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TRAVELING WAVES FOR A NONLOCAL DISPERSAL SIR MODEL EQUIPPED DELAY AND GENERALIZED INCIDENCE

YANG YANG, YUN-RUI YANG* AND XIN-JUN JIAO

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ABSTRACT. In this paper, the existence and non-existence of traveling wave solutions are established for a nonlocal dispersal SIR model equipped delay and generalized incidence. In addition, the existence and asymptotic behaviors of traveling waves under critical wave speed are also contained. Especially, the boundedness of traveling waves is obtained completely without imposing additional conditions on the nonlinear incidence.

1. INTRODUCTION

Since traveling waves of reaction-diffusion equations are often used to describe many propagation phenomena in nature [1,2,5,7,8], such as species invasion, phase transition, epidemic transmission in biology, ecology, epidemiology and so on, wide attention has been attracted to the investigations of traveling waves. In particular, much focus has been drawn to the famous SIR epidemic models [3, 6, 9, 13]. For example, in 2012, Wang et al. [13] considered the SIR model equipped the standard incidence

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} - \frac{\alpha u_1(x,t) u_2(x,t)}{u_1(x,t)+u_2(x,t)}, \\ \frac{\partial u_2(x,t)}{\partial t} = d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + \frac{\alpha u_1(x,t) u_2(x,t)}{u_1(x,t)+u_2(x,t)} - \nu u_2(x,t), \\ \frac{\partial u_3(x,t)}{\partial t} = d_3 \frac{\partial^2 u_3(x,t)}{\partial x^2} + \nu u_2(x,t), \end{cases} \tag{1.1}$$

where u_1, u_2 and u_3 are the size of susceptible, infectious and cured(removal) individuals, $d_i > 0$ $(i = 1, 2, 3)$, $\alpha > 0$ and $\nu > 0$ represent their ability of mobility, infection and recovery, respectively. Based on the work of [14], Wang et al. [13] obtained that (1.1) has a traveling wave solution while the wave speed $c > c_* = 2\sqrt{d_2(\alpha - \nu)}$ and the basic reproduction number $R_0 = \frac{\alpha}{\nu} > 1$. Moreover, the non-existence was also contained when $R_0 \in (1, +\infty)$ with $c \in (0, c_*)$ and $R_0 \in (0, 1)$ by two-side Laplace transform [15].

As is well known, for a long range diffusion such as population ecology, neurology and epidemiology, the flow of individuals is not only limited to the same one point, but is affected by other points around it. Therefore, the nonlocal dispersal is more realistic than the local diffusion [4, 10, 20], which can be expressed by a convolution term $L[u](x,t) = J*u(x,t) - u(x,t) = \int_{\mathbb{R}} J(x-\eta)(u(\eta,t)d\eta - u(x,t))d\eta$, where $u(x,t)$ denotes the density of individuals and $J(x-\eta)$ is the probability distribution of individuals which jump from location η to location x. Then $J * u(x,t) = \int_{\mathbb{R}} J(x -$

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$\eta)u(\eta,t)d\eta$ denotes the rate at which individuals are arriving at location x from all other locations, while the term $-u(x,t) = -\int_{\mathbb{R}} J(x-\eta)u(x,t)d\eta$ is the rate at which they are leaving location x to travel to all other locations. Thus, the nonlocal dispersal $L[u](x,t)$ can be biologically interpreted as the net increasing rate of $u(x,t)$. In 2014, by the same method in [13], Li and Yang [11] inspected the nonlocal dispersal situation of (1.1):

(1.2)

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1 \int_{\mathbb{R}} J(x-\eta)(u_1(\eta,t)-u_1(x,t))d\eta - \frac{\alpha u_1(x,t)u_2(x,t)}{u_1(x,t)+u_2(x,t)}, \\ \frac{\partial u_2(x,t)}{\partial t} = d_2 \int_{\mathbb{R}} J(x-\eta)(u_2(\eta,t)-u_2(x,t))d\eta + \frac{\alpha u_1(x,t)u_2(x,t)}{u_1(x,t)+u_2(x,t)} - \nu u_2(x,t), \\ \frac{\partial u_3(x,t)}{\partial t} = d_3 \int_{\mathbb{R}} J(x-\eta)(u_3(\eta,t)-u_3(x,t))d\eta + \nu u_3(x,t), \end{cases}$$

where $\int_{\mathbb{R}} J(x-\eta)u(\eta,t)d\eta := J*u$ denotes the normal convolution. For other related works, one can refer to [17, 18].

Furthermore, the state of time delay exists universally in the objective material world [12, 16]. In addition, the general incidence is more extensive to illustrate the disease spread process than the special standard incidence. For the above reasons, Zhang et al. [19] considered the following SIR model

(1.3)

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = & d_1 \int_{\mathbb{R}} J(x-\eta)(u_1(\eta,t)-u_1(x,t))d\eta - f(u_1(x,t))g(u_2(x,t-\tau)), \\ \frac{\partial u_2(x,t)}{\partial t} = & d_2 \int_{\mathbb{R}} J(x-\eta)(u_2(\eta,t)-u_2(x,t))d\eta \\ & +f(u_1(x,t))g(u_2(x,t-\tau)) - \nu u_2(x,t), \\ \frac{\partial u_3(x,t)}{\partial t} = & d_3 \int_{\mathbb{R}} J(x-\eta)(u_3(\eta,t)-u_3(x,t))d\eta + \nu u_2(x,t). \end{cases}$$

They showed that there is a number $c_* > 0$ such that traveling wave solutions $(U(x+ct), V(x+ct))$ of (1.3) conforming to $U(\pm\infty) = U_{\pm\infty}$, $V(\pm\infty) = 0$ and $U(-\infty) > U(+\infty)$ exist when V is a bounded function and $R_0 := \frac{f(U_{-\infty})g'(0)}{\nu} > 1$ with $c > c_*$, but for $R_0 \in (0,1)$ and $R_0 \in (1,+\infty)$ with $c \in (0, c_*)$, there are no traveling waves.

Although there have been many excellent results as mentioned above, it is necessary to indicate the core problem that (i) the boundedness of traveling waves is not obtained easily by constructing bounded invariant cones due to the shortage of natural upper bound of nonlinear incidence $g(u_2)$, which is different from the standard incidence case $\frac{u_2}{u_1+u_2} < 1$. On the other hand, (ii) it is extremely tough to investigate the existence and asymptotic behaviors with $c = c_*$ because of the absence of order-preserving quality of semi-flow of (1.3) and the inferior smoothness of solutions for the import of nonlocal dispersal.

In order to solve the first problem (i), Zhang et al. [19] obtained the boundedness and asymptotic behaviors of traveling waves when $c > c_*$ and $R_0 > 1$ by assuming that $f(U_{-\infty})g(V_0) \leq \nu V_0$ holds for some $V_0 \in \mathbb{R}$. Similarly, owing to the same difficulties, Zou and Wu [21] only obtained the boundedness and asymptotic behaviors under the large wave speed and a specific assumption.

However, the scope of incidence functions is not extensive since the strict condition in [19] and there is still not result of existence of traveling waves under critical wave speed. Fortunately, Yang and Li [18] recently considered a SIR model equipped bilinear function $\alpha u_1 u_2$ and established the boundedness and asymptotic behaviors of traveling waves for $c \geq c_*$ and $R_0 > 1$ by some limit discussions and a series of analyses without imposing additional conditions upon incidence function.

Based on the above fact and motivated by the idea in [17, 18], in this paper, we illustrate the existence, boundedness and asymptotic behaviors of traveling waves of system (1.3) for non-critical and critical wave speed, respectively, which complete and improve the works in [19, 21]. In this sense, the above two difficulties we mentioned in (i) and (ii) are solved. Moreover, we extend the delay-free case in [17, 18] to the case with time delay and generalize the bilinear incidence to a more general case.

Below, the following assumptions are always valid for the whole paper:

- (A_1)**:** $f(\cdot) \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, $f(0) = 0$ and $f'(U) > 0$ for all $U \geq 0$, where $\mathbb{R}^+ = [0, +\infty)$;
- (A_2)**:** $g(\cdot) \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, $g(0) = 0$ and $g'(V) > 0$, $g''(V) \leq 0$ for all $V \geq 0$;
- (A_3)**:** $J(\cdot) \in C^1(\mathbb{R}, \mathbb{R}^+)$, $J(-x) = J(x)$ and $\int_{\mathbb{R}} J(x)dx = 1$;
  Moreover, J is compactly supported.

The remaining part of this paper is designed as follows. In section 2, we complete the existence results of traveling waves when $R_0 > 1$ with $c > c_*$ in [21] by some analytical techniques. In addition, the boundedness of traveling waves is also included. In section 3, the existence and asymptotic behaviors of traveling waves when $R_0 > 1$ and $c = c_*$ are established by a prior estimate and some technical analyses. In section 4, a new way is given to derive the non-existence of traveling waves for $R_0 > 1$ and $c < c_*$.

2. Boundedness and existence of traveling waves with $c > c_*$

In this section, the boundedness and existence of traveling wave solutions of (1.3) are established for $R_0 = \frac{f(U_{-\infty})g'(0)}{\nu} > 1$ with $c > c_*$.

Noticing that the first two equation of (1.3) are independent of the function u_3, we focus only on the solutions with the profile of $(U(x+ct), V(x+ct)) = (U(\xi), V(\xi))$ of the following system

(2.1)
$$\begin{cases} cU'(\xi) = d_1 \int_{\mathbb{R}} J(\xi - \eta)(U(\eta) - U(\xi))d\eta - f(U(\xi))g(V(\xi - c\tau)), \\ cV'(\xi) = d_2 \int_{\mathbb{R}} J(\xi - \eta)(V(\eta) - V(\xi))d\eta + f(U(\xi))g(V(\xi - c\tau)) - \nu V(\xi) \end{cases}$$

conforming to

$$(U(-\infty), V(-\infty)) = (U_{-\infty}, 0), \ (U(+\infty), V(+\infty)) = (U_{+\infty}, 0), \tag{2.2}$$

where $\xi = x + ct$. Next, the following two important conclusions in [21] are needed:

Proposition 2.1. [[21], Lemma 2.1] *Assume that $R_0 = \frac{f(U_{-\infty})g'(0)}{\nu} > 1$. Then some positive pair of (c_*, λ_*) exists for the following equations*

$$\triangle(\lambda_*, c_*) = 0, \quad \left.\frac{\partial \triangle(\lambda, c)}{\partial \lambda}\right|_{(\lambda_*, c_*)} = 0,$$

where $\triangle(\lambda, c) = d_2 \int_{\mathbb{R}} J(\eta)(e^{-\lambda\eta} - 1)d\eta + f(U_{-\infty})g'(0)e^{-\lambda c\tau} - \nu - c\lambda$. Moreover,
(1) if $c > c_*$, the equation $\triangle(\lambda, c) = 0$ admits two positive real roots $\lambda_1(c) < \lambda_2(c) < +\infty$ conforming to that $\triangle(\lambda, c) > 0$ in $(0, \lambda_1(c)) \cup (\lambda_2(c), +\infty)$ and $\triangle(\lambda, c) < 0$ in $(\lambda_1(c), \lambda_2(c))$;
(2) if $0 < c < c_*$, then $\triangle(\lambda, c) > 0$ for all $\lambda \in [0, +\infty)$.

Proposition 2.2. [[21], Theorem 2.1] *If $(A_1) - (A_3)$ hold and $R_0 > 1$, then,*

- **(i) :** *for $c > c_*$, the system (1.3) admits some $(U(\xi), V(\xi))$ according with $U(-\infty) = U_{-\infty} > 0$ and $V(-\infty) = 0$;*
- **(ii) :** *if $\limsup_{\xi \to +\infty} V(\xi) < +\infty$, then $V(+\infty) = 0$ and $U(+\infty) < U_{-\infty}$;*

(iii): *if* $c > \max\{c_*, \frac{3}{2}d_2k_1\}$*, then* $\limsup_{\xi\to+\infty} V(\xi) < +\infty$*, where* $k_1 := U_{-\infty}\int_{\mathbb{R}} J(\eta)|\eta|d\eta$.

According to Proposition 2.2, $\limsup_{\xi\to+\infty} V(\xi) < +\infty$ holds for $R_0 > 1$ with $c > c_*$ when $\frac{3}{2}d_2k_1 \le c_*$. To perfect Proposition 2.2, we complete the case of $\frac{3}{2}d_2k_1 > c_*$ and give out the proof of $\limsup_{\xi\to+\infty} V(\xi) < +\infty$ for $R_0 > 1$ with $c_* < c < \frac{3}{2}d_2k_1 := c_1$. For the proof, we first establish the following lemmas and the boundedness of $V(\xi)$.

Lemma 2.1. *If* $R_0 > 1$ *and* $c > c^*$*, then* $0 < U(\xi) < U_{-\infty}$ *and* $V(\xi) > 0$ *on* $\mathbb{R}$.

Proof. Firstly, from Lemma 2.5 and Theorem 2.1 in [21], we obtain that

$$0 \le U(\xi) \le U_{-\infty}, \quad 0 \le \max\{e^{\lambda_1\xi}(1 - Me^{\eta_0\xi}), 0\} \le V(\xi) \le e^{\lambda_1\xi}, \tag{2.3}$$

for some number $M > 0, \eta_0 > 0$.

Secondly, if there is some $\xi_0 \in \mathbb{R}$ with $U(\xi_0) = 0$, then $U'(\xi_0) = 0$. By (2.1) and (A_1), we have

$$0 = cU'(\xi_0) = d_1\int_{\mathbb{R}} J(\eta - \xi_0)(U(\eta) - U(\xi_0))d\eta - f(U(\xi_0))g(V(\xi_0 - c\tau)) \ge 0, \tag{2.4}$$

Denote $R_J > 0$ as the radius of the support set of J. It follows that $U(\xi) \equiv 0$ for $\xi \in [\xi_0 - R_J, \xi_0 + R_J]$ by (2.4). Applying the above facts to $U(\xi_0 - R_J) = U(\xi_0 + R_J) = 0$, we obtain $U(\xi) \equiv 0$ for $\xi \in [\xi_0 - 2R_J, \xi_0 + 2R_J]$ and consequently $U(\xi) \equiv 0$ for $\xi \in \mathbb{R}$, which is contradictory to $U(-\infty) > 0$. Therefore, U is positive on $\mathbb{R}$.

Finally, we can prove similarly that $V > 0$ and $U < U_{-\infty}$ for $\xi \in \mathbb{R}$. This proof is complete. □

Lemma 2.2. *Let* $K(\xi) = \int_{\mathbb{R}} J(\xi - \eta)\frac{V(\eta)}{V(\xi)}d\eta$ *and* $\omega(\xi) = \frac{V'(\xi)}{V(\xi)}$*. Then,* K *and* ω *are both bounded for* $c_* < c \le c_1$.

Proof. According to (2.1), we have

$$\omega(\xi) = dK(\xi) - \rho + \frac{1}{c}\frac{f(U(\xi))g(V(\xi - c\tau))}{V(\xi)} \ge dK(\xi) - \rho. \tag{2.5}$$

where $\rho = \frac{d_2+\nu}{c}$, $d = \frac{d_2}{c}$. Let $H(\xi) = e^{\rho\xi + \int_0^\xi \omega(s)ds}$. Then, it can be derived from (2.5) that

$$\frac{V(\xi - y)}{V(\xi)} = e^{\int_\xi^{\xi-y}\omega(s)ds} = e^{\rho y}\frac{H(\xi - y)}{H(\xi)} \tag{2.6}$$

and thus

$$H'(\xi) = (\rho + \omega(\xi))H(\xi) \ge d\int_{\mathbb{R}} J(\eta)e^{\rho\eta}\frac{H(\xi - \eta)}{H(\xi)}d\eta \cdot H(\xi) \ge 0. \tag{2.7}$$

Therefore, H is non-decreasing and $\lim_{\xi\to-\infty} H(\xi) = 0$. Choose a number $R_1 > 0$ with $2R_1 < R_J$. By an integral process for (2.7) from $-\infty$ to ξ, it holds that

$$\begin{aligned} H(\xi) &\ge d\int_{-\infty}^{\xi}\int_{\mathbb{R}} J(\eta)e^{\rho\eta}H(\theta - \eta)d\eta d\theta \\ &\ge d\int_{\mathbb{R}}\int_{\xi - R_1}^{\xi} J(\eta)e^{\rho\eta}H(\theta - \eta)d\theta d\eta \ge dR_1\int_{\mathbb{R}} J(\eta)e^{\rho\eta}H(\xi - R_1 - \eta)d\eta \end{aligned}$$

and thus

$$\int_{\mathbb{R}} J(\eta)e^{\rho\eta}\frac{H(\xi-R_1-\eta)}{H(\xi)}d\eta \leq \frac{1}{dR_1}. \tag{2.8}$$

By a similar integral process for (2.7) from $\xi - R_1$ to ξ, we find that

(2.9)

$$\begin{aligned} H(\xi) &\geq d\int_{\mathbb{R}}\int_{\xi-R_1}^{\xi} J(\eta)e^{\rho\eta}H(\theta-\eta)d\theta d\eta + H(\xi-R_1) \\ &\geq dR_1\int_{-\infty}^{-2R_1} J(\eta)e^{\rho\eta}H(\xi-R_1-\eta)d\eta \geq dR_1\int_{-\infty}^{-2R_1} J(\eta)e^{\rho\eta}d\eta\cdot H(\xi+R_1). \end{aligned}$$

Defining $k_2 := (dR_1\int_{-\infty}^{-2R_1} J(\eta)e^{\rho\eta}d\eta)^{-1}$, then

$$H(\xi+R_1) \leq k_2 H(\xi) \quad \text{for } \xi\in\mathbb{R}, \tag{2.10}$$

By (2.6), (2.8) and (2.10), we have

$$|K(\xi)| \leq k_2\int_{\mathbb{R}} J(\eta)e^{\rho\eta}\frac{H(\xi-R_1-\eta)}{H(\xi)}d\eta \leq \frac{k_2}{dR_1}. \tag{2.11}$$

On the other hand, it is obvious that

$$g(V) = g(0) + g'(\hat{V})V \leq g'(0)V \tag{2.12}$$

for some $\hat{V}\in(0,V)$ by (A_2). From (2.5), (2.11), (2.12) and $c_* < c \leq c_1$, it follows that

$$\begin{aligned} |\omega(\xi)| &\leq d\cdot\frac{k_2}{dR_1} + \rho + \frac{f(U_{-\infty})g'(0)}{c}\cdot\frac{V(\xi-c\tau)}{V(\xi)} \\ &= \frac{k_2}{R_1} + \rho + \frac{f(U_{-\infty})g'(0)}{c}e^{\int_{\xi}^{\xi-c\tau}\omega(s)ds} \\ &\leq \frac{k_2}{R_1} + \rho + \frac{f(U_{-\infty})g'(0)}{c_*}e^{\rho c_1\tau}. \end{aligned}$$

This proof is complete. □

Lemma 2.3. *Let $c_k\in(c_*,c_1)$ and $\{(c_k,U_k,V_k)\}$ represent a sequence of solution of (2.1). If there is a sequence $\{\xi_k\}$ satisfying $V_k(\xi_k) = \max_{\xi\in[\xi_k-R_J,\xi_k+R_J]} V_k(\xi)$ and*
$\lim\limits_{k\to+\infty} V_k(\xi_k) = +\infty$ for all $k\in N$, then $\lim\limits_{k\to+\infty} U_k(\xi_k) = 0$.

Proof. Suppose that there exist some sequence $\{\xi_k\}$ and a number $\delta_1 > 0$ satisfying $\lim\limits_{k\to+\infty} V_k(\xi_k) = +\infty$, $V_k(\xi_k) = \max_{\xi\in[\xi_k-R_J,\xi_k+R_J]} V_k(\xi)$ and $U_k(\xi_k) \geq \delta_1$ for all $k\in N$.

From (2.1) and (2.3), it holds that

$$U_k'(\xi) \leq \frac{d_1}{c_k}\int_{\mathbb{R}} J(\eta)|U_k(\xi-\eta)-U_k(\xi)|d\eta \leq \frac{d_1U_{-\infty}}{c_*}, \quad k\in N. \tag{2.13}$$

Denoting $k_3 := \frac{\delta_1 c_*}{2d_1U_{-\infty}}$ and by an integral process for (2.13) from ξ to ξ_k, it follows that

$$U_k(\xi) \geq U_k(\xi_k) - \int_{\xi}^{\xi_k}\frac{d_1U_{-\infty}}{c_*}d\eta \geq \delta_1 - k_3\frac{d_1U_{-\infty}}{c_*} = \frac{\delta_1}{2}, \tag{2.14}$$

$$\xi\in[\xi_k-k_3,\xi_k], \quad k\in N.$$

In view of the fact that $V_k(\xi_k) = \max_{\xi\in[\xi_k-R_J,\xi_k+R_J]} V_k(\xi)$, we obtain that $V_k'(\xi_k) = 0$ for all $k \in N$ and therefore

$$f(U_k(\xi_k))g(V_k(\xi_k - c\tau)) > \nu V_k(\xi_k) \to +\infty \quad \text{as } k \to +\infty \tag{2.15}$$

by using (2.2). Since $f(U_k(\xi_k)) \le f(U_{-\infty})$ and $g \in C^1(\mathbb{R})$, (2.15) implies that

$$g(V_k(\xi_k - c\tau)) \to +\infty \text{ and } V_k(\xi_k - c\tau) \to +\infty \quad \text{as } k \to +\infty. \tag{2.16}$$

Moreover, by Lemma 2.2 and (2.6), there exists some $C_0 > 0$ conforming to

$$\frac{V_k(\xi_k - c\tau)}{V_k(\xi - c\tau)} = e^{\int_{\xi-c\tau}^{\xi_k-c\tau}\omega(s)ds} \le e^{C_0 k_3}, \quad \xi \in [\xi_k - k_3, \xi_k], \ k \in N. \tag{2.17}$$

Thus, it follows from (2.16) and (2.17) that

$$\min_{\xi\in[\xi_k-k_3,\xi_k]} V_k(\xi - c\tau) \ge e^{-C_0k_3} V_k(\xi_k - c\tau) \to +\infty \quad \text{as } k \to +\infty$$

and

$$\min_{\xi\in[\xi_k-k_3,\xi_k]} g(V_k(\xi - c\tau)) \to +\infty \quad \text{as } k \to +\infty. \tag{2.18}$$

By (2.14), (2.18) and (2.2), we obtain

$$\begin{aligned}\max_{\xi\in[\xi_k-k_3,\xi_k]} U_k'(\xi) &\le \frac{d_1 U_{-\infty}}{c_k} - \frac{1}{c_k}\min_{\xi\in[\xi_k-k_3,\xi_k]} f(U_k(\xi))g(V_k(\xi - c\tau)) \\ &\le \frac{d_1 U_{-\infty}}{c_*} - \frac{1}{c_1} f(\frac{\delta_1}{2}) \min_{\xi\in[\xi_k-k_3,\xi_k]} g(V_k(\xi - c\tau)) \to -\infty \quad \text{as } k \to +\infty,\end{aligned}$$

which implies some $k_0 > 0$ exists satisfying

$$U_k'(\xi) \le -\frac{U_{-\infty}}{k_3}, \quad k \ge k_0, \ \xi \in [\xi_k - k_3, \xi_k]. \tag{2.19}$$

Integrating on both sides of (2.19) from $\xi_k - k_3$ to ξ_k, we have

$$U_k(\xi_k) \le U_k(\xi_k - k_3) - k_3 \cdot \frac{U_{-\infty}}{k_3} \le U_{-\infty} - U_{-\infty} = 0, \quad k \ge k_0.$$

This contradicts with the inequation $U_k(\xi) > 0$ on $\mathbb{R}$. The proof is complete. □

Lemma 2.4. *Suppose that* $\limsup_{\xi\to+\infty} V(\xi) = +\infty$. *Then* $V(\xi)$ *satisfies* $V(+\infty) = +\infty$.

Proof. Assume that $V_0 = \liminf_{\xi\to+\infty} V(\xi)$ is finite and take some sequence $\{\xi_k\}$ satisfying $\lim_{k\to+\infty} \xi_k = \infty$ and $\lim_{k\to+\infty} V(\xi_k) = V_0$. Moreover, choose a sequence $\{\eta_k\}$ with $\eta_k \in [\xi_k, \xi_{k+1}]$ and $V(\eta_k) = \max_{\xi\in[\xi_k,\xi_{k+1}]} V(\xi)$ and make a general assumption that $V(\xi_k) < V_0 + 1$ for $k \in N$. Then, from $\limsup_{\xi\to+\infty} V(\xi) = +\infty$, it follows that

$$V(\eta_k) \to +\infty \textit{ as } k \to +\infty. \tag{2.20}$$

Consequently, we can assume that $V(\eta_k) \ge (V_0 + 1)e^{k_4 R_J}$, where $k_4 = \sup_{\xi\in\mathbb{R}} |\omega(\xi)|$. By (2.6), we find that

$$V(\xi) = e^{-\int_\xi^{\eta_k}\omega(s)ds} V(\eta_k) \ge e^{-k_4R_J} V(\eta_k) \ge V_0 + 1, \quad \xi \in [\eta_k - R_J, \eta_k + R_J].$$

Therefore, $[\eta_k - R_J, \eta_k + R_J] \subset (\xi_k, \xi_{k+1})$ and hence $\lim_{k\to+\infty} U(\eta_k) = 0$ due to Lemma 2.3. Furthermore, it follows from (2.1), (2.6), (2.12) and (2.20) that

$$\begin{aligned}
0 &= cV'(\eta_k)\\
&= d_2\int_{\mathbb{R}} J(\eta)(V(\eta_k-\eta)-V(\eta_k))d\eta + f(U(\eta_k))g(V(\eta_k-c\tau)) - \nu V(\eta_k)\\
&\le (f(U(\eta_k))g'(0)e^{\int_{\eta_k}^{\eta_k-c\tau}\omega(s)ds} - \nu)V(\eta_k)\\
&\le (f(U(\eta_k))g'(0)e^{c\tau k_4} - \nu)V(\eta_k) \to -\infty \ \ as\ k\to+\infty.
\end{aligned}$$

This is a contradiction and the proof is thus finished. □

Next, we display and prove the primary results of this section.

Theorem 2.5. (Boundedness) *For every $c > c_*$, the solution V is a bounded function.*

Proof. By Proposition 2.2, it is sufficient to verify $\limsup_{\xi\to+\infty} V(\xi) < +\infty$ for $c_* < c \le c_1$. Suppose that $\limsup_{\xi\to+\infty} V(\xi) = +\infty$ for a contrary.

Denote $B(\xi) = -(d_2+\nu) + \frac{f(U(\xi))g(V(\xi-c\tau))}{V(\xi)}$. By Lemma 2.4 and Lemma 2.3, it can be seen that $(U(+\infty), V(+\infty)) = (0, +\infty)$. Therefore, from Lemma 2.2, it follows that

$$\left|\frac{f(U(\xi))g(V(\xi-c\tau))}{V(\xi)}\right| \le f(U(\xi))g'(0)e^{\rho c_1\tau} \to 0 \ \text{ as } \xi\to+\infty$$

and $B(+\infty) = -(d_2+\nu)$. By (2.5) and Proposition 3.7 in [20], the limit $\lim_{\xi\to+\infty}\omega(\xi)$ exists and belongs to the solution set of the following equation

$$P(\lambda, c) := d_2\int_{\mathbb{R}} J(\eta)e^{-\lambda\eta}d\eta - \nu - c\lambda - d_2.$$

On the other hand, the equation $P(\lambda, c) = 0$ admits a unique positive real root λ_0 by a similar calculation to Theorem 2.6 in [18]. Therefore, $\omega(+\infty) = \lambda_0$ due to the positivity of V and $V(+\infty) = +\infty$. Notice that $\frac{\partial^2 P}{\partial\lambda^2} = d_2\int_{\mathbb{R}} J(\eta)\eta^2 e^{-\lambda\eta}d\eta > 0$ and λ_2 satisfies

$$d_2(\int_{\mathbb{R}} J(\eta)e^{-\lambda\eta}d\eta - 1) - c\lambda - \nu = -f(U_{-\infty})g'(0)e^{-\lambda c\tau} < 0.$$

It is clear that $\lambda_2 < \lambda_0$. Moreover, by $\lim_{\xi\to+\infty}\omega(\xi) = \lambda_0$, we can take a number $\xi_* \in \mathbb{R}$ complying with $\omega(\xi) = \frac{V'(\xi)}{V(\xi)} \ge \lambda_2$ for $\xi > \xi_*$. Recalling $\lambda_2 > \lambda_1$ and (2.3), it follows that

$$0 < V(\xi_*) \le V(\xi)e^{-\lambda_2(\xi-\xi_*)} \le e^{(\lambda_1-\lambda_2)\xi+\lambda_2\xi_*} \to 0 \ \text{ as } \xi\to+\infty,$$

which raises a contradiction. This ends the proof. □

Theorem 2.6. (Existence) *If $R_0 = \frac{f(U_{-\infty})g'(0)}{\nu} > 1$ with $c > c^*$, then some $(U(\xi), V(\xi))$ exists for (2.1) conforming to $(U(\pm\infty), V(\pm\infty)) = (U_{\pm\infty}, 0)$ for some number $U_{+\infty} < U_{-\infty}$. Furthermore, $0 < U(\xi) < U_{-\infty}$, $0 < V(\xi) < +\infty$ for $\xi \in \mathbb{R}$ and*

(2.21)
$$\lim_{\xi\to-\infty}\frac{V(\xi)}{e^{\lambda_1\xi}} = 1,\ \int_{\mathbb{R}} f(U(\theta))g(V(\theta-c\tau))d\theta < +\infty,\ \int_{\mathbb{R}} V(\theta)d\theta = \frac{c(U_{-\infty}-U_{+\infty})}{\nu}.$$

Proof. By Proposition 2.2, Lemma 2.1 and Theorem 2.5, it is enough to prove that $(U(\xi), V(\xi))$ satisfies (2.21).

The rest of proofs are divided into the following three steps.

Step 1. $\int_{\mathbb{R}} f(U(\theta))g(V(\theta - c\tau))d\theta < +\infty$ *and* $\lim_{\xi\to-\infty} e^{-\lambda_1\xi}V(\xi) = 1.$

For $\xi < \frac{1}{\eta}\ln\frac{1}{M}$, from (2.3), we find that $1 \geq e^{-\lambda_1\xi}V(\xi) \geq (1 - Me^{\eta_0\xi}) \to 1$ as $\xi \to -\infty$. Therefore, $\lim_{\xi\to-\infty} e^{-\lambda_1\xi}V(\xi) = 1$ by Squeeze theorem.

Notice that

$$\begin{aligned}\left|\int_z^x (J(\eta) * U(\theta) - U(\theta))d\theta\right| &= \left|\int_z^x \int_{\mathbb{R}} J(\eta)\eta \int_0^1 U'(\theta - t\eta)dtd\eta d\theta\right| \\ &= \left|\int_{\mathbb{R}} J(\eta)\eta \int_0^1 (U(z - t\eta) - U(x - t\eta))dtd\eta\right| \leq k_1.\end{aligned} \tag{2.22}$$

Taking $z \to -\infty$ and $x \to +\infty$ in (2.22), we have

$$\begin{aligned}&\lim_{\substack{z\to-\infty\\x\to+\infty}} \left|\int_z^x (J(\eta) * U(\theta) - U(\theta))d\theta\right| \\ &= \lim_{\substack{z\to-\infty\\x\to+\infty}} \left|\int_{\mathbb{R}} J(\eta)\eta \int_0^1 (U(z - t\eta) - U(x - t\eta))dtd\eta\right| \leq k_1.\end{aligned} \tag{2.23}$$

Moreover, by $0 < U(\xi) < U_{-\infty}$, $U(-\infty) = U_{-\infty}$, (2.23) and (2.1), we obtain

$$\begin{aligned}\int_{-\infty}^x f(U(\theta))g(V(\theta - c\tau))d\theta =& d_1 \int_{-\infty}^x \int_{-\infty}^{+\infty} J(\eta)(U(\theta - \eta) - U(\theta))d\eta d\theta \\ &- c[U(x) - U_{-\infty}] \leq k_1 + cU_{-\infty}.\end{aligned} \tag{2.24}$$

Therefore, by taking $x \to +\infty$, it follows that

$$\int_{\mathbb{R}} f(U(\theta))g(V(\theta - c\tau))d\theta \leq k_1 + cU_{-\infty} < +\infty.$$

Step 2. $\int_{\mathbb{R}} V(\theta)d\theta = \frac{c(U_{-\infty} - U_{+\infty})}{\nu}$.

From (2.1), (2.22), (2.23), (2.24) and Proposition 2.2 , it follows that

$$\begin{aligned}\nu\int_{\mathbb{R}} V(\theta)d\theta &= d_2 \int_{\mathbb{R}}\int_{\mathbb{R}} J(\eta)(V(\theta - \eta) - V(\theta))d\eta d\theta \\ &\quad + \int_{\mathbb{R}} f(U(\theta))g(V(\theta - c\tau))d\theta - c[V(+\infty) - V(-\infty)] \\ &= d_2 \int_{\mathbb{R}} J(\eta)\eta \int_0^1 (V(-\infty) - V(+\infty))dtd\eta - c[V(+\infty) - V(-\infty)] \\ &\quad + d_1 \int_{\mathbb{R}} J(\eta)\eta \int_0^1 (U(-\infty) - U(+\infty))dtd\eta - c[U(+\infty) - U(-\infty)] \\ &= c(U_{-\infty} - U_{+\infty}).\end{aligned}$$

This completes the proof. □

Up to now, by constructing the boundedness of $V(\xi)$, we obtain a more general existence result and thus improve and complete the results in [19, 21]. Next, we illustrate the existence under critical speed for further improvement.

3. Existence of traveling waves with $c = c_*$

In this section, an approximating method is applied to establish the existence of solutions of (2.1) when $R_0 > 1$ with $c = c_*$. For this proof, a prior estimate is needed in the followings.

Lemma 3.1. *Assume $\{c_k\} \subset (c_*, c_* + 1)$ is a decreasing sequence with $c_k \to c_*$ as $k \to +\infty$ and let $(c_k, U_k,\ V_k)$ be a solution of (2.1) for $k \in N$. Then, $\| U_k \|_{C^{1,1}(\mathbb{R})}$ and $\| V_k \|_{C^{1,1}(\mathbb{R})}$ are both uniformly bounded.*

Proof. Firstly, we prove the uniform boundedness of $\{U_k\}$ and $\{V_k\}$. It is obvious that $\{U_k\}$ is uniformly bounded due to (2.3). Suppose that there is a sequence $\{\xi_k\}$ satisfying $\lim\limits_{k\to+\infty} V_k(\xi_k) = +\infty$ for a contradiction.

Take a sequence $\{\eta_k\}$ with $V_k(\eta_k) = \max\limits_{\xi\in\mathbb{R}} V_k(\xi)$. Then, $V_k'(\eta_k) = 0$ and $\lim\limits_{k\to+\infty} V_k(\eta_k) = +\infty$. By Lemma 2.3, we find $\lim\limits_{k\to+\infty} U_k(\eta_k) = 0$. Therefore, from (2.1), (2.12) and Lemma 2.2, it follows that

$$\begin{aligned}
0 &= c_k V_k'(\eta_k) \\
&= d_2 \int_{\mathbb{R}} J(\eta)(V_k(\eta_k - \eta) - V_k(\eta_k))d\eta + f(U_k(\eta_k))g(V_k(\eta_k - c\tau)) - \nu V_k(\eta_k) \\
&\le f(U_k(\eta_k))g'(0)V_k(\eta_k - c_k\tau) - \nu V_k(\eta_k) \le (f(U_k(\eta_k))g'(0)e^{c_k\tau\rho} - \nu)V_k(\eta_k) \\
&\le (f(U_k(\eta_k))g'(0)e^{(c_*+1)\tau\rho} - \nu)V_k(\eta_k) \to -\infty, \quad as \quad k \to +\infty,
\end{aligned}$$

which deduces a contradiction. Hence, $\{V_k\}$ is uniformly bounded on $\mathbb{R}$.

Secondly, according to the above discussions and (2.1), it holds that $\{U_k'\}$ and $\{V_k'\}$ are both uniformly bounded. By a similar discussion to Lemma 2.6 in [19], it can be derived that $U_k,\ V_k,\ U_k'(\xi)$ and $V_k'(\xi)$ are all Lipschitz continuous. The proof is finished. □

Next, we demonstrate the existence, boundedness, positivity and asymptotic behavior of traveling waves as followings.

Theorem 3.2. *If $R_0 = \frac{f(U_{-\infty})g'(0)}{\nu} > 1$ with $c = c_*$. Then, there is a solution $(U_*(\xi), V_*(\xi))$ of (2.1) conforming to $(U_*(\pm\infty), V_*(\pm\infty)) = (U_{\pm\infty}, 0)$ for some positive number $U_{+\infty} < U_{-\infty}$. Furthermore, $0 < U_*(\xi) < U_{-\infty},\ 0 < V_*(\xi) < +\infty$ for $\xi \in \mathbb{R}$ and*

$$\lim_{\xi\to-\infty} \frac{V_*(\xi)}{e^{\lambda_1\xi}} = 1, \quad \int_{\mathbb{R}} f(U_*(\theta))g(V_*(\theta - c\tau))d\theta < +\infty, \quad \int_{\mathbb{R}} V_*(\theta)d\theta = \frac{c_*(U_{-\infty} - U_{+\infty})}{\nu}.$$

Proof. According to Theorem 2.6, it can be seen that (U_k, V_k) satisfies (2.1), (2.2), (2.21), $0 < U_k(\xi) < U_{-\infty},\ V_k(\xi) > 0$ for all $\xi \in \mathbb{R}$. In addition, by Lemma 3.1 and Arzela-Ascoli theorem, some subsequence of $\{(U_k, V_k)\}$ can be extracted, still denoted by $\{(U_k, V_k)\}$, satisfying $\lim\limits_{k\to+\infty}(U_k, V_k) = (U_*, V_*)$ in $C^1_{loc}(\mathbb{R})$.

Therefore, from the compactness of J and a limiting process, it follows that (U_*, V_*) satisfies (2.1). Moreover, by the properties of (U_k, V_k) and a similar discussion to Theorem 2.6, we find that $0 \le U_*(\xi) \le U_{-\infty},\ 0 \le V_*(\xi) < \infty$ on $\mathbb{R}$ and

$$\int_{\mathbb{R}} f(U_*(\theta))g(V_*(\theta - c_*\tau))d\theta < +\infty, \quad \int_{\mathbb{R}} V_*(\theta)d\theta < +\infty.$$

Consequently, $V_*'(\xi)$ is bounded on $\mathbb{R}$ and $V_*(\pm\infty) = 0$.

The rest of proofs are divided into the following three steps.

Step 1. *$U_*(\pm\infty) = U_{\pm\infty}$ for some $U_{+\infty} < U_{-\infty}$ and $\int_{\mathbb{R}} V_*(\theta)d\theta = \frac{c_*(U_{-\infty}-U_{+\infty})}{\nu}$.*

Assume that $\underline{U}_* := \liminf_{\xi\to-\infty} U_*(\xi) < U_{-\infty}$. Then, there exists a sequence $\{\xi_n\}$ with $\lim_{n\to+\infty} \xi_n = -\infty$ and $\lim_{n\to+\infty} U_*(\xi_n) = \underline{U}_*$. Denote that $((U_*)_n, (V_*)_n)(\xi) = (U_*, V_*)(\xi+\xi_n)$. As $\| (U_*)_n \|_{C^{1,1}(\mathbb{R})}$ is uniformly bounded, we can assume generally that $\lim_{n\to+\infty} (U_*)_n(\xi) = U_\infty(\xi)$ in $C^1_{loc}(\mathbb{R})$. Meanwhile, it follows from $V_*(-\infty) = 0$ that $\lim_{n\to+\infty} (V_*)_n(\xi) = 0$ in $C^1_{loc}(\mathbb{R})$. Therefore, combining the fact that $g(0) = 0$ with (2.1), we obtain

$$c_* U'_\infty(\xi) = d_1(J * U_\infty(\xi) - U_\infty(\xi)), \quad \xi \in \mathbb{R}. \tag{3.1}$$

From Proposition 3.6 in [20], we find that $U_\infty(\xi)$ is a constant function. Furthermore, $U_\infty(\xi) \equiv \underline{U}_*$ in $\mathbb{R}$ by $U_\infty(0) = \underline{U}_*$, which implies $\lim_{n\to+\infty} (U_*)_n(\xi) \equiv \underline{U}_*$ in $C^1_{loc}(\mathbb{R})$.

Notice that the solution (c_k, U_k, V_k) satisfies

$$c_k U'_k(\xi) = d_1 \int_{\mathbb{R}} J(\eta)(U_k(\xi-\eta) - U_k(\xi))d\eta - f(U_k(\xi))g(V_k(\xi - c\tau)), \\ \xi \in \mathbb{R},\ k \in N. \tag{3.2}$$

By an integral process for (3.2) from $-\infty$ to ξ_n and combining (2.22) with $U_k(-\infty) = U_{-\infty}$ for all $n \in N$, it follows that

$$\begin{aligned} & c_k[U_k(\xi_n) - U_{-\infty}] \\ = \;& d_1 \int_{\mathbb{R}} J(\eta)\eta \int_0^1 (U_{-\infty} - U_k(\xi_n - t\eta))dtd\eta - \int_{-\infty}^{\xi_n} f(U_k(\xi))g(V_k(\xi - c\tau))d\xi. \end{aligned}$$

Owing to U_* and V_* are both bounded in $\mathbb{R}$, passing to $k \to +\infty$ and $n \to +\infty$ on the above equation, we find that $0 > c_*(\underline{U}_* - U_{-\infty}) = 0$, which deduces a contradiction. Therefore, $U_*(-\infty) = U_{-\infty}$.

The remaining proofs in this step are similar to (ii) of Theorem 2.1 in [21] and Step 2 of Theorem 2.6 in this paper, so we omit them here.

Step 2. The functions U_* and V_* are both positive on $\mathbb{R}$.

Suppose there is a number $\xi_0 \in \mathbb{R}$ with $U_*(\xi_0) = 0$, and thereby $U'_*(\xi_0) = 0$. By a similar discussion to (2.4), it leads to $U_*(\xi) \equiv 0$ on $\mathbb{R}$. Consequently, there is a contradiction with the fact $U_*(-\infty) \neq U_*(+\infty)$. Hence, $U_*(\xi) > 0$ in $\mathbb{R}$.

For the proof of positivity of $V_*(\xi)$, we still assume some η_0 exists satisfying $V_*(\eta_0) = 0$. According to (2.1) and $V_*(\xi) \geq 0$ in $\mathbb{R}$, we obtain

$$0 = c_* V'_*(\eta_0) = d_2 \int_{\mathbb{R}} J(\eta)V_*(\eta_0 - \eta)d\eta + f(U_*(\eta_0))g(V_*(\eta_0 - c\tau)) \geq 0.$$

Therefore, $V_*(\xi) \equiv 0$ for a similar argument as (2.4). Moreover, by (2.1),

$$c_* U'_*(\xi) = d_1 \int_{\mathbb{R}} J(\eta)(U_*(\xi - \eta) - U_*(\xi))d\eta, \quad \xi \in \mathbb{R}.$$

On the other hand, $U_*(\xi)$ is a constant function by the same discussions as (3.1), which raises a contradiction for the reason of $U_*(-\infty) \neq U_*(+\infty)$.

Step 3. $U_*(\xi) < U_{-\infty}$ for $\xi \in \mathbb{R}$.

Assume that $U_*(\gamma_0) = U_{-\infty}$ for some $\gamma_0 \in \mathbb{R}$. By the fact that $U_*(\xi) \le U_{-\infty}$ and (2.1), it follows that

$$\begin{aligned} 0 = c_* U_*'(\gamma_0) &= d_1 \int_{\mathbb{R}} J(\eta)(U_*(\gamma_0 - \eta) - U_*(\gamma_0))d\eta - f(U_*(\gamma_0))g(V_*(\gamma_0 - c_*\tau)) \\ &\le -f(U_*(\gamma_0))g(V_*(\gamma_0 - c_*\tau)), \end{aligned}$$

which is impossible since the positivity of U_* and V_*. This finishes the proof. □

4. Nonexistence of traveling waves

In this section, we prove the nonexistence of solution of (2.1) by a different approach which depends closely on the conclusions in Section 2.

Theorem 4.1. *Suppose* $R_0 = \frac{f(U_{-\infty})g'(0)}{\nu} > 1$. *For every speed* $c < c_*$ *with* $c \ne 0$ *and any positive number* μ_0, *there are no solutions of* (2.1) *satisfying*

$$U(-\infty) = U_{-\infty}, \quad V(-\infty) = 0, \quad \lim_{\xi \to -\infty} e^{-\mu_0 \xi} V(\xi) = 1. \tag{4.1}$$

Proof. For the first case $0 < c < c_*$, we assume that (2.1) admits some positive solution $(U(\xi), V(\xi))$ conforming to (4.1) with speed c for a contradiction. Take a sequence $\{\xi_n\}$ with $\lim\limits_{n\to+\infty} \xi_n = -\infty$ and denote

$$U_n(\xi) := U(\xi_n + \xi), \quad V_n(\xi) := \frac{V(\xi_n + \xi)}{V(\xi_n)}, \quad G_n(V_n(\xi - c\tau)) := \frac{g(V_n(\xi - c\tau)V(\xi_n))}{V(\xi_n)}.$$

By (2.1) and $U(-\infty) = U_{-\infty}$, we find that $(U_n(\xi), V_n(\xi))$ satisfies $\lim\limits_{n\to+\infty} U_n(\xi) = U_{-\infty}$ in $C^1_{loc}(\mathbb{R})$ and

$$cV_n'(\xi) = d_2 \int_{\mathbb{R}} J(\eta)V_n(\xi - \eta)d\eta - (d_2 + \nu)V_n(\xi) + f(U_n(\xi))G_n(V_n(\xi - c\tau)). \tag{4.2}$$

Define $\omega(\xi) = \frac{V'(\xi)}{V(\xi)}$. It is clear that $\omega(\xi)$ is bounded in $\mathbb{R}$ by a similar discussion to Lemma 2.2. Moreover, $V_n(\xi)$, $V_n'(\xi)$ and $V_n''(\xi)$ are all locally uniformly bounded on $\mathbb{R}$ by the fact that $V_n(\xi) = e^{\int_{\xi_n}^{\xi_n+\xi} \omega(s)ds}$ and (4.2). Therefore, there is some subsequence of $\{V_n\}$, still denoted by $\{V_n\}$, conforming to $\lim\limits_{n\to+\infty} V_n(\xi) = \widetilde{V}(\xi)$ in $C^1_{loc}(\mathbb{R})$. Furthermore, $\widetilde{V}(\xi) \ge 0$ in $\mathbb{R}$ and $\widetilde{V}(0) = 1$. Meanwhile, from the fact that $V(-\infty) = 0$, $g(0) = 0$ and Taylor's formula, it follows that

$$\begin{aligned} \frac{g(V(\xi_n + \xi - c\tau))}{V(\xi_n)} &= g'(0)\frac{V(\xi_n + \xi - c\tau)}{V(\xi_n)} + \frac{1}{V(\xi_n)} o(V^2(\xi_n + \xi - c\tau)) \\ &= g'(0)\frac{V(\xi_n + \xi - c\tau)}{V(\xi_n)} + \frac{V(\xi_n + \xi - c\tau)}{V(\xi_n)} o(V(\xi_n + \xi - c\tau)) \\ &= g'(0)V_n(\xi - c\tau) + V_n(\xi - c\tau)o(V(\xi_n + \xi - c\tau)). \end{aligned} \tag{4.3}$$

Consequently, by the boundedness of $V_n(\xi)$ and (4.3), it leads to

$$\lim_{n\to+\infty} \frac{g(V(\xi_n + \xi - c\tau))}{V(\xi_n)} = g'(0)\widetilde{V}(\xi - c\tau).$$

According to (4.2), we obtain

$$c\widetilde{V}'(\xi) = d_2 \int_{\mathbb{R}} J(\eta)\widetilde{V}(\xi - \eta)d\eta - (d_2 + \nu)\widetilde{V}(\xi) + f(U_{-\infty})g'(0)\widetilde{V}(\xi - c\tau). \tag{4.4}$$

Next, we claim that $\widetilde{V}(\xi) \neq 0$. Otherwise, the equation $\widetilde{V}(\xi_0) = 0$ holds for some number $\xi_0 \in \mathbb{R}$ and it follows from (4.4) that

$$0 = c\widetilde{V}'(\xi_0) = d_2 \int_{\mathbb{R}} J(\eta)\widetilde{V}(\xi_0 - \eta)d\eta - (d_2 + \nu)\widetilde{V}(\xi_0) + f(U_{-\infty})g'(0)\widetilde{V}(\xi_0 - c\tau) \geq 0.$$

Therefore, $\widetilde{V}(\xi) \equiv 0$ in $\mathbb{R}$, which contradicts with $\widetilde{V}(0) = 1$.

Let $\widetilde{\omega}(\xi) = \widetilde{V'}(\xi)/\widetilde{V}(\xi)$. Since $\lim_{\xi \to -\infty} e^{-\mu_0 \xi} V(\xi) = 1$, it leads to

$$\widetilde{V}(\xi) = e^{\mu_0 \xi} \cdot \lim_{n \to +\infty} \frac{e^{-\mu_0(\xi + \xi_n)} V(\xi + \xi_n)}{e^{-\mu_0 \xi_n} V(\xi_n)} = e^{\mu_0 \xi} \tag{4.5}$$

and thereby $\widetilde{\omega}(\xi) \equiv \mu_0$ in $\mathbb{R}$. Meanwhile, dividing both sides of (4.4) by $\widetilde{V}(\xi)$ and combining (2.6) with (4.5), we have

$$\begin{aligned} c\widetilde{\omega}(\xi) &= d_2 \int_{\mathbb{R}} J(\eta) e^{\int_\xi^{\xi - \eta} \widetilde{\omega}(s)ds} d\eta - (d_2 + \nu) + f(U_{-\infty})g'(0)\frac{\widetilde{V}(\xi - c\tau)}{\widetilde{V}(\xi)} \\ &= d_2 \int_{\mathbb{R}} J(\eta) e^{\int_\xi^{\xi - \eta} \widetilde{\omega}(s)ds} d\eta - (d_2 + \nu) + f(U_{-\infty})g'(0)e^{-\mu_0 c\tau}. \end{aligned}$$

Therefore, according to Proposition 3.7 in [20], the limits $\lim_{\xi \to \pm\infty} \widetilde{\omega}(\xi)$ both exist and belong to the root set of the following equation

$$\triangle_1(\lambda, c) := d_2 \int_{-\infty}^{+\infty} J(\eta) e^{-\lambda \eta} d\eta - c\lambda + f(U_{-\infty})g'(0)e^{-\mu_0 c\tau} - \nu - d_2.$$

However, $\triangle_1(\mu_0, c) = \triangle(\mu_0, c) > 0$ when $c < c_*$, which implies a contradiction.

For the second case $c < 0$, denote $\varphi(\theta) = U(-\xi)$ and $\phi(\theta) = V(-\xi)$. Then, $(\varphi(\theta), \phi(\theta))$ satisfies

$$|c|\phi'(\theta) = d_2 \left(\int_{\mathbb{R}} J(\eta)\phi(\theta - \eta)d\eta - \phi(\theta) \right) + f(\varphi(\theta))g(\phi(\theta - |c|\tau)) - \nu\phi(\theta). \tag{4.6}$$

Applying a similar discussion as the above case $0 < c < c_*$, it can be obtained that the nonexistence of traveling waves with $c < 0$. This accomplishes the proof. ☐

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Yang Yang, School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou, Gansu 730070, China
Email address: 15095339103@163.com

Yun-Rui Yang, School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou, Gansu 730070, China
Email address: lily1979101@163.com

Xin-Jun Jiao, School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou, Gansu 730070, China
Email address: jxj19948@163.com