x)|^{r'})(1-\varphi_\delta)^{(s-1)r'}dx+Ck\int_\Omega(|\nabla\varphi_\delta^+|^r+|\nabla\varphi_\delta^+|^r)dx\\ \le& Ck(\int_\Omega(|\nabla u_n|^{(p-1)r'}+|b(x)|^{r'})(1-\varphi_\delta)^{(s-1)r'}dx+\delta).\end{aligned}\tag{45}$$

Combine (42)-(45), by (7) and $\{g_n\}\in L^1(\Omega)$, $b\in L^{p'}(\Omega)$, we have

$$\begin{aligned}&\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{\theta(p-1)}}d\mu+k^{q+1}\mu(\{|u_n|\ge k\})\\ \le& Ck(\int_\Omega|\nabla u_n|^{(p-1)r'}(1-\varphi_\delta)^{(s-1)r'}dx+\delta+\mu(\Omega).\end{aligned}\tag{46}$$

For a fixed $\sigma\ge 0$, thanks to (46), we get

$$\begin{aligned}&\mu(\{|\nabla u_n|>\sigma\})\\ =&\mu(\{|\nabla u_n|>\sigma,|u_n|<k\})+\mu(\{|\nabla u_n|>\sigma,|u_n|\ge k\})\\ \le&\frac{1}{\sigma^p}\int_\Omega|\nabla T_k(u_n)|^pd\mu+\mu(\{|u|>k\})\\ \le&\frac{(1+k)^{\theta(p-1)}}{\sigma^p}\int_\Omega\frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{\theta(p-1)}}d\mu+\mu(\{|u|>k\})\\ \le& C\left(\int_\Omega|\nabla u_n|^{(p-1)r'}(1-\varphi_\delta)^{(s-1)r'}dx+\delta+\mu(\Omega)\right)\left(\frac{(1+k)^{1+\theta(p-1)}}{\sigma^p}+\frac{1}{k^q}\right),\end{aligned}$$

which implies

$$\mu|\{|\nabla u_n| > \sigma\}| \tag{47}$$
$$\leq C\sigma^{-\frac{pq}{q+1+\theta(p-1)}}\left(\int_\Omega |\nabla u_n|^{(p-1)r'}(1-\varphi_\delta)^{(s-1)r'}dx + \delta + \mu|\Omega|\right).$$

Let

$$(p-1)r' < \eta < \frac{pq}{q+1+\theta(p-1)}. \tag{48}$$

Clearly, such η exists by (10). In view of (47)-(48), we have

$$\int_\Omega |\nabla u_n|^\eta d\mu \leq C\left(\int_\Omega |\nabla u_n|^{(p-1)r'}(1-\varphi_\delta)^{(s-1)r'}dx + \delta + \mu(\Omega)\right).$$

By the Holder's inequality,

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{(p-1)r'}(1-\varphi_\delta)^{(s-1)r'}dx\\ \leq& C\left(\int_\Omega |\nabla u_n|^\eta d\mu\right)^{\frac{(p-1)r'}{\eta}}\\ \leq& C\left(\int_\Omega |\nabla u_n|^{(p-1)r'}(1-\varphi_\delta)^{(s-1)r'}dx + \delta + \mu|\Omega|\right)^{\frac{(p-1)r'}{\eta}}.\end{aligned}$$

By Lemma 2.1, $1-\varphi_\delta$ is zero both on a neighbourhood of K^+ and K^-. Hence

$$\int_\Omega |\nabla u_n|^{(p-1)r'}(1-\varphi_\delta)^{(s-1)r'}dx \leq C(\delta + \mu|\Omega|) \leq C(\delta). \tag{49}$$

Using (46) and (49), we conclude that

$$\int_\Omega |\nabla T_k(u_n)|^p dx \leq Ck^{1+\theta(p-1)}. \tag{50}$$

According to Lemma 2.5, we have $|\nabla u_n| \in M^s(\Omega)$, where $s = \frac{pq}{q+1+\theta(p-1)}$.

By (50) and Lemma 2.6, there exists a subsequence, still denoted by u_n, which converges to a measurable function u almost everywhere in Ω. So $T_k(u_n) \to T_k(u)$ in Ω for every $k > 0$.

Since $T_k(u_n) \in W_0^{1,p}(\Omega)$, by the weak lower semi-continuity of the norm, $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. Thus u has an gradient ∇u in the sense of Definition 2.2, as a consequence of the a priori estimates on ∇u_n and (4), we have

$$a(x, u_n, \nabla u_n) \to a(x, u, \nabla u) \textit{ strongly in } (L^s(\Omega))^N, \tag{51}$$

for every $s < \frac{pq}{(q+1+\theta(p-1))(p-1)}$.

Step 2 (Energy estimates). Let $\psi_\delta = \psi_\delta^+ + \psi_\delta^-$, where ψ_δ^+ and ψ_δ^- are as in Lemma 2.1. Then

$$\int_{\{u_n > 2m\}} u_n^q(1-\psi_\delta)dx = \omega(n, m, \delta), \tag{52}$$

and

$$\int_{\{u_n < -2m\}} |u_n|^q(1-\psi_\delta)dx = \omega(n, m, \delta). \tag{53}$$

Choose $\beta_m(u_n)(1-\psi_\delta)$ as test function in the weak formulation of (9), where

$$\beta_m(s) = \begin{cases} \frac{s}{m} - 1, & m < s \le 2m, \\ 1, & s > 2m, \\ 0, & s \le m. \end{cases}$$

We obtain

$$\frac{1}{m}\int_{\{m<u_n<2m\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n(1-\psi_\delta)dx \qquad (A)$$
$$-\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\psi_\delta\beta_m(u_n)dx \qquad (B)$$
$$+\int_\Omega |u_n|^{q-1}u_n\beta_m(u_n)(1-\psi_\delta)dx \qquad (C)$$
$$=\int_\Omega f_n^+\beta_m(u_n)(1-\psi_\delta)dx \qquad (D)$$
$$-\int_\Omega f_n^-\beta_m(u_n)(1-\psi_\delta)dx \qquad (E)$$
$$+\int_\Omega g_n\beta_m(u_n)(1-\psi_\delta)dx. \qquad (F)$$

Since (A) and $-(E)$ are non-negative, we can get rid of them. And since $\beta_m(u_m)$ converges to $\beta_m(u)$ almost everywhere in Ω and in the weak$*$topology of $L^\infty(\Omega)$, $\beta_m(u_n)$ converges to zero in the weak$*$topology of $L^\infty(\Omega)$ as $m\to\infty$, we have

$$-(B) = \int_\Omega a(x,u,\nabla u)\cdot\nabla\psi_\delta\beta_m(u)dx + \omega(n) = \omega(n,m),$$

and

$$(C) \ge \int_{\{u_n>2m\}} u_n^q(1-\psi_\delta)dx.$$

By $\psi_\delta = \psi_\delta^+ + \psi_\delta^-$ and (6),

$$(D) \le \int_\Omega f_n^+(1-\psi_\delta)dx = \int_\Omega (1-\psi_\delta^+)d\lambda^+ - \int_\Omega \psi_\delta^- d\lambda^- + \omega(n) = \omega(n,\delta),$$

and

$$(F) = \omega(n,m).$$

We get (52), the proof of (53) is identical.

Step 3 (Passing to the limit). Now we show that u is an entropy solution to (8) with datum g. Choose $T_k(u_n-\varphi)(1-\psi_\delta)$ as test function in the weak formulation of (9), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n-\varphi)(1-\psi_\delta)dx \qquad (A)$$
$$-\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\psi_\delta T_k(u_n-\varphi)dx \qquad (B)$$
$$+\int_\Omega |u_n|^{q-1}u_nT_k(u_n-\varphi)(1-\psi_\delta)dx \qquad (C)$$
$$=\int_\Omega f_n^+T_k(u_n-\varphi)(1-\psi_\delta)dx \qquad (D)$$

$$-\int_{\Omega} f_n^- T_k(u_n-\varphi)(1-\psi_\delta)dx \qquad (E)$$
$$+\int_{\Omega} g_n T_k(u_n-\varphi)(1-\psi_\delta)dx. \qquad (F)$$

By (13),

$$(A)=\int_{\{|u_n-\varphi|<k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n(1-\psi_\delta)dx$$
$$-\int_{\{|u_n-\varphi|<k\}} a(x,u_n,\nabla u_n)\cdot\nabla\varphi(1-\psi_\delta)dx,$$

while

$$\int_{\{|u_n-\varphi|<k\}} a(x,u_n,\nabla u_n)\cdot\nabla\varphi(1-\psi_\delta)dx$$
$$=\int_{\{|u-\varphi|<k\}} a(x,u,\nabla u)\cdot\nabla\varphi dx+\omega(n,\delta).$$

The Fatou lemma implies

$$\int_{\{|u-\varphi|<k\}} a(x,u,\nabla u)\cdot\nabla u dx$$
$$\leq \liminf_{n\to\infty}\int_{\{|u_n-\varphi|<k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n dx.$$

Using (13), (51), we have

$$-(B)=\int_{\Omega} a(x,u,\nabla u)\cdot\nabla\psi_\delta T_k(u-\varphi)dx+\omega(n)=\omega(n,\delta).$$

While

$$(F)=\int_{\Omega} gT_k(u-\varphi)dx+\omega(n,\delta),$$

and

$$|(D)|+|(E)|=\int_{\Omega}(f_n^+ + f_n^-)T_k(u_n-\varphi)(1-\psi_\delta)dx$$
$$\leq k\int_{\Omega}(f_n^+ + f_n^-)(1-\psi_\delta)dx=\omega(n,\delta).$$

So that we only need to deal with (C). Let $m>k+\|\varphi\|_{L^\infty(\Omega)}$ be fixed,

$$(C)=\int_{\{-2m\leq u_n\leq 2m\}} |u_n|^{q-1}u_n T_k(u_n-\varphi)(1-\psi_\delta)dx \qquad (G)$$
$$+k\int_{\{u_n>2m\}} u_n^q(1-\psi_\delta)dx+k\int_{\{u_n<-2m\}} |u_n|^q(1-\psi_\delta)dx. \qquad (H)$$

By (52) and (53), we get

$$(H)=\omega(n,m,\delta),$$

and

$$(G)=\int_{\Omega}|u|^{q-1}uT_k(u-\varphi)(1-\psi_\delta)dx+\omega(n,m)$$
$$=\int_{\Omega}|u|^{q-1}uT_k(u-\varphi)dx+\omega(n,m,\delta).$$

Summing up the result of (A)-(H), we have

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-\varphi)dx+\int_\Omega |u|^{q-1}uT_k(u-\varphi)dx\le\int_\Omega gT_k(u-\varphi)dx.$$

Thus u is the entropy solution of (8).

Finally we prove (10). Choose $\varphi\in C_c^\infty(\Omega)$ as test function in the weak formulation of (9), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi dx+\int_\Omega |u_n|^{q-1}u_n\varphi dx=\int_\Omega (f_n+g_n)\varphi dx.$$

Thanks to the assumptions of f_n, g_n and by (51),

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{q-1}u_n\varphi dx=-\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi dx+\int_\Omega g\varphi dx+\int_\Omega \varphi d\lambda. \tag{54}$$

Since the entropy solution of (8) is also a distributional solution of the same problem, for the same φ,

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi dx+\int_\Omega |u|^{q-1}u\varphi dx=\int_\Omega g\varphi dx. \tag{55}$$

Together with (54) and (55), we find

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{q-1}u_n\varphi dx=\int_\Omega |u|^{q-1}u\varphi dx+\int_\Omega \varphi d\lambda.$$

Thus (11) holds for every $\varphi\in C_c^\infty(\Omega)$. Since $|u_n|^{q-1}u_n$ is bounded in $L^1(\Omega)$, (11) can be extended by density to the functions in $C_c(\Omega)$. □

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References

[1] A. Alvino, L. Boccardo, V. Ferone, L. Orsina and G. Trombetti, Existence results for nonlinear elliptic equations with degenerate coercivity, *Ann. Mat. Pura. Appl. (4)*, **182** (2003), 53–79. MR 1970464

[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vázquez, An L^1 theory of existence and uniqueness of nonlinear elliptic equations, *Ann. Scuola. Norm. Sup. Pisa Cl. Sci. (4)*, **22** (1995), 241–273. MR 1354907

[3] P. Bénilan, H. Brézis and M. Crandall, A semilinear equation in $L^1(\mathbb{R}^N)$, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **2** (1975), 523–555. MR 390473

[4] L. Boccardo, Some elliptic problems with degenerate coercivity, *Adv. Nonlinear Stud.*, **6** (2006), 1–12. MR 2196888

[5] L. Boccardo, Some cases of weak continuity in nonlinear Dirichlet problems, *J. Funct. Anal.*, **277** (2019), 3673–3687. MR 4001084

[6] L. Boccardo and H. Brézis, Some remarks on a class of elliptic equations with degenerate coercivity, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, **6** (2003), 521–530. MR 2014815

[7] L. Boccardo, G. Croce and L. Orsina, Nonlinear degenerate elliptic problems with $W_0^{1,1}(\Omega)$ solutions, *Manuscripta Math.*, **137** (2012), 419–439. MR 2875285

[8] L. Boccardo, T. Gallouët and L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **13** (1996), 539–551. MR 1409661

[9] H. Brézis, Nonlinear elliptic equations involving measures, in *Contributions to Nonlinear Partial Differential Equations*, Res. Notes Math., 89, Pitman, Boston, MA, 1983, 82–89. MR 730798

[10] G. Cirmi, On the existence of solutions to non-linear degenerate elliptic equations with measures data, *Ricerche Mat.*, **42** (1993), 315–329. MR 1283363

[11] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, Renormalized solutions for elliptic equations with general measure data, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **28** (1999), 741–808. MR 1760541

[12] D. Giachetti and M. Porzio, Exitence results for some nonuniformly elliptic equations with irregular data, *J. Math. Anal. Appl.*, **257** (2001), 100–130. MR 1824669

[13] D. Giachetti and M. Porzio, Elliptic equations with degenerate coercivity: Gradient regularity, *Acta. Math. Sin. (Engl. Ser.)*, **19** (2003), 349–370. MR 1987800

[14] S. Huang, Quasilinear elliptic equations with exponential nonlinearity and measure data, *Math. Methods Appl. Sci.*, **43** (2020), 2883–2910.

[15] S. Huang, T. Su, X. Du and X. Zhang, Entropy solutions to noncoercive nonlinear elliptic equations with measure data, *Electron. J. Differential Equations*, **2019** (2019), 1–22. MR 3992308

[16] S. Huang and Q. Tian, Marcinkiewicz estimates for solution to fractional elliptic Laplacian equation, *Comput. Math. Appl.*, **78** (2019), 1732–1738. MR 3988064

[17] S. Huang and Q. Tian, Harnack-type inequality for fractional elliptic equations with critical exponent, *Math. Methods Appl. Sci.*, (2020), 1–18.

[18] S. Huang, Q. Tian, J. Wang and J. Mu, Stability for noncoercive elliptic equations, *Electron. J. Differential Equations*, **2016** (2016), 1–11. MR 3547431

[19] H.-F. Huo, Q. Yang and H. Xiang, Dynamics of an edge-based SEIR model for sexually transmitted diseases, *Math. Biosci. Eng.*, **17** (2020), 669–699. MR 4045022

[20] J. Leray and J. L. Lions, Quelques résultats de Višik sur les problèmes elliptiques semi-linéaires par les méthodes de Minty-Browder, *Bull. Soc. Math. France*, **93** (1965), 97–107. MR 194733

[21] X. Li and S. Huang, Stability and bifurcation for a single-species model with delay weak kernel and constant rate harvesting, *Complexity*, **2019** (2019).

[22] L. Orsina and A. Prignet, Non-existence of solutions for some nonlinear elliptic equations involving measures, *Proc. Roy. Soc. Edinburgh Sect. A*, **130** (2000), 167–187. MR 1742585

[23] L. Orsina and A. Porretta, Strong stability results for nonlinear elliptic equations with respect to very singular perturbation of the data, *Commum. Contemp. Math.*, **3** (2001), 259–285. MR 1831931

[24] L. Orsina and A. Prignet, Strong stability results for solutions of elliptic equations with power-like lower order terms and measure data, *J. Funct. Anal.*, **189** (2002), 549–566. MR 1892181

[25] M. M. Porzio and F. Smarrazzo, Radon measure-valued solutions for some quasilinear degenerate elliptic equations, *Ann. Mat. Pura. Appl. (4)*, **194** (2015), 495–532. MR 3322432

[26] Q. Tian and Y. Xu, Effect of the domain geometry on the solutions to fractional Brezis-Nirenberg problem, *J. Funct. Spaces*, **2019** (2019), 4pp. MR 3985641

[27] Y. Ye, H. Liu, Y. Wei, M. Ma and K. Zhang, Dynamic study of a predator-prey model with weak Allee effect and delay, *Adv. Math. Phys.*, **2019** (2019), 15pp. MR 3994110

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