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A CONJECTURE ON CLUSTER AUTOMORPHISMS OF CLUSTER ALGEBRAS

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Abstract. A cluster automorphism is a $\mathbb{Z}$-algebra automorphism of a cluster algebra $\mathcal{A}$ satisfying that it sends a cluster to another and commutes with mutations. Chang and Schiffler conjectured that a cluster automorphism of $\mathcal{A}$ is just a $\mathbb{Z}$-algebra homomorphism of a cluster algebra sending a cluster to another. The aim of this article is to prove this conjecture.

1. Introduction

Cluster algebras were invented by Fomin and Zelevinsky in a series of papers [9, 2, 10, 11]. A cluster algebra is a $\mathbb{Z}$-subalgebra of an ambient field $\mathcal{F} = \mathbb{Q}(u_1, \cdots, u_n)$ generated by certain combinatorially defined generators (i.e., *cluster variables*), which are grouped into overlapping *clusters*. Many relations between cluster algebras and other branches of mathematics have been discovered, for example, Poisson geometry, discrete dynamical systems, higher Teichmüller spaces, representation theory of quivers and finite-dimensional algebras.

We first recall the definition of cluster automorphisms, which were introduced by Assem, Schiffler and Shamchenko in [1].

Definition 1.1 ([1]). Let $\mathcal{A} = \mathcal{A}(\mathbf{x}, B)$ be a cluster algebra, and $f : \mathcal{A} \to \mathcal{A}$ be an automorphism of $\mathbb{Z}$-algebras. f is called a **cluster automorphism** of $\mathcal{A}$ if there exists another seed $(\mathbf{z}, B')$ of $\mathcal{A}$ such that

(1) $f(\mathbf{x}) = \mathbf{z}$;
(2) $f(\mu_x(\mathbf{x})) = \mu_{f(x)}(\mathbf{z})$ for any $x \in \mathbf{x}$.

Cluster automorphisms and their related groups were studied by many authors, and one can refer to [6, 7, 8, 14, 13, 4, 5, 16] for details.

The following very insightful conjecture on cluster automorphisms is by Chang and Schiffler, which suggests that we can weaken the conditions in Definition 1.1. In particular, it suggests that the second condition in Definition 1.1 can be obtained from the first one and the assumption that f is a $\mathbb{Z}$-algebra homomorphism.

Conjecture 1. [5, Conjecture 1] *Let $\mathcal{A}$ be a cluster algebra, and $f : \mathcal{A} \to \mathcal{A}$ be a $\mathbb{Z}$-algebra homomorphism. Then f is cluster automorphism if and only if there exist two clusters $\mathbf{x}$ and $\mathbf{z}$ such that $f(\mathbf{x}) = \mathbf{z}$.*

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The following is our main result, which affirms the Conjecture 1.

Theorem 3.6 *Let $\mathcal{A}$ be a cluster algebra, and $f : \mathcal{A} \to \mathcal{A}$ be a $\mathbb{Z}$-algebra homomorphism. Then f is a cluster automorphism if and only if there exist two clusters $\mathbf{x}$ and $\mathbf{z}$ such that $f(\mathbf{x}) = \mathbf{z}$.*

2. Preliminaries

In this section, we recall basic concepts and important properties of cluster algebras. In this paper, we focus on cluster algebras without coefficients (that is, with trivial coefficients). For a positive integer n, we will always denote by $[1, n]$ the set $\{1, 2, \dots, n\}$.

Recall that B is said to be **skew-symmetrizable** if there exists an positive diagonal integer matrix D such that BD is skew-symmetric.

Fix an ambient field $\mathcal{F} = \mathbb{Q}(u_1, u_2, \dots, u_n)$. A **labeled seed** is a pair $(\mathbf{x}, B)$, where $\mathbf{x}$ is an n-tuple of free generators of $\mathcal{F}$, and B is an $n \times n$ skew-symmetrizable integer matrix. For $k \in [1, n]$, we can define another pair $(\mathbf{x}', B') = \mu_k(\mathbf{x}, B)$, where

(1) $\mathbf{x}' = (x'_1, \dots, x'_n)$ is given by
$$x'_k = \frac{\prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]_+}}{x_k}$$
and $x'_i = x_i$ for $i \neq k$;

(2) $B' = \mu_k(B) = (b'_{ij})_{n \times n}$ is given by
$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+, & \text{otherwise.} \end{cases}$$

where $[x]_+ = \max\{x, 0\}$. The new pair $(\mathbf{x}', B') = \mu_k(\mathbf{x}, B)$ is called the **mutation** of $(\mathbf{x}, B)$ at k. We also denote $B' = \mu_k(B)$.

It can be seen that $(\mathbf{x}', B')$ is also a labeled seed and μ_k is an involution.

Let $(\mathbf{x}, B)$ be a labeled seed. $\mathbf{x}$ is called a **labeled cluster**, elements in $\mathbf{x}$ are called **cluster variables**, and B is called an **exchange matrix**. The **unlabeled seeds** are obtained by identifying labeled seeds that differ from each other by simultaneous permutations of the components in $\mathbf{x}$, and of the rows and columns of B. We will refer to unlabeled seeds and unlabeled clusters simply as **seeds** and **clusters** respectively, when there is no risk of confusion.

Lemma 2.1 ([2]). *Let B be an $n \times n$ skew-symmetrizable matrix. Then $\mu_k(B) = (J_k + E_k)B(J_k + F_k)$, where*

(1) J_k denotes the diagonal $n \times n$ matrix whose diagonal entries are all $1'$s, except for -1 in the k-th position;
(2) E_k is the $n \times n$ matrix whose only nonzero entries are $e_{ik} = [-b_{ik}]_+$;
(3) F_k is the $n \times n$ matrix whose only nonzero entries are $f_{kj} = [b_{kj}]_+$.

Definition 2.2 ([9, 11]). (1) Two labeled seeds $(\mathbf{x}, B)$ and $(\mathbf{x}', B')$ are said to be **mutation equivalent** if $(\mathbf{x}', B')$ can be obtained from $(\mathbf{x}, B)$ by a sequence of mutations;

(2) Let T_n be an n-regular tree and valencies emitting from each vertex are labelled by $1, 2, \dots, n$. A **cluster pattern** is an n-regular tree T_n such that for each vertex $t \in \mathrm{T}_n$, there is a labeled seed $\Sigma_t = (\mathbf{x}_t, B_t)$ and for

each edge labelled by k, two labeled seeds in the endpoints are obtained from each other by seed mutation at k. We always write

$$\mathbf{x}_t = (x_{1;t}, x_{2;t}, \ldots, x_{n;t}), \quad B_t = (b_{ij}^t).$$

The cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, B_{t_0})$ associated with the initial seed $(\mathbf{x}_{t_0}, B_{t_0})$ is a $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by cluster variables appeared in $\mathrm{T}_n(\mathbf{x}_{t_0}, B_{t_0})$, where $\mathrm{T}_n(\mathbf{x}_{t_0}, B_{t_0})$ is the cluster pattern with $(\mathbf{x}_{t_0}, B_{t_0})$ lying in the vertex $t_0 \in \mathrm{T}_n$.

Theorem 2.3 (Laurent phenomenon and positivity [11, 15, 12])**.** *Let $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, B_{t_0})$ be a cluster algebra. Then each cluster variable $x_{i,t}$ is contained in*

$$\mathbb{Z}_{\geq 0}[x_{1;t_0}^{\pm 1}, x_{2;t_0}^{\pm 1}, \ldots, x_{n;t_0}^{\pm 1}].$$

3. The proof of main result

In this section, we will give our main result, which affirms the Conjecture 1.

Lemma 3.1. *Let $0 \neq B$ be a skew-symmetrizable integer matrix. If B' is obtained from B by a sequence of mutations and $B = aB'$ for some $a \in \mathbb{Z}$, then $a = \pm 1$ and $B = \pm B'$.*

Proof. Since B' is obtained from B by a sequence of mutations, there exist integer matrices E and F such that $B' = EBF$, by Lemma 2.1. If $B = aB'$, then we get $B' = aE(B')F$. Also, we can have

$$B' = aE(B')F = a^2E^2(B')F^2 = \cdots = a^sE^s(B')F^s,$$

where $s \geq 0$. By $B \neq 0$, we know $a \neq 0$. Thus $\frac{1}{a^s}B' = E^s(B')F^s$ holds for any $s \geq 0$.

Assume by contradiction that $a \neq \pm 1$, then when s is large enough, $\frac{1}{a^s}B'$ will not be an integer matrix. But $E^s(B')F^s$ is always an integer matrix. This is a contradiction. So we must have $a = \pm 1$ and thus $B = \pm B'$. $\square$

A square matrix A is **decomposable** if there exists a permutation matrix P such that PAP^{T} is a block-diagonal matrix, and **indecomposable** otherwise.

Lemma 3.2. *Let $0 \neq B$ be an indecomposable skew-symmetrizable matrix. If B' is obtained from B by a sequence of mutations and $B = B'A$ for some integer diagonal matrix $A = diag(a_1, \cdots, a_n)$, then $A = \pm I_n$ and $B = \pm B'$.*

Proof. If there exists i_0 such that $a_{i_0} = 0$, then the i_0-th column vector of B is zero, by $B = B'A$. This contradicts that B is indecomposable and $B \neq 0$. So each a_{i_0} is nonzero for $i_0 = 1, \cdots, n$.

Let $D = diag(d_1, \cdots, d_n)$ be a skew-symmetrizer of B. By $B = B'A$ and $AD = DA$, we know that

$$BD = B'AD = (B'D)A.$$

By the definition of mutation, we know that D is also a skew-symmetrizer of $\mu_k(B)$, $k = 1, \cdots, n$. Since B' is obtained from B by a sequence of mutations, we get that D is a skew-symmetrizer of B'. Namely, we have that both $B'D$ and $BD = (B'D)A$ are skew-symmetric. Since $0 \neq B$ is indecomposable, we must have $a_1 = \cdots = a_n$. So $A = aI_n$ for some $a \in \mathbb{Z}$, and $B = aB'$. Then by Lemma 3.1, we can get $A = \pm I_n$ and $B = \pm B'$. $\square$

Lemma 3.3. *Let $B = diag(B_1, \cdots, B_s)$, where each B_i is a nonzero indecomposable skew-symmetrizable matrix of size $n_i \times n_i$. If B' is obtained from B by a sequence of mutations and $B = B'A$ for some integer diagonal matrix $A = diag(a_1, \cdots, a_n)$, then $a_j = \pm 1$ for $j = 1, \cdots, n$.*

Proof. By the definition of mutation, we know that B' has the form of $B' = diag(B'_1, \cdots, B'_s)$, where each B'_i is obtained from B_i by a sequence of mutations. We can write A as a block-diagonal matrix $A = diag(A_1, \cdots, A_s)$, where A_i is a $n_i \times n_i$ integer diagonal matrix. By $B = B'A$, we know that $B_i = B'_i A_i$. Then by Lemma 3.2, we have $A_i = \pm I_{n_i}$ and $B_i = \pm B'_i$ for $i = 1, \cdots, s$. In particular, we get $a_j = \pm 1$ for $j = 1, \cdots, n$. □

Lemma 3.4. *Let $\mathcal{A} = \mathcal{A}(\mathbf{x}, B)$ be a cluster algebra, and $f : \mathcal{A} \to \mathcal{A}$ be a $\mathbb{Z}$-homomorphism of $\mathcal{A}$. If there exists another seed $(\mathbf{z}, B')$ of $\mathcal{A}$ such that such that $f(\mathbf{x}) = \mathbf{z}$, then $f(\mu_x(\mathbf{x})) = \mu_{f(x)}(\mathbf{z})$ for any $x \in \mathbf{x}$.*

Proof. After permutating the rows and columns of B, it can be written as a block-diagonal matrix as follows.

$$B = diag(B_1, B_2, \cdots, B_s),$$

where B_1 is an $n_1 \times n_1$ zero matrix and B_j is nonzero indecomposable skew-symmetrizable matrix of size $n_j \times n_j$ for $j = 2, \cdots, s$.

Without loss of generality, we assume that $f(x_i) = z_i$ for $1 \le i \le n$.

Let x'_k and z'_k be the new obtained variables in $\mu_k(\mathbf{x}, B)$ and $\mu_k(\mathbf{z}, B')$. So we have

$$x_k x'_k = \prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+}, \quad \text{and} \quad z_k z'_k = \prod_{i=1}^n z_i^{[b'_{ik}]_+} + \prod_{i=1}^n z_i^{[-b'_{ik}]_+}.$$

Thus

$$\begin{aligned} f(x'_k) &= f\left(\frac{\prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+}}{x_k}\right) \\ &= \frac{\prod_{i=1}^n z_i^{[b_{ik}]_+} + \prod_{i=1}^n z_i^{[-b_{ik}]_+}}{z_k} \\ &= \frac{\prod_{i=1}^n z_i^{[b_{ik}]_+} + \prod_{i=1}^n z_i^{[-b_{ik}]_+}}{\prod_{i=1}^n z_i^{[b'_{ik}]_+} + \prod_{i=1}^n z_i^{[-b'_{ik}]_+}} z'_k. \end{aligned}$$

Note that the above expression is the expansion of $f(x'_k)$ with respect to the cluster $\mu_k(\mathbf{z})$. By

$$f(x'_k) \in f(\mathcal{A}) = \mathcal{A} \subset \mathbb{Z}[z_1^{\pm 1}, \ldots, (z'_k)^{\pm 1}, \ldots, z_n^{\pm 1}],$$

we can get

$$\frac{\prod_{i=1}^n z_i^{[b_{ik}]_+} + \prod_{i=1}^n z_i^{[-b_{ik}]_+}}{\prod_{i=1}^n z_i^{[b'_{ik}]_+} + \prod_{i=1}^n z_i^{[-b'_{ik}]_+}} \in \mathbb{Z}[z_1^{\pm 1}, \cdots, z_{k-1}^{\pm 1}, z_{k+1}^{\pm 1}, \cdots, z_n^{\pm 1}].$$

Since both $\prod_{i=1}^{n} z_i^{[b_{ik}]_+} + \prod_{i=1}^{n} z_i^{[-b_{ik}]_+}$ and $\prod_{i=1}^{n} z_i^{[b'_{ik}]_+} + \prod_{i=1}^{n} z_i^{[-b'_{ik}]_+}$ is not divided by any z_i, we actually have

$$\frac{\prod_{i=1}^{n} z_i^{[b_{ik}]_+} + \prod_{i=1}^{n} z_i^{[-b_{ik}]_+}}{\prod_{i=1}^{n} z_i^{[b'_{ik}]_+} + \prod_{i=1}^{n} z_i^{[-b'_{ik}]_+}} \in \mathbb{Z}[z_1, \cdots, z_{k-1}, z_{k+1}, \cdots, z_n].$$

So for each k, there exists an integer $a_k \in \mathbb{Z}$ such that $(b_{1k}, b_{2k}, \ldots, b_{nk})^{\mathrm{T}} = a_k(b'_{1k}, b'_{2k}, \ldots, b'_{nk})^{\mathrm{T}}$. Namely, we have $B = B'A$, where $A = diag(a_1, \cdots, a_n)$. Note that B has the form of

$$B = diag(B_1, B_2, \cdots, B_s),$$

where B_1 is an $n_1 \times n_1$ zero matrix and B_j is a nonzero indecomposable skew-symmetrizable matrix of size $n_j \times n_j$ for $j = 2, \cdots, s$. Applying Lemma 3.3 to the skew-symmetrizable matrix $diag(B_2, \cdots, B_s)$, we can get $a_j = \pm 1$ for $n_1 + 1, \cdots, n$. Since the first n_1 column vectors of both B and B' are zero vectors, we can just take $a_1 = \cdots = a_{n_1} = 1$. So for each k, we have $a_k = \pm 1$ and

$$(b_{1k}, b_{2k}, \ldots, b_{nk})^{\mathrm{T}} = a_k(b'_{1k}, b'_{2k}, \ldots, b'_{nk})^{\mathrm{T}} = \pm(b'_{1k}, b'_{2k}, \ldots, b'_{nk})^{\mathrm{T}}.$$

Hence,

$$\frac{\prod_{i=1}^{n} z_i^{[b_{ik}]_+} + \prod_{i=1}^{n} z_i^{[-b_{ik}]_+}}{\prod_{i=1}^{n} z_i^{[b'_{ik}]_+} + \prod_{i=1}^{n} z_i^{[-b'_{ik}]_+}} = 1.$$

Thus we get

$$f(x'_k) = \frac{\prod_{i=1}^{n} z_i^{[b_{ik}]_+} + \prod_{i=1}^{n} z_i^{[-b_{ik}]_+}}{\prod_{i=1}^{n} z_i^{[b'_{ik}]_+} + \prod_{i=1}^{n} z_i^{[-b'_{ik}]_+}} z'_k = z'_k.$$

So $f(\mu_x(\mathbf{x})) = \mu_{f(x)}(\mathbf{z})$ for any $x \in \mathbf{x}$. □

Lemma 3.5. *Let $\mathcal{A} = \mathcal{A}(\mathbf{x}, B) \subseteq \mathcal{F}$ be a cluster algebra, and f be an automorphism of the ambient field $\mathcal{F}$. If there exists another seed $(\mathbf{z}, B')$ of $\mathcal{A}$ such that $f(\mathbf{x}) = \mathbf{z}$ and $f(\mu_x(\mathbf{x})) = \mu_{f(x)}(\mathbf{z})$ for any $x \in \mathbf{x}$. Then*

(i) f is an automorphism of $\mathcal{A}$;

(ii) f is a cluster automorphism of $\mathcal{A}$.

Proof. (i) Since f is an automorphism of the ambient field $\mathcal{F}$, we know that f is injective.

Since f commutes with mutations, we know that f restricts to a surjection on $\mathcal{X}$, where $\mathcal{X}$ is the set of cluster variables of $\mathcal{A}$. Because $\mathcal{A}$ is generated by $\mathcal{X}$, we get that f restricts to an epimorphism of $\mathcal{A}$.

Hence, f is an automorphism of $\mathcal{A}$.

(ii) follows from (i) and the definition of cluster automorphisms. □

Theorem 3.6. *Let $\mathcal{A}$ be a cluster algebra, and $f : \mathcal{A} \to \mathcal{A}$ be a $\mathbb{Z}$-algebra homomorphism. Then f is a cluster automorphism if and only if there exist two clusters $\mathbf{x}$ and $\mathbf{z}$ such that $f(\mathbf{x}) = \mathbf{z}$.*

Proof. "*Only if part*": It follows from the definition of cluster automorphism.

"*If part*": It follows from Lemma 3.4 and Lemma 3.5. □

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