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ASYMPTOTIC SPREADING IN A DELAYED DISPERSAL PREDATOR-PREY SYSTEM WITHOUT COMPARISON PRINCIPLE

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Abstract. This paper deals with the initial value problem of a predator-prey system with dispersal and delay, which does not admit the classical comparison principle. When the initial value has nonempty compact support, the initial value problem formulates that two species synchronously invade the same habitat in population dynamics. By constructing proper auxiliary equations and functions, we confirm the faster invasion speed of two species, which equals to the minimal wave speed of traveling wave solutions in earlier works.

1. Introduction

Spatial propagation dynamics of parabolic type systems has been widely investigated in literature, and two important indices on spatial propagation are minimal wave speed and spreading speed. Here, the minimal wave speed is the threshold on the existence of specific traveling wave solutions and the spreading speed of a nonnegative function $u(x,t), x\in\mathbb{R}, t>0$ is defined as follows [1].

Definition 1.1. Assume that $u(x,t)$ is a nonnegative function for $x\in\mathbb{R}, t>0$. Then $\widehat{c}$ is called the **spreading speed** of $u(x,t)$ if

- **a):** $\lim_{t\to\infty}\sup_{|x|>(\widehat{c}+\epsilon)t} u(x,t)=0$ for any given $\epsilon>0$;
- **b):** $\liminf_{t\to\infty}\inf_{|x|<(\widehat{c}-\epsilon)t} u(x,t)>0$ for any given $\epsilon\in(0,\widehat{c})$.

From the viewpoint of mathematical biology, the above speed characterizes the spatial expansion of the individuals [28, 34]. In the past decades, some important results on these two thresholds have been established for monotone semiflows, see [12, 18, 26, 37, 38] and a survey paper by Zhao [43]. When some special cooperative systems are concerned, it has been proven that all components governed by a system have the same spreading speed that is also the minimal wave speed of traveling wave solutions [18, 26, 38]. At the same time, it has been shown that different components may have different spreading speeds in several noncooperative systems [19, 20, 21, 22, 30, 33], and at least the spreading speed of one species equals to the minimal wave speed of traveling wave solutions.

Recently, Li et al. [17] investigated the following nonmonotone system

$$(1)\qquad \begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1[J_1*u_1](x,t)+r_1u_1(x,t)F_1(u_1,u_2)(x,t),\\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2[J_2*u_2](x,t)+r_2u_2(x,t)F_2(u_1,u_2)(x,t),\end{cases}$$

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where $x \in \mathbb{R}, t > 0$, $(u_1, u_2) \in \mathbb{R}^2$, r_1, r_2, d_1, d_2 are positive constants, F_1, F_2 are defined by

$$\begin{aligned} F_1(u_1,u_2)(x,t) &= 1 - u_1(x,t) \\ &\quad -b_1 \int_{-\tau}^{0} u_1(x,t+s)d\eta_{11}(s) - a_1 \int_{-\tau}^{0} u_2(x,t+s)d\eta_{12}(s), \\ F_2(u_1,u_2)(x,t) &= 1 - u_2(x,t) \\ &\quad -b_2 \int_{-\tau}^{0} u_2(x,t+s)d\eta_{22}(s) + a_2 \int_{-\tau}^{0} u_1(x,t+s)d\eta_{21}(s) \end{aligned}$$

with constants $b_1 \geq 0, b_2 \geq 0, a_1 \geq 0, a_2 \geq 0, \tau > 0$ and

$$\eta_{ij}(s) \text{ is nondecreasing on } [-\tau, 0] \text{ and } \eta_{ij}(0) - \eta_{ij}(-\tau) = 1,\ i,j = 1,2.$$

In this system, $[J_1 * u_1](x,t), [J_2 * u_2](x,t)$ reflecting the spatial dispersal indicate the long distance effect and nonadjacent contact among individuals [4, 13, 14], and are defined by

$$[J_i * u_i](x,t) = \int_{\mathbb{R}} J_i(x-y)[u_i(y,t) - u_i(x,t)]dy, i = 1,2,$$

where $J_i, i = 1,2$, play the role of probability kernel functions about the random walk of individuals and satisfy the following assumptions:

(J1): J_i is nonnegative and continuous for each $i = 1,2$;
(J2): for any $\lambda \in \mathbb{R}$, $\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy < \infty, i = 1,2$;
(J3): $\int_{\mathbb{R}} J_i(y)dy = 1,\ J_i(y) = J_i(-y), y \in \mathbb{R}, i = 1,2$.

Clearly, (1) is a predator-prey system in population dynamics. In Li et al. [17], Yu and Yuan [41], Zhang et al. [42], the authors investigated its traveling wave solutions connecting $(0,0)$ with the positive steady state, which reflect the process that these two species invade a new habitat from the viewpoint of biology invasion. In particular, Li et al. [17] obtained the minimal wave speed defined by $c^* = \max\{c_1^*, c_2^*\}$ with

$$c_i^* = \inf_{\lambda > 0} \frac{d_i \left[\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy - 1\right] + r_i}{\lambda}, i = 1,2.$$

From the viewpoint of initial value problem, let any fixed time be the initial time, the traveling wave solutions in [17, 41, 42] indicate the initial size of habitat of both species is infinite, which contradicts to some natural phenomena because the initial invasion often begins in finite domain. The purpose of this paper is to explore the dynamics when the initial habitats of two invaders are finite and investigate the long time behavior of

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1[J_1 * u_1](x,t) + r_1 u_1(x,t) F_1(u_1,u_2)(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2[J_2 * u_2](x,t) + r_2 u_2(x,t) F_2(u_1,u_2)(x,t), \\ u_i(x,s) = \phi_i(x,s), x \in \mathbb{R}, t > 0, s \in [-\tau, 0], i = 1,2, \end{cases} \tag{2}$$

in which $\phi_i(x,s)$ satisfies

(I): For $i = 1,2$, $\phi_i(x,s), x \in \mathbb{R}, s \in [-\tau, 0]$, is nonnegative, bounded and continuous such that

$$\phi_i(x,s) = 0, |x| > L, s \in [-\tau, 0], \phi_i(x_i, 0) > 0$$

for some $L > 0, x_i \in \mathbb{R}$. Moreover, they satisfy

$$0 \le \phi_1(x,s) \le 1, 0 \le \phi_2(x,s) \le 1 + a_2, x \in \mathbb{R}, s \in [-\tau, 0].$$

Since (2) involves delay effect of intraspecific competition if $b_1 + b_2 > 0$, it does not satisfy the comparison principle of classical predator-prey systems or monotone semiflows [40]. Therefore, the spreading speeds can not be investigated by the abstract results mentioned above. In this paper, we shall estimate the spreading speeds of these two species. By constructing proper auxiliary equations and functions, we confirm that either the predator or the prey invades the new habitat at the rough speed c^* while the spreading speed of the other species may be smaller than c^*.

2. Main results

In this section, we shall give and prove the main results on (2). Before giving the main results, we first define some positive constants as follows

$$\begin{aligned} c_1 &= \inf_{\lambda>0} \frac{d_1\left[\int_{\mathbb{R}} J_1(y)e^{\lambda y}dy - 1\right] + r_1(1 - a_1(1+a_2))}{\lambda}, \\ c_2 &= \inf_{\lambda>0} \frac{d_2\left[\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1\right] + r_2(1+a_2)}{\lambda}, \\ c_2^* &= \frac{d_2\left[\int_{\mathbb{R}} J_2(y)e^{\lambda_2 y}dy - 1\right] + r_2}{\lambda_2}, \end{aligned}$$

in which the existence and uniqueness of $\lambda_2 > 0$ is due to (J2)-(J3) and the convex of $d_2\left[\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1\right] - c\lambda + r_2$ in $\lambda > 0$ for every $c > 0$. Using these constants, we present the following conclusion.

Theorem 2.1. *Assume that the mild solution $(u_1(x,t), u_2(x,t))$ is defined by (2). Then it satisfies*

$$(0,0) \le (u_1(x,t), u_2(x,t)) \le (1, 1+a_2), x \in \mathbb{R}, t > 0. \tag{3}$$

Moreover, $(u_1(x,t), u_2(x,t))$ satisfies the following properties.

- **(1):** *If $c_1 > c_2$ is true, then c^* is the spreading speed of $u_1(x,t)$ while the spreading speed of $u_2(x,t)$ is not larger than c_2.*
- **(2):** *Further suppose that*

$$d_1\left[\int_{\mathbb{R}} J_1(y)e^{\lambda_2 y}dy - 1\right] - c_2^*\lambda_2 + r_1 \le 0. \tag{4}$$

  If $c_1^* \le c_2^*$ is true, then c^* is the spreading speed of $u_2(x,t)$ while the spreading speed of $u_1(x,t)$ is not larger than c_1^*.

We now prove the above theorem by several lemmas. Let X be the Banach space of uniformly continuous and bounded functions equipped with supremum norm. For each $i \in \{1,2\}$, we see that $d_i[J_i * v](x) : X \to X$ is a bounded linear operator by (J1), so

$$\frac{\partial u_i(x,t)}{\partial t} = d_i[J_i * u_i](x,t), u_i(x,0) \in X$$

generates a positive C_0 semigroup $T_i(t) : X \to X, t \ge 0$ (see [39, Lemma 3.1]), and the mild solution of the above initial value problem is denoted as

$$u_i(x,t) = T_i(t)u_i(x,0) = T_i(t-s)u_i(x,s)$$

for any $0 \le s < t < \infty$. Moreover, for any given kernel function J satisfying (J1)-(J3), it also generates a positive C_0 semigroup $T(t) : X \to X, t \ge 0$. Consider the following initial value problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d[J * u](x,t) + u(x,t)\left[r - u(x,t)\right], \\ u(x,0) = \chi(x) \in X, x \in \mathbb{R}, \end{cases} \tag{5}$$

where J satisfies (J1)-(J3), $d > 0$ and $r > 0$ are constants. Also define

$$c' = \inf_{\lambda > 0} \frac{d\left[\int_{\mathbb{R}} J(y)e^{\lambda y} dy - 1\right] + r}{\lambda}. \tag{6}$$

By Jin and Zhao [15], we have the following conclusion.

Lemma 2.2. *Assume that $0 \le \chi(x) \le 1$. Then (5) admits a solution $u(\cdot, t) \in X$ for all $t > 0$, which also satisfies*

$$u(x,t) = T(t-s)u(x,s) + \int_s^t T(t-\theta)[u(x,\theta)[r - u(x,\theta)]]d\theta$$

for $x \in \mathbb{R}, 0 \le s < t < \infty$. If $w(\cdot,t) \in X, t \ge 0$ is nonnegative and bounded such that

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} \ge (\le) d[J * w](x,t) + w(x,t)\left[r - w(x,t)\right], x \in \mathbb{R}, t > 0, \\ w(x,0) \ge (\le)\chi(x), x \in \mathbb{R}, \end{cases}$$

or

$$w(x,t) \ge (\le) T(t-s)w(x,s) + \int_s^t T(t-\theta)[w(x,\theta)[r - w(x,\theta)]]d\theta$$

for $x \in \mathbb{R}, 0 \le s < t < \infty$, then

$$w(x,t) \ge (\le) u(x,t), x \in \mathbb{R}, t > 0.$$

If $\chi(x)$ has nonempty support, then for any $c < c'$, we have

$$\liminf_{t \to \infty} \inf_{|x| < ct} u(x,t) = \limsup_{t \to \infty} \sup_{|x| < ct} u(x,t) = r.$$

If $\chi(x)$ has compact support, then

$$\lim_{t \to \infty} \sup_{|x| > ct} u(x,t) = 0, c > c'.$$

Remark 1. By the positivity of the semigroup, if

$$\begin{cases} \frac{\partial w_i(x,t)}{\partial t} \ge d[J * w_i](x,t) + w_i(x,t)\left[r - w_i(x,t)\right], x \in \mathbb{R}, t > 0, \\ w_i(x,0) \ge \chi(x), x \in \mathbb{R} \end{cases}$$

for $i \in \{1,2,3\}$, then

$$\min\{w_1(x,t), w_2(x,t), w_3(x,t)\} \ge u(x,t), x \in \mathbb{R}, t > 0.$$

On the existence of mild solution of (1), we have the following result.

Lemma 2.3. *The positive mild solution $(u_1(\cdot,t), u_2(\cdot,t)) \in X^2$ exists for all $t > 0$ and satisfies (3).*

Proof. The local existence is evident by the theory of abstract functional differential equations [27], here the mild solution is defined by

$$\begin{aligned} u_1(x,t) &= T_1(t-\theta)u_1(x,\theta) + \int_\theta^t T_1(t-s)\left[r_1 u_1(x,s)F_1(u_1,u_2)(x,s)\right] ds, \\ u_2(x,t) &= T_2(t-\theta)u_2(x,\theta) + \int_\theta^t T_2(t-s)\left[r_2 u_2(x,s)F_2(u_1,u_2)(x,s)\right] ds \end{aligned}$$

for $0 \le \theta < t < T$ with some $T \in (0,\infty]$. If $T=\infty$, then the global existence is true.

Further by the quasipositivity in u_1F_1, u_2F_2, we see the mild solution is nonnegative. If $u_1(x,t)$ only exists for $t \in [0,T)$ with bounded T such that

$$\lim_{t\to T-} \sup_{x\in\mathbb{R}} u_1(x,t) = \infty,$$

then

$$u_1(x,t) \le T_1(t-\theta)u_1(x,\theta) + \int_\theta^t T_1(t-s)\left[r_1u_1(x,s)[1-u_1(x,s)]\right] ds$$

for $0 \le \theta < t < T$, and the comparison principle (Lemma 2.2) implies

$$0 \le u_1(x,t) \le 1, x \in \mathbb{R}, t \in [0,T).$$

A contradiction occurs. The proof is complete by similar discussion on $u_2(x,t)$. □

To continue the discussion, we investigate the following scalar equation

$$(7) \quad \begin{cases} \dfrac{\partial v(x,t)}{\partial t} = d[J*v](x,t) + rv(x,t)\left[1 - v(x,t) - b\int_{-\tau}^0 v(x,t+s)d\eta(s)\right], \\ v(x,s) = \nu(x,s), \end{cases}$$

where $x\in\mathbb{R}, t>0, s\in[-\tau,0]$, J satisfies (J1)-(J3), $d>0, r>0, b\ge 0$,

$$\eta(s) \text{ is nondecreasing on } [-\tau,0] \text{ such that } \eta(0)-\eta(-\tau)=1.$$

Furthermore, $\nu(x,s)\ge 0$ is uniformly continuous and bounded. Evidently, the global existence of mild solution of (7) is true by Lemma 2.3.

Lemma 2.4. *Assume that $v(x,t)$ is the mild solution defined by (7). If $\nu(x,0)$ admits nonempty compact support such that $0\le\nu(x,0)\le 1, x\in\mathbb{R}$, then its spreading speed is c' defined by (6).*

Proof. We now prove it by the idea in Liu and Pan [25]. If

$$b\int_{-\tau}^0 v(x,t+s)d\eta(s) = bv(x,t),$$

then the result is clear by Lemma 2.2. Otherwise, the positivity implies that

$$v(x,t) \le T(t-s)v(x,s) + \int_s^t T(t-\theta)[rv(x,\theta)[1-v(x,\theta)]]d\theta$$

for any $0 \le s < t < \infty$, then Lemma 2.2 implies $v(x,t)\le 1$ and

$$\lim_{t\to\infty} \sup_{|x|>ct} v(x,t) = 0, c > c'.$$

For any fixed $c<c'$, it suffices to prove that

$$\liminf_{t\to\infty} \inf_{|x|<ct} v(x,t) > 0.$$

For the purpose, we select $\epsilon > 0$ such that

$$c < \inf_{\lambda > 0} \frac{d\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + r(1-4\epsilon)}{\lambda},$$

and $\tau' \in (0, \tau)$ such that

$$b\int_{-\tau'}^{0} d\eta(s) < \epsilon.$$

If $b\int_{-\tau}^{-\tau'} v(x,t+s)d\eta(s) \le 2\epsilon$, then

$$rv(x,t)\left[1 - v(x,t) - b\int_{-\tau}^{0} v(x,t+s)d\eta(s)\right] \ge rv(x,t)[1 - 3\epsilon - v(x,t)].$$

When $b\int_{-\tau}^{-\tau'} v(x,t+s)d\eta(s) > 2\epsilon$, there exists $s_0 \in [-\tau, -\tau']$ such that

$$v(x,s_0) \ge \frac{2\epsilon}{b\tau},$$

and the uniform continuity implies

$$v(y,s_0) \ge \frac{\epsilon}{b\tau}, |x-y| \le \sigma$$

for some $\sigma > 0$. Consider the initial value problem

$$\begin{cases} \dfrac{\partial \underline{v}(x,t)}{\partial t} = d[J * \underline{v}](x,t) + r\underline{v}(x,t)\left[1 - b - \underline{v}(x,t)\right], \\ \underline{v}(x,0) = \underline{\nu}(x), \end{cases} \tag{8}$$

where $\underline{\nu}(x)$ is a continuous function satisfying

(1): $\underline{\nu}(x) = \frac{\epsilon}{b\tau}, |x| \le \sigma/2$;
(2): $\underline{\nu}(x) = 0, |x| \ge \sigma$;
(3): $\underline{\nu}(x)$ is even and decreasing in $x \in [\sigma/2, \sigma]$.

By the positivity of $T(t)$ and the property of continuous functions, we see that $\underline{v}(0,t)$ is positive in $t > 0$, and there exists $\rho > 0$ such that $\underline{v}(0,t) > \rho, t \in [\tau', \tau]$ and so

$$b\int_{-\tau}^{0} v(x,t+s)d\eta(s) \le b = \frac{b\rho}{\rho} \le \frac{b}{\rho}v(x,t).$$

That is, $v(x,t)$ satisfies

$$v(x,t) \ge T(t-s)v(x,s) + \int_s^t T(t-\theta)[rv(x,\theta)[1 - 3\epsilon - (1 + b/\rho)v(x,\theta)]]d\theta$$

for all $0 \le s < t < \infty$. The proof is complete by Lemma 2.2. □

Lemma 2.5. *If $c_1 > c_2$ is true, then c^* is the spreading speed of $u_1(x,t)$ while the spreading speed of $u_2(x,t)$ is not larger than c_2.*

Proof. By (3), u_2 satisfies

$$u_2(x,t) \le T_2(t-\theta)u_2(x,\theta) + \int_\theta^t T_2(t-s)\left[r_2u_2(x,s)[1 + a_2 - u_2(x,s)]\right]ds$$

for $0 \le \theta < t < \infty$, so Lemma 2.2 implies that the spreading speed of $u_2(x,t)$ is not larger than c_2, which also leads to

$$\limsup_{t\to\infty} \sup_{3|x| > (2c_2 + c_1)t} u_2(x,t) = 0. \tag{9}$$

Again by (3), we see that

$$\begin{aligned} u_1(x,t) &\geq T_1(t-\theta)u_1(x,\theta) + \int_\theta^t T_1(t-s)\left[r_1 u_1(x,s)\times\right. \\ &\left.[1-a_1(1+a_2) - u_1(x,s) - b_1\int_{-\tau}^0 u_1(x,s+\gamma)d\eta_{11}(\gamma)]\right] ds \end{aligned}$$

for all $0 \le \theta < t < \infty$, and Lemma 2.4 or its proof implies

$$\liminf_{t\to\infty} \inf_{3|x|<(c_2+2c_1)t} u_1(x,t) > 0. \tag{10}$$

We now verify that c^* is the spreading speed of $u_1(x,t)$. Because of Lemma 2.2 and

$$u_1(x,t) \le T_1(t-\theta)u_1(x,\theta) + \int_\theta^t T_1(t-s)\left[r_1 u_1(x,s)[1-u_1(x,s)]\right] ds$$

for all $0 \le \theta < t < \infty$, it suffices to confirm that

$$\liminf_{t\to\infty} \inf_{|x|<ct} u_1(x,t) > 0 \tag{11}$$

for any given $c < c^*$. We now fix $3c > c_2 + 2c_1$ and $\epsilon > 0$ such that

$$c < \inf_{\lambda>0} \frac{d\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + r(1-2\epsilon)}{\lambda}.$$

By (9) and (10), there exists $T > 0$ such that

$$\sup_{2|x|>(c_2+c_1)t} u_2(x,t) < \epsilon, t \ge T$$

and

$$\inf_{t>T} \inf_{2|x|\le(c_2+c_1)t} u_1(x,t) > 0,$$

which implies that there exists $M > 0$ depending on ϵ such that

$$\begin{aligned} u_1(x,t) &\geq T_1(t-\theta)u_1(x,\theta) + \int_\theta^t T_1(t-s) \\ &\left[r_1 u_1(x,s)[1-\epsilon - Mu_1(x,s) - b_1\int_{-\tau}^0 u_1(x,s+\gamma)d\eta_{11}(\gamma)]\right] ds \end{aligned}$$

for any $T \le \theta < t$. Clearly, the spreading speed is not less than that of

$$\frac{\partial v(x,t)}{\partial t} = d[J * v](x,t) + rv(x,t)\left[1-\epsilon - Mv(x,t) - b\int_{-\tau}^0 v(x,t+s)d\eta(s)\right]$$

by repeating the proof of Lemma 2.4. The proof is complete. □

Lemma 2.6. *Assume that (4) is true. If $c_1^* \le c_2^*$ is true, then c^* is the spreading speed of $u_2(x,t)$ while the spreading speed of $u_1(x,t)$ is not larger than c_1^*.*

Proof. By the positivity of (3), $u_1(x,t)$ satisfies

$$u_1(x,t) \le T_1(t-\theta)u_1(x,\theta) + \int_\theta^t T_1(t-s)\left[r_1 u_1(x,s)[1-u_1(x,s)\right] ds$$

for any $0 \le s < t$, the spreading speed of $u_1(x,t)$ is not larger than c_1^* by Lemma 2.2.

On the other hand, $u_2(x,t)$ satisfies

$$u_2(x,t) \geq T_2(t-\theta)u_2(x,\theta) + \int_\theta^t T_2(t-s)\left[r_2u_2(x,s)\left[1-u_2(x,s)-b_2\int_{-\tau}^0 u_2(x,s+\gamma)d\eta_{22}(\gamma)\right]\right]ds$$

for any $0 \leq s < t$, and Lemma 2.4 implies that the spreading speed of $u_2(x,t)$ is not less than c^*. Now, we shall prove that

$$\limsup_{t\to\infty}\sup_{|x|>ct} u_2(x,t) = 0$$

for any fixed $c > c^*$. By the positivity, we obtain

$$u_2(x,t) \leq T_2(t-\theta)u_2(x,\theta) + \int_\theta^t T_2(t-s)\left[r_2u_2(x,s)\left[1-u_2(x,s)+a_2\int_{-\tau}^0 u_1(x,s+\gamma)d\eta_{21}(\gamma)\right]\right]ds,$$

and the result is true if the spreading speed of the following equation is c^*

$$\begin{aligned}\frac{\partial \underline{u}_2(x,t)}{\partial t} &= d_2[J_2 * \underline{u}_2](x,t) \\ &\quad + r_2\underline{u}_2(x,t)\left[1-\underline{u}_2(x,t)+a_2\int_{-\tau}^0 \overline{u}_1(x,s)d\eta_{21}(s)\right],\end{aligned} \tag{12}$$

where $\underline{u}_2(x,0) = u_2(x,0)$, $\overline{u}_1(x,s) = \phi_1(x,s), s \leq 0$, and $\overline{u}_1(x,t), t > 0$ is defined by

$$\overline{u}_1(x,t) = T_1(t)\overline{u}_1(x,0) + \int_0^t T_1(t-s)\left[r_1\overline{u}_1(x,s)[1-\overline{u}_1(x,s)]\right]ds.$$

The main reason why the above claim is true is that the above equation (12) is monotone and admits comparison principle.

By Remark 1 and (4), we see that there exists $T_1 > 0$ such that

$$\overline{u}_1(x,t) \leq \min\left\{e^{\lambda_2(x+c_2t+T_1)}, e^{\lambda_2(-x+c_2t+T_1)}, 1\right\}, x \in \mathbb{R}, t \geq 0.$$

In fact, let $\widehat{u}_1(x,t) = e^{\lambda_2(x+c_2t+T_1)}$ or $e^{\lambda_2(-x+c_2t+T_1)}$, then

$$\frac{\partial \widehat{u}_1(x,t)}{\partial t} \geq d_1[J_1 * \widehat{u}_1](x,t) + r_1\widehat{u}_1(x,t),$$

and

$$\frac{\partial \widehat{u}_1(x,t)}{\partial t} \geq d_1[J_1 * \widehat{u}_1](x,t) + r_1\widehat{u}_1(x,t)[1-\widehat{u}_1(x,t)]$$

if $\widehat{u}_1(x,t) = 1$. Further select $T_2 \geq T_1$ such that

$$e^{\lambda_2(T_2-T_1)} > a_2$$

and

$$\min\left\{e^{\lambda_2(x+T_2-c_2\tau)}, e^{\lambda_2(-x+T_2-c_2\tau)}, 1+a_2\right\} \geq \sup_{s\in[-\tau,0]}\phi_2(x,s), x \in \mathbb{R}.$$

Then Remark 1 implies that

$$u_2(x,t) \leq \min\left\{e^{\lambda_2(x+c_2t+T_2)}, e^{\lambda_2(-x+c_2t+T_2)}, 1+a_2\right\}, x \in \mathbb{R}, t \geq 0$$

because $\widehat{u}_2(x,t) = e^{\lambda_2(x+c_2t+T_2)}$ $\left(e^{\lambda_2(-x+c_2t+T_2)}, 1+a_2\right)$ implies

$$\frac{\partial \widehat{u}_2(x,t)}{\partial t} \geq d_2[J_2 * \widehat{u}_2](x,t) + r_2\widehat{u}_2(x,t)\left[1-\widehat{u}_2(x,t)+a_2\int_{-\tau}^0 \overline{u}_1(x,s)d\eta_{21}(s)\right].$$

The proof is complete. □

3. Discussion

The propagation dynamics of predator-prey systems has important ecology background, one typical case is the evolution of insect herbivores and lupins on Mount St Helens, see [11, 29]. Another related topic is the asymptotic spreading in epidemic models because of the similar monotone conditions. In literature, much attention has been paid to the traveling wave solutions since the work of Dunbar [10]. However, there are a few results on asymptotic spreading of predator-prey systems, see part results by Ducrot [7, 8], Ducrot et al. [9], Lin [19], Lin et al. [23], Pan [30, 32], Wang and Zhang [36].

The model in this paper admits nonlocal dispersal and time delays, the mechanism has significant biological reasons and other backgrounds [4, 5, 13, 14, 28], and the monotone case has been widely studied, see some recent works by [2, 3, 6]. Similar to that in [16, 24], the model in this paper is not a monotone system [35] and time delay is not small enough. When the diffusion is of the classical Ficker type in (1), Pan [31] studied its minimal wave speed of traveling wave solutions connecting trivial steady state with the positive one.

In this paper, we show that c^* may be the spreading speed of u_1 or u_2, which is the minimal wave speed in [17]. It is possible to study the asymptotic spreading of the model in [31] by our idea. Both thresholds in this paper and [17] formulate that two species invade a new habitat. From our results, we see that the predator and the prey may have different spreading speeds, but it is difficult to estimate these spreading speeds. To answer these questions, more properties on the nonlocal operator and delayed systems are necessary. We shall further investigate these questions in our future research.

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