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THE GLOBAL CONSERVATIVE SOLUTIONS FOR THE GENERALIZED CAMASSA-HOLM EQUATION

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Abstract. This paper deals with the continuation of solutions to the generalized Camassa-Holm equation with higher-order nonlinearity beyond wave breaking. By introducing new variables, we transform the generalized Camassa-Holm equation to a semi-linear system and establish the global solutions to this semi-linear system, and by returning to the original variables, we obtain the existence of global conservative solutions to the original equation. We introduce a set of auxiliary variables tailored to a given conservative solution, which satisfy a suitable semi-linear system, and show that the solution for the semi-linear system is unique. Furthermore, it is obtained that the original equation has a unique global conservative solution. By Thom's transversality lemma, we prove that piecewise smooth solutions with only generic singularities are dense in the whole solution set, which means the generic regularity.

1. Introduction

In this paper we consider the continuation of solutions for the generalized Camassa-Holm (g-CH) equation

$$(1)\quad \begin{cases} u_t - u_{xxt} + \frac{(m+2)(m+1)}{2} u^m u_x = \left(\frac{m}{2} u^{(m-1)} u_x^2 + u^m u_{xx} \right)_x, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where m is a positive integer.

The equation (1) was first proposed by Hakkaev and Kirchev in [18], and the local well-posedness of the Cauchy problem (1) was studied for the Sobolev spaces H^s with $s > \frac{3}{2}$. Under suitable assumptions and energy conservations, the orbital stability and instability of solitary wave solutions were considered. In [20], the authors established the local well-posedness to (1) for a range of Besov spaces and proved that its solutions are analytic in both variables. The persistence property of strong solutions for (1) was investigated in weighted L^p spaces [23]. And it was

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shown that the equation is well-posed in Sobolev spaces H^s $(s > \frac{3}{2})$ for both the periodic and the nonperiodic case in the sense of Hadamard [21]. Moreover, the nonuniform dependence and Hölder continuous to (1)were discussed.

In fact, the equation (1) is a natural generalization of the famous Camassa-Holm (CH) model

$$u_t + u_{xxt} + 3uu_x = uu_{xxx} + 2u_x u_{xx}. \tag{2}$$

The Camassa-Holm equation first arisen in the context of hereditary symmetries was studied by Fokas and Fuchssteiner [14], but did not receive much attention until Camassa and Holm [9, 7] derived it as a model of shallow water waves over a flat bottom. It has bi-Hamiltonian structure, infinitely many conservation laws and is completely integrable [8, 7, 14, 19]. In addition, the stability of the smooth solitons and the orbital stability of the peaked solitons to (1) were established in [10] and [11] respectively. Particularly, the Camassa-Holm equation possesses solutions with presence of wave breaking (that is, the solution remains bounded while its slope becomes unbounded in finite time [6, 12]). When these two waves collide at some time, the combined wave forms an infinite slope. After the collision, there are two things that happen: either two waves pass through each other with total energy preserved; or annihilate each other with a lose of energy. The solutions in the first case is called conservative, and the second case is called dissipative.

So far, the continuation of the solutions after wave breaking has been studied widely. Bressan and Constantin proved that the solution of the Camassa-Holm equation can be continued as either global conservative or global dissipative solutions [2, 3]. Notice that, the conservative solutions are about preservation of the H^1 norm, while dissipative solutions are characterized by a sudden drop in the norm at blow-up. Afterwards, the uniqueness of the conservative solution and the dissipative solution for the Camassa-Holm equation were obtained [4, 16]. Recently, the generic regularity of conservative solutions to Camassa-Holm equation was discussed in [17]. It is worth mentioning that the $H^1(\mathbb{R})$ norm conserved quantity plays a key role in the process of studying the conservative and dissipative solutions.

In the case of a more general Camassa-Holm equation, the global existence and uniqueness to the solution are established in [22, 25]. Moreover, for the Camassa-Holm equation with a forcing term ku, Zhu obtained the global existence and uniqueness [26].

Similar to the well-known Camassa-Holm equation, the system (1) also models the peculiar wave breaking phenomena [24]. Therefore, in this paper, we still focus on the conservative case of equation (1.1) in H^1 space, including the existence, uniqueness and generic regularity problems. Setting $G(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{R})$. Thus the equation (1.1) can be rewritten as the following integral-differential form:

$$\begin{cases} u_t + u^m u_x = -P_x, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3}$$

where

$$P = G * \left(\frac{m}{2} u^{(m-1)} u_x^2 + \frac{m(m+3)}{2(m+1)} \right). \tag{4}$$

Motivated by [2, 4, 17], in this paper, we consider the global weak conservative solutions defined by as follows.

Definition 1.1. Let $u_0 \in H^1(\mathbb{R})$, there exists a family of Radon measure $\{\mu_{(t)}, t \in \mathbb{R}\}$, depending continuously on time w.r.t. the topology of weak convergence of measures, such that the following properties hold.

(i) The map $t \to u(., t)$ is Lipschitz continuous from $[0, T]$ into $L^2(\mathbb{R})$ with the initial data $u_0 \in H^1(\mathbb{R})$.

(ii) The solution $u = u(x, t)$ satisfies the initial data $u_0 \in L^2(\mathbb{R})$. For any test function $\phi \in \mathcal{C}_c{}^1(\Omega)$ with $\Omega = \{(x,t) | x \in \mathbb{R}, t \in [0, +\infty)\}$, one has

$$\begin{aligned}\int\int_{\Omega}\Big(&-u_x(\phi_t + u^m\phi_x) + (-\frac{m}{2}u^m u_x^2 - \frac{m(m+3)}{2(m+1)}u^{m+1} + P)\phi\Big)dxdt \\ &+ \int_{\mathbb{R}} u_{0x}\phi(0,x)dx = 0.\end{aligned} \tag{5}$$

For a solution $u = u(t, x)$, we say that u is conservative, which means that the balance law (10) is satisfied in the following sense.

There exists a family of Radon measure $\{\mu_{(t)}, t \in \mathbb{R}^+\}$, depending continuously on time w.r.t. the topology of weak convergence of measures. For any $t \in \mathbb{R}^+$, the absolutely continuous measure $\mu_{(t)}$ has density $u_x^2(t, \cdot)$ w.r.t. Lebesgue measure. Moreover, for any test function $\phi \in \mathcal{C}_c{}^1(\Omega)$, the family $\{\mu_{(t)}; t \in \mathbb{R}\}$ supplies a measure-valued solution to the balance law

$$\begin{aligned}\int_0^{\infty}\Big(&\int_{\mathbb{R}}(\phi_t + u^m\phi_x)d\mu_{(t)} + \int_{\mathbb{R}} 2u_x\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)dx\Big)dt \\ &- \int_{\mathbb{R}} u_{0x}^2\phi(0,x)dx = 0.\end{aligned} \tag{6}$$

Based on the characteristic, by introducing a new variables, we transform the equation (1) to a semi-linear system, and prove the semi-linear system has global solutions. Then by a reverse transformation, one can get the conservative solutions for equation (1). Our results are stated as follows.

Theorem 1.2. *Let $u_0 \in H^1(\mathbb{R})$. Then the generalized Camassa-Holm equation (3) has global conservative solution $u = u(x,t)$ defined on $\mathbb{R} \times (0, +\infty)$. Moreover, the solution has the following properties.*

(i) $u(x,t)$ is 1/2-Hölder continuous on both t and x.

(ii) The function u provides a solution to the Cauchy problem (3) in the sense of Definition 1.1.

(iii) There exists a null set $\mathcal{N} \subset \mathbb{R}$ with meas$\mathcal{N} = 0$ such that for any $t \notin \mathcal{N}$, the measure $\mu_{(t)}$ is absolutely continuous and has density $u_x^2(t, \cdot)$ w.r.t. the Lebesgue measure.

(iv) The energy $u^2 + u_x^2$ coincides a.e. with a constant, that is,

$$E(t) = E(0) \quad \textit{for } t \notin \mathcal{N}, \quad E(t) < E(0) \textit{ for } t \in \mathcal{N}.$$

(v) The continuous dependence of solutions to system (3) holds with the initial data belongs to $H^1(\mathbb{R})$. More precisely, given a sequence of initial data $\{u_{0n}\}$ satisfy $||u_{0n} - u_0||_{H^1(\mathbb{R})} \to 0$, then the corresponding solutions $u_n(t,x)$ converge to $u(t,x)$ uniformly for $(t,x) \in [0,T] \times \mathbb{R}$.

Theorem 1.3. *Given any initial data $u_0 \in H^1(\mathbb{R})$, the Cauchy problem (3) has a unique conservative solution.*

Remark 1. In fact, we know the process for the proof of the existence is an inverse, but the method here is an irreversible.

By virtue of the analysis of solutions along characteristic, we show that piecewise smooth solutions with only generic singularities are dense in the whole solution set. Using the Thom's transversality Lemma [1, 15], we give the following generic regularity result.

Theorem 1.4. *For any $T > 0$, there exists an open dense set of initial data $\mathcal{D} \subset C^3(\mathbb{R}) \cap H^1(\mathbb{R})$, such that for any $u_0 \in \mathcal{D}$, the conservative solution $u = u(t, x)$ of the equation (3) is twice continuously differentiable in the complement of finitely many characteristic curves, within the domain $[0, T] \times \mathbb{R}$.*

Remark 2. The generic regularity is very interesting since it reflects the structure of singularities. Similar issue was first established for the variational wave equation[5], and later this method was applied to the Camassa-Holm equation [17].

This paper is organized as follows. In Section 2, we give the energy conservation laws and introduce a new set of independent and dependent variables. In Section 3, we first obtain a global conservative solution of the semi-linear system (27), and then by inverse transformation we prove the existence of the global conservative solution to equation (1). In Section 4, we establish the uniqueness of the characteristic curve through each initial point, and by considering the dynamics of a conservative solution along a characteristic, we obtain the proof of the uniqueness for the global conservation solution. In Section 5, the generic regularity of conservative solutions to equation (1) is investigated.

2. Preliminary

2.1. **The basic equations.** For smooth solutions, we claim that the total energy

$$E(t) = \int_{\mathbb{R}} (u^2 + u_x^2) dx \tag{7}$$

is constant in time. In fact, by using $\partial_x^2 G * f = G * f - f$ and differentiating the equation (3) with respect to x, we have

$$u_{xt} + u^m u_{xx} + m u^{m-1} u_x^2 = \frac{m}{2} u^{m-1} u_x^2 + \frac{m(m+3)}{2(m+1)} u^{m+1} - P. \tag{8}$$

Multiplying (3) by u, and (8) by u_x, one get

$$(\frac{u^2}{2})_t + (\frac{1}{m+1} u^{m+1})_x + u P_x = 0, \tag{9}$$

$$(\frac{u_x^2}{2})_t + (\frac{u^m u_x}{2})_x = \left(\frac{m(m+3)}{2(m+1)} u^{m+1} - P \right) u_x. \tag{10}$$

It follows from (9)-(10) that

$$\frac{d}{dt} E(t) = \frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) dx = 0. \tag{11}$$

Therefore, the conservation law is given by

$$E(t) := \int_{\mathbb{R}} (u^2 + u_x^2) dx = E(0). \tag{12}$$

Since P, P_x are both defined as convolutions, by Young's inequality and Sobolev's inequality $\|u\|_{L^\infty(\mathbb{R})} \le \|u\|_{H^1(\mathbb{R})} = E(0)^{\frac{1}{2}}$, it implies that

$$\begin{aligned}&\|P(t)\|_{L^\infty},\ \|P_x(t)\|_{L^\infty}\\&\le \|\frac{1}{2}e^{-|x|}\|_{L^\infty}\|\frac{m}{2}u^{m-1}u_x^2+\frac{m(m+3)}{2(m+1)}u^{m+1}\|_{L^1}\le \frac{m(m+3)}{4(m+1)}E(0)^{\frac{m+1}{2}},\end{aligned}\tag{13}$$

and

$$\begin{aligned}&\|P(t)\|_{L^2},\ \|P_x(t)\|_{L^2}\\&\le \|\frac{1}{2}e^{-|x|}\|_{L^2}\|\frac{m}{2}u^{m-1}u_x^2+\frac{m(m+3)}{2(m+1)}u^{m+1}\|_{L^1}\le \frac{m(m+3)}{2(m+1)}E(0)^{\frac{m+1}{2}}.\end{aligned}\tag{14}$$

2.2. **A new set of independent and dependent variables.** Let $\tilde{u}=u_0(x)\in H^1(\mathbb{R})$ be the initial data. Considering the energy variable $\xi\in\mathbb{R}$, the non-decreasing map $\xi\mapsto\tilde{y}(\xi)$ is defined by

$$\int_0^{\tilde{y}(\xi)}\left(1+\tilde{u}_x^2\right)dx=\xi.\tag{15}$$

Then the characteristic map $t\mapsto y(t,\xi)$ satisfies

$$\frac{\partial}{\partial t}y(t,\xi)=u^m(t,y(t,\xi)),\quad y(0,\xi)=\tilde{y}(\xi).\tag{16}$$

And the new variables $\theta=\theta(t,\xi)$ and $h=h(t,\xi)$ are introduced as

$$\theta\doteq 2\arctan u_x,\quad h\doteq 1+u_x^2\cdot\frac{\partial y}{\partial\xi},\tag{17}$$

$$h(0,\xi)\equiv 1,\tag{18}$$

$$\frac{1}{1+u_x^2}=\cos^2\frac{\theta}{2},\quad \frac{u_x}{1+u_x^2}=\frac{1}{2}\sin\theta,\ \frac{u_x^2}{1+u_x^2}=\sin^2\frac{\theta}{2},\tag{19}$$

$$\frac{\partial y}{\partial\xi}=\frac{h}{(1+u_x^2)}=\cos^2\frac{\theta}{2}\cdot h,\tag{20}$$

$$y(t,\bar{\xi})-y(t,\xi)=\int_\xi^{\bar{\xi}}\cos^2\frac{\theta(t,s)}{2}\cdot h(t,s)ds.\tag{21}$$

Furthermore, we get

$$\begin{aligned}&P(t,\xi)=\frac{1}{2}\int_{-\infty}^{\infty}\exp\{-|y(t,\xi)-x|\}\left(\frac{m(m+3)}{2(m+1)}u^{m+1}+\frac{m}{2}u^{m-1}u_x^2\right)dx,\\&P_x(t,\xi)\\&=\frac{1}{2}(\int_{y(t,\xi)}^{\infty}-\int_{-\infty}^{y(t,\xi)})\exp\{-|y(t,\xi)-x|\}\left(\frac{m(m+3)}{2(m+1)}u^{m+1}+\frac{m}{2}u^{m-1}u_x^2\right)dx.\end{aligned}$$

It follows from identities (18) to (21) that an expression for P and P_x in terms of the new variable ξ

$$\begin{aligned}P(t,\xi)=&\frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\left|\int_\xi^{\bar{\xi}}\left(h\cdot\cos^2\frac{\theta}{2}\right)(s)ds\right|\right\}\\&\cdot\left[h\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2}+\frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)\right](\bar{\xi})d\bar{\xi},\end{aligned}\tag{22}$$

$$
\begin{aligned}
P_x(t,\xi) =& \frac{1}{2}\left(\int_{\xi}^{\infty} - \int_{-\infty}^{\xi}\right) \exp\left\{ - \left| \int_{\xi}^{\bar{\xi}} \left(h \cdot \cos^2 \frac{\theta}{2}\right)(s) ds \right| \right\} \\
& \cdot \left[h \left(\frac{m(m+3)}{2(m+1)} u^{m+1} \cos^2 \frac{\theta}{2} + \frac{m}{2} u^{m-1} \sin^2 \frac{\theta}{2} \right) \right] (\bar{\xi}) d\bar{\xi},
\end{aligned} \tag{23}
$$

By (4) and (16), the evolution equation for u takes the form

$$
\frac{\partial}{\partial t} u(t,\xi) = u_t + u_y y_t = u_t + u^m u_x = -P_x(t,\xi), \tag{24}
$$

where P_x is given in (23).

By the definition of variable h, it follows that

$$
\int_{\xi^1}^{\xi^2} h(t,\xi) d\xi = \int_{y(t,\xi^1)}^{y(t,\xi^2)} \left(1 + u_x^2(t,x)\right) dx.
$$

By using (16) and $P_{xx} = P - \frac{m}{2} u^{m-1} u_x^2 - \frac{m(m+3)}{2(m+1)} u^{m+1}$, we have

$$
\begin{aligned}
\frac{d}{dt} \int_{\xi^1}^{\xi^2} h(t,\xi) d\xi &= \int_{y(t,\xi^1)}^{y(t,\xi^2)} \{[(1+u_x^2)]_t + [u(1+u_x^2)]_x\} dx \\
&= \int_{y(t,\xi^1)}^{y(t,\xi^2)} \left(\frac{m(m+3)}{m+1} u^{m+1} + m u^{m-1} - 2P \right) u_x dx.
\end{aligned}
$$

Differentiating with respect to ξ, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t} h(t,\xi) &= \left(\frac{m(m+3)}{m+1} u^{m+1} + m u^{m-1} - 2P \right) \frac{u_x}{1+u_x^2} \cdot h \\
&= \left(\frac{m(m+3)}{m+1} u^{2(m+1)} + \frac{m}{2} u^{m-1} - P \right) \sin\theta \cdot h.
\end{aligned} \tag{25}
$$

Using (17) and (19), we see

$$
\begin{aligned}
\frac{\partial}{\partial t} \theta(t,\xi) &= \frac{2}{1+u_x^2} (u_{xt} + u^m u_{xx}) \\
&= \frac{2}{1+u_x^2} \left(-\frac{m}{2} u^{m-1} (u_x)^2 + \frac{m(m+3)}{2(m+1)} u^{m+1} - P \right) \\
&= \left(\frac{m(m+3)}{m+1} u^{m+1} - 2P \right) \cos^2 \frac{\theta}{2} - m u^{m-1} \sin^2 \frac{\theta}{2},
\end{aligned} \tag{26}
$$

where P is defined by (22).

3. Global conservative solutions

3.1. Global solutions of semi-linear system. According to (24)-(26), we obtain the following semi-linear system

$$
\begin{cases}
u_t = -P_x, \\
\theta_t = \left(\frac{m(m+3)}{m+1} u^{m+1} - 2P \right) \cos^2 \frac{\theta}{2} - m u^{m-1} \sin^2 \frac{\theta}{2}, \\
h_t = \left(\frac{m(m+3)}{2(m+1)} u^{m+1} + \frac{m}{2} u^{m-1} - P \right) \sin\theta \cdot h
\end{cases} \tag{27}
$$

with the initial data

$$
\begin{cases}
u(0,\xi) = \tilde{u}(\tilde{y}(\xi)) \\
\theta(0,\xi) = 2 \arctan \tilde{u}_x(\tilde{y}(\xi)) \\
h(0,\xi) = 1,
\end{cases} \tag{28}
$$

where P and P_x are give by (22)-(23). System (27) can be regarded as an ODE in the Banach space

$$X \doteq H^1(\mathbb{R}) \times [L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})] \times L^\infty(\mathbb{R}), \tag{29}$$

with $\|(u,\theta,h)\| \doteq \|u\|_{H^1} + \|\theta\|_{L^2} + \|\theta\|_{L^\infty} + \|h\|_{L^\infty}$.

In light of the standard theory of ODE in the Banach space, we can establish that all functions on the right-hand side of (27) are locally Lipschitz continuous, this implies the local existence of solutions to the system (27)-(28).

Lemma 3.1. *Given initial data $\tilde{u} \in H^1(\mathbb{R})$, the Cauchy problem (27)-(28) has a unique solution defined on any given time interval $[0,T]$ with $T \geq 0$.*

Proof. Set any bounded domain $\Lambda \subset X$ defined by

(30)
$$\Lambda = (u,\theta,h) = \{\|u\|_{H^1} \leq \gamma, \|\theta\|_{L^2} \leq \delta, \|\theta\|_{L^\infty} \leq \frac{3\pi}{2}, h^- \leq h(x) \leq h^+ a.e. x \in \mathbb{R}\},$$

for any positive constants γ, δ, h^-, h^+. In view of the Sobolev's inequality

$$\|u\|_{L^\infty} \leq \|u\|_{H^1}, \tag{31}$$

and the uniform boundedness of θ, h, it is easy to see that

$$\frac{m(m+3)}{m+1}u^{m+1}\cos^2\frac{\theta}{2},\ mu^{m-1}\sin^2\frac{\theta}{2},\ \left(\frac{m(m+3)}{m+1}u^{m+1} + \frac{m}{2}u^{m-1}\right)\sin\theta \cdot h$$

are Lipschitz continuous from Λ into $L^2 \cap L^\infty$. The next aim is to prove the maps

$$(u,\theta,h) \mapsto (P, P_x) \tag{32}$$

are Lipschitz continuous from Λ into $L^2 \cap L^\infty$. Actually, we only need to show that these maps are Lipschitz continuous from Λ into H^1. To this, we first observe that

$$\begin{aligned}
&\text{measure}\left\{\xi \in \mathbb{R}; |\frac{\theta(\xi)}{2}| \geq \frac{\pi}{4}\right\} \leq \text{measure}\left\{\xi \in \mathbb{R}; \sin^2\frac{\theta(\xi)}{2} \geq \frac{1}{4}\right\} \\
&\leq 4\int_{\{\xi\in\mathbb{R};\sin^2\frac{\theta(\xi)}{2}\geq\frac{1}{4}\}} \sin^2\frac{\theta(\xi)}{2} d\xi \\
&\leq \frac{1}{4}\int_{\{\xi\in\mathbb{R};\sin^2\frac{\theta(\xi)}{2}\geq\frac{1}{4}\}} \frac{\theta(\xi)}{2} d\xi \leq \frac{1}{4}\delta^2
\end{aligned}$$

for $(u,\theta,h) \in \Lambda$. Then we have

(33)
$$\int_\xi^{\bar{\xi}} \cos^2\frac{\theta(\xi)}{2} h(\xi')d\xi \geq \int_{\{\xi\in[\xi,\bar{\xi}]\frac{\theta(\xi)}{2}\leq\frac{\pi}{4}\}} \frac{h^-}{2} d\xi \geq \frac{h^-}{2}(\bar{\xi} - \xi - \frac{1}{4}\delta^2) \text{ for any } \xi \in \bar{\xi},$$

which guarantees that exponential term in the (22)-(23) for P andP_x decreases quickly as $|\bar{\xi} - \xi| \to \infty$. Taking

$$\Gamma(\epsilon) \doteq \min\left\{1, \exp\left(\frac{1}{8}\delta^2 h^- - \frac{h^-}{2}|\epsilon|\right)\right\}, \tag{34}$$

we see

$$\|\Gamma\|_{L^1} = \left(\int_{\epsilon\leq\frac{1}{4}\delta^2} + \int_{\epsilon\geq\ \frac{1}{4}\delta^2}\right)\Gamma(\epsilon)d\epsilon = \frac{1}{2}\delta^2 + \frac{h^-}{4}. \tag{35}$$

Next we show that $P, P_x \in H^1$, namely,

$$P,\ P_\xi,\ P_x,\ P_{x\xi} \in L^2. \tag{36}$$

Since the estimates for P and P_x are similar, we only need consider a priori bounds on P_x. From the definition of P_x in (23), it follows that

$$|P_x(\xi)| \leq \frac{h^+}{2}\left|\Gamma * \left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)\right|. \tag{37}$$

A standard properties of convolutions ensures that

$$\begin{aligned} ||P_x||_{L^2} &\leq \frac{h^+}{2}||\Gamma||_{L^1}\left(\frac{m(m+3)}{2(m+1)}\|u^{m+1}\|_{L^2} + \frac{m}{8}||u^{m-1}\theta^2||_{L^2}\right) \\ &\leq \frac{Ch^+}{2}||\Gamma||_{L^1}\left(||u||^m_{L^\infty}||u||_{L^2} + ||u||^{m-2}_{L^\infty}||u||_{L^2}||\theta^2||_{L^2}\right) < \infty, \end{aligned} \tag{38}$$

where $C = \frac{m(m+3)}{2(m+1)}$. Next we observe that

$$\begin{aligned} P_{x\xi} &= \left[h\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)\right](\xi) \\ &+ \frac{1}{2}\left(\int_\xi^\infty - \int_{-\infty}^\xi\right)\exp\left\{-\left|\int_\xi^{\bar{\xi}}\left(h\cdot\cos^2\frac{\theta}{2}\right)(s)ds\right|\right\}\left[\cos^2\frac{\theta}{2}h(\xi)\right] \\ &sign(\xi-\bar{\xi})\cdot\left[h\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)\right](\bar{\xi})d\bar{\xi}. \end{aligned} \tag{39}$$

Since

$$\begin{aligned} |P_{x\xi}(\xi)| &\leq h^+\left|\frac{m(m+3)}{2(m+1)}u^{m+1}(\xi) + \frac{m}{8}u^{m-1}(\xi)\theta^2(\xi)\right| \\ &+ \frac{h^+}{2}\left|\Gamma * \left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)(\xi)\right|, \end{aligned} \tag{40}$$

this implies

$$\begin{aligned} ||P_{x\xi}||_{L^2} &= h^+\left(\frac{m(m+3)}{2(m+1)}||u^{m+1}||_{L^2} + \frac{m}{8}||u^{m-1}\theta^2||_{L^2}\right) \\ &+ \frac{h^+}{2}||\Gamma||_{L^1}\left(\frac{m(m+3)}{2(m+1)}u^{m+1}_{L^2} + \frac{m}{8}||u^{m-1}\theta^2||_{L^2}\right) \\ &\leq \frac{Ch^+}{2}||\Gamma||_{L^1}\left(||u||^m_{L^\infty}||u||_{L^2} + ||u||^{m-2}_{L^\infty}||u||_{L^2}||\theta^2||_{L^2}\right) \\ &\leq C(h^+ + \frac{h^+}{2})||\Gamma||_{L^1} < \infty. \end{aligned} \tag{41}$$

A similar argument leads to $P, P_\xi \in L^2$.

To show that the maps given in (32) are Lipschitz continuous. It suffices to verify that partial derivatives

$$\frac{\partial_P}{\partial_u},\ \frac{\partial_P}{\partial_\theta},\ \frac{\partial_P}{\partial_h},\ \frac{\partial_{P_x}}{\partial_u},\ \frac{\partial_{P_x}}{\partial_\theta},\ \frac{\partial_{P_x}}{\partial_h}, \tag{42}$$

are uniformly bounded for $(u, \theta, h) \in \Lambda$. We observe that above derivatives are bounded linear operators from appropriate spaces into $H^1(\mathbb{R})$. For the sake of illustration, we just give a detailed estimate for $\frac{\partial_{P_x}}{\partial_u}$, the boundedness of other derivatives can be obtained in a similar way.

For $(u,\theta,h)\in\Lambda$, the partial derivative $\frac{\partial_{P_x}}{\partial_u}$ and $\frac{\partial(\partial_\xi P_x)}{\partial_u}$ are linear operator, defined by

$$\left[\frac{\partial_{P_x}(u,\theta,h)}{\partial_u}u^*\right](\xi)=\frac{1}{2}\left(\int_\xi^\infty-\int_{-\infty}^\xi\right)\exp\left\{-\left|\int_\xi^{\bar\xi}\left(h\cdot\cos^2\frac{\theta}{2}\right)(s)ds\right|\right\}\cdot\left[hu^*\left(\frac{m(m+3)}{2}u^m\cos^2\frac{\theta}{2}+\frac{m(m-1)}{2}u^{m-2}\sin^2\frac{\theta}{2}\right)\right](\bar\xi)d\bar\xi \tag{43}$$

and

$$\begin{aligned}&\left[\frac{\partial(\partial_\xi\partial_{P_x})(u,\theta,h)}{\partial_u}u^*\right](\xi)\\&=-\left[hu^*\left(\frac{m(m+3)}{2}u^m\cos^2\frac{\theta}{2}+\frac{m(m-1)}{2}u^{m-2}\sin^2\frac{\theta}{2}\right)\right](\xi)\\&+\frac{1}{2}\left(\int_\xi^\infty-\int_{-\infty}^\xi\right)\exp\left\{-\left|\int_\xi^{\bar\xi}\left(h\cdot\cos^2\frac{\theta}{2}\right)(s)ds\right|\right\}\left(h\cdot\cos^2\frac{\theta}{2}\right)\\&sign(\bar\xi-\xi)\left(\frac{m(m+3)}{2}u^m\cos^2\frac{\theta}{2}+\frac{m(m-1)}{2}u^{m-2}\sin^2\frac{\theta}{2}\right)u^*(\bar\xi)d\bar\xi.\end{aligned} \tag{44}$$

In view of $||u^*||_{L^\infty}\le||u^*||_{H^1}$, the above operators norm can be estimated as follows:

$$\begin{aligned}\left\|\frac{\partial_{P_x}}{\partial_u}u^*\right\|_{L^2}&\le\frac{h^+}{2}\left\|\Gamma*\left(\frac{m(m+3)}{2}u^m\cos^2\frac{\theta}{2}+\frac{m(m-1)}{2}u^{m-2}\sin^2\frac{\theta}{2}\right)\right\|_{L^2}||u^*||_{L^\infty}\\&\le\frac{h^+}{2}||\Gamma||_L^1\left(\frac{m(m+3)}{2}\|u^m\|_{L^2}+\frac{m(m-1)}{8}||u^{m-2}\theta^2||_{L^2}\right)||u^*||_{H^1}\end{aligned} \tag{45}$$

and

$$\begin{aligned}\left\|\frac{\partial(\partial_\xi\partial_{P_x})}{\partial_u}u^*\right\|_{L^2}&\le\frac{h^+}{2}\left\|\frac{m(m+3)}{2}u^m\cos^2\frac{\theta}{2}+\frac{m(m-1)}{2}u^{m-2}\sin^2\frac{\theta}{2}\right\|_{L^2}||u^*||_{H^1}\\&+\frac{(h^+)^2}{2}\left\|\Gamma*\left(\frac{m(m+3)}{2}u^m\cos^2\frac{\theta}{2}+\frac{m(m-1)}{2}u^{m-2}\sin^2\frac{\theta}{2}\right)\right\|_{L^2}||u^*||_{H^1}.\end{aligned} \tag{46}$$

By (45) and (46), we obtain $\frac{\partial P_x}{\partial_u}$ is a bounded linear operator from $H^1(\mathbb{R})$ into $H^1(\mathbb{R})$. And the boundedness of other partial derivatives in (42) can be proved by the same arguments. which means that the maps (32) are Lipschitz continuous. □

Next we show that the local solutions of the system (27) can be extended globally in time.

Lemma 3.2. *Given initial data $\tilde u\in H^1(\mathbb{R})$, the Cauchy problem (27))-(28) has a unique solution defined for all time $T>0$.*

Proof. To extend the local solutions of the system (27) to global solutions, we only need to prove that

$$||u||_{H^1}+||\theta||_{L^2}+||\theta||_{L^\infty}+||h||_{L^\infty}+\left\|\frac{1}{h}\right\|_{L^\infty}<\infty \tag{47}$$

for all $T < \infty$. We first claim

$$u_\xi = \frac{h}{2}\sin\theta. \tag{48}$$

In fact, recalling (27), (22) and (23), one can get

$$u_{\xi t} = h\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2} - P\cos^2\frac{\theta}{2}\right) = u_{t\xi}. \tag{49}$$

Moreover, from (19)) and ((20) we have

$$u_\xi = \frac{u_x}{1+{u_x}^2} = \frac{1}{2}\sin\theta,\ \ h=1$$

at $t = 0$. This implies that (48) holds for all t, as long as the solution defined.

Next we prove

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^2\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)hd\xi = 0. \tag{50}$$

Using (27), a direct calculation yields that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left(u^2\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)hd\xi\\
&= \int_{\mathbb{R}} h\{(u^2\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2})\left(\frac{m(m+3)}{m+1}u^{m+1} + mu^{m-1} - 2P\right)\\
&\cos\frac{\theta}{2}\sin\frac{\theta}{2} - 2uP_x\cos^2\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}(1-u^2)\\
&\left[\left(\frac{m(m+3)}{m+1}u^{m+1} - 2P\right)\cos^2\frac{\theta}{2} - mu^{m-1}\sin^2\frac{\theta}{2}\right]\}d\xi\\
&= \int_{\mathbb{R}} h\{\frac{2m(m+2)}{m+1}u^{m+1}\cos\frac{\theta}{2}\sin\frac{\theta}{2} - 2P\\
&\cos\frac{\theta}{2}\sin\frac{\theta}{2} - 2uP_x\cos^2\frac{\theta}{2}\}d\xi.
\end{aligned} \tag{51}$$

By (20), we have

$$P_\xi = hP_x\cos^2\frac{\theta}{2},$$

which implies

$$(uP)_\xi = (P\sin\frac{\theta}{2}\cos\frac{\theta}{2} + uP_x\cos^2\frac{\theta}{2})h \tag{52}$$

and

$$\frac{2m(m+2)}{m+1}u^{m+1}\cos\frac{\theta}{2}\sin\frac{\theta}{2}h = \frac{2m(m+2)}{m+1}u^{m+1}u_\xi = \left(\frac{2m}{m+1}u^{m+2}\right)_\xi. \tag{53}$$

From (52) and (53), one has

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^2\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)d\xi = 0,$$

namely,

$$\int_{\mathbb{R}}\left(u^2\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)d\xi = E(0) = E_0. \tag{54}$$

If the solution is well defined, we obtain a priori bound on $||u(t)||_{L^\infty(\mathbb{R})}$ as follows

$$\sup_{\xi\in\mathbb{R}} |u^2(t,\xi)| \le 2\int_{\mathbb{R}} |uu_\xi| d\xi \le 2\int_{\mathbb{R}} |u|\cdot\left|\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right| hd\xi \le E_0. \tag{55}$$

According to (22), (23) and (54) we know

(56)

$$\begin{aligned} ||P(t)||_{L^\infty},\ ||P_x(t)||_{L^\infty} &\le \frac{1}{2}\left\| h\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)\right\|_{L^1} \\ &\le \frac{m(m+3)}{4(m+1)}E_0^{\frac{m+1}{2}}. \end{aligned}$$

Using the third equation in (27), together with (55) and (56), we conclude

$$|h_t| \le \left(\frac{3m(m+3)}{8(m+1)}E_0^{\frac{m+1}{2}} + \frac{m}{4}E_0^{\frac{m-1}{2}}\right)h.$$

Therefore, it follows that

$$e^{-\left(\frac{3m(m+3)}{8(m+1)}E_0^{\frac{m+1}{2}} + \frac{m}{4}E_0^{\frac{m-1}{2}}\right)t} \le h(t) \le e^{\left(\frac{3m(m+3)}{8(m+1)}E_0^{\frac{m+1}{2}} + \frac{m}{4}E_0^{\frac{m-1}{2}}\right)t}. \tag{57}$$

Also, we observe that there exists a suitable constant $A = A(E_0)$ such that

$$\|\theta\|_{L^\infty} \le e^{At}. \tag{58}$$

Multiplying the first equation of (27) by u and integrating, one has

$$\left|\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|_{L^2}{}^2\right)\right| \le \|u(t)\|_{L^\infty}\|P_x\|_{L^1}.$$

Similarly, we deduce

$$\left|\frac{d}{dt}\left(\frac{1}{2}\|u_\xi(t)\|_{L^2}{}^2\right)\right| \le \|u_\xi(t)\|_{L^\infty}\|\partial_\xi P_x\|_{L^1}.$$

Therefore, we obtain that u and u_ξ are uniformly bound on $[0,T]$ by (48), (55)and (57). A bound on the L^1 norms of P_x and $\partial_x P_x$ yield that $\|u\|_{H^1}$ is bounded for any $T \le \infty$. Therefore, we only consider estamates $\|\partial_x P_x\|_{L^1}$, $\|P_x\|_{L^1}$. In fact, for $\xi > \bar{\xi}$, we find that

(59)

$$\begin{aligned} \int_{\bar{\xi}}^{\xi} h\cos^2\frac{\theta}{2}(s)ds &\ge \int_{\{s\in[\bar{\xi},\xi],|\frac{\theta}{2}(\bar{\xi})|\ge\frac{\pi}{4}\}} h\cos^2\frac{\theta}{2}(s)ds \ge \int_{\{s\in[\bar{\xi},\xi],|\frac{\theta}{2}(\bar{\xi})|\ge\frac{\pi}{4}\}} \frac{h}{2}(s)ds \\ &\ge \frac{h^-}{2}(\xi-\bar{\xi}) - \int_{\{s\in[\bar{\xi},\xi],|\frac{\theta}{2}(\bar{\xi})|\ge\frac{\pi}{4}\}} \frac{h}{2}(s)ds \ge \frac{h^-}{2}(\xi-\bar{\xi}) \\ &\quad - \int_{\{s\in[\bar{\xi},\xi],|\frac{\theta}{2}(\bar{\xi})|\ge\frac{\pi}{4}\}} h\sin^2\frac{\theta}{2}(s)ds \ge \frac{h^-}{2}(\xi-\bar{\xi}) - 2E_0, \end{aligned}$$

where $h^- = e^{-\left(\frac{3m(m+3)}{8(m+1)}E_0^{\frac{m+1}{2}} + \frac{m}{4}E_0^{\frac{m-1}{2}}\right)t} = e^{-Bt}$. Denoting

$$\Gamma_1(\epsilon) \doteq \min\left\{1, e^{(2E_0 - \frac{h^-|\xi|}{2})}\right\},$$

it follows from (39) that

$$\begin{aligned}\|\partial_\xi(\partial_x P(t))\|_{L^1} \le |P_{x\xi}(\xi)| &= \left\| h\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)(\xi)\right\|_{L^1} \\ &\quad + \left\|\frac{h^+}{2}\right\|_{L^\infty}\left\|\Gamma_1 * \left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)(\xi)\right\|_{L^1} \\ &\le \left(1+\frac{1}{2}\|h\|_{L^\infty}\|\Gamma_1\|_{L^1}\right)\left\| h\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)(\xi)\right\|_{L^1} \\ &\le \frac{m(m+3)}{2(m+1)}[1+2e^{2Bt}(2E_0+1)]E_0^{\frac{m+1}{2}},\end{aligned}$$

where

$$\|\Gamma_1\|_{L^1} = 4e^{Bt}(2E_0+1).$$

Multiplying the second equation (27) by θ, we see that

$$\begin{aligned}\left|\frac{d}{dt}\left(\frac{1}{2}\|\theta(t)\|_{L^2}{}^2\right)\right| &\le \int_{\mathbb{R}}\left|\left(\frac{m(m+3)}{m+1}u^{m+1} - 2P\right)\theta\right|d\xi + \int_{\mathbb{R}}\frac{m}{8}u^{m-1}|\theta|^3 d\xi \\ &\le \|\theta\|_{L^\infty}\left(\frac{m(m+3)}{m+1}\|u\|_{L^2}^2\|u^{m-1}\|_{L^\infty} + 2\|P\|_{L^1}\right) + \frac{m}{8}\|u^{m-1}\|_{L^\infty}\|\theta\|_{L^\infty}\|\theta\|_{L^2}^2.\end{aligned}$$

From this, we prove that $\|\theta\|_{L^2}$ remains bounded on bounded intervals of time. This complete the proof of Lemma 3.2. □

3.2. **Global existence of solutions to the equation (1).** Let (u,θ,h) be a global solution to (27), and

$$y(t,\xi) \doteq \tilde{y}(\xi) + \int_0^t u^m(\tau,\xi)d\tau. \tag{60}$$

Then for each fixed ξ, the functions $t \mapsto y(t,\xi)$ provides a solution to the Cauchy problem

$$\frac{\partial}{\partial t}y(t,\xi) = u^m(t,\xi), y(0,\xi) = \tilde{y}(\xi). \tag{61}$$

Now we claim that

$$u(t,x) \doteq u(t,\xi) \text{ if } y(t,\xi) = x, \tag{62}$$

is a weak solutions of equation (3).

Proof of Theorem 1.2. The proof is divided into the following steps.

Step 1. It is clearly that we have the uniform bound

$$|u(t,\xi)| \le E_0^{\frac{1}{2}}.$$

From (60), we have the estimate

$$\tilde{y}(\xi) - E_0^{\frac{m}{2}}t \le y(t,\xi) \le \tilde{y}(\xi) + E_0^{\frac{m}{2}}t, t \ge 0.$$

The definition of ξ in (15) implies

$$\lim_{\xi\to\pm\infty} y(t,\xi) = \pm\infty.$$

Hence the image of the map $(t,\xi) \mapsto (t, y(t,\xi))$ is the entire plane $\mathbb{R}^2$.

Step 2. We check

$$y_\xi = h\cos^2\frac{\theta}{2} \quad \text{for all } t \ge 0 \text{ and a.e } \xi \in \mathbb{R}. \tag{63}$$

Indeed, thanks to system (27), by a straightforward computation we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(h\cos^2\frac{\theta}{2}\right)(t,\xi) &= -h\theta_t \sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2} + h_t\cos^2\frac{v}{2}\\ &= mu^{m-1}h\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\\ &= (u^m)_\xi(t,\xi).\end{aligned}$$

On the other hand, (60) implies

$$\frac{\partial}{\partial t}y_\xi(t,\xi) = (u^m)_\xi(t,\xi).$$

Since the function $x \mapsto 2\arctan\tilde{u}_x(x)$ is measurable, hence (63) is true for almost every ξ at $t=0$, then the above calculation (63) remains true for all $t \geq 0$, and $y(t,\xi)$ is non-deceasing. Moreover, if $\xi < \bar{\xi}$ but $y(t,\xi) = y(t,\bar{\xi})$, then

$$\int_\xi^{\bar{\xi}} y_\xi(t,s)ds = \int_\xi^{\bar{\xi}} h(t,s)\cos^2\frac{\theta}{2}ds = 0.$$

Hence $\cos\frac{\theta}{2} \equiv 0$ throughout the interval of the integration. By (48), we have

$$u(t,\bar{\xi}) - u(t,\xi) = \int_\xi^{\bar{\xi}} \frac{h(t,s)}{2}\sin\theta(t,s)\cos\frac{\theta}{2}ds = 0.$$

This shows that the map $(t,x) \longmapsto u(t,y(\xi))$ is well defined for all $t \geq 0$ and $x \in \mathbb{R}$.

Step 3. Recalling the basic relations, we have

$$\frac{\partial u(t,\xi)}{\partial \xi} = \frac{h(t,\xi)}{2}\sin\theta(t,\xi),\ \frac{\partial y(t,\xi)}{\partial\xi} = h(t,\xi)\cos^2\frac{\theta(t,\xi)}{2}. \tag{64}$$

In addition, if $x = y(t,\xi),\ \cos\theta(t,\xi) \neq -1$, and

$$u_x(t,x) = \frac{\sin\theta(t,\xi)}{1+\cos\theta(t,\xi)}, \tag{65}$$

then for any time t, it follows from (64) and (65) that

$$\begin{aligned}&\int_{\mathbb{R}} \left(u^2(t,x) + u_x^2(t,x)\right)dx\\ &= \int_{\mathbb{R}\cap\cos\frac{\theta}{2}\neq -1}\left(u^2(t,\xi)\cos^2\frac{\theta(t,\xi)}{2} + \sin^2\frac{\theta(t,\xi)}{2}\right)h(t,\xi)d\xi \leq E_0,\end{aligned} \tag{66}$$

which implies $E(t) = E_0$. Since the boundedness of $\|P_x\|_{L^\infty}$, we obtain

$$\frac{du(t,y(t,\xi))}{dt} = u_t \leq \infty.$$

On the basis of the Sobolev inequality, $u(t,x)$ is Hölder continuous with exponent $\frac{1}{2}$ on both x and x.

Step 4. We are ready to show that the Lipschitz continuity of $u(t,x)$ with values in $L^2(\mathbb{R})$. Consider any interval $[\tau,\tau+h]$, given a point x, we choose $\xi\in\mathbb{R}$ such that the characteristic $t\mapsto y(t,\xi)$ passes through the point (τ,x). By (27) and (55), it follows that

$$\begin{aligned}|u(\tau+s,x)-u(\tau,x)| &\le |u(\tau+s,x)-u(\tau+s,y(\tau+s,\xi))| \\ &\quad + |u(\tau+s,y(\tau+s,\xi))-u(\tau,x)| \\ &\le \sup_{|y-x|\le E_0^{\frac{m}{2}}s} |u(\tau+s,y)-u(\tau+s,x)| + \int_\tau^{\tau+s} |P_x(t,\xi)|dt.\end{aligned}$$

Integrating over $\mathbb{R}$, using the boundedness of $||P_x||_{L^2(\mathbb{R})}$ and $\|u_x\|_{L^2}$, we deduce that

$$\begin{aligned}&\int_{\mathbb{R}} |u(\tau+s,y)-u(\tau,y)|^2dx \\ &\le 2\int_{\mathbb{R}}\left(\int_{x-E_0^{\frac{m}{2}}s}^{x+E_0^{\frac{m}{2}}s} |u_x(\tau+s,y)|dy\right)^2 dx \\ &\qquad + 2\int_{\mathbb{R}}\left(\int_\tau^{\tau+s} |P_x(t,\xi)|dt\right)^2 h(\tau,\xi)\cos^2\frac{\theta(\tau,\xi)}{2}d\xi \\ &\le 4E_0^{\frac{m}{2}}s\int_{\mathbb{R}}\int_{x-E_0^{\frac{m}{2}}s}^{x+E_0^{\frac{m}{2}}s} |u_x(\tau+s,y)|^2dydx + 2s\|h\|_{L^\infty}\int_{\mathbb{R}}\int_\tau^{\tau+s}|P_x(t,\xi)|^2dtd\xi \\ &\le 8E_0^m s^2\|u_x(\tau+s)||^2_{L^2(\mathbb{R})} + 2s\|h\|_{L^\infty}||P_x(t)||^2_{L^2(\mathbb{R})}dt \le Cs,\end{aligned}$$

where the constant C depending only on T. The above inequality implies that the map $t\mapsto u(t)$ is Lipschitz continuous for the variable x.

Step 5. Define $\Omega=[0,\infty)\times\mathbb{R}$ and $\Omega'=\Omega\cap\{(t,y)|\cos^2\frac{\theta(\tau,\xi)}{2}\doteq 0\}$, for any text function $\phi(x,t)\in C_c^1(\Omega)$, we have the following weak form

(67)

$$\begin{aligned}0 &= \int\int_\Omega \left\{u_{\xi t}\phi_t + h\phi\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2}+\frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}-P\cos^2\frac{\theta}{2}\right)\right\}\xi dt \\ &= \int\int_\Omega \left\{-u_\xi\phi_t\phi_t + h\phi\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2}+\frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}-P\cos^2\frac{\theta}{2}\right)\right\}d\xi dt \\ &= \int\int_{\Omega'} \left\{-u_\xi\phi_t\phi_t + h\phi\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2}+\frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}-P\cos^2\frac{\theta}{2}\right)\right\}\xi dt \\ &= \int\int_\Omega \left\{-u_x(\phi_t+u^m\phi_x) + \phi\left(-\frac{m(m+3)}{2(m+1)}u^{m+1}-\frac{m}{2}u^{m-1}u_x^2+P\right)\right\}dxdt,\end{aligned}$$

which proves that (5) holds. Let us introduce the Radon measures $\{\mu_{(t)}, t\in\mathbb{R}^+\}$, for any Lebesgue measurable set $\{x\in\mathcal{A}\}$ in $\mathbb{R}$, assuming the corresponding pre-image set of transformation is $\{\xi\in\mathcal{F}(\mathcal{A})\}$, one has

$$\mu_t(\mathcal{A}) = \int_{\mathcal{F}(\mathcal{A})}\sin^2\frac{\theta}{2}(t,\xi)d\xi.$$

By (63), the measure $\mu_{(t)}$ is absolutely continuous and has density $u_x^2(t,\cdot)$ w.r.t. Lebesgue measure. It is easy to check that (6) is right. Indeed, from (3.1) we have

(68)

$$\begin{aligned}-\int_{\mathbb{R}^+}\left\{\int(\phi_t+u^m\phi_x)d\nu_{(t)}\right\}dt &= -\int\int_\Omega \sin^2\frac{\theta}{2}h\phi_t d\xi dt = \int\int_\Omega(\sin^2\frac{\theta}{2}h)_t\phi d\xi dt \\ &= \int\int_\Omega\left(\frac{m(m+3)}{m+1}u^{m+1}-2P\right)\sin\frac{\theta}{2}\cos\frac{\theta}{2}h\phi d\xi dt\end{aligned}$$

$$= \int\int_{\Omega}\left(\frac{m(m+3)}{m+1}u^{m+1} - 2P\right)u_x\phi dxdt.$$

Step 6. Ultimately, we show that for almost every $t \in \mathbb{R}^+$, the singular part of v_t is concentrated on the set where $u = 0$. The proof is similar to the argument in [2]. Note that when blow up occurs, $\cos\frac{\theta}{2} = 0$, it follows that $\theta_t = -mu^{m-1}$, which implies $\theta_t \neq 0$ only when $m \neq 0$ or $u \neq 0$. Moreover, the proof in the seventh step is different from the Camassa-Holm equation. □

4. The uniqueness of conservative solutions for equation (3)

4.1. **Uniqueness of characteristics.** Let $u = u(t, x)$ be a conservative solution of equation (1). We introduce the new coordinates (t, β), and define $x(t, \beta)$ is the unique point x such that

$$x(t,\beta) + \mu_{(t)}\{(-\infty, x)\} \leq \beta \leq x(t,\beta) + \mu_{(t)}\{(-\infty, x]\} \tag{69}$$

for any time t and $\beta \in \mathbb{R}$. When the measure $\mu_{(t)}$ is absolutely continuous with density u_x^2 w.r.t Lebesgue measure, the above definition gives that

$$x(t,\beta) + \int_{-\infty}^{x(t,\beta)} u_x^2(t,\xi)d\xi = \beta. \tag{70}$$

Next, we will give following Lemma which is helpful to prove the Lipschitz continuity of x and u as functions of the variables t, β.

Lemma 4.1. *Assume that $u = u(t, x)$ is a conservative solution of (1). For every $t \geq 0$, then the maps $\beta \mapsto x(t, \beta)$ and $\beta \mapsto u(t, \beta) \doteq u(t, x(t, \beta))$, which are implicitly defined by (69), are Lipschitz continuous. Moreover, The map $t \mapsto x(t, \beta)$ is also Lipschitz continuous with a constant depending only on $\|u_0\|_{H^1}$.*

Proof of Lemma 4.1. We split the proof into three steps.

Step 1. For any time $t \geq 0$, the map

$$x \mapsto \beta(t, x)$$

is right continuous and strictly increasing. Thus, the inverse $\beta \mapsto x(t, \beta)$ is well defined, continuous, nondecreasing. If $\beta_1 < \beta_2$, we see that

$$x(t,\beta_2) - x(t,\beta_1) + \mu_{(t)}\{(x(t,\beta_2), x(t,\beta_1))\} \leq \beta_2 - \beta_1,$$

which implies

$$x(t,\beta_2) - x(t,\beta_1) \leq \beta_2 - \beta_1, \tag{71}$$

and the map $\beta \mapsto x(t, \beta)$ is Lipschitz continuous.

Step 2. For the map $\beta \mapsto u(t, \beta)$, as $\beta_1 < \beta_2$, it follows from (69) and (71) that

$$\begin{aligned} |u(t, x(t,\beta_2)) - u(t, x(t,\beta_1))| &\leq \int_{x(t,\beta_1)}^{x(t,\beta_2)} |u_x|dx \leq \int_{x(t,\beta_1)}^{x(t,\beta_2)} \frac{1}{2}(1+u_x^2)dy \\ &\leq \frac{1}{2}\left[x(t,\beta_2) - x(t,\beta_1) + \mu_{(t)}\{(x(t,\beta_2), x(t,\beta_1))\}\right] \\ &\leq \frac{1}{2}(\beta_2 - \beta_1). \end{aligned} \tag{72}$$

Hence, the map $\beta \mapsto u(t, \beta)$ is Lipschitz continuous.

Step 3. Now we claim the Lipschitz continuity of the map $t \mapsto x(t,\beta)$. Assume $x(\tau,\beta) = y$, since the family of measure $\mu_{(t)}$ satisfies the balance law (6), we infer the source term $2u_x\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)$ satisfies

$$\|2u_x\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)\|_{L^1} \le 2\big(\frac{m(m+3)}{2(m+1)}\|u\|^m_{L^\infty}\|u\|_{L^2} + \|P\|_{L^2}\big)\|u_x\|_{L^2} \le C_s. \tag{73}$$

For $t \ge \tau$, it follows from (73) that

$$\mu_{(t)}\{(-\infty, y - C_\infty(t-\tau))\} \le \mu_{(\tau)}\{(-\infty, y)\} + C_s(t-\tau),$$

where the constant C_s depending only on the $H^1(\mathbb{R})$ norm of u and m. Denoting $y^-(t) \doteq y - (C_\infty + C_s)(t-\tau)$, we have

$$\begin{aligned} y^-(t) + \mu_{(t)}\{(-\infty, y^-(t)]\} &\le y - (C_\infty + C_s)(t-\tau) + \mu_{(\tau)}\{(-\infty, y)\} + C_s(t-\tau) \\ &\le y + \mu_{(\tau)}\{(-\infty, y)\} \le \beta, \end{aligned}$$

which implies $x(t,\beta) \ge y^-(t)$ for all $t > \tau$. A similar argument yields

$$x(t,\beta) \le y^+(t) \doteq y + (C_\infty + C_s)(t-\tau).$$

This proves the uniform Lipschitz continuity of the map $t \mapsto x(t,\beta)$. □

Lemma 4.2. *Let $u = u(t,x)$ be the conservative solution of equation (1). Then there exists a unique Lipschitz continuous map $t \mapsto x(t)$ for any $\tilde{y} \in \mathbb{R}$, where the map satisfies*

$$\frac{d}{dt}x(t) = u^m(t,x), \;\; x(0) = \tilde{y}, \tag{74}$$

and

$$\frac{d}{dt}\int_{-\infty}^{x(t,\beta)} u_x^2 = \int_{-\infty}^{x(t,\beta)} 2u_x\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)dx. \tag{75}$$

Moreover, we have

$$u(t,x(t)) - u(\tau,x(\tau)) = -\int_\tau^t P_x(s,x(s))ds, \tag{76}$$

for any $0 \le \tau \le t$.

Proof. **Step 1.** According to the adapted coordinates (t,β), we write the characteristic beginning with $\tilde{y}$ in the form $t \mapsto x(t) = x(t,\beta(t))$. $\beta(\cdot)$ is a map to be determined. Together with (74) and (75), we obtain

$$\begin{aligned} x(t) + \int_{-\infty}^{x(t)} u_x^2(t,y)dy &= \tilde{y} + \int_{-\infty}^{\tilde{y}} u_{0,x}^2 dy \\ &+ \int_0^t \Big(u^m(s,x(s)) + \int_{-\infty}^{x(s)} 2u_x\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)(s,y)dy\Big)ds. \end{aligned} \tag{77}$$

For convenience, let

$$G(t,\beta) \doteq \int_{-\infty}^{x(s)} 2u_x\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)dx \tag{78}$$

and

$$\tilde{\beta} = \tilde{y} + \int_{-\infty}^{\tilde{y}} u_{0,x}^2(y)dy. \tag{79}$$

Therefore, we can rewrite the equation (77) as follow

$$\beta(t) = \tilde{\beta} + \int_0^t G(s, \beta(s))ds, \text{ for all } t > 0. \tag{80}$$

Step 2. For every fix $t \geq 0$, in view of the maps $x \mapsto u(t,x),\ x \mapsto P(t,x) \in H^1(\mathbb{R})$, and the function $\beta \mapsto G(t,\beta)$ defined by (78) is uniformly bounded and absolutely continuous. Furthermore, we have

$$\begin{aligned} G_\beta &= 2u_x\left(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\right)x_\beta \\ &= \frac{2u_x\left(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\right)}{1+u_x^2} \in [-C, C] \end{aligned} \tag{81}$$

for some constant C, which depends only on the H^1 norm of u. Consequently, the function G in (80) is uniformly Lipschitz continuous w.r.t. β.

Step 3. Based on the Lipschitz continuity of the function G, applying the standard fixed point theory, one can get the existence of a unique solution for the integral (80). More details can refer to [4].

Step 4. Owing to the previous construction, we conclude that the map $t \mapsto x(t) \doteq x(t, \beta(t))$ is a unique solution for equation (77). $\beta(t)$ and $x(t)$ are differentiable almost everywhere because of the Lipschitz continuity of $\beta(t)$ and $x(t) = x(t, \beta(t))$, so we only consider the time where $x(t)$ is differentiable. We prove that (74) holds at almost every time. Suppose, on the contrary, $\dot{x}(\tau) \doteq u^m(\tau, x(\tau))$. Without loss of generality, let

$$\dot{x}(\tau) \doteq u^m(\tau, x(\tau)) + 2\varepsilon_0 \tag{82}$$

for some $\varepsilon_0 > 0$. The case $\varepsilon_0 < 0$ can be proved by similar approach. For $t \in (\tau, \tau+\delta]$, choosing $\delta > 0$ small enough, we find

$$x^+(t) \doteq x(\tau) + (t-\tau)[u^m(\tau, x(\tau)) + \varepsilon_0] < x(t). \tag{83}$$

We also observe that (6) still holds for any test ϕ with compact support.

For any $\epsilon > 0$, we consider the test functions as

$$\rho^\epsilon(s,y) = \begin{cases} 0 & \text{if } y \leq -\epsilon^{-1}, \\ (y+\epsilon^{-1}) & \text{if } -\epsilon^{-1} \leq y \leq 1-\epsilon^{-1}, \\ 1 & \text{if } 1-\epsilon^{-1} \leq y \leq x^+(s), \\ 1-\epsilon^{-1}(y-x(s)) & \text{if } x^+(s) \leq y \leq x^+(s)+\epsilon, \\ 0 & \text{if } y \geq x^+(s)+\epsilon, \end{cases}$$

$$\chi^\epsilon(s,y) = \begin{cases} 0 & \text{if } s \leq \tau-\epsilon, \\ \epsilon^{-1}(s-\tau+\epsilon) & \text{if } \tau-\epsilon \leq s \leq \tau, \\ 1 & \text{if } \tau \leq s \leq t, \\ 1-\epsilon^{-1}(s-t) & \text{if } t \leq s < t+\epsilon, \\ 0 & \text{if } s \geq t+\epsilon. \end{cases} \tag{84}$$

We define

$$\psi^\epsilon(s,y) \doteq \min\{\rho^\epsilon(s,y), \chi^\epsilon(s)\}. \tag{85}$$

Using ψ^ϵ as a test function in (6), it follows that

$$\int\int\left[u_x^2\psi_t^\epsilon+u^mu_x^2\psi_x^\epsilon+2u_x\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P\right)\psi^\epsilon\right]dxdt=0. \tag{86}$$

If $t\to\tau$, we have

$$\lim_{\epsilon\to0}\int_\tau^t\int_{x^+(s)-\epsilon}^{x^+(s)+\epsilon}u_x^2(\psi_t^\epsilon+u^m\psi_x^\epsilon)dyds\geq 0. \tag{87}$$

Actually, for $s\in[\tau+\epsilon,t-\epsilon]$, one has

$$0=\psi_t^\epsilon+[u^m(\tau,x(\tau))+\varepsilon_0]\psi_x^\epsilon\leq\psi_t^\epsilon+u^m(s,x)\psi_x^\epsilon, \tag{88}$$

where we use the fact that $u^m(s,x)<u^m(\tau,x(\tau))+\varepsilon_0$ and $\psi_x^\epsilon\leq 0$.

Due to the family of measures $\mu_{(t)}$ depend continuously on t in the topology of weak convergence, taking the limit of (86) as $\epsilon\to0$, we have

$$\begin{aligned}0=&\int_{-\infty}^{x(\tau)}u_x^2(\tau,y)dy-\int_{-\infty}^{x^+(t)}u_x^2(t,y)dy\\&+\int_\tau^t\int_{-\infty}^{x^+(s)}2u_x\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P\right)dyds\\&+\lim_{\epsilon\to0}\int_\tau^t\int_{x^+(s)-\epsilon}^{x^+(s)+\epsilon}u_x^2(\psi_t^\epsilon+u^m\psi_x^\epsilon)dyds\\\geq&\int_{-\infty}^{x(\tau)}u_x^2(\tau,y)dy-\int_{-\infty}^{x^+(t)}u_x^2(t,y)dy\\&+\int_\tau^t\int_{-\infty}^{x^+(s)}2u_x\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P\right)dyds,\end{aligned}$$

which yields

$$\begin{aligned}&\int_{-\infty}^{x^+(t)}u_x^2(t,y)dy\geq\int_{-\infty}^{x(\tau)}u_x^2(\tau,y)dy+\int_\tau^t\int_{-\infty}^{x^+(s)}2u_x\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P\right)dyds\\&=\int_{-\infty}^{x(\tau)}u_x^2(\tau,y)dy+\int_\tau^t\int_{-\infty}^{x(s)}2u_x\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P\right)dyds+o_1(t-\tau).\end{aligned}$$

Note that the last term is higher order infinitesimal, satisfying $\frac{o_1(t-\tau)}{t-\tau}\to0$ as $t\to\tau$. Indeed,

$$\begin{aligned}|o_1(t-\tau)|&=\left|\int_\tau^t\int_{x^+(s)}^{x^+(s)+\epsilon}2u_x\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P\right)dxdyds\right|\\&\leq\|2\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P\right)\|_{L^\infty}\int_\tau^t\int_{x^+(s)}^{x^+(s)+\epsilon}|u_x|dyds\\&\leq\|2\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P\right)\|_{L^\infty}\int_\tau^t(x(s)-x^+(s))^{\frac12}\|u_x(s,\cdot)\|_{L^2}ds\\&\leq C\cdot(t-\tau)^{\frac32}.\end{aligned}$$

On the other hand, together with (78) and (80), we see

$$\beta(t)=\beta(\tau)+(t-\tau)\left[u^m(\tau,x(\tau)+\int_{-\infty}^{x(\tau)}2u_x\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P\right)dy\right]+o_2(t-\tau), \tag{89}$$

with $\frac{o_2(t-\tau)}{t-\tau}$ as $t \to \tau$. For t sufficiently close to τ, we have

$$
\begin{aligned}
\beta(t) &= x(t) + \int_{-\infty}^{x(t)} (u_x^2(t,y)dy \\
&> x(\tau) + (t-\tau)[u^m(\tau, x(\tau)) + \varepsilon_0] + \int_{-\infty}^{x^+(t)} u_x^2(t,y)dy \\
&\geq x(\tau) + (t-\tau)[u^m(\tau, x(\tau)) + \varepsilon_0] + \int_{-\infty}^{x(\tau)} u_x^2(\tau,y)dy \\
&\quad + \int_\tau^t \int_{-\infty}^{x(s)} 2u_x\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)dyds + o_1(t-\tau).
\end{aligned} \tag{90}
$$

By (89) and (90), we have

(91)

$$
\begin{aligned}
&\beta(t) + (t-\tau)\Big[u^m(\tau, x(\tau)) + \int_{-\infty}^{x(\tau)} 2u_x\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)dy\Big] + o_2(t-\tau) \\
&\geq \Big[x(\tau) + \int_{-\infty}^{x(\tau)} u_x^2(\tau,y)dy\Big] + (t-\tau)[u^m(\tau, x(\tau)) + \varepsilon_0] \\
&+ \int_\tau^t \int_{-\infty}^{x(s)} 2u_x\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)dyds + o_1(t-\tau).
\end{aligned}
$$

Dividing both sides by $t-\tau$ and letting $t \to \tau$, we get a contradiction, namely, (74) holds.

Step 5. Now we prove (75). By (3), one has

$$
\int_0^\infty \int \Big[u\phi_t + \frac{u^{m+1}}{m+1}\phi_x + P_x\phi\Big]dxdt + \int u_0(x)\phi(0,x)dx = 0, \tag{92}
$$

for every test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. Let $\phi = \psi_x$, where $\psi \in \mathcal{C}_c^\infty$. Due to the fact that the map $x \mapsto u(t,x)$ is absolutely continuous, integrating by part w.r.t. x, then we get

$$
\int_0^\infty \int \Big[u_x\psi_t + u^m u_x\psi_x + P_x\psi_x\Big]dxdt + \int u_{0,x}(x)\psi(0,x)dx = 0. \tag{93}
$$

By an approximation argument, we find the identity (93) still holds for any test function ψ which is Lipschitz continuous with compact support. consider the function

$$
\eta(s,y) = \begin{cases} 0 & if\ y \leq -\epsilon^{-1}, \\ (y+\epsilon^{-1}) & if\ -\epsilon^{-1} \leq y \leq 1-\epsilon^{-1}, \\ 1 & if\ 1-\epsilon^{-1} \leq y \leq x(s), \\ 1-\epsilon^{-1}(y-x(s)) & if\ x(s) \leq y \leq x(s)+\epsilon, \\ 0 & if\ y \geq x(s)+\epsilon, \end{cases}
$$

for any $\epsilon > 0$ sufficiently small. We define

$$
\varphi^\epsilon(s,y) = \min\{\eta^\epsilon(s,y), \chi^\epsilon(s)\}, \tag{94}
$$

with $\chi^\epsilon(s)$ defined in (84). We use the test function $\psi = \varphi^\epsilon$ in (93). And let $\epsilon \to 0$. Since the function P_x is continuous, we have

$$\begin{aligned}\int_{-\infty}^{x(t)} u_x(t,y)dy = \int_{-\infty}^{x(\tau)} u_x(\tau,y)dy - \int_\tau^t P_x(s,x(s))ds \\ + \lim_{\epsilon\to 0}\int_{\tau-\epsilon}^{t+\epsilon}\int_{x(s)}^{x(s)+\epsilon} u_x^2(\varphi_t^\epsilon + u^m\varphi_x^\epsilon)dyds.\end{aligned} \tag{95}$$

It suffices to prove that the last term of the limit in (95) is zero. The Cauchy's inequality implies

$$\begin{aligned}&\Big|\int_\tau^t\int_{x(s)}^{x(s)+\epsilon} u_x(\varphi_t^\epsilon + u^m\varphi_x^\epsilon)dyds\Big| \\ &\le \int_\tau^t\left(\int_{x(s)}^{x(s)+\epsilon}|u_x|^2dy\right)^{\frac{1}{2}}\left(\int_{x(s)}^{x(s)+\epsilon}(\varphi_t^\epsilon + u^m\varphi_x^\epsilon)^2dy\right)^{\frac{1}{2}}ds,\end{aligned} \tag{96}$$

where $u_x \in L^2$. For each $\epsilon > 0$, denoting

$$\varsigma_\epsilon(s) \doteq \big(\sup_{x\in\mathbb{R}}\int_x^{x+\epsilon} u_x^2(s,y)dy\big)^{\frac{1}{2}}, \tag{97}$$

we see that all functions ς_ϵ are uniformly bounded and $\varsigma_\epsilon(t) \to 0$ pointwise at a.e. time t as $\epsilon \to 0$. Therefore, it follows from the dominated convergence theorem that

$$\lim_{\epsilon\to 0}\int_\tau^t\left(\int_{x(s)}^{x(s)+\epsilon}|u_x(s,y)|^2dx\right)^{\frac{1}{2}}ds \le \lim_{\epsilon\to 0}\int_\tau^t \varsigma_\epsilon(s)ds = 0. \tag{98}$$

On the other hand, for every time $s \in [\tau, t]$, we obtain

$$\varphi_x^\epsilon(s,y) = \epsilon^{-1},\ \varphi_t^\epsilon(s,y) + u^m(s,x(s))\varphi_x^\epsilon(s,y) = 0,$$

for $x(s) < y < x(s) + \epsilon$. This yields

$$\begin{aligned}&\int_{x(s)}^{x(s)+\epsilon}|\varphi_t^\epsilon(s,y) + u^m(s,x(s))\varphi_x^\epsilon(s,y)|^2dy \\ &= \epsilon^{-2}\int_{x(s)}^{x(s)+\epsilon}|u^m(s,y) - u^m(s,x(s))|^2dy \\ &\le \epsilon^{-1}\cdot\left(\max_{x(s)\le y\le x(s)+\epsilon}|u^m(s,y) - u^m(s,x(s))|\right)^2 \\ &\le m^2\epsilon^{-1}\cdot\left(\int_{x(s)}^{x(s)+\epsilon}|u^{m-1}u_x(s,y)|dy\right)^2 \\ &\le \epsilon^{-1}m^2(\|u\|_{L^\infty}^{m-1}\epsilon^{\frac{1}{2}}\cdot\|u_x(s)\|_{L^2})^2 \le m^2\|u(s)\|_{H^1}^{2m}.\end{aligned} \tag{99}$$

By (98) and (99), one has the integral in (96) approaches to zero as $\epsilon \to 0$. We now estimate the integral near the corners of the domain,

$$\begin{aligned}&\left|\left(\int_{\tau-\epsilon}^\tau + \int_t^{t+\epsilon}\right)\int_{x(s)}^{x(s)+k\epsilon} u_x(\psi_t^\epsilon + u^m\varphi_x^\epsilon)dxds\right| \\ &\le \left(\int_{\tau-\epsilon}^\tau + \int_t^{t+\epsilon}\right)\left(\int_{x(s)}^{x(s)+\epsilon}|u_x|^2dx\right)^{\frac{1}{2}}\left(\int_{x(s)}^{x(s)+\epsilon}(\psi_t^\epsilon + u^m\varphi_x^\epsilon)^2dx\right)^{\frac{1}{2}}ds \\ &\le 2\epsilon\cdot\|u(s)\|_{H^1}\left(\int_{x(s)}^{x(s)+\epsilon}4\epsilon^{-2}\|u\|_{L^\infty})^{2m}dx\right)^{\frac{1}{2}} \le C\epsilon^{\frac{1}{2}} \to 0\end{aligned} \tag{100}$$

as $\epsilon \to 0$. We conclude

$$\lim_{\epsilon\to 0}\int_{\tau-\epsilon}^{t+\epsilon}\int_{x(s)}^{x(s)+\epsilon} u_x^2(\psi_t^\epsilon + u^m u\varphi_x^\epsilon)dxds = 0. \tag{101}$$

Therefore, using (95), we deduce (76).

Step 6. Finally, the uniqueness of the solution $x(t)$ is clear. □

Lemma 4.3. *If $u = u(t,x)$ is a conservative solution of equation (3), then the map $(t,\beta) \mapsto u(t,\beta) \doteq u(t,x(t,\beta))$ is Lipschitz continuous, where the Lipschitz constant depending only on the norm $\|u_0\|_{H^1}$.*

Proof. By (72), (76) and (80), we have

$$\begin{aligned}
&\Big|u(t,x(t,\tilde\beta)) - u(\tau,\tilde\beta)\Big| \\
&\le \big|u(t,x(t,\tilde\beta)) - u(t,x(t,\beta(t)))\big| + \big|u(t,x(t,\beta(t))) - u(t,x(\tau,\beta(\tau)))\big| \\
&\le \frac{1}{2}|\beta(t)-\tilde\beta| + C(t-\tau) \le C(t-\tau),
\end{aligned} \tag{102}$$

where C is a constant depending only on $\|u_0\|_{H^1}$. □

Lemma 4.4. *Let u be a conservative solution to the equation (3). If $t \mapsto \beta(t;\tau,\tilde\beta)$ is the solution to the integral equation*

$$\beta(t) = \tilde\beta + \int_\tau^t G(\tau,\beta(\tau))d\tau, \tag{103}$$

where the G is defined in (78), then there exists a constant C, such that for any two initial data $\tilde\beta_1, \tilde\beta_2$ and any $t,\tau \ge 0$ the corresponding solutions satisfy

$$|\beta(t;\tau,\tilde\beta_1) - \beta(t;\tau,\tilde\beta_2)| \le e^{C|t-\tau|}|\tilde\beta_1 - \tilde\beta_2|. \tag{104}$$

Proof. Using the Lipschitz continuity of G with respect to β, the lemma can be proved. We omit here for brevity. □

Lemma 4.5. *Suppose $u \in H^1(\mathbb{R})$. Then P_x is absolutely continuous and satisfies*

$$P_{xx} = P - \frac{m}{2}u^{m-1}u_x^2 - \frac{m(m+3)}{2(m+1)}u^{m+1}. \tag{105}$$

Proof. The function $\phi(x) = \frac{1}{2}e^{-|x|}$ satisfies the distributional identity

$$D_x^2\phi = \phi - \delta_0.$$

Here δ_0 denotes a unit Dirac mass at the origin. For every function $f \in L^1(\mathbb{R})$, the convolution satisfies

$$D_x^2(\phi * f) = \phi * f - f.$$

Choosing $f = \frac{m}{2}u^{m-1}u_x^2 + \frac{m(m+3)}{2(m+1)}u^{m+1}$, we obtain the result. □

4.2. **Uniqueness of conservative solutions for equation (1).** In this subsection, we mainly prove the uniqueness of conservative solutions for equation (1).

Proof of Theorem 1.3. The proof is divided into following steps.

Step 1. It follows from Lemma 4.1 and Lemma 4.3 that the map $(t,\beta)\mapsto(x,u)(t,\beta)$ is Lipschitz continuous. By a similar approach, we find the maps $\beta\mapsto G(t,\beta)\doteq G(t,x(\beta))$ and $\beta\mapsto P_x(t,\beta)\doteq P_x(t,x(t,\beta))$ are also Lipschitz continuous. Thanks to the Rademacher's theorem, the partial derivatives $x_t, x_\beta, u_t, u_\beta$ and $P_{x,\beta}$ exist almost everywhere. And for these derivatives, a.e. point (t,β) is a Lebesgue one. Recalling that $t\mapsto\beta(t,\tilde\beta)$ the unique solution of the equation (80), for a.e $\tilde\beta$, from Lemma 4.4 we can draw the following conclusion.

(GC) For a.e. $t>0$, except a measure zero set $\mathcal{N}\in\mathbb{R}^+$, the point $(t,\beta(t,\tilde\beta))$ is a Legbesgue point with respect to the partial derivatives $x_t, x_\beta, u_t, u_\beta, G_\beta, P_{x,\beta}$. And $x_\beta(t,\beta(t,\tilde\beta))>0$ for a.e. $t>0$.

If the above condition is true, we say that $t\mapsto\beta(t,\tilde\beta)$ is a **good characteristic**.

Step 2. We find an ODE to describe a change in these two quantities u_β and x_β along a good characteristic. Denote $t\mapsto\beta(t;\tau,\tilde\beta)$ to be the solution of (103). For $\tau,t\notin\mathcal{N}$, let $\beta(\cdot;\tau,\tilde\beta)$ be a good characteristic. Differentiating (103) w.r.t $\tilde\beta$ we obtain

$$\frac{\partial\beta(t)}{\partial\tilde\beta}=1+\int_\tau^t G_\beta(s,\beta(s;\tau,\tilde\beta))\cdot\frac{\partial}{\partial\tilde\beta}\beta(s;\tau,\tilde\beta)ds. \tag{106}$$

Next, differentiating w.r.t. $\tilde\beta$ the identity

$$x(t,\beta(t;\tau,\tilde\beta))=x(\tau,\tilde\beta)+\int_\tau^t u^m(s,x(s,\beta(t;\tau,\tilde\beta)))ds,$$

we have

$$\begin{aligned}&x_\beta(t,\beta(t;\tau,\tilde\beta))\cdot\frac{\partial}{\partial\tilde\beta}\beta(t;\tau,\tilde\beta)\\&=x_\beta(\tau,\tilde\beta)+\int_\tau^t u^m_\beta(s,\beta(s;\tau,\tilde\beta))\cdot\frac{\partial}{\partial\tilde\beta}\beta(t;s,\tilde\beta)ds.\end{aligned} \tag{107}$$

Finally, by (76), differentiating w.r.t. $\tilde\beta$, we have

$$u_\beta(t,\beta(t;\tau,\tilde\beta))\cdot\frac{\partial}{\partial\tilde\beta}\beta(t;\tau,\tilde\beta)=u_\beta(\tau,\tilde\beta)+\int_\tau^t P_{x,\beta}(s,\beta(s;\tau,\tilde\beta))\cdot\frac{\partial}{\partial\tilde\beta}\beta(t;s,\tilde\beta)ds. \tag{108}$$

Together with (106), (107) and (108) yield the following ODE system:

$$\left\{\begin{array}{l}\frac{d}{dt}\left[\frac{\partial}{\partial\tilde\beta}\beta(t;\tau,\tilde\beta)\right]=G_\beta(t,\beta(t;\tau,\tilde\beta))\cdot\frac{\partial}{\partial\tilde\beta}\beta(t;\tau,\tilde\beta),\\ \frac{d}{dt}\left[x_\beta(t,\beta(t;\tau,\tilde\beta))\cdot\frac{\partial}{\partial\tilde\beta}\beta(t;\tau,\tilde\beta)\right]=(u^m)_\beta(t,\beta(t;\tau,\tilde\beta))\cdot\frac{\partial}{\partial\tilde\beta}\beta(t;\tau,\tilde\beta),\\ \frac{d}{dt}\left[u_\beta(t,\beta(t;\tau,\tilde\beta))\cdot\frac{\partial}{\partial\tilde\beta}\beta(t;\tau,\tilde\beta)\right]=P_{x,\beta}(\beta(t;\tau,\tilde\beta))\cdot\beta(t;\tau,\tilde\beta).\end{array}\right. \tag{109}$$

The quantities within square brackets on the left hand sides of (109) are absolutely continuous. By the above system and using Lemma 4.5, along a good characteristic, we deduce

$$\begin{cases} \frac{d}{dt}x_\beta + G_\beta x_\beta &= mu^{m-1}u_\beta, \\ \frac{d}{dt}u_\beta + G_\beta u_\beta &= \Big(\frac{m}{2}u^{m-1}u_x^2 + \frac{m(m+3)}{2(m+1)}u^{m+1} - P\Big)x_\beta \\ &= \Big(\frac{m(m+3)}{2(m+1)}u^{m+1} + \frac{m}{2}u^{m-1}(\frac{1}{x_\beta} - 1) - P\Big)x_\beta \\ &= \Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P - \frac{m}{2}u^{m-1}\Big)x_\beta + \frac{m}{2}u^{m-1}. \end{cases} \tag{110}$$

Step 3. Now we return to the original coordinates (t, x) and deduce an evolution equation for u_x along a "good" characteristic curve.

Fixed a point $(\tau, \tilde{x})$ for $\tau \notin \mathcal{N}$. Suppose that $\tilde{x}$ is a Lebesgue point for the map $x \mapsto u_x(\tau, x)$. Let $\tilde{\beta}$ be such that $\tilde{x} = x(\tau, \tilde{\beta})$. Assume that $t \mapsto \beta(t; \tau, \tilde{\beta})$ is a good characteristic, so that **(GC)** holds. Notice that

$$u_x^2(\tau, x) = \frac{1}{x_\beta(\tau, \tilde{\beta})} - 1 \geq 0, \;\; x_\beta(\tau, \tilde{\beta}) > 0.$$

If $x_\beta > 0$, along the characteristic though $(\tau, \tilde{x})$, it follows that

$$u_x\Big(t, x(t, \beta(t; \tau, \tilde{\beta}))\Big) = \frac{u_\beta(t, \beta(t; \tau, \tilde{\beta}))}{x_\beta(t, \beta(t; \tau, \tilde{\beta}))}. \tag{111}$$

From (110), we obtain that the map $t \mapsto u_x(t, x(t, \beta(t; \tau\tilde{\beta})))$ is absolutely continuous (as long as $x_\beta \neq 0$) and satisfies

(112)

$$\begin{aligned} &\frac{d}{dt}u_x(t, x(t, \beta(t; \tau\tilde{\beta}))) = \frac{d}{dt}(\frac{u_\beta}{x_\beta}) \\ &= \frac{x_\beta\{\Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P - \frac{m}{2}u^{m-1}\Big)x_\beta + \frac{m}{2}u^{m-1}x_\beta - u_\beta G_\beta\} - u_\beta\{mu^{m-1}u_\beta - x_\beta G_\beta\}}{x_\beta^2} \\ &= \frac{m(m+3)}{2(m+1)}u^{m+1} - P - \frac{m}{2}u^{m-1} + \frac{mu^{m-1}}{2x_\beta} - \frac{u_\beta G_\beta}{x_\beta} - \frac{mu^{m-1}u_\beta^2}{x_\beta^2} + \frac{u_\beta G_\beta}{x_\beta} \\ &= \frac{m(m+3)}{2(m+1)}u^{m+1} - P - \frac{m}{2}u^{m-1} + \frac{mu^{m-1}}{2x_\beta} - \frac{mu^{m-1}u_\beta^2}{x_\beta^2}. \end{aligned}$$

Thus, for $x_\beta > 0$, one has

$$\begin{aligned} &\frac{d}{dt}\arctan u_x(t, x(t, \beta(t; \tau\tilde{\beta}))) = \frac{1}{1 + u_x^2} \cdot \frac{d}{dt}u_x \\ &= \Big(\frac{m(m+3)}{2(m+1)}u^{m+1} - P - \frac{m}{2}u^{m-1} + \frac{mu^{m-1}}{2x_\beta} - \frac{mu^{m-1}u_\beta^2}{x_\beta^2}\Big)x_\beta \\ &= (\frac{m(m+3)}{2(m+1)}u^{m+1} - P - \frac{m}{2}u^{m-1})x_\beta + \frac{mu^{m-1}}{2} - \frac{mu^{m-1}u_\beta^2}{x_\beta} \\ &= (\frac{m(m+3)}{2(m+1)}u^{m+1} - mu^{m-1}u_x^2 - P - \frac{m}{2}u^{m-1})x_\beta + \frac{mu^{m-1}}{2}. \end{aligned} \tag{113}$$

Step 4. Introduce the function

$$\theta \doteq \begin{cases} 2\arctan u_x \text{ if } 0 < x_\beta \leq 1, \\ \pi \text{ if } x_\beta = 0. \end{cases} \tag{114}$$

This yields

$$x_\beta = \frac{1}{1+u_x^2} = \cos^2\frac{\theta}{2},\ \frac{u_x}{1+u_x^2} = \frac{1}{2}\sin\theta,\ \frac{u_x^2}{1+u_x^2} = \sin^2\frac{\theta}{2}. \tag{115}$$

where θ can be seen as a map taking values in the unit circle $\Omega \doteq [-\pi,\pi]$ with endpoints identified. We say that this map $t \mapsto \theta(t) \doteq \theta(t, x(t, \beta(t;\tau\tilde{\beta})))$ is absolutely continuous and satisfies

$$\frac{d}{dt}\theta(t) = \theta_t = \left(\frac{m(m+3)}{m+1}u^{m+1} - 2P\right)\cos^2\frac{\theta}{2} - \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}, \tag{116}$$

along each good characteristic. In fact, for simplicity, denote by $x_\beta(t), u_\beta(t)$ and $u_x(t) = \frac{u_\beta(t)}{x_\beta(t)}$ the values of x_β, u_β and u_x along this particular characteristic. From (GC), for a.e. $t > 0$, we have $x_\beta(t) > 0$. Assume that τ is any time where $x_\beta(\tau) > 0$, we find a neighborhood $I = [\tau-\delta, \tau+\delta]$ satisfies $x_\beta(\tau) > 0$ on I. It follows from (113) and (115) that $v = 2\arctan(\frac{u_\beta}{x_\beta})$ is absolutely continuous restricted to I and satisfies (116). To prove our previous conclusion, we need to prove that $t \mapsto v(t)$ is continuous on the null set $\mathcal{N}$ of times at $x_\beta(t) = 0$. Let $x_\beta(t_0) = 0$. By the following identity

$$u_x^2(t) = \frac{1 - x_\beta(t)}{x_\beta(t)}, \tag{117}$$

which is valid as long as $x_\beta > 0$, we have $u_x^2 \to \infty$ as $t \to t_0$ and $x_\beta(t) \to 0$, which denotes $\theta(t) = 2\arctan u_x(t) \to \pm\pi$. Since we identify the points $\pm\pi$ in Ω , so we establish the continuity of θ for all $t \geq 0$. This completes our conclusion.

Step 5. If $u = u(t,x)$ is a conservation solution, in terms of the variables t, β, the quantities x, u, θ, we deduce

$$\begin{cases} \frac{d}{dt}\beta(t,\tilde{\beta}) = G(t, \beta(t,\tilde{\beta})), \\ \frac{d}{dt}x(t,\beta(t,\tilde{\beta})) = u^m(t,\beta(t,\tilde{\beta})), \\ \frac{d}{dt}u(t,\beta(t,\tilde{\beta})) = -P_x(t,\beta(t,\tilde{\beta})), \\ \frac{d}{dt}v(t,\beta(t,\tilde{\beta})) = \left(\frac{m(m+3)}{m+1}u^{m+1} - 2P\right)\cos^2\frac{\theta}{2} - \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}. \end{cases} \tag{118}$$

Recalling the definition of P and G, in term of the variable β, the function P and P_x have representations as follows

$$\begin{aligned} P(x(\beta)) = {} & \frac{1}{2}\int_{-\infty}^{\infty} \exp\left\{-\left|\int_\beta^{\beta'} \cos^2\frac{v(s)}{2}ds\right|\right\} \\ & \cdot\left[\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)(\beta')\right]d\beta', \end{aligned} \tag{119}$$

$$\begin{aligned} P_x(x(\beta)) = {} & \frac{1}{2}\left(\int_\xi^{\infty} - \int_{-\infty}^{\xi}\right) \exp\left\{-\left|\int_\xi^{\beta'} \cos^2\frac{v(s)}{2}ds\right|\right\} \\ & \cdot\left[\left(\frac{m(m+3)}{2(m+1)}u^{m+1}\cos^2\frac{\theta}{2} + \frac{m}{2}u^{m-1}\sin^2\frac{\theta}{2}\right)(\beta')\right]d\beta'. \end{aligned} \tag{120}$$

For every $\tilde{\beta} \in \mathbb{R}$, the initial condition is

$$\begin{cases} \beta(0,\tilde{\beta}) = \tilde{\beta}, \\ x(0,\tilde{\beta}) = x(0,\tilde{\beta}), \\ u(0,\tilde{\beta}) = u_0(x(0,\tilde{\beta})), \\ \theta(0,\tilde{\beta}) = 2\arctan u_{0,x}(x(0,\tilde{\beta})). \end{cases} \tag{121}$$

Since the Lipschitz continuity of coefficients, the Cauchy problem (118), (121) has a unique solution with initial data condition (121), which is globally defined for all $t \geq 0, x \in \mathbb{R}$.

Step 6. Suppose that $u, \bar{u}$ are two conservative solutions of equation (3) with the same initial data $u_0 \in H^1(\mathbb{R})$. For a.e. $t > 0$, the Lipschitz continuous maps $\beta \mapsto x(t,\beta)$, $\beta \mapsto \bar{x}(t,\beta)$ are strictly increasing. Therefore, the above maps have continuous inverses, i.e. $x \mapsto \beta^*(t,x)$, $x \mapsto \bar{\beta}^*(t,x)$. In summary, the map $(t,\beta) \mapsto\mapsto (x,u,\theta)(t,\beta)$ is uniquely resolved by the initial data u_0. Accordingly,

$$x(t,\beta) = \bar{x}(t,\beta),\ u(t,\beta) = \bar{u}(t,\beta).$$

In turn, for a.e. $t > 0$, we conclude

$$u(t,x) = u(t,\beta^*(t,x)) = \bar{u}(t,\bar{\beta}^*(t,x)) = \bar{u}(t,x).$$

□

5. Generic regularity

To prove the singularities of the solution for u to (3) in $t-x$ plane, we need to consider the level sets $\{\theta(t,\xi) = \pi\}$. According to the fact that u, θ and h are smooth, the generic structure of these level sets can be studied by Thom's transversality theorem [1, 15]. Our aim is to establish several families of perturbations for a given solution of (27). To this, we introduce the following lemma.

Lemma 5.1. *Let (u,θ,h) be a smooth solution of the semilinear system for (27). Given a point $(t_0,\xi_0) \in \mathbb{R}_+ \times \mathbb{R}$.*
(1) If $(\theta,\theta_\xi,\theta_{\xi\xi})(t_0,\xi_0) = (\pi,0,0)$, then there exists a $3-$parameter family of smooth solutions $(u^\lambda,\theta^\lambda,h^\lambda)$, depending smoothly on $\lambda \in \mathbb{R}^3$, such that the following holds.
i) When $\lambda = 0 \in \mathbb{R}^3$, one goes back to the original solution, namely, $(u^0,\theta^0) = (u,\theta)$.
ii) At the point (t_0,ξ_0), when $\lambda = 0$ one can obtain

$$rank\ D_\lambda(v^\lambda, \theta^\lambda_\xi, \theta^\lambda_{\xi\xi}) = 3.$$

2) If $(\theta,\theta_\xi,\theta_t) = (\pi,0,0)$, then there exists a $3-$parameter family of smooth solutions $(u^\lambda,\theta^\lambda,h^\lambda)$, depending smoothly on $\lambda \in \mathbb{R}^3$, satisfying (i)-(ii).

Proof. Let (u,θ,h) be a smooth solution of the semilinear system (27). Given a point (t_0,ξ). Taking derivatives to the equation of θ in the semilinear system (27), one has

(122)
$$\frac{\partial}{\partial_t}\theta_\xi(t,\xi) = \left(\frac{m(m+3)}{2}u^m u_\xi - P_\xi\right)(1+\cos\theta) - \left(\frac{m(m+3)}{2(m+1)}u^{m+1} - P + \frac{m}{2}u^{m-1}\right)\sin\theta\theta_\xi - \frac{m(m-1)}{2}u^{m-2}u_\xi(1-\cos\theta),$$

(123)
$$\frac{\partial}{\partial_t}\theta_t(t,\xi) = \left(\frac{m(m+3)}{2}u^m P_x - P_t\right)(1+\cos\theta) - \left(\frac{m(m+3)}{2(m+1)}u^{m+1} - P + \frac{m}{2}u^{m-1}\right)\sin\theta\left[\left(\frac{m(m+3)}{m+1}u^{m+1} - 2P\right)\right.$$

$$\cos^2\frac{\theta}{2} - mu^{m-1}\sin^2\frac{\theta}{2}\Big] + \frac{m(m-1)}{2}u^{m-2}P_x(1-\cos\theta),$$

(124)

$$\begin{aligned}\frac{\partial}{\partial_t}\theta_{\xi\xi}(t,\xi) &= \left(\frac{m^2(m+3)}{2}u^{m-1}u_\xi^2 + \frac{m(m+3)}{2}u^m u_{\xi\xi} - P_{\xi\xi}\right)(1+\cos\theta)\\ &- \left(m(m+3)u^m u_\xi - 2P_\xi + \frac{m(m-1)}{2}u^{m-2}u_\xi\right)\sin\theta\theta_\xi\\ &- \left(\frac{m(m+3)}{2(m+1)}u^{m+1} - P + \frac{m}{2}u^{m-1}(\cos\theta\theta_\xi^2 + \sin\theta\theta_{\xi\xi})\right)\\ - \frac{m(m-1)}{2}u^{m-2}u_\xi\sin\theta\theta_\xi &- \frac{m(m-1)}{2}u^{m-2}u_\xi^2 - \frac{m(m-1)(m-2)}{2}u^{m-3}u_\xi^2,\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial_t}h_\xi(t,\xi) &= \left(\frac{m(m+3)}{2(m+1)}u^{m+1} + \frac{m}{2}u^{m-1} - P\right)\sin\theta h_\xi)\\ &\left(\frac{m(m+3)}{2(m+1)}u^{m+1} + \frac{m}{2}u^{m-1} - P\right)\cos\theta\theta_\xi h \qquad (125)\\ &\left(\frac{m(m+3)}{2}u^m u_\xi + \frac{m(m-1)}{2}u^{m-2}u_\xi - P_\xi\right)\sin\theta h,\end{aligned}$$

with

$$u_\xi = \frac{1}{2}\sin\theta h, \quad u_{\xi\xi} = \frac{1}{2}\sin\theta h_\xi + \frac{1}{2}\cos\theta h\theta_\xi. \qquad (126)$$

We consider the families $(\bar{u}^\lambda, \bar{\theta}^\lambda, \bar{h}^\lambda)$ of perturbations of the initial data as

$$\bar{u}^\lambda = \bar{u}(\xi) + \sum_{i=1,2,3}\lambda_i U_i(\xi), \qquad (127)$$

$$\bar{\theta}^\lambda = \bar{\theta}(\xi) + \sum_{i=1,2,3}\lambda_i \Theta_i(\xi), \qquad (128)$$

$$\bar{h}^\lambda = \bar{h}(\xi) + \sum_{i=1,2,3}\lambda_i H_i(\xi). \qquad (129)$$

Together with (27), ((122), (124) and (125) form a complete system. □

Next, the following lemma will be used to get the rank which we desired.

Lemma 5.2 ([5]). *Consider the following ODE system*

$$\frac{d}{d_t}u^\epsilon = g(u^\epsilon), \quad u^\epsilon(0) = u_0 + \epsilon_1 v_1 + \cdots + \epsilon_k v_k, \qquad (130)$$

where $u^\epsilon(t) : \mathbb{R} \to \mathbb{R}^n$ and g is a Lipschitz continuous function. The system is well-posed in $[0,T)$. If the matrix

$$D_\epsilon u_0^\epsilon = (v_1, v_2, \cdots, v_k) \in \mathbb{R}^{n\times k}, \qquad (131)$$

and the rank of this matrix is

$$rank(D_\epsilon u_0^\epsilon) = l. \qquad (132)$$

Then $rank(D_\epsilon u^\epsilon(t)) = l$ for any $t \in [0,T)$.

Proof Lemma 5.1. By (27), (122) and (124), we achieve an ODE system as follows,

$$\frac{\partial}{\partial_t}\begin{pmatrix} u \\ \theta \\ h \\ \theta_\xi \\ \theta_{\xi\xi} \end{pmatrix} =$$

$$\begin{pmatrix} -P_x \\ \left(\frac{m(m+3)}{m+1}u^{m+1}-2P\right)\cos^2\frac{\theta}{2}-mu^{m-1}\sin^2\frac{\theta}{2} \\ \left(\frac{m(m+3)}{2(m+1)}u^{m+1}+\frac{m}{2}u^{m-1}-P\right)\sin\theta\cdot h \\ \left(\frac{m(m+3)}{2}u^m u_\xi - P_\xi\right)(1+\cos\theta) \\ -\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P+\frac{m}{2}u^{m-1}\right)\sin\theta\theta_\xi \\ -\frac{m(m-1)}{2}u^{m-2}u_\xi(1-\cos\theta) \\ \left(\frac{m^2(m+3)}{2}u^{m-1}u_\xi^2+\frac{m(m+3)}{2}u^m u_{\xi\xi}-P_{\xi\xi}\right)(1+\cos\theta) \\ -\left(m(m+3)u^m u_\xi - 2P_\xi+\frac{m(m-1)}{2}u^{m-2}u_\xi\right)\sin\theta\theta_\xi \\ -\left(\frac{m(m+3)}{2(m+1)}u^{m+1}-P+\frac{m}{2}u^{m-1}(\cos\theta\theta_\xi^2+\sin\theta\theta_{\xi\xi})\right) \\ -\frac{m(m-1)}{2}u^{m-2}u_\xi\sin\theta\theta_\xi-\frac{m(m-1)}{2}u^{m-2}u_\xi^2-\frac{m(m-1)(m-2)}{2}u^{m-3}u_\xi^2 \end{pmatrix}. \tag{133}$$

Then we establish a family of solutions $(\bar{u}^\lambda, \bar{v}^\lambda, \bar{h}^\lambda)$ of perturbations with the initial data given in (127)-(129). Differentiating w.r.t. λ, one has

$$\frac{\partial}{\partial_t}\begin{pmatrix} D_\lambda\bar{u}^\lambda \\ D_\lambda\bar{\theta}^\lambda \\ D_\lambda\bar{h}^\lambda \\ D_\lambda\bar{\theta}_\xi^\lambda \\ D_\lambda\bar{\theta}_{\xi\xi}^\lambda \end{pmatrix} = \begin{pmatrix} D_\lambda g_1^\lambda \\ D_\lambda g_2^\lambda \\ D_\lambda g_3^\lambda \\ D_\lambda g_4^\lambda \\ D_\lambda g_5^\lambda \end{pmatrix}, \tag{134}$$

where $g_1^\lambda, \cdots g_5^\lambda$ are the perturbation of the right-hand-side of (134). Therefore, we have

$$\begin{aligned} &\frac{\partial}{\partial_t}\begin{pmatrix} D_\lambda\bar{u}^\lambda \\ D_\lambda\bar{\theta}^\lambda \\ D_\lambda\bar{h}^\lambda \\ D_\lambda\bar{\theta}_\xi^\lambda \\ D_\lambda\bar{\theta}_{\xi\xi}^\lambda \end{pmatrix} \\ &= \begin{pmatrix} D_u g_1^\lambda & D_\theta g_1^\lambda & D_h g_1^\lambda & D_{\theta_\xi} g_1^\lambda & D_{\theta_{\xi\xi}} g_1^\lambda \\ D_u g_2^\lambda & D_\theta g_2^\lambda & D_h g_2^\lambda & D_{\theta_\xi} g_2^\lambda & D_{\theta_{\xi\xi}} g_2^\lambda \\ D_u g_3^\lambda & D_\theta g_3^\lambda & D_h g_3^\lambda & D_{\theta_\xi} g_3^\lambda & D_{\theta_{\xi\xi}} g_3^\lambda \\ D_u g_4^\lambda & D_\theta g_4^\lambda & D_h g_4^\lambda & D_{\theta_\xi} g_4^\lambda & D_{\theta_{\xi\xi}} g_4^\lambda \\ D_u g_5^\lambda & D_\theta g_5^\lambda & D_h g_5^\lambda & D_{\theta_\xi} g_5^\lambda & D_{\theta_{\xi\xi}} g_5^\lambda \end{pmatrix} \cdot \begin{pmatrix} D_{\lambda_1}\bar{u}^\lambda & D_{\lambda_2}\bar{u}^\lambda & D_{\lambda_3}\bar{u}^\lambda \\ D_{\lambda_1}\bar{\theta}^\lambda & D_{\lambda_2}\bar{\theta}^\lambda & D_{\lambda_3}\bar{\theta}^\lambda \\ D_{\lambda_1}\bar{h}^\lambda & D_{\lambda_2}\bar{h}^\lambda & D_{\lambda_3}\bar{h}^\lambda \\ D_{\lambda_1}\bar{\theta}_\xi^\lambda & D_{\lambda_2}\bar{\theta}_\xi^\lambda & D_{\lambda_3}\bar{\theta}_\xi^\lambda \\ D_{\lambda_1}\bar{\theta}_{\xi\xi}^\lambda & D_{\lambda_2}\bar{\theta}_{\xi\xi}^\lambda & D_{\lambda_3}\bar{\theta}_{\xi\xi}^\lambda \end{pmatrix}. \end{aligned} \tag{135}$$

Based on Lemma 5.2, we need to explain the Lipschitz continuity of g_i^λ $(i = 1\cdots 5)$. Since the function (u, θ, h) is smooth, we only need to prove the Lipschitz continuity of the nonlocal term of P and P_x. In Section 3, we have obtained the boundedness of

$|\frac{\partial P}{\partial u}|, |\frac{\partial P}{\partial \theta}|, |\frac{\partial P}{\partial h}|, |\frac{\partial P_x}{\partial u}|, |\frac{\partial P_x}{\partial \theta}|, |\frac{\partial P_x}{\partial h}|$. Choosing suitable perturbation $\Theta_i(i = 1, 2, 3)$, when $\lambda = 0$, we have

$$rank D_\lambda \begin{pmatrix} \theta \\ \theta_\xi \\ \theta_{\xi\xi} \end{pmatrix} = 3. \tag{136}$$

The system (27) combine with (122), (123) forms a complete system. By choosing suitable perturbation $\Theta_i(i = 1, 2, 3)$ observe that

$$rank D_\lambda \begin{pmatrix} \theta \\ \theta_\xi \\ \theta_t \end{pmatrix} = 3, \tag{137}$$

while $\lambda = 0$. □

Next, we are going to investigate smooth solutions to the semi-linear system (27), and determine the generic structure of level sets $\{\theta(t, \xi) = \pi\}$. We give the key lemma to prove Theorem 1.4.

Lemma 5.3 ([5]). *Given a compact domain*

$$\mathcal{D} := \{(t, \xi); 0 \le t \le T, |\xi| \le M\},$$

let $\mathcal{W}$ be the family of all C^2 solutions (u, θ, h) for the semi-linear system (27), with $h > 0$ for all $(t, \xi) \in [0, T] \times \mathbb{R}^+$. Moreover, let $\mathcal{W}' \subset \mathcal{W}$ be the subfamily of all solutions (u, θ, h), such that for $(t, \xi) \in \mathcal{D}$, the value

$$(\theta, \theta_\xi, \theta_{\xi\xi}) = (\pi, 0, 0), \ (\theta, \theta_\xi, \theta_t) = (\pi, 0, 0) \tag{138}$$

cannot be obtained. Then $\mathcal{W}'$ is a relatively open and dense subset of $\mathcal{W}$, in the topology induced by $C^2(\mathcal{D})$.

Proof of Theorem 1.4. **Step 1.** For convenience, we denote the space

$$\mathcal{M} := C^3(\mathbb{R}^+) \cap H^1(\mathbb{R}^+),$$

with the norm

$$\|u_0\|_{\mathcal{M}} := \|u_0\|_{C^3} + \|u_0\|_{H^1}.$$

Given a initial data $u_0^* \in \mathcal{M}$, and we introduce the open ball

$$B_\delta := \{u_0 \in \mathcal{M}; \|u_0 - u_0^*\|_{\mathcal{M}} < \delta\}.$$

By the definition of the space of $\mathcal{M}$, it follows that $u_0(x) \to 0$ and $u_{0,x}(x) \to 0$. Therefore, we choose $\kappa > 0$ big enough such that $u_0(x)$ and $u_{0x}(x)$ are uniformly bounded for $|x| > \kappa$. By a standard comparison argument on the domain $\{(t, x); t \in [0, T], |x| \ge \kappa + \|u\|_{L^\infty}^m\}$, we see that the partial derivative u_x is uniformly bounded. This implies the singularity of $u(t, x)$ in set $[0, T] \times \mathbb{R}$ only appears on the compact set $\Delta := [0, T] \times [-r - \|u\|_{L^\infty}^m T, r + \|u\|_{L^\infty}^m T]$, where $\|u\|_{L^\infty} := \max\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}\}$. In (t, x) plane, we take a domain $\mathcal{D}$ such that $\Delta \subset \mathcal{J}(\mathcal{D})$, where $\mathcal{J}$ is a map from (t, ξ) to $(t, x(t, \xi))$.

We define the subset $\Gamma \subset B_\delta$ as follows: $u_0 \in \Gamma$ if $u_0 \in B_\delta$ and for the corresponding solution (u, θ, h) of (27), the values (138) are never attained for $(t, x) \in \Delta$.

Step 2. We now claim the set Γ is open, in the topology of C^3. Take a sequence of initial data $(u_0^n)_{n\ge1}$ such that the sequence converges to u_0. From the definition of Γ, there exists a point (t^n, ξ^n) such that

$$(\theta^n, \theta^n_\xi, \theta^n_{\xi\xi})(t^n, \theta^n) = (\pi, 0, 0), \ (t^n, x^n(t^n, \xi^n)) \in \Delta$$

for all $n \geq 1$. Since the domain Δ is compact, we can take a subsequence (t^n, ξ^n), which converges to some point (t, ξ). It follows from continuity that

$$(\theta, \theta_\xi, \theta_{\xi\xi})(t, \xi = (\pi, 0, 0),\ (t, x(t, \xi)) \in \Delta.$$

This denotes $u_0 \notin \Gamma$. Using similar procedure, other case $(\theta, \theta_\xi, \theta_t) = (\pi, 0, 0)$ can be proved. So Γ is open.

Step 3. We explain that Γ is dense in B_δ. Let $u_0 \in B_\delta$, by a small perturbation, we assume $u_0 \in C^\infty$. From Lemma 5.3, we construct a sequence of solutions (u^n, θ^n, h^n) of (3.1), such that

i) for every $n \geq 1$, the values in (138) are never attained for any $(t, \xi) \in \mathcal{D}$.

ii) The $C^k(k > 1)$ norm of the difference satisfies

$$\lim_{n\to\infty} \|(u^n - u, \theta^n - \theta, h^n - h, x^n - x)\|_{C^k(I)} = 0,$$

for every bounded set $I \subset [0, T] \times \mathbb{R}^+$. When $t = 0$, the corresponding sequence of initial value satisfies

$$\lim_{n\to\infty} \|u_0^n - u_0\|_{C^k[a,b]} = 0$$

for every bounded set $[a, b] \in \mathbb{R}^+$.

Introducing a cutoff function

$$p(x) = \begin{cases} 1, & if|x| \leq r, \\ 0, & if|x| \geq r+1, \end{cases} \tag{139}$$

where $r \gg \kappa + \|u\|_{L^\infty}^{m-1} T$ is large enough. For every $n \geq 1$, let the initial data

$$\tilde{u}_0^v := p u_0^n + (1 - p) u_0.$$

We obtain

$$\lim_{n\to\infty} \|\tilde{u}_0^n - u_0\|_{\mathcal{M}} = 0.$$

Furthermore, choosing $r > 0$ sufficiently large for any $(t, x) \in \Delta$, we have

$$\tilde{u}^n(t, x) = u^n(t, x).$$

It is obvious that $\tilde{u}^n(t, x)$ is C^2 on the outer domain. Therefore, $\tilde{u}^n(t, x) \in \Gamma$ for every $n \geq 1$ sufficiently large. Thus Γ is dense in B_δ.

Step 4. Finally, we prove that, for every initial data $u_0 \in \Gamma$, the solution of (3) is piecewise C^2 on the domain $[0, T] \times \mathbb{R}^+$. By previous argument, we only study the singularity of u on the inner domain Δ. For every point $(t_0, \xi_0) \in \mathcal{D}$, two cases can appear.

Case I. $\theta(t_0, \xi_0) \neq \pi$. By the coordinate change $x_\xi = h \cos^2 \frac{\theta}{2}$, we know that the map $(t, \xi) \mapsto (t, x)$ is locally invertible in a neighborhood of (t_0, ξ_0). Then we conclude that the function u is C^2 in a neighborhood of the point $(t_0, x(t_0, \xi_0))$.

Case II. $\theta(t_0, \xi_0) = \pi$. From (138), $\theta_t(t_0, \xi_0) \neq 0$ or $\theta_\xi(t_0, \xi_0) \neq 0$. Thanks to the implicit function theorem, we derive that the set

$$\mathcal{W}^\theta := \{(t, \xi) \in \Delta; \theta(t, \xi) = \pi\}$$

is the union of finitely many C^2 curves. □

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