ELECTRONIC RESEARCH ARCHIVE
Volume 28 Number 1, March 2020 Pages 157–163
eISSN: 2688-1594 AIMS (2020)
doi:10.3934/era.2020010

RICCI SOLITONS OF THE $\mathbb{H}^2 \times \mathbb{R}$ LIE GROUP

LAKEHAL BELARBI*

(Communicated by Abdon Atangana)

Abstract. In this work we consider the three-dimensional Lie group denoted by $\mathbb{H}^2 \times \mathbb{R}$, equipped with left-invariant Riemannian metric. The existence of non-trivial (i.e., not Einstein) Ricci solitons on three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$ is proved. Moreover, we show that there are not gradient Ricci solitons.

1. Introduction

The notion of Ricci solitons is introduced by Hamilton in [16], which is a naturel generalization of Einstein metrics. A Ricci solitons is a pseudo-Riemannian metric g on a smooth manifold $\mathcal{M}$ such that there exists a smooth vector field X on $\mathcal{M}$ satisfying the following equation:

$$\mathcal{L}_X g + Ric = \lambda g, \tag{1}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, Ric denotes the Ricci tensor and λ is a real number. A Ricci soliton is said to be a shrinking, steady or expanding, respectively, if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. Moreover, we say that a Ricci soliton (M, g) is a *gradient Ricci soliton* if it admits a vector field X satisfying $X = \text{grad}\, h$, for some potential function h.

The description of Ricci solitons can be regarded as a first step in understanding the Ricci flow, since they are the fixed points of the flow. Moreover, they are important in understanding singularities of the Ricci flow. Under suitable conditions, type I singularity models correspond to shrinking solitons, type II models correspond to steady Ricci solitons, while type III models correspond to expanding Ricci solitons.

In the special case that $\mathcal{M}$ is a Lie group and g is a left-invariant metric, we say that g is a left-invariant Ricci soliton on $\mathcal{M}$ if the above equation (1) holds.

Study of Ricci soliton, over different geometric spaces is one of interesting topics in geometry and mathematical physics. In particular, it has become more important after Grigori Perelman applied Ricci solitons to solve the long standing Poincaré conjecture. Ricci solitons correspond to self-similar solutions of Hamilton's Ricci flow [17], play a fundamental role in the formation of singularities of the flow and have been studied by several authors (see [12], [13]). They can be viewed as fixed points of the Ricci flow, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. Ricci solitons are of interests to physicists as well and are called quasi-Einstein metrics in physics literature. In fact, Theoretical

Received by the editors October, 2019 and, in revised form, February, 2020.
2010 *Mathematics Subject Classification.* 53C50, 53B30.
Key words and phrases. $\mathbb{H}^2 \times \mathbb{R}$ Lie group, Ricci soliton, Riemannian metrics.
* Corresponding author: Lakehal Belarbi.

physicists have been looking into the equation of Ricci solitons in relation with String Theory. A seminal contribution in this direction is due to Friedan ([15]).

Lorentzian Ricci solitons have been intensively studied, showing many essential differences with respect to the Riemannian case (see [2], [4], [3], [8], [23],[22]). In fact, although there exist three-dimensional Riemannian homogeneous Ricci solitons [1], [20], there are no left-invariant Riemannian Ricci solitons on three-dimensional Lie groups [14] (see also [18] and [24]). Moreover, the Lorentzian case is much richer, allowing the existence of expanding, steady and shrinking left-invariant Ricci soliton [7]. In [22] prove that Lorentzian and Riemannian five-dimensional solvable Lie groups admit different vector fields resulting in expanding Ricci solitons. Also has proved that those Ricci solitons are not gradient.

In this paper, we consider the left-invariant Riemannian metric admitted by the three-dimensional Lie group denoted by $\mathbb{H}^2 \times \mathbb{R}$ and we prove the existence of vector field for which the soliton equation (1) holds. In Section 3, Ricci solitons of three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$ are characterized via a system of partial differential equations. In particular, we show that three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$ admit a vector fields in expanding Ricci solitons. Finally, we show that there are not gradient Ricci solitons.

2. Connection and curvature of the $\mathbb{H}^2 \times \mathbb{R}$ Lie group

Let $\mathbb{H}^2$ be represented by the upper half-plane model $\{(x, y) \in \mathbb{R}^2 | y > 0\}$ equipped with the metric $g_{\mathbb{H}^2} = \frac{1}{y^2}(dx^2 + dy^2)$. The space $\mathbb{H}^2$, with the group structure derived by the composition of proper affine map, is a Lie group and the metric $g_{\mathbb{H}^2}$ is left invariant. Therefore the Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with respect to the operation

$$(x, y, z) \star (x', y', z') = (x'y + x, yy', z + z')$$

and the left invariant product metric

$$g = \frac{1}{y^2}(dx^2 + dy^2) + dz^2. \tag{2}$$

With respect to the metric g an orthonormal basis of left invariant vector fields on $\mathbb{H}^2 \times \mathbb{R}$ is

$$E_1 = y\frac{\partial}{\partial x}, E_2 = y\frac{\partial}{\partial y}, E_3 = \frac{\partial}{\partial z}. \tag{3}$$

From this, the Lie brackets are given by

$$[E_1, E_2] = -E_1, [E_2, E_3] = 0, [E_3, E_1] = 0.$$

Throughout the paper, we shall endow the three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$ with left-invariant Riemannian g.

We will denote by ∇ the Levi-Civita connection of $(\mathbb{H}^2 \times \mathbb{R}, g)$, by R its curvature tensor, taken with the sign convention:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X z - \nabla_{[X,Y]} Z,$$

and by Ric the Ricci tensor of $(\mathbb{H}^2 \times \mathbb{R}, g)$, which is defined by

$$Ric\,(X, Y) = \sum_{k=1}^{3} g\,(E_k, E_k)\, g(R(E_k, X)Y, E_k),$$

where $\{E_k\}_{k=1,..,3}$ is an orthonormal basis.

The Levi-Civita connection ∇ of the $\mathbb{H}^2 \times \mathbb{R}$ Lie group with respect to this frame (3) is

$$\begin{cases} \nabla_{E_1} E_1 = E_2, \nabla_{E_1} E_2 = -E_1, \nabla_{E_1} E_3 = 0 \\ \nabla_{E_2} E_1 = 0, \nabla_{E_2} E_2 = 0, \nabla_{E_2} E_3 = 0 \\ \nabla_{E_3} E_1 = 0, \nabla_{E_3} E_2 = 0, \nabla_{E_3} E_3 = 0. \end{cases} \tag{4}$$

The non-vanishing curvature tensor R components are computed as

$$R(E_1, E_2)E_1 = E_2, R(E_1, E_2)E_2 = -E_1. \tag{5}$$

The Ricci curvature components $\{Ric_{ij}\}$ are computed as

$$Ric_{11} = Ric_{22} = -1,\ Ric_{12} = Ric_{13} = Ric_{23} = Ric_{33} = 0. \tag{6}$$

The scalar curvature τ of the $\mathbb{H}^2 \times \mathbb{R}$ Lie group is constant and we have

$$\tau = trRic = \sum_{i=1}^{3} g(E_i, E_i) Ric(E_i, E_i) = -2. \tag{7}$$

3. Ricci solitons of 3-dimensional $\mathbb{H}^2 \times \mathbb{R}$ Lie group

In this section we analyze the existence of Ricci solitons on three-dimensional Riemannian Lie group $(\mathbb{H}^2 \times \mathbb{R}, g)$ equipped with the left-invariant Riemannian metric (2).

Let $X = f_1 E_1 + f_2 E_2 + f_3 E_3$ be an arbitrary vector field on $(\mathbb{H}^2 \times \mathbb{R}, g)$, where $f_1, .., f_3$ are smooth functions of the variables x, y, z. We will denote the coordinate basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ by $\{\partial_x, \partial_y, \partial_z\}$.

The Lie derivative of the metric (2) with respect to X is given by :

$$\begin{cases} (\mathcal{L}_X g)(E_1, E_1) = -2(f_2 - \partial_x f_1), \\ (\mathcal{L}_X g)(E_1, E_2) = f_1 + \partial_y f_1 + \partial_x f_2, \\ (\mathcal{L}_X g)(E_1, E_3) = \partial_z f_1 + \partial_x f_3, \\ (\mathcal{L}_X g)(E_2, E_2) = 2\partial_y f_2, \\ (\mathcal{L}_X g)(E_2, E_3) = \partial_z f_2 + \partial_y f_3, \\ (\mathcal{L}_X g)(E_3, E_3) = 2\partial_z f_3. \end{cases} \tag{8}$$

Thus, by using (2), (6) and (8) in (1), a standard calculation gives that the three-dimensional Lie group $\left(\mathbb{H}^2 \times \mathbb{R}, g\right)$ is a Ricci soliton if and only if the following system holds,

$$\begin{cases} -2(f_2 - \partial_x f_1) - 1 = \lambda, \\ f_1 + \partial_y f_1 + \partial_x f_2 = 0, \\ \partial_z f_1 + \partial_x f_3 = 0, \\ 2\partial_y f_2 - 1 = \lambda, \\ \partial_z f_2 + \partial_y f_3 = 0, \\ 2\partial_z f_3 = \lambda. \end{cases} \tag{9}$$

We deriving the third equation in (9) with respect to z, we get

$$\partial_z^2 f_1 = 0. \tag{10}$$

The equation (10) yields that

$$f_1 = \varphi(x, y)z + \psi(x, y), \tag{11}$$

where φ and ψ are smooth functions.

Deriving the first equation with respect to y and used the fourth equation in (9), we find

$$\partial_x \partial_y f_1 = \frac{1}{2}(1+\lambda). \tag{12}$$

Next, deriving the second equation with respect to y in (9) and used equation in (12), we find

$$\partial_y f_1 + \partial_y^2 f_1 = 0. \tag{13}$$

Replacing f_1 in equations (12) and (13), we get

$$\begin{cases} (\partial_y \varphi + \partial_y^2 \varphi) z + \partial_y \psi + \partial_y^2 \psi = 0, \\ \partial_x \partial_y \varphi z + \partial_x \partial_y \psi = \frac{1}{2}(1+\lambda). \end{cases} \tag{14}$$

By derivation of the equations (14) with respect to z, we get

$$\begin{cases} \partial_y \varphi + \partial_y^2 \varphi = 0, \\ \partial_y \psi + \partial_y^2 \psi = 0, \\ \partial_x \partial_y \varphi = 0, \\ \partial_x \partial_y \psi = \frac{1}{2}(1+\lambda). \end{cases} \tag{15}$$

We deriving the second equation with respect to x and used the fourth equation in (15), we find

$$\lambda = -1 \tag{16}$$

Integrating the first and second equations in (15), we find

$$\begin{cases} \varphi(x,y) = \alpha_1 e^{-y} + \varphi_1(x), \\ \psi(x,y) = \alpha_2 e^{-y} + \psi_1(x), \end{cases} \tag{17}$$

where $\alpha_i \in \mathbb{R}$, and φ_1, ψ_1 are smooth functions with only one variable x. Thus

$$f_1 = (\alpha_1 e^{-y} + \varphi_1(x)) z + \alpha_2 e^{-y} + \psi_1(x). \tag{18}$$

Next, we replace f_1 in the second equation in (9), we get

$$(\varphi_1(x) + \varphi_1''(x)) z + \psi_1(x) + \psi_1''(x) = 0. \tag{19}$$

We derive equation (19) with respect to z, we get

$$\begin{cases} \varphi_1(x) + \varphi_1''(x) = 0, \\ \psi_1(x) + \psi_1''(x) = 0. \end{cases} \tag{20}$$

By integration of (20) with respect to x, we find that

$$\begin{cases} \varphi_1(x) = \alpha_3 \cos(x) + \alpha_4 \sin(x), \\ \psi_1(x) = \alpha_5 \cos(x) + \alpha_6 \sin(x), \end{cases} \tag{21}$$

where $\alpha_i \in \mathbb{R}$. Thus

$$f_1 = (\alpha_1 e^{-y} + \alpha_3 \cos(x) + \alpha_4 \sin(x)) z + \alpha_2 e^{-y} + \alpha_5 \cos(x) + \alpha_6 \sin(x). \tag{22}$$

From the first equation in (9), gives

$$f_2 = (-\alpha_3 \sin(x) + \alpha_4 \cos(x)) z - \alpha_5 \sin(x) + \alpha_6 \cos(x). \tag{23}$$

The last equation in (9) gives

$$f_3 = -\frac{1}{2}z + \xi(x, y), \tag{24}$$

where ξ is a smooth function depending to x and y. We replacing f_1, f_2, and f_3 in the third and fifth equations in (9), we get

$$\begin{cases} \partial_x \xi = \alpha_1 e^{-y} + \alpha_3 \cos(x) + \alpha_4 \sin(x), \\ \\ \partial_y \xi = -\alpha_3 \sin(x) + \alpha_4 \cos(x). \end{cases} \tag{25}$$

Integrating the first equation in (25) with respect to x, we get

$$\xi(x, y) = -\alpha_1 x e^{-y} - \alpha_3 \sin(x) + \alpha_4 \cos(x) + \alpha_7,$$

and replace ξ in the second equation in (25), we find that

$$\alpha_1 = \alpha_3 = \alpha_4 = 0.$$

finally for arbitrary reals constants α_i we have

$$\begin{cases} f_1 = \alpha_2 e^{-y} + \alpha_5 \cos(x) + \alpha_6 \sin(x), \\ \\ f_2 = -\alpha_5 \sin(x) + \alpha_6 \cos(x), \\ \\ f_3 = -\frac{1}{2}z + \alpha_7, \end{cases} \tag{26}$$

Thus, it easily follows that the vector field $X = f_1E_1 + f_2E_2 + f_3E_3$ where $f_1, .., f_3$ are given by (26) satisfies (9). Note that $\lambda = -1$. Summarizing, we proved that the three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$ admits appropriate vector fields for which (1) holds, obtaining the following result.

Theorem 3.1. *Let $(\mathbb{H}^2 \times \mathbb{R}, g)$ be the three-dimensional Lie group equipped with the left-invariant Riemannian metric g given by (2). Then, $\left(\mathbb{H}^2 \times \mathbb{R}, g\right)$ is an Expanding Ricci soliton.*

Now, let $X = \text{grad } h$ be an arbitrary gradient vector field on $\left(\mathbb{H}^2 \times \mathbb{R}, g\right)$ with potential function h, X is then given by

$$\text{grad}h \quad = \quad y^2 \partial_x h \partial_x + y^2 \partial_y h \partial_y + \partial_z h \partial_z. \tag{27}$$

From (26) it follows that $\left(\mathbb{H}^2 \times \mathbb{R}, g\right)$ is gradient soliton if and only if the potential function h satisfy the following systems

$$\begin{cases} \partial_x h = \frac{\alpha_2}{y} e^{-y} + \frac{\alpha_5}{y} \cos(x) + \frac{\alpha_6}{y} \sin(x), \\ \\ \partial_y h = -\frac{\alpha_5}{y} \sin(x) + \frac{\alpha_6}{y} \cos(x), \\ \\ \partial_z h = -\frac{1}{2}z + \alpha_7. \end{cases} \tag{28}$$

Hence, with direct integration we prove that

$$h(x, y, z) = \ln(y)[\alpha_5 \sin(x) - \alpha_6 \cos(x)] - \frac{1}{4}z^2 + \alpha_7 z + \alpha_8, \ \ \alpha_i \in \mathbb{R}. \tag{29}$$

But the function h does not verify the first equation in (28), we proved the following result.

Corollary 1. *Let $(\mathbb{H}^2 \times \mathbb{R}, g)$ be the three-dimensional Lie group equipped with the left-invariant Riemannian metric g given by (2). Then, $\left(\mathbb{H}^2 \times \mathbb{R}, g\right)$ is not gradient Ricci soliton.*

4. Conclusion

Our study on the Ricci solitons of three-dimensional Lie group denoted by $\mathbb{H}^2 \times \mathbb{R}$, equipped with left-invariant Riemannian metric.The existence of non-trivial (i.e., not Einstein) Ricci solitons on three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$ is proved. More precisely, we proved that there are not gradient Ricci solitons.

Acknowledgments

We would like to thank the referee for valuable suggestions regarding both the contents and exposition of this article.

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Lakehal Belarbi, Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (U.M.A.B.), B.P.227,27000, Mostaganem, Algeria

Email address: lakehalbelarbi@gmail.com