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ON THE TIME DECAY IN PHASE–LAG THERMOELASTICITY WITH TWO TEMPERATURES

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Abstract. The aim of this paper is to study the time decay of the solutions for two models of the one-dimensional phase-lag thermoelasticity with two temperatures. The first one is obtained when the heat flux vector and the inductive temperature are approximated by a second-order and first-order Taylor polynomial, respectively. In this case, the solutions decay in a slow way. The second model that we consider is obtained taking first-order Taylor approximations for the inductive thermal displacement, the inductive temperature and the heat flux. The decay is, therefore, of exponential type.

1. Introduction

The Fourier formulation to describe heat conduction is widely used by mathematicians, physicists and engineers. For this model, the heat flux is proportional to the gradient of the temperature. Unfortunately, this formulation jointly with the usual energy equation

$$c\dot{\theta} + \text{ div } \mathbf{q} = 0, \quad (c > 0) \tag{1}$$

leads to the instantaneous propagation of heat, a drawback of the model because this fact is incompatible with real observations. In the above equation $\mathbf{q} = (q_i)$ is the heat flux vector and θ is the temperature. In order to overcome this drawback, alternative proposals have been stated.

In 1995, Tzou proposed a theory in which the heat flux and the gradient of the temperature have a delay in the constitutive equations [32]. When this consideration is taken into account, it is usual to speak of phase-lag theories. In that case, the constitutive equations are given by:

$$q_i(\mathbf{x}, t + \tau_1) = -k\theta_{,i}(\mathbf{x}, t + \tau_2), \qquad k > 0, \tag{2}$$

where τ_1 and τ_2 are the delay parameters which are assumed to be positive. As usual, the notation $\theta_{,i}$ means the derivative of θ with respect to the variable x_i, and

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repeated subscripts means summation. The derivative with respect to the time is denoted using a dot over the function.

This equation suggests that the temperature gradient established across a material volume at position $\mathbf{x}$ and time $t+\tau_2$ results in a heat flux to flow at a different time $t+\tau_1$. These delays can be understood in terms of the microstructure of the material. This theory is usually known as dual-phase-lag.

In 2007, Choudhuri [7] suggested an extension of Tzou's theory in which the heat flux is described using the following constitutive equations:

$$q_i(\mathbf{x}, t+\tau_1) = -k_1 \alpha_{,i}(\mathbf{x}, t+\tau_3) - k_2 \theta_{,i}(\mathbf{x}, t+\tau_2), \tag{3}$$

where $\dot{\alpha} = \theta$. The variable α is called the *thermal displacement*, and was introduced by Green and Naghdi [10, 11]. The parameter τ_3 is another delay parameter. Choudhuri's version is commonly known as three-dual-phase-lag.

It is worth noting that both proposals, those of Tzou and Choudhuri, lead to *ill-posed* problems in the sense of Hadamard. To be more precise, it has been shown that combining equation (2) (or (3)) with the energy equation (1) leads to the existence of a sequence of elements in the point spectrum such that its real part tends to infinity [8].

These two aforementioned theories have several derivations when the heat flux and the gradients of the temperature and the thermal displacement are replaced by Taylor approximations. In fact, one can think that Choudhuri's proposal aims to recover Green and Naghdi theories when different Taylor approximations are considered. This new approach gives rise to different equations (depending on the selected Taylor approximation) to describe heat conduction that have been analyzed by many authors (see, for example, [1, 3, 12, 19, 23, 26, 27, 28, 29, 30, 31, 34]).

In order to obtain a heat conduction theory with delays but without such an explosive behavior, Quintanilla [24, 25] combined the delay parameters of Tzou and Choudhuri with the two-temperatures theory proposed by Chen and Gurtin [4, 5, 6, 33]. The basic constitutive equations read

$$q_i(\mathbf{x}, t+\tau_1) = -k_1 \beta_{,i}(\mathbf{x}, t+\tau_3) - k_2 T_{,i}(\mathbf{x}, t+\tau_2), \tag{4}$$

where $\alpha = \beta - m\Delta\beta$, $\theta = T - m\Delta T$ and m is a positive constant.

This theory has also been extended to the thermoelasticity context [24, 25]. To do so, one must assume the equation of motion

$$t_{ji,j} = \rho \ddot{u}_i, \tag{5}$$

the energy equation

$$\dot{\eta} = -q_{i,i} \tag{6}$$

and the constitutive equations

$$\begin{aligned} t_{ji} &= 2\mu e_{ij} + \lambda e_{rr}\delta_{ij} + a\,\theta\delta_{ij} \\ \eta &= -a\, e_{ii} + c\theta \end{aligned} \tag{7}$$

where t_{ji} represents the stress tensor, η is the entropy, (u_i) is the displacement vector, e_{ij} is the strain tensor, λ and μ are the Lamé constants and a is related with the thermal expansion constant and ρ and c are the mass density and the thermal capacity, respectively.

It is worth noting that these new thermomechanical theories have attracted a lot of attention recently [2, 9, 14, 20, 21, 34].

In this work we restrict our attention to the homogeneous one-dimensional case. Therefore, the system of equations that we want to study is given by

$$\begin{aligned} t_x &= \rho \ddot{u} \\ \dot{\eta} &= -q_x \end{aligned} \tag{8}$$

with the following constitutive equations:

$$\begin{aligned} & q(t+\tau_1) = -k_1 \beta_x (t+\tau_3) - k_2 T_x (t+\tau_2) \\ & t = \mu u_x + a\theta \\ & \eta = -a u_x + c\theta \end{aligned} \tag{9}$$

In this paper we assume that the delay parameters τ_1, τ_2 and τ_3 are nonnegative and, in each section, we will impose several conditions on them to guarantee the stability or instability of the solutions. A similar assumption is made on k_1 and k_2.

In a recent paper [15], it was proved that the Lord-Shulman thermoelasticity combined with the two-temperatures theory leads to the slow decay of the solutions, that is: the decay cannot be controlled in a uniform way by means of a negative exponential. Nevertheless, the Green-Lindsay thermoelastic theory with two-temperatures leads to the exponential decay. The aim of this paper is to continue this line of research. To be more precise, in this work we consider two third-order in time heat conduction models with two temperatures. The first one comes from the dual-phase-lag theory (see approach [24]), and the other one from the three-dual-phase-lag theory (see [25]). We prove the slow decay of solutions for the first model and the exponential decay for the second.

The plan of the paper is the following. In the next section we consider the dual-phase-lag thermoelasticity taking a second-order Taylor approximation for the heat flux and a first-order approximation for the inductive temperature. We first prove the well-posedness of the problem using the semigroup arguments in a convenient Hilbert space. Then we show the slow decay of the solutions by proving that elements of the point spectrum can be found as close as desired to the imaginary axis. In Section 3 we study the three-dual-phase-lag thermoelasticity introducing first-order Taylor approximations for the heat flux, for the inductive thermal displacement and for the inductive temperature. By using semigroup arguments, we prove again the existence and uniqueness of solutions. Next we show the exponential decay of the solutions by means of the semigroup of linear operators theory.

2. First case: Dual-phase-lag

In this first case we assume that $k_1 = 0$ and $k_2 > 0$ in the basic constitutive equations. Notice that taking $k_1 = 0$ is equivalent to consider the dual-phase-lag case. We take a second-order Taylor approximation for the heat flux q_i and a first-order Taylor approximation for the inductive temperature T.

$$\begin{aligned} & q(x, t+\tau_1) \approx q(x) + \tau_1 \dot{q}(x) + \tfrac{\tau_1^2}{2} \ddot{q}(x), \\ & T(x, t+\tau_2) \approx T(x) + \tau_2 \dot{T}(x). \end{aligned} \tag{10}$$

Replacing the above expressions into the constitutive equations, we obtain the following system of equations for our model:

$$\begin{cases} \rho \ddot{u} = \mu\, u_{xx} + a\, \theta_x \\ c\Big(\dot{\theta} + \tau_1 \ddot{\theta} + \tfrac{\tau_1^2}{2} \dddot{\theta}\Big) = k_2 T_{xx} + k_2 \tau_2 \dot{T}_{xx} + a\Big(\dot{u}_x + \tau_1 \ddot{u}_x + \tfrac{\tau_1^2}{2} \dddot{u}_x\Big) \\ \theta = T - m T_{xx}. \end{cases} \tag{11}$$

Here u is the displacement, θ is the temperature and T is the inductive temperature. As usual, ρ represents the mass density, μ is the elasticity, c is the thermal capacity, k_2 plays a similar role to the thermal conductivity, a is the coupling coefficient between the displacement and the temperature, τ_1 and τ_2 are two delay parameters and m is a positive constant related with the two temperatures theory. We also assume that $2\tau_2 > \tau_1$. This assumption comes from a previous work and it is related with the exponential stability of the heat equation (see [17]). To be precise, the exponential stability of solutions was proved assuming this condition. It would be interesting to know if this property is conserved when the elasticity is also considered.

We study the system in $[0, \pi] \times [0, \infty)$.

To have a well-posed problem we need to impose initial and boundary conditions. We assume null Dirichlet boundary conditions, that is,

$$u(0,t) = u(\pi,t) = T(0,t) = T(\pi,t) = 0 \text{ for } t \in [0,\infty). \tag{12}$$

As far as the initial conditions are concerned, we assume that

$$\begin{array}{l} u(x,0) = u_0(x), \dot{u}(x,0) = v_0(x), \\ \theta(x,0) = \theta_0(x), \dot{\theta}(x,0) = \phi_0(x), \ddot{\theta}(x,0) = \psi_0(x) \text{ for } x \in (0,\pi). \end{array} \tag{13}$$

We study the problem determined by system (11), the boundary conditions (12) and the initial conditions (13).

We will transform the given problem into an abstract problem involving a convenient Hilbert space.

First, we note that $Id - m\partial_{xx} : T \to T - m\partial_{xx}T = \theta$ is an isomorphism on $W^{2,2} \cap W_0^{1,2}$ and takes values in L^2, where $W^{2,2}, W_0^{1,2}$ and L^2 are the usual Hilbert spaces. We shall denote by $\Phi(\theta) = T$ the inverse operator.

From the definition of θ and in view of the boundary conditions we see that

$$\|\theta\|^2 = \|T\|^2 + 2m\|T_x\|^2 + m^2\|T_{xx}\|^2. \tag{14}$$

Therefore, the L^2 norm of θ is equivalent to the $W^{2,2}$ norm of T.

2.1. **On the well-posedness.** Let us denote $v = \dot{u}$, $\phi = \dot{\theta}$ and $\psi = \dot{\phi}$. We introduce also the following notation:

$$\hat{f} = f + \tau_1 \dot{f} + \frac{\tau_1^2}{2}\ddot{f}.$$

Therefore, system (11) may be written as

$$\begin{cases} \dot{\hat{u}} = \hat{v} \\ \dot{\hat{v}} = \frac{1}{\rho}\left(\mu \hat{u}_{xx} + a(\theta_x + \tau_1\phi_x + \frac{\tau_1^2}{2}\psi_x)\right) \\ \dot{\theta} = \phi \\ \dot{\phi} = \psi \\ \dot{\psi} = \frac{2}{c\tau_1^2}\left(k_2\Phi(\theta)_{xx} + k_2\tau_2\Phi(\phi)_{xx}\right) + \frac{2a}{c\tau_1^2}\hat{v}_x - \frac{a}{\tau_1^2}\phi - \frac{2}{\tau_1}\psi \end{cases} \tag{15}$$

If the above system is solved, then we will find u from $\hat{u}$ solving a second-order ordinary differential equation.

To ease the notation, we remove the hat from variables u and v and rewrite the system as follows:

$$\text{(16)}\qquad \begin{cases} \dot{u} = v \\ \dot{v} = \frac{1}{\rho}\left(\mu u_{xx} + a(\theta_x + \tau_1\phi_x + \frac{\tau_1^2}{2}\psi_x)\right) \\ \dot{\theta} = \phi \\ \dot{\phi} = \psi \\ \dot{\psi} = \frac{2}{c\tau_1^2}\left(k_2\Phi(\theta)_{xx} + k_2\tau_2\Phi(\phi)_{xx}\right) + \frac{2a}{c\tau_1^2}v_x - \frac{a}{\tau_1^2}\phi - \frac{2}{\tau_1}\psi \end{cases}$$

To prove the existence and uniqueness of solutions we consider the Hilbert space

$$\mathcal{H} = \{U = (u, v, \theta, \phi, \psi) : u \in W_0^{1,2}, v, \theta, \phi, \psi \in L^2\}$$

with the inner product defined by

(17)

$$\begin{aligned}\langle U, U^*\rangle = \frac{1}{2}\int_0^\pi &\Big(\rho v\overline{v^*} + \mu u_x\overline{u_x^*} + c(\theta + \tau_1\phi + \frac{\tau_1^2}{2}\psi)\overline{(\theta^* + \tau_1\phi^* + \frac{\tau_1^2}{2}\psi^*)} \\ &+ k_2(\tau_1+\tau_2)\Big(\Phi(\theta)_x\overline{\Phi(\theta^*)_x} + m\Phi(\theta)_{xx}\overline{\Phi(\theta^*)_{xx}}\Big) \\ &+ \frac{k_2\tau_1^2\tau_2}{2}\Big(\Phi(\phi)_x\overline{\Phi(\phi^*)_x} + m\Phi(\phi)_{xx}\overline{\Phi(\phi^*)_{xx}}\Big) \\ &+ \frac{k_2\tau_1^2}{2} \\ &\Big(\Phi(\theta)_x\overline{\Phi(\phi^*)_x} + \Phi(\phi)_x\overline{\Phi(\theta^*)_x} + m\Phi(\theta)_{xx}\overline{\Phi(\phi^*)_{xx}} + m\Phi(\phi)_{xx}\overline{\Phi(\theta^*)_{xx}}\Big)\Big)\, dx.\end{aligned}$$

Here, and from now on, the bar means the conjugate complex. Notice that the norm induced by this inner product is equivalent to the usual one in $\mathcal{H}$.

To propose a synthetic expression to the above problem, we define the matrix operator:

$$\text{(18)}\qquad \mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \frac{\mu}{\rho}D^2 & 0 & \frac{a}{\rho}I & \frac{a\tau_1}{\rho}D & \frac{a\tau_1^2}{2\rho}D \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & \frac{2a}{c\tau_1^2}D & \frac{2k_2}{c\tau_1^2}D^2\Phi & \frac{2k_2\tau_2}{c\tau_1^2}D^2\Phi - \frac{2}{\tau_1^2}I & -\frac{2}{\tau_1}I \end{pmatrix}.$$

Here I is the identity operator and D denotes the derivative with respect to x.

Therefore, our problem can be written as

$$\text{(19)}\qquad \frac{dU}{dt} = \mathcal{A}U, \;\; U(0) = (u_0, v_0, \theta_0, \phi_0, \psi_0).$$

Notice that the domain of $\mathcal{A}$ is the set $\mathcal{D} = \{U \in \mathcal{H}$ such that $\mathcal{A}U \in \mathcal{H}\}$, which is a dense subspace of $\mathcal{H}$.

Lemma 2.1. *The operator $\mathcal{A}$ is dissipative. That is:*

$$\Re\langle \mathcal{A}U, U\rangle \le 0$$

for every $U \in D$.

Proof. If we take into account the evolution equations and the boundary conditions we see that

$$\Re\langle \mathcal{A}U,U\rangle = -k_2\int_0^\pi \big(|\Phi(\theta)_x|^2 + m|\Phi(\theta)_{xx}|^2\big)dx \\ -\frac{k_2\tau_1}{2}(2\tau_2-\tau_1)\int_0^\pi \big(|\Phi(\phi)_x|^2 + m|\Phi(\phi)_{xx}|^2\big)dx.$$

As we assume that $2\tau_2 > \tau_1$, the lemma is proved. □

Lemma 2.2. *0 belongs to the resolvent of $\mathcal{A}$.*

Proof. We have to prove that for any $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$ the equation $\mathcal{A}U = F$ has a solution. If we write this equation term by term we get:

$$\left.\begin{aligned} v &= f_1 \\ \mu u_{xx} + a\big(\theta_x + \tau_1\phi_x + \tfrac{\tau_1^2}{2}\psi_x\big) &= \rho f_2 \\ \phi &= f_3 \\ \psi &= f_4 \\ k_2\Phi(\theta)_{xx} + k_2\tau_2\Phi(\phi)_{xx} + av_x - c\phi - c\tau_1\psi &= \tfrac{c\tau_1^2}{2}f_5 \end{aligned}\right\} \tag{20}$$

We obtain v, ϕ and ψ straight away. Therefore, we have to solve the system given by

$$\left.\begin{aligned} \mu u_{xx} + a\theta_x &= \rho f_2 - a\tau_1 f_{3,x} - a\tfrac{\tau_1^2}{2}f_{4,x} \\ k_2\Phi(\theta)_{xx} &= \tfrac{c\tau_1^2}{2}f_5 - af_{1,x} + cf_3 + c\tau_1 f_4 - k_2\tau_2\Phi(f_3)_{xx} \end{aligned}\right\} \tag{21}$$

If we assume homogeneous boundary conditions on $\Phi(\theta)$, we can solve the second equation of the above system. Once we have θ, substituting it in the first equation we obtain u.

It is also clear that the inequality $\|U\| \le K\|F\|$ holds for a positive constant K independent of U. □

As a consequence of the above lemmas and the Lumer–Phillips corollary to the Hille–Yosida Theorem (see [18], page 136) we obtain the well–posedness.

Theorem 2.3. *The operator $\mathcal{A}$ generates a contractive semigroup in $\mathcal{H}$, and for each $U(0) \in \mathcal{D}$ there exists a unique solution $U(t) \in C^1([0,\infty),\mathcal{H}) \cap C^0([0,\infty),\mathcal{D})$ to the problem determined by the system (11) with boundary conditions (12) and initial conditions (13).*

Remark 1. The continuous dependence of solutions on initial data and supply terms (in case they were assumed) can also be obtained.

These facts prove that the problem is well-posed in the sense of Hadamard.

Remark 2. We could have also considered the problem determined by system (11) with the following boundary conditions:

$$u(0,t) = u(\pi,t) = T_x(0,t) = T_x(\pi,t) = 0 \text{ for } t \in [0,\infty). \tag{22}$$

In this case, the operator Φ acts on

$$L^2_\star = \left\{\theta \in L^2 : \int_0^\pi \theta dx = 0\right\},$$

and takes values in $W^{2,2} \cap L^2_\star \cap \{T : T_x(0) = T_x(\pi) = 0\}$. Nevertheless, Φ is still an isomorphism and the equality (14) also holds.

2.2. **On the stability.** We will prove that a uniform rate of decay of exponential type cannot be obtained for the solutions of system (11) with the initial conditions (13) and the boundary conditions (22).

Theorem 2.4. *Let (u, T) be a solution of the problem determined by (11), (13) and (22). Then (u, T) decays in a slow way.*

Proof. We will prove that there exists a solution of system (11) of the form

$$u = K_1 e^{\omega t} \sin(nx), \quad T = K_2 e^{\omega t} \cos(nx),$$

such that $\Re(\omega) > -\epsilon$ for all positive ϵ. Hence, a solution ω as close as desired to the imaginary axis can be found. Imposing that u and T are as above and replacing them in (11) the following homogeneous system in the unknowns K_1 and K_2 is obtained:

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\begin{aligned} A_1 &= an\left(mn^2+1\right) \\ A_2 &= \mu n^2 + \omega^2 \rho \\ A_3 &= c\omega\left(mn^2+1\right)\left(\tau_1^2\omega^2 + 2\tau_1\omega + 2\right) + 2k_2 n^2(\tau_2\omega + 1) \\ A_4 &= -an\omega\left(\tau_1^2\omega^2 + 2\tau_1\omega + 2\right) \end{aligned}$$

This system will have nontrivial solutions if and only if the determinant of the coefficients matrix is equal to zero. We denote by $p(x)$ the determinant once ω is replaced by x. Straightforward calculations (made using Mathematica) show that $p(x)$ is a fifth degree polynomial:

$$p(x) = a_0 x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5,$$

where

$$\begin{aligned} a_0 &= c\tau_1^2 \rho(mn^2+1) \\ a_1 &= 2c\tau_1\rho(mn^2+1) \\ a_2 &= n^2\tau_1^2\left(mn^2+1\right)\left(a^2+c\mu\right) + 2\varrho\left(cmn^2 + c + k_2 n^2 \tau_2\right) \\ a_3 &= 2n^2\left(\tau_1\left(mn^2+1\right)\left(a^2+c\mu\right) + k_2\varrho\right) \\ a_4 &= 2n^2\left(\left(mn^2+1\right)\left(a^2+c\mu\right) + k_2\mu n^2\tau_2\right) \\ a_5 &= 2k_2\mu n^4 \end{aligned}$$

To prove that $p(x)$ has roots as close as we want to the complex axis, we will show that for any $\epsilon > 0$ there are roots of $p(x)$ located on the right-hand side of the vertical line $\Re(z) = -\epsilon$. This fact will be shown if the polynomial $p(x-\epsilon)$ has a root with positive real part. To prove that, we use the Routh–Hurwitz theorem. It assesses that, if $b_0 > 0$, then all the roots of polynomial

$$b_0 x^5 + b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5$$

have negative real part if, and only if, all the leading minors of the matrix

$$\begin{pmatrix} b_1 & b_0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ 0 & 0 & b_5 & b_4 & b_3 \\ 0 & 0 & 0 & 0 & b_5 \end{pmatrix}$$

are positive. We denote by L_i, for $i = 1, 2, 3, 4, 5$, the leading minors of this matrix.

Direct computations show that the second leading minor, L_2, is a sixth degree polynomial on n:

$$L_2 = -2cm^2\tau_1^4\epsilon\rho\left(a^2 + c\mu\right)n^6 + R(n),$$

where $R(n)$ is a polynomial on n of degree 4. Thus, for n large enough, L_2 is negative and $p(x-\epsilon)$ has a root with positive real part. (We have used Mathematica to compute L_2.)

This argument shows that the solutions of system (11) decay in a slowly way, or, in other words, that a uniform rate of decay of exponential type for all the solutions can not be obtained. □

We want to point out that this result differs considerably from the one known for the usual dual-phase-lag thermoelasticity: if only one temperature is considered, the solutions decay in an exponential way [26]. Let us also highlight the following fact. If in the second equation of system (11) variable T is replaced by θ, then the resulting equation would be hyperbolic and, therefore, in some sense we can say that the second equation of system (11) is a combination of an hyperbolic equation with the two temperatures theory. We have seen that this combination, when it is coupled in the usual way with the elasticity, drives the solutions of the system to the slow decay. This result was already observed in [15] for another combination of an hyperbolic equation with the two temperatures theory.

3. Second case: Three-dual-phase-lag

We consider now another system. We assume that $k_1 > 0$ and $k_2 > 0$. We take a first-order Taylor approximation for q_i, for β and for T. Therefore, we assume that

$$\begin{aligned} q(x, t+\tau_1) &\approx q(x) + \tau_1\dot{q}(x),\\ \beta(x, t+\tau_3) &\approx \beta(x) + \tau_3\dot{\beta}(x),\\ T(x, t+\tau_2) &\approx T(x) + \tau_2\dot{T}(x). \end{aligned} \tag{23}$$

Substituting these expressions into the constitutive equations, we obtain the following system:

$$\left\{\begin{array}{l} \rho\ddot{u} = \mu\, u_{xx} + a\,\theta_x \\ c\Big(\ddot{\theta} + \tau_1\dddot{\theta}\Big) = k_1 T_{xx} + \tau_4\dot{T}_{xx} + k_2\tau_2\ddot{T}_{xx} + a\Big(\ddot{u}_x + \tau_1\dddot{u}_x\Big) \\ \theta = T - mT_{xx}, \end{array}\right. \tag{24}$$

where $\tau_4 = \tau_3 k_1 + k_2$.

As in the previous case, ρ, μ and m are positive. The delay parameters are also positive and $\tau_4 - k_1\tau_1 > 0$. This last assumption comes again from the exponential stability of the heat equation (see [17]). The sign of a is not relevant, but it must be different from zero. The parameters k_1 and k_2 are also assumed positive.

We assume the same boundary and initial conditions as in the previous section.

As before, we introduce suitable notation. In this case we write $\hat{f} = f + \tau_1\dot{f}$. Moreover, we use $w = \dot{v}$. Therefore, we consider the system given by

$$\left\{\begin{array}{l} \dot{\hat{v}} = \hat{w} \\ \dot{\hat{w}} = \frac{1}{\rho}\left(\mu\hat{v}_{xx} + a(\phi_x + \tau_1\psi_x)\right) \\ \dot{\theta} = \phi \\ \dot{\phi} = \psi \\ \dot{\psi} = \frac{1}{c\tau_1}\left(k_1\Phi(\theta)_{xx} + \tau_4\Phi(\phi)_{xx} + k_2\tau_2\Phi(\psi)_{xx}\right)\right) + \frac{a}{c\tau_1}\hat{w}_x - \frac{1}{c}\psi \end{array}\right. \tag{25}$$

If this system is solved, then we can also find v by solving an ordinary differential equation.

To ease the notation we remove the hat from the variables and we concentrate in the following system of equations:

$$\text{(26)}\qquad \begin{cases} \dot{v} = w \\ \dot{w} = \frac{1}{\rho}\left(\mu v_{xx} + a(\phi_x + \tau_1 \psi_x)\right) \\ \dot{\theta} = \phi \\ \dot{\phi} = \psi \\ \dot{\psi} = \frac{1}{c\tau_1}\left(k_1 \Phi(\theta)_{xx} + \tau_4 \Phi(\phi)_{xx} + k_2\tau_2 \Phi(\psi)_{xx})\right) + \frac{a}{c\tau_1} w_x - \frac{1}{c}\psi \end{cases}$$

3.1. **On the well-posedness.** To prove the existence and uniqueness of solutions we consider the Hilbert space

$$\mathcal{H} = \{U = (v, w, \theta, \phi, \psi) : v \in W_0^{1,2}, w, \theta, \phi, \psi \in L^2\}$$

with the inner product defined by

$$\begin{aligned} \langle U, U^* \rangle = & \frac{1}{2}\int_0^\pi \Big(\rho w \overline{w^*} + \mu v_x \overline{v_x^*} + c(\phi + \tau_1 \psi)(\overline{\phi^* + \tau_1 \psi^*}) \\ & + k_1 \Big(\Phi(\theta)_x + \tau_1 \Phi(\phi)_x\Big)\Big(\overline{\Phi(\theta^*)_x + \tau_1 \Phi(\phi^*)_x}\Big) \\ & + m k_1 \Big(\Phi(\theta)_{xx} + \tau_1 \Phi(\phi)_{xx}\Big)\Big(\overline{\Phi(\theta^*)_{xx} + \tau_1 \Phi(\phi^*)_{xx}}\Big) \\ & + (\tau_1(\tau_4 - k_1\tau_1) + k_2\tau_2)\Big(\Phi(\phi)_x \overline{\Phi(\phi^*)_x} + m\Phi(\phi)_{xx}\overline{\Phi(\phi^*)_{xx}}\Big)\Big)\, dx. \end{aligned} \qquad (27)$$

Again, this inner product defines a norm which is equivalent to the usual norm in $\mathcal{H}$.

We abuse the notation a little bit and denote the following matrix operator again by $\mathcal{A}$:

$$\text{(28)}\qquad \mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \frac{\mu}{\rho}D^2 & 0 & 0 & \frac{a}{\rho}D & \frac{a\tau_1}{\rho}D \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & \frac{a}{c\tau_1}D & \frac{k_1}{c\tau_1}D^2\Phi & \frac{\tau_4}{c\tau_1}D^2\Phi & \frac{k_2\tau_2}{c\tau_1}D^2\Phi - \frac{1}{c}I \end{pmatrix}.$$

Therefore, our problem can be written as

$$\text{(29)}\qquad \frac{dU}{dt} = \mathcal{A}U, \;\; U(0) = (v_0, w_0, \theta_0, \phi_0, \psi_0).$$

Notice that the domain of $\mathcal{A}$ is the set $\mathcal{D} = \{U \in \mathcal{H}$ such that $\mathcal{A}U \in \mathcal{H}\}$, which is a dense subspace of $\mathcal{H}$.

Lemma 3.1. *The operator $\mathcal{A}$ is dissipative. That is:*

$$\Re\langle \mathcal{A}U, U\rangle \leq 0$$

for every $U \in D$.

Proof. In view of the evolution equations and the boundary conditions we see that

$$\begin{aligned} \Re\langle \mathcal{A}U, U\rangle = & -(\tau_4 - k_1\tau_1)\int_0^\pi \left(|\Phi(\phi)_x|^2 + m|\Phi(\phi)_{xx}|^2\right)dx \\ & - k_2\tau_1\tau_2 \int_0^\pi \left(|\Phi(\psi)_x|^2 + m|\Phi(\psi)_{xx}|^2\right)dx. \end{aligned}$$

As we assume that $\tau_4 > k_1\tau_1$, k_2, τ_1 and τ_2 positive, the lemma is proved. □

Lemma 3.2. 0 *belongs to the resolvent of* $\mathcal{A}$.

Proof. We have to prove that for any $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$ the equation $\mathcal{A}U = F$ has a solution. If we write this equation term by term we get:

$$\left.\begin{aligned} w &= f_1 \\ \mu v_{xx} + a\big(\phi_x + \tau_1 \psi_x\big) &= \rho f_2 \\ \phi &= f_3 \\ \psi &= f_4 \\ k_1 \Phi(\theta)_{xx} + \tau_4 \Phi(\phi)_{xx} + k_2 \tau_2 \Phi(\psi)_{xx} + a w_x - \tau_1 \psi &= c\tau_1 f_5 \end{aligned}\right\} \tag{30}$$

As in the previous section, we obtain w, ϕ and ψ straight away. Therefore, we have to solve the system given by

$$\left.\begin{aligned} \mu v_{xx} &= \rho f_2 - a(f_{3,x} + \tau_1 f_{4,x}) \\ k_1 \Phi(\theta)_{xx} &= c\tau_1 f_5 - a f_{1,x} + \tau_1 f_4 - \tau_4 \Phi(f_3)_{xx} - k_2 \tau_2 \Phi(f_4)_{xx} \end{aligned}\right\} \tag{31}$$

If we assume homogeneous Dirichlet boundary conditions, we can solve this system as in the previous case. Hence, we can get θ by means of the isomorfism. Again, an inequality of the type $\|U\| \leq K\|F\|$ can be obtained. $\square$

As a consequence of the above lemmas and the Lumer–Phillips corollary to the Hille–Yosida Theorem we obtain the well–posedness in the sense of Hadamard.

Theorem 3.3. *The operator* $\mathcal{A}$ *generates a contractive semigroup,* $S(t) = \{e^{t\mathcal{A}}\}$, *in* $\mathcal{H}$, *and for each* $U(0) \in \mathcal{D}$ *there exists a unique solution* $U(t) \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), \mathcal{D})$ *to the problem determined by the system (24) with boundary conditions (12) and initial conditions (13).*

3.2. **On the stability.** We have now the basic tools to prove the main result of this section. Before doing this, we recall the caractherization stated in the book of Liu and Zheng that ensures the exponential decay (see [13], [16] or [22]).

Theorem 3.4. *Let* $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ *be a* C_0*-semigroup of contractions on a Hilbert space. Then* $S(t)$ *is exponentially stable if and only if the following two conditions are satisfied:*

(i) $i\mathbb{R} \subset \rho(\mathcal{A})$, *(here* $\rho(\mathcal{A})$ *means the resolvent of* $\mathcal{A}$*).*
(ii) $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Theorem 3.5. *The operator* $\mathcal{A}$ *defined at (28) generates a semigroup which is exponentially stable.*

Proof. Following the arguments given by Liu and Zheng ([16], page 25), the proof consists of the following steps:

(i) Since 0 is in the resolvent of $\mathcal{A}$, using the contraction mapping theorem, we have that for any real λ such that $|\lambda| < ||\mathcal{A}^{-1}||^{-1}$, the operator $i\lambda\mathcal{I} - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - \mathcal{I})$ is invertible. Moreover, $||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1}, ||\mathcal{A}^{-1}||^{-1})$.

(ii) If $\sup\{||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||, |\lambda| < ||\mathcal{A}^{-1}||^{-1}\} = M < \infty$, then by the contraction theorem, the operator

$$i\lambda\mathcal{I} - \mathcal{A} = (i\lambda_0\mathcal{I} - \mathcal{A})\Big(\mathcal{I} + i(\lambda - \lambda_0)(i\lambda_0\mathcal{I} - \mathcal{A})^{-1}\Big),$$

is invertible for $|\lambda - \lambda_0| < M^{-1}$. It turns out that, by choosing λ_0 as close to $||\mathcal{A}^{-1}||^{-1}$ as we can, the set $\{\lambda, |\lambda| < ||\mathcal{A}^{-1}||^{-1} + M^{-1}\}$ is contained in the resolvent of $\mathcal{A}$ and $||(i\lambda\mathcal{I}-\mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1}-M^{-1}, ||\mathcal{A}^{-1}||^{-1}+M^{-1})$.

(iii) Let us assume that the intersection of the imaginary axis and the spectrum is not empty. Then there exists a real number ϖ with $||\mathcal{A}^{-1}||^{-1} \leq |\varpi| < \infty$ such that the set $\{i\lambda, |\lambda| < |\varpi|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{||(i\lambda\mathcal{I}-\mathcal{A})^{-1}||, |\lambda| < |\varpi|\} = \infty$. Therefore, there exist a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of vectors $\mathcal{U}_n = (v_n, w_n, \theta_n, \phi_n, \psi_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$\|(i\lambda_n\mathcal{I}-\mathcal{A})\mathcal{U}_n\| \to 0. \tag{32}$$

If we write (32) in components, we obtain the following conditions:

$$i\lambda_n v_n - w_n \to 0, \text{ in } W^{1,2} \tag{33}$$

$$i\lambda_n w_n - \frac{1}{\rho}\Big(\mu D^2 v_n + a(D\phi_n + \tau_1 D\psi_n)\Big) \to 0, \text{ in } L^2 \tag{34}$$

$$i\lambda_n \theta_n - \phi_n \to 0, \text{ in } L^2 \tag{35}$$

$$i\lambda_n \phi_n - \psi_n \to 0, \text{ in } L^2 \tag{36}$$

$$\begin{aligned} i\lambda_n\psi_n - \frac{1}{c\tau_1}\Big(k_1 D^2\Phi(\theta_n) + \tau_4 D^2\Phi(\phi_n) + k_2\tau_2 D^2\Phi(\psi_n) + aDw_n\Big) \\ + \frac{1}{c}\psi_n \to 0, \text{ in } L^2 \end{aligned} \tag{37}$$

In view of the dissipative term of the operator (see the proof of Lemma 3.1), we see that

$$D\Phi(\phi_n), D^2\Phi(\phi_n), D\Phi(\psi_n), D^2\Phi(\psi_n) \to 0 \text{ in } L^2.$$

Therefore, from (14) we conclude that $\phi_n \to 0$ in $W^{2,2}$ and, consequently, $\theta_n, \psi_n \to 0$ in L^2. Now, from (37) we obtain

$$i\lambda_n\psi_n - \frac{a}{c\tau_1}Dw_n \to 0, \text{ in } L^2,$$

which, after simplifying, implies that

$$i\psi_n - \frac{a}{c\tau_1}Dv_n \to 0, \text{ in } L^2.$$

Then, as a is different from 0, $Dv_n \to 0$ in L^2. Finally, from (33) and (34) we conclude that $w_n \to 0$ in $W^{1,2}$. These behaviors contradict the hypothesis that U_n has unit norm.

We can prove the second condition of the Theorem 3.4 following a similar argument. □

Remark 3. The analysis proposed in this section can be easily adapted to the boundary conditions

$$u(0,t) = u(\pi,t) = T_x(0,t) = T_x(\pi,t) = 0.$$

That means that we can obtain the existence, the uniqueness and the exponential decay of the solutions for the problem determined by system (24) with the initial conditions given by (13) and the above boundary conditions.

We also point out that this behavior differs from the one obtained in the previous section. This is because now the heat conduction is described by the combination of a *parabolic* equation with the two temperatures theory.

4. Conclusions

In this paper we have analised two systems of equations for the phase-lag thermoelasticicty with two temperatures, theory which is currently being studied from different points of view. We have focused on the decay of solutions. To be precise, we have analysed the following situations:

- Dual-phase-lag. We have introduced a second order Taylor approximation for the heat flux and a first-order approximation for the inductive temperature and we have obtained system (11). The second equation of this system can be seen as a combination of a hyperbolic equation with the two temperatures theory. We have proved that the solutions of this thermoelastic system of equations decay in a slow way.
- Three-dual-phase-lag. In this case we have considered first order Taylor approximations for the heat flux, for the inductive temperature and for the inductive thermal displacement, and we have obtained system (24). In this case, the second equation can be seen as the combination of a parabolic equation with the two temperatures theory. We have proved that the solutions of this thermoelastic system of equations decay exponentially.

These results seem to suggest a different behavior for the decay of the solutions depending on the type of equation (hyperbolic or parabolic) that is combined with the two temperatures theory.

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