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Theory article

$C^{1,\alpha}$-regularity theory for weak solutions of a general quasilinear elliptic equation

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Abstract: In this paper, we establish $C^{1,\alpha}$-regularity theory with an accurate estimate

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\left(b^{-1}\left(\|f\|_{L^\infty(B_1)}\right) + \|u\|_{W^{1,B}(B_1)}\right),$$

where $b(t) = ta(t)$, for weak solutions of the following general quasilinear elliptic equation with Orlicz growth in divergence form:

$$-\text{div}\left(a(|\nabla u|)\nabla u\right) = f \in L^\infty_{loc}(\Omega) \quad \text{in} \;\; \Omega \subset \mathbb{R}^n$$

for $n \geq 2$. Its prototypes are the nonhomogeneous elliptic p-Laplacian equations with and without a logarithmic term, respectively. Meanwhile, we also present the local optimal $(1 + s_a')$-cap continuity for the above problem.

Keywords: $C^{1,\alpha}$; regularity; divergence; weak solutions; Orlicz; quasilinear; elliptic
Mathematics Subject Classification: 35B65, 35J60

1. Introduction

In this paper we focus on investigating the sharp $C^{1,\alpha}$-regularity estimates for weak solutions of the following quasilinear elliptic equation with Orlicz growth and a bounded source in divergence form:

$$-\text{div}\left(a(|\nabla u|)\nabla u\right) = f \in L^\infty_{loc}(\Omega) \quad \text{in} \;\; \Omega \subset \mathbb{R}^n, \tag{1.1}$$

where $n \geq 2$ and the function $a(t) : [0, +\infty) \longrightarrow [0, +\infty)$ is of class C^1 and satisfies

$$0 < i_a := \inf_{t>0} \frac{ta'(t)}{a(t)} \leq \sup_{t>0} \frac{ta'(t)}{a(t)} =: s_a < \infty. \tag{1.2}$$

In fact, if $a(t) = t^{p-2}$, then $i_a = s_a = p - 2$ for $p > 2$, and (1.1) reduces to the nonhomogeneous classical elliptic p-Laplacian equation in divergence form:

$$- \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f. \tag{1.3}$$

It may be worthwhile to remark that another special case of such problems is the nonhomogeneous elliptic p-Laplacian equation with logarithmic growth in divergence form:

$$- \operatorname{div}\Big(|\nabla u|^{p-2}\ln(1 + |\nabla u|)\nabla u\Big) = f.$$

It should be pointed out that the methods and conclusions established in this paper can be extended to the more general quasilinear elliptic equation given by

$$-\operatorname{div} A(x, \nabla u) = f \in L^{\infty}_{loc}(\Omega) \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where the vector field $A(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular in the variable ξ satisfying

$$\begin{aligned} &|A(x,\xi)| + \left|D_\xi A(x,\xi)\right| |\xi| \le Ca(|\xi|)|\xi|, \\ &D_\xi A(x,\xi)\eta \cdot \eta \ge Ca(|\xi|)|\eta|^2, \\ &|A(x_1,\xi) - A(x_2,\xi)| \le w\,(|x_1 - x_2|)\, a(|\xi|)|\xi| \end{aligned}$$

with a Hölder continuous function w and the function $a(t)$ satisfying (1.2). For the sake of simplicity and clarity, we focus on the constant-coefficient case (1.1) as the main object of our study in this paper.

The classical elliptic p-Laplacian equation arises from the generalization of the linear Laplace equation and the investigation of nonlinear variational problems. It first emerged in the context of nonlinear potential theory and calculus of variations, with its roots tracing back to efforts aimed at describing phenomena involving nonlinear diffusion, elasticity, and fluid dynamics, where linear models fail to capture the underlying physics. To derive the p-Laplacian equation

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \tag{1.4}$$

for $p > 1$, we consider the problem of minimizing the following nonlinear energy functional involving the gradient of a function v in a domain Ω,

$$E(v) := \frac{1}{p}\int_\Omega |\nabla v|^p \, dx \quad \text{for any} \quad v \in W^{1,p}(\Omega).$$

In the regularity theory of partial differential equations (PDEs), the primary objective is to investigate the smoothness of solutions to a given PDE. Among the most fundamental tools for this purpose are L^p-type estimates and Hölder estimates, which connect weak and classical solutions, quantify smoothness, and facilitate both theoretical insights and practical applications. To date, extensive research has been devoted to L^p-type estimates for quasilinear elliptic equations of the p-Laplacian type under various coefficient and domain assumptions (see [1–4]). Over the past two to three decades, numerous scholars have explored the corresponding regularity theory for equations that are more general than the classical p-Laplacian equation (see [5] and the references therein). More precisely, Cianchi and Maz'ya [6] established global Lipschitz regularity for weak solutions to

$$\operatorname{div}\left(b'\left(|\nabla u|\right)\nabla u\right) = f \quad \text{ in } \Omega \tag{1.5}$$

under the condition given in (1.2). Additionally, Baroni [7] obtained pointwise gradient estimates via linear Riesz potentials for solutions to the following nonlinear elliptic equations with a measure-valued right-hand side:

$$\operatorname{div}\left(b'\left(|\nabla u|\right)\nabla u\right)=\mu.$$

Moreover, we also studied L^p-type estimates for weak solutions to (1.1)–(1.2) in our previous work [8].

Earlier studies have investigated the $C^{1,\alpha}$ regularity estimates for weak solutions of the elliptic p-Laplace equation (1.4) as well as for more general cases involving variable coefficients. Recently, several scholars [9–12] have conducted more in-depth research on the corresponding $C^{1,\alpha}$ estimates of weak solutions for more general p-Laplacian-type equations. In particular, when $f\equiv n$, we know that

$$v(x)=\frac{p-1}{p}|x|^{\frac{p}{p-1}}=\frac{p-1}{p}|x|^{1+\frac{1}{p-1}}\in C^{1+\frac{1}{p-1}}\quad \text{for }\; p>1$$

is a solution of the elliptic p-Laplacian equation (1.1)

$$\operatorname{div}\left(|\nabla v|^{p-2}\,\nabla v\right)=n\in L^{\infty}.$$

In light of this fact, if we establish $C^{1,\alpha}$-regularity estimates for the elliptic p-Laplacian equations (1.3), then α must be smaller than $1/(p-1)$. A longstanding conjecture in elliptic regularity theory asks: Is every $W^{1,p}$-function with bounded p-Laplacian locally of class $C^{p'}=C^{1,1/(p-1)}$? Very recently, Araújo, Teixeira, and Urbano [13] established the planar version of $C^{p'}$-regularity: for the degenerate elliptic p-Laplacian equation

$$-\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)=f(x)\quad \text{in }\; \Omega\subset\mathbb{R}^2$$

with a bounded source term $f\in L^{\infty}$, they proved that its weak solutions are locally of class $C^{p'}=C^{1,1/(p-1)}$. In their work, they provided precise control over a new oscillation property of weak solutions, expressed in terms of the magnitude of their gradients, and further improved $C^{1,\alpha}$ regularity estimates via geometric iteration. Moreover, several experts [14–17] have further explored high-dimensional cases as well as p-Laplacian and $p(x)$-Laplacian types problems. We also mention the recent work [18] of Di Fazio, Teymurazyan, and Urbano, who established $C^{1,\alpha}$ regularity for p-Laplacian–type equations with data in Morrey spaces $L^{1,\lambda}(\Omega)$, $n-1<\lambda<n$ via Fefferman–Phong inequality. Their framework handles low-integrability right-hand sides but is specific to p-Laplacian structure. Our work differs by treating general Orlicz growth conditions with bounded sources $f\in L^{\infty}_{loc}(\Omega)$, and establishing the optimal $(1+s_a')$-cap continuity. The two approaches are complementary. To be precise, this paper aims to explore the interior $C^{1,\alpha}$-regularity estimates with an accurate estimate (1.12) and local optimal $(1+s_a')$-cap continuity for weak solutions to the problems (1.1)–(1.2). Moreover, we emphasize that the proof is significantly influenced by the works in [13–17].

In this work, we define

$$b(t):=ta(t)\quad \text{and}\quad B(t):=\int_0^t b(\tau)d\tau\quad \text{for } t\geq 0. \tag{1.6}$$

From condition (1.2), it is straightforward to verify that

$$b(t) \text{ is strictly increasing and continuous on } [0,+\infty), \tag{1.7}$$

and

$$B(t) \text{ is increasing on } [0, +\infty). \tag{1.8}$$

Furthermore, we can confirm that

$$\theta^{i_a} a(t) \le a(\theta t) \le \theta^{s_a} a(t) \quad \text{for every } \ \theta \ge 1. \tag{1.9}$$

This result can be proven as follows: from (1.2), we obtain

$$\begin{aligned}
\frac{1}{t}\ln\theta^{i_a} &= \int_1^\theta \frac{i_a}{\theta t} d\theta \\
&\le \int_1^\theta \frac{a'(\theta t)}{a(\theta t)} d\theta = \frac{1}{t}\int_1^\theta d\left(\ln\left(a(\theta t)\right)\right) = \frac{1}{t}\ln\left(\frac{a(\theta t)}{a(t)}\right) \\
&\le \int_1^\theta \frac{s_a}{\theta t} d\theta = \frac{1}{t}\ln\theta^{s_a}.
\end{aligned}$$

For convenience, we first recall some definitions and fundamental results concerning general Orlicz spaces, which are widely used in analysis as one of the most natural generalizations of Sobolev spaces. For the function $B(t)$ defined in (1.6) above, it is straightforward to verify that $B(t)$ satisfies the Δ_2 condition (written as $B \in \Delta_2$),

$$B(2t) \le KB(t) \quad \text{for any } t > 0,$$

as well as the ∇_2 condition (written as $B \in \nabla_2$):

$$B(t) \le \frac{B(\theta t)}{2\theta} \quad \text{for any } t > 0,$$

where $K > 0$ and $\theta > 1$ are two given constants.

Definition 1.1. *The Orlicz class $K^B(\Omega)$ consists of all measurable functions $g : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega B(|g|)\, dx < \infty.$$

The Orlicz space $L^B(\Omega)$ is the linear hull of $K^B(\Omega)$. Furthermore, we define the Orlicz–Sobolev space $W^{1,B}(\Omega)$ as

$$W^{1,B}(\Omega) := \left\{u \in L^B(\Omega) \mid \nabla u \in L^B(\Omega)\right\},$$

and $W_0^{1,B}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,B}(\Omega)$.

As usual, solutions to (1.1) are understood in the weak sense. We now present the definition of a weak solution.

Definition 1.2. *A function $u \in W_{loc}^{1,B}(\Omega)$ is called a local weak solution of (1.1) if, for all $\varphi \in C_0^\infty(\Omega)$, the following holds:*

$$\int_\Omega \left(a(|\nabla u|)\nabla u \cdot \nabla\varphi - f\varphi\right) dx = 0.$$

Before we state the main results of this work, we first recall that in [19] we have proved $C^{1,\alpha}_{loc}$ for some $\alpha \in (0,1)$ depending on n, i_a, s_a for weak solutions of a class of the homogeneous quasilinear elliptic equation

$$\operatorname{div}\left(a\left(|\nabla u|\right)\nabla u\right) = 0 \quad \text{in } \Omega. \tag{1.10}$$

However, a minor limitation remains: although we have established that α lies within the relatively wide interval $(0,1)$, it is challenging to directly link it to the optimal exponent in the aforementioned conjecture. This work is primarily devoted to extending the homogeneous case (1.10) to the inhomogeneous case (1.2) with $f \in L^\infty$. We note that several new technical tools, including the corrector lemma and the handling of the source term in the scaling arguments, are employed to address the inhomogeneous term. To better elaborate on the conclusions of our paper, we define the maximal C^{1,α_M} regularity for weak solutions of (1.10) as corresponding to the maximal $\alpha_M \in (0,1]$, where α_M is given by

$$\alpha_M := \sup\left\{\alpha \in (0,1) \,\middle|\, \begin{array}{l} h \text{ belongs to } C^{1,\alpha}_{loc}(\Omega) \quad \text{for every local weak solution} \\ h \text{ of the problem (1.10) with the condition (1.2)} \end{array}\right\}.$$

More precisely, we shall obtain the following local $C^{1,\alpha}$ estimates for weak solutions of (1.1).

Theorem 1.3. *If $u \in W^{1,B}_{loc}(\Omega)$ is a local weak solution of (1.1) with the assumption (1.2), then for any positive number α satisfying*

$$\alpha \in (0, s'_a] \cap (0, \alpha_M) \quad \textit{for} \qquad s'_a =: \frac{1}{1+s_a}, \tag{1.11}$$

there exists a positive constant C depending on n, i_a, s_a, α such that

$$\|u\|_{C^\alpha(B_{1/2})} + \|\nabla u\|_{C^\alpha(B_{1/2})} \le C\Big(b^{-1}\left(\|f\|_{L^\infty(B_1)}\right) + \|u\|_{W^{1,B}(B_1)}\Big). \tag{1.12}$$

It is particularly worth noting that for special p-harmonic type equations, that is, when $i_a = s_a = p-2$, then s'_a is actually equal to $p' - 1 = 1/(p-1)$. Furthermore, from Theorem 1.3, we easily derive the conclusion that $\alpha \le s'_a = 1/(p-1)$, which is the expected optimal exponent in the longstanding conjecture for an elliptic p-Laplacian equation. Moreover, we can also prove the following local optimal $(1+s'_a)$-cap continuity for weak solutions of (1.1).

Theorem 1.4. *If $u \in W^{1,B}_{loc}(\Omega)$ is a local weak solution of (1.1) in Ω with the assumption (1.2), and u can be touched from below at an interior point x_0 in Ω by a $(1+s'_a)$-cap φ, that is, φ satisfies*

$$|\varphi(x) - \varphi(x_0)| \le [\varphi]_{1+s'_a} r^{1+s'_a} \quad \textit{for any } \; x \in B_r(x_0) \subset\subset \Omega \tag{1.13}$$

with a positive constant $[\varphi]_{1+s'_a}$, then u also satisfies a $(1+s'_a)$-cap with the estimate

$$|u(x) - u(x_0)| \le C r^{1+s'_a},$$

where the constant C depends on n, i_a, s_a, $\|u\|_{W^{1,B}(B_1)}$, $\|f\|_{L^\infty(B_1)}$ and $[\varphi]_{1+s'_a}$.

2. Auxiliary lemmas

In this section, we shall introduce several auxiliary lemmas, which will be used in proving the conclusions of Theorems 1.3 and 1.4. In the main conclusions of this paper, we will focus on establishing the $C^{1,\alpha}$ regularity theory for weak solutions to the general quasilinear elliptic equation with Orlicz growth and a bounded source in divergence form (1.1). Our approach hinges on two precise controls regarding the new oscillation estimates for weak solutions of (1.1):

$$\sup_{B_r} \left|u(x) - u(0)\right| \lesssim r^{1+\alpha} + |\nabla u(0)|\, r$$

and

$$\sup_{B_r} \left|u(x) - u(0) - \nabla u(0) \cdot x\right| \lesssim r^{1+\alpha}.$$

Essentially, the two estimates outlined above bear a close resemblance to the classical Taylor expansion in that they both serve as systematic tools to approximate and analyze the behavior of functions. Moreover, these estimates can reveal some essential qualities or characteristics of the solutions to the problem (1.1).

Now, we aim to show that if u is a weak normalized solution to

$$-\text{div}\,(a(|\nabla u|)\nabla u) = f \quad \text{in } \ B_1,$$

where f is a bounded function belonging to $L^\infty(B_1)$, then it is possible to find a corrector function ξ that is continuously differentiable (of class C^1) and satisfies the condition that its C^1-norm over the smaller ball $B_{1/2}$ is sufficiently small such that when this corrector ξ is added to the solution u, the resulting function $u + \xi$ becomes a solution to the homogeneous form of the original equation. In more specific terms, we can verify that the weak solution u of (1.1) can be approximated by a C^1 corrector ξ.

Lemma 2.1. *Let $u \in W^{1,B}_{loc}(\Omega)$ be a local weak solution of (1.1) in $\Omega \supset B_1$ with the assumption (1.2). For any $\epsilon > 0$, there exists a positive constant $\delta = \delta(\epsilon) > 0$ such that if*

$$\|f\|_{L^\infty(B_1)} \le \delta \quad \textit{and} \quad \|u\|_{L^\infty(B_1)} \le 1,$$

then we can find a corrector $\xi \in C^1(B_{1/2})$ with

$$|\xi(x)| + |\nabla\xi(x)| \le \epsilon \quad \textit{in } \ B_{1/2}, \tag{2.1}$$

such that

$$-\textit{div}\,(a(|\nabla(u+\xi)|)\nabla(u+\xi)) = 0 \quad \textit{in } \ B_{1/2}. \tag{2.2}$$

Proof. We argue by contradiction. If the result does not hold, then there would exist $\epsilon_0 > 0$ and two sequences of $\{u_j\}_{j=1}^\infty, \{f_j\}_{j=1}^\infty$ satisfying

$$\int_\Omega \left(a\left(\left|\nabla u_j\right|\right)\nabla u_j \cdot \nabla\varphi - f_j\varphi\right)dx = 0 \quad \text{for any } \varphi \in C_0^\infty(\Omega), \tag{2.3}$$

$$\|u_j\|_{L^\infty(B_1)} \le 1,$$
$$\|f_j\|_{L^\infty(B_1)} \le \frac{1}{j},$$

such that

$$|\xi(x)| + |\nabla\xi(x)| > \epsilon_0 \tag{2.4}$$

for any weak solution $\xi \in C^1(B_{1/2})$ satisfying the homogeneous equation

$$-\mathrm{div}\left(a\left(\left|\nabla\left(u_j+\xi\right)\right|\right)\nabla\left(u_j+\xi\right)\right) = 0 \quad \text{in } B_{1/2}.$$

From the standard regularity estimate for weak solution of (2.3) (see [7], Theorem 1.2), we can conclude that $|\nabla u_j| \le L/2$ for some $L > 0$ and any $x \in B_{3/4}$ and extract a subsequence of $\{u_j\}_{j=1}^\infty$, still denoted by $\{u_j\}_{j=1}^\infty$, such that

$$u_j \longrightarrow u_\infty \quad \text{in } C^1(B_{1/2}) \quad \text{as } j \to +\infty. \tag{2.5}$$

Although Theorem 1.2 in [7] addresses Riesz potential estimates, we note that the L^∞-boundedness of f_j and the uniform bound on u_j yield the gradient boundedness. Then, (2.5) implies that

$$\int_{B_{1/2}} a(|\nabla u_\infty|)\nabla u_\infty \cdot \nabla\varphi dx = 0 \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

That is to say, $u_j + (u_\infty - u_j) = u_\infty$ is a weak solution of the homogeneous equation

$$\begin{aligned}&-\mathrm{div}\left(a\left(\left|\nabla\left(u_j+(u_\infty-u_j)\right)\right|\right)\nabla\left(u_j+(u_\infty-u_j)\right)\right)\\ &= -\mathrm{div}\left(a(|\nabla u_\infty|)\nabla u_\infty\right) = 0 \quad \text{in} \quad B_{1/2}.\end{aligned}$$

Therefore, from (2.4) and (2.5), we can get a contradiction and then prove the desired result. □

Based on an iteration reasoning, we shall establish the following crucial lemma, which allows us to prove the $C^{1,\alpha}$ estimates for $\alpha \in (0, s_a'] \cap (0, \alpha_M)$ with $s_a' =: 1/(1+s_a)$. It should be also noted that, specifically when $i_a = s_a = p-2$, s_a' is actually equal to $p'-1 = 1/(p-1)$. This is also consistent with the objective of the $C^{p'} = C^{1,1/(p-1)}$ conjecture for the elliptic p-Laplacian equation.

Lemma 2.2. *Let $u \in W_{loc}^{1,B}(\Omega)$ be a local weak solution of (1.1) in $\Omega \supset B_1$ with the assumption (1.2). There exist three positive constants δ_0, C, and $\lambda_0 \in (0, 1/2)$, depending on n, s_a, i_a and the Orlicz function $a(t)$, such that if*

$$\|f\|_{L^\infty(B_1)} \le \delta_0 \quad \textit{and} \quad \|u\|_{L^\infty(B_1)} \le 1,$$

then we have

$$\sup_{B_r}|u(x)-u(0)| \le C\left(r^{1+\alpha} + |\nabla u(0)|\, r\right) \quad \textit{for any } r \in (0, \lambda_0].$$

Proof. Let $x \in B_{\lambda_0} \subset B_{1/2}$ and $\epsilon = \lambda_0^{1+\alpha}/2$, which determines the smallness assumption on the constant $\delta_0 > 0$. Then we can apply Lemma 2.1 to find that

$$\begin{aligned}
&|u(x) - (u(0) + \nabla u(0) \cdot x)| \\
&\leq |(u + \xi)(x) - ((u + \xi)(0) + \nabla (u + \xi)(0) \cdot x)| + |\xi(x)| + |\xi(0)| + |\nabla \xi(0) \cdot x| \\
&\leq C \lambda_0^{1+\alpha'_M} + \epsilon \\
&= C \lambda_0^{\alpha'_M - \alpha} \lambda_0^{1+\alpha} + \frac{1}{2} \lambda_0^{1+\alpha} \\
&\leq \lambda_0^{1+\alpha} \quad \text{for } x \in B_{\lambda_0}
\end{aligned} \tag{2.6}$$

by choosing λ_0 small enough satisfying $C\lambda_0^{\alpha'_M - \alpha} \leq 1/2$, because $(u + \xi) \in C^{1,\alpha'_M}$ for some $\alpha'_M \in (\alpha, \alpha_M)$. Moreover, we use (2.6) to obtain that

$$|u(x) - u(0)| \leq \lambda_0^{1+\alpha} + |\nabla u(0)|\lambda_0 \quad \text{for} \quad x \in B_{\lambda_0}. \tag{2.7}$$

Now, we define

$$\mu := \lambda_0^{\alpha+1} + |\nabla u(0)|\lambda_0,$$

$$u_\mu(x) := \frac{u(\lambda_0 x) - u(0)}{\mu} \quad \text{for} \quad x \in B_1,$$

$$a_\mu(t) := \frac{a\left(\frac{\mu t}{\lambda_0}\right)}{a\left(\frac{\mu}{\lambda_0}\right)},$$

and

$$f_\mu := \frac{\lambda_0 f(\lambda_0 x)}{\frac{\mu}{\lambda_0} a\left(\frac{\mu}{\lambda_0}\right)}.$$

And then, from (1.2), (1.9), (1.11), and (2.6) we can check that

$$\|u_\mu\|_{L^\infty(B_1)} \leq 1,$$

$$u_\mu(0) = 0,$$

$$\left|\nabla u_\mu(0)\right| = \frac{\lambda_0}{\mu} |\nabla u(0)|,$$

$$|f_\mu| \leq \left| \frac{\lambda_0 f(\lambda_0 x)}{\frac{\mu}{\lambda_0} a\left(\frac{\mu}{\lambda_0}\right)} \right| \leq \left| \frac{\lambda_0 f(\lambda_0 x)}{\lambda_0^\alpha a\left(\lambda_0^\alpha\right)} \right| \leq \left| \lambda_0^{1-\alpha-\alpha s_a} f(\lambda_0 x) \right| \leq |f(\lambda_0 x)| \leq \delta_0$$

in view of the fact that $\alpha < 1/(1 + s_a)$, a_μ satisfies (1.2), and u_μ satisfies

$$-\operatorname{div}\left(a_\mu\left(\left|\nabla u_\mu\right|\right) \nabla u_\mu\right) = f_\mu \quad \text{in} \ B_1.$$

Similarly to (2.7), we apply Lemma 2.1 to deduce that

$$\left|u_\mu(x) - u_\mu(0)\right| \leq \lambda_0^{1+\alpha} + |\nabla u_\mu(0)|\lambda_0 \quad \text{for any} \quad x \in B_{\lambda_0},$$

which implies that

$$\begin{aligned} |u(\lambda_0 x) - u(0)| &= \mu \left|\frac{u(\lambda_0 x) - u(0)}{\mu}\right| \\ &\leq \mu\left(\lambda_0^{1+\alpha} + |\nabla u_\mu(0)|\lambda_0\right) \\ &\leq \mu\left(\lambda_0^{1+\alpha} + \frac{\lambda_0^2}{\mu}|\nabla u(0)|\right) \\ &\leq \lambda_0^{2(1+\alpha)} + |\nabla u(0)|\,\lambda_0^2 + |\nabla u(0)|\,\lambda_0^{2+\alpha} \\ &\leq \lambda_0^{2(1+\alpha)} + |\nabla u(0)|\,\lambda_0^2 \end{aligned}$$

for any $x \in B_{\lambda_0}$. That is to say, we prove that

$$\sup_{B_{\lambda_0^2}} |u(x) - u(0)| \leq \lambda_0^{2(1+\alpha)} + |\nabla u(0)|\,\lambda_0^2. \tag{2.8}$$

Actually, similarly to the previous procedure, we can prove

$$\sup_{B_{\lambda_0^i}} |u(x) - u(0)| \leq \lambda_0^{i(1+\alpha)} + |\nabla u(0)|\,\lambda_0^i \quad \text{for any integer} \quad i \geq 3 \tag{2.9}$$

by defining

$$\mu_i := \lambda_0^{(i-1)(\alpha+1)} + |\nabla u(0)|\lambda_0^{i-1},$$

$$u_{\mu_i}(x) := \frac{u\left(\lambda_0^{i-1}x\right) - u(0)}{\mu_i} \quad \text{for} \quad x \in B_1,$$

$$a_{\mu_i}(t) := \frac{a\left(\frac{\mu_i t}{\lambda_0^{i-1}}\right)}{a\left(\frac{\mu_i}{\lambda_0^{i-1}}\right)},$$

and

$$f_{\mu_i} := \frac{\lambda_0^{i-1} f(\lambda_0^{i-1}x)}{\frac{\mu_i}{\lambda_0^{i-1}} a\left(\frac{\mu_i}{\lambda_0^{i-1}}\right)}.$$

Let $0 < r \leq \lambda_0 < 1$ and then $\lambda_0^{k+1} < r \leq \lambda_0^k$ for some positive integer $k \in \mathbb{N}$. Therefore, from (2.7)–(2.9), we conclude that

$$\begin{aligned} \sup_{B_r} |u(x) - u(0)| &\leq \sup_{B_{\lambda_0^k}} |u(x) - u(0)| \\ &\leq \lambda_0^{k(1+\alpha)} + |\nabla u(0)|\,\lambda_0^k \\ &= \lambda_0^{-(1+\alpha)}\lambda_0^{(k+1)(1+\alpha)} + |\nabla u(0)|\,\lambda_0^{k+1}\lambda_0^{-1} \\ &\leq C\left(r^{1+\alpha} + |\nabla u(0)|\,r\right). \end{aligned}$$

This completes the proof. □

Next, we shall present the following important lemma, which will be used in proving the second result among the main results, Theorem 1.4.

Lemma 2.3. *For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $v(x)$ satisfies $-div\ (a(|\nabla v|)\nabla v) = f$ in B_1 with $|v(x)| \leq 1$ for $x \in B_1$ and*

$$\|f\|_{L^\infty(B_1)} + \left(v(0) - \inf_{B_1} v\right) \leq \delta,$$

then we have

$$\underset{B_{1/2}}{osc}\, v \leq \epsilon.$$

Proof. We prove it by contradiction. If the statement were false, then there would exist $\epsilon_0 > 0$ and two sequences of functions $\left\{v_j\right\}_{j=1}^\infty$ and $\left\{f_j\right\}_{j=1}^\infty$ satisfying

$$\left|v_j\right| \leq 1, \tag{2.10}$$

$$\left\|f_j\right\|_{L^\infty(B_1)} + \left(v_j(0) - \inf_{B_1} v_j\right) \leq \frac{1}{j}, \tag{2.11}$$

$$-\text{div}\left(a(|\nabla v_j|)\nabla v_j\right) = f_j(x) \ \text{ in } B_1, \tag{2.12}$$

$$\underset{B_{1/2}}{\text{osc}}\, v_j \geq \epsilon_0. \tag{2.13}$$

From the standard local gradient estimate (see [7], Theorem 1.2), we can extract a subsequence of $\{v_j\}_{j=1}^\infty$, still denoted by $\{v_j\}_{j=1}^\infty$, such that $v_j \to \bar{v}$ as $j \to +\infty$, and $\bar{v}$ solves the homogeneous equation

$$-\text{div}\ (a(|\nabla\bar{v}|)\nabla\bar{v}) = 0 \quad \text{in } B_{1/2}. \tag{2.14}$$

From (2.11), we know that $\bar{v}(x)$ attains its minimum value at 0. Without loss of generality, we may as well assume that $|\nabla\bar{v}| > 0$. If not, similarly to Lemma 2.2 in [19], we can consider the approximation problem

$$-\text{div}\left(a(\sqrt{\epsilon + |\nabla\bar{v}|^2})\nabla\bar{v}\right) = 0.$$

It follows from (2.14) that

$$\sum_{i,j=1}^n a_{ij}\bar{v}_{x_ix_j} := \sum_{i,j=1}^n \left(\delta_{ij} + \frac{a'\ (|\nabla\bar{v}|)}{a\ (|\nabla\bar{v}|)} \frac{\bar{v}_{x_i}\bar{v}_{x_j}}{|\nabla\bar{v}|}\right)\bar{v}_{x_ix_j} = 0 \quad \text{in } B_{1/2}. \tag{2.15}$$

Actually, from the condition (1.2), we can check that

$$\begin{aligned}
\sum_{i,j=1}^n a_{ij}\xi_i\xi_j &= |\xi|^2 + \frac{|\nabla\bar{v}|a'\ (|\nabla\bar{v}|)}{a\ (|\nabla\bar{v}|)} \frac{(\xi\cdot\bar{v})^2}{|\nabla\bar{v}|^2} \\
&\geq |\xi|^2 + i_a\frac{(\xi\cdot\bar{v})^2}{|\nabla\bar{v}|^2} \\
&\geq |\xi|^2
\end{aligned}$$

and similarly,

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j &\leq |\xi|^2 + s_a \frac{(\xi\cdot\bar{v})^2}{|\nabla\bar{v}|^2} \\ &\leq (1+s_a)\,|\xi|^2.\end{aligned}$$

Because $v(x) = \bar{v}(x) - \bar{v}(0) \geq 0$ in B_1 also satisfies (2.15), from Theorem 8.18 in [20], we have

$$0 = \inf_{B_{1/2}} (\bar{v}(x) - \bar{v}(0)) \geq C \int_{B_1} (\bar{v}(x) - \bar{v}(0))\, dx \geq 0,$$

which implies that $\bar{v}(x) \equiv \bar{v}(0)$ in $B_{1/2}$ and then leads to a contradiction with (2.13). Thus, we finish the proof. □

3. Final proof

Finally, we shall combine the previous lemmas to finish the main results of this paper. We first provide the proof of Theorem 1.3.

Proof. Without loss of generality, we may assume that $u(0) = 0$. Now, we define

$$u_\lambda(x) = \frac{u(x)}{\lambda},$$

$$f_\lambda(x) = \frac{f(x)}{b(\lambda)},$$

and

$$a_\lambda(t) = \frac{a(\lambda t)}{a(\lambda)},$$

where

$$b(\lambda) = \lambda a(\lambda)$$

and

$$\lambda = b^{-1}\left(\frac{1}{\delta_0}\|f\|_{L^\infty(B_1)}\right) + \|u\|_{W^{1,B}(B_1)}.$$

Then, a_λ satisfies the condition (1.2), and u_λ is still a weak solution of

$$-\operatorname{div}\left(a_\lambda(|\nabla u_\lambda|)\,\nabla u_\lambda\right) = f_\lambda \quad \text{in } B_1 \tag{3.1}$$

with the estimates

$$\|f_\lambda\|_{L^\infty(B_1)} \leq \delta_0 \qquad \text{and} \qquad \|u_\lambda\|_{W^{1,B}(B_1)} \leq 1.$$

So, it is sufficient to prove the following inequality to finish the proof:

$$\|u\|_{C^\alpha(B_{1/2})} + \|\nabla u\|_{C^\alpha(B_{1/2})} \leq C.$$

Therefore, from now on, we may as well assume that the center of the ball is the ordinate origin 0 by translation transformation,

$$\|f\|_{L^\infty(B_1)} \le \delta_0 \qquad \text{and} \qquad \|u\|_{W^{1,B}(B_1)} \le 1.$$

Furthermore, we can suppose that $\|u\|_{L^\infty(B_1)} \le 1$ by Theorem 1.4 in [7]. Actually, we shall analyze the oscillation decay around points where the gradients are small and large: $|\nabla u(0)| \lesssim r$ and $r \lesssim |\nabla u(0)|$. If $|\nabla u(0)| \le r^\alpha$ for any $r \in (0, \lambda_0)$, we can directly use Lemma 2.2 to get that

$$\sup_{B_r} \left| u(x) - u(0) \right| + \sup_{B_r} \left| u(x) - (u(0) + \nabla u(0) \cdot x) \right| \le C r^{1+\alpha} \tag{3.2}$$

for any $r \in (0, \lambda_0)$. Next, we shall claim that there exist three positive constants δ_0, C, and $\tilde{\lambda}$, depending on n, s_a, i_a, such that if $|\nabla u(0)| > r^\alpha$ for any $0 < r \le \tilde{\lambda}$, then for any $0 < r \le \tilde{\lambda}$, we have

$$\sup_{B_r} \left| u(x) - u(0) \right| \le C r^\alpha \tag{3.3}$$

and

$$\sup_{B_r} \left| u(x) - u(0) - \nabla u(0) \cdot (x - 0) \right| \le C r^{1+\alpha}. \tag{3.4}$$

Actually, for $x \in B_1$, we define

$$\mu := |\nabla u(0)|^{\frac{1}{\alpha}},$$

$$u_\mu(x) := \frac{u(\mu x) - u(0)}{\mu^{1+\alpha}},$$

$$a_\mu(t) := \frac{a(\mu^\alpha t)}{a(\mu^\alpha)},$$

and

$$f_\mu := \frac{f(\mu x)}{\mu^{\alpha-1} a(\mu^\alpha)}.$$

It is easy to check that a_μ still satisfies the assumption (1.2), $u_\mu(0) = 0$,

$$|\nabla u_\mu(0)| = \frac{|\nabla u(0)|}{\mu^\alpha} = 1,$$

and u_μ satisfies

$$-\operatorname{div}\left(a_\mu\left(\left| \nabla u_\mu \right| \right) \nabla u_\mu \right) = f_\mu \quad \text{in } B_1. \tag{3.5}$$

Now, we divide into two cases.
Case 1: Assume $\mu \le \lambda_0 \le 1$. Then, we find that

$$|\nabla u(0)| = \mu^\alpha.$$

From (3.2), we know that

$$\sup_{B_\mu}\left|u(x)-u(0)\right|+\sup_{B_\mu}\left|u(x)-(u(0)+\nabla u(0)\cdot x)\right|\le C\mu^{1+\alpha} \tag{3.6}$$

for any $\mu \le \lambda_0 \le 1$. Moreover, by applying the C^0-estimates to ∇u_μ (see Theorem 1.4 in [7]), there exists $\tau_* > 0$ such that

$$\operatorname*{osc}_{B_{\tau_*}} |\nabla u_\mu| < \frac{1}{2}.$$

Because $|\nabla u_\mu(0)| = 1$, we deduce that

$$\frac{1}{2} \le |\nabla u_\mu(x)| \le \frac{3}{2} \quad \text{in} \quad B_{\tau_*}.$$

Because at this moment, the nonlinear PDE (3.5) can be viewed as essentially a linear equation, we can obtain the $C^{1,\beta}$ estimates for the linear equation (see Theorem 3.1 in [17]):

$$\sup_{B_r} |\nabla u_\mu(x) - \nabla u_\mu(0)| \le Cr^\beta \tag{3.7}$$

and

$$\sup_{B_r} |u_\mu(x) - u_\mu(0) - \nabla u_\mu(0)\cdot x| \le Cr^{1+\beta} \tag{3.8}$$

for any $0 < r \le \lambda_*$ with some $\lambda_* > 0$ and any $\beta \in (0, 1)$. For the corresponding proof, we may employ Campanato's embedding theorem, with detailed derivations referred to the proof of Theorem 3.1 in [17] (see §5, Appendix). Furthermore, the two inequalities above imply that

$$\sup_{B_{\mu r}} |u(x) - u(0)| \le C\mu^{1+\alpha} r^\beta \le C(\mu r)^\alpha$$

and

$$\sup_{B_{\mu r}} |u(x) - u(0) - \nabla u(0)\cdot(x-0)| \le C\mu^{1+\alpha} r^{1+\beta} \le C(\mu r)^{1+\alpha}$$

for any $0 < r \le \lambda_*$. Here, we select some $\beta \in (\alpha, 1)$. That is to say,

$$\sup_{B_r} |u(x) - u(0)| \le Cr^\alpha$$

and

$$\sup_{B_r} |u(x) - u(0) - \nabla u(0)\cdot(x-0)| \le Cr^{1+\alpha}$$

for any $0 < r \le \lambda_*\mu$. On the one hand, if $\lambda_*\mu < r \le \mu \le \lambda_0 < 1$, (3.6) implies that

$$\sup_{B_r}\left|u(x)-u(0)\right| \le C\mu^{1+\alpha} \le C\mu^\alpha \le C\left(\frac{1}{\lambda_*}\right)^\alpha r^\alpha \le Cr^\alpha$$

and then

$$\sup_{B_r}\left|u(x)-(u(0)-\nabla u(0)\cdot x)\right|\leq C\mu^{1+\alpha}\leq C\left(\frac{1}{\lambda_*}\right)^{1+\alpha}r^{1+\alpha}\leq Cr^{1+\alpha}.$$

Case 2: Assume $\mu > \lambda_0$. Then, the definition of μ implies that

$$|\nabla u(0)| = \mu^\alpha > \lambda_0^\alpha.$$

Because the argument applied previously in Case 1 for the function u_μ can also be directly used to the function u for the fixed constant $\lambda_0^\alpha > 0$, we can directly get the desired results for (3.3)–(3.4) for any $0 < r < \lambda'$ with some positive constant λ' and then obtain the conclusions for (3.3)–(3.4) by choosing $\tilde{\lambda} := \min\{\lambda', \lambda_0\}$. Let $x, y \in B_{1/2}$ and $|x-y| \leq \tilde{\lambda} < 1$. Then from (3.2)–(3.4), we deduce that

$$\left|u(x)-u(y)\right|\leq C|x-y|^\alpha.$$

And then, we can conclude that

$$\sup_{\substack{x,y\in\Omega' \\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\alpha}\leq C$$

holds true by a standard finite covering argument, which implies that $\|u\|_{C^\alpha(B_{1/2})} \leq C$. Furthermore, without loss of generality, we may as well assume that $x' = (x_1, x_2, ..., x_{i-1}, x_i', x_{i+1}, ..., x_n) \in \Omega'$ and $x'' = (x_1, x_2, ..., x_{i-1}, x_i'', x_{i+1}, ..., x_n) \in \Omega'$. At this time, we can obtain that $|x'-x''| = |x_i - x_i'| \leq \tilde{\lambda} < 1$,

$$u(x'')=u(x')-\frac{\partial u(x')}{\partial x_i}(x_i'-x_i'')+O\left(|x_i'-x_i''|^{\alpha+1}\right),$$

and

$$u(x')=u(x'')+\frac{\partial u(x'')}{\partial x_i}(x_i'-x_i'')+O\left(|x_i'-x_i''|^{\alpha+1}\right),$$

which implies that

$$\left|\frac{\partial u(x')}{\partial x_i}-\frac{\partial u(x'')}{\partial x_i}\right||x'-x''|=O\left(|x'-x''|^{\alpha+1}\right).$$

Therefore, we can reach the conclusion that

$$\sup_{\substack{x,y\in\Omega' \\ x\neq y}}\frac{\left|\frac{\partial u(x)}{\partial x_i}-\frac{\partial u(y)}{\partial x_i}\right|}{|x-y|^\alpha}\leq C$$

by a standard finite covering argument. Thus, we can get the conclusion $\|\nabla u\|_{C^\alpha(B_{1/2})} \leq C$ and then finish the final proof of Theorem 1.3. □

Finally, we shall finish the proof of the second main result in this paper: Theorem 1.4.

Proof. Without loss of generality, we can assume that $x_0 = 0$ and $u(0) = 0$ by translations. Let $\varphi(x)$ be the $(1 + s_a')$-cap touching u at 0 from below and $v(x) = u(\lambda x)$ for $\lambda \in (0, 1)$. Then, one can verify that $v(x)$ is touched from below by the $(1 + s_a')$-cap $\varphi_\lambda(x) := \varphi(\lambda x)$, and $v_\lambda(x)$ satisfies

$$-\operatorname{div}\,(a_\lambda(|\nabla v_\lambda|)\nabla v_\lambda) = f_\lambda \quad \text{in } \ B_1,$$

where

$$a_\lambda(t) = \frac{a\left(\frac{t}{\lambda}\right)}{a\left(\frac{1}{\lambda}\right)},$$

$$v_\lambda(x) = u(\lambda x) - u(0) = u(\lambda x),$$

and

$$f_\lambda = \frac{\lambda^2}{a\left(\frac{1}{\lambda}\right)} f(\lambda x).$$

Choose $\epsilon_* = 2^{-(1+s_a')}$ and then fix the closeness number $\delta_* > 0$ in Lemma 2.3 satisfying

$$|f_\lambda| = \left| \frac{\lambda^2}{a\left(\frac{1}{\lambda}\right)} f(\lambda x) \right| \le C\lambda^{2+i_a} \le \frac{\delta_*}{2}$$

by (1.9) and choosing λ small enough. Similarly, we have

$$\sup_{B_r} \varphi_\lambda \le [\varphi]_{1+s_a'} r^{1+s_a'} \lambda^{1+s_a'} \le \frac{r^{1+s_a'}\delta_*}{2} \quad \text{for any } \ 0 < r \le 1. \tag{3.9}$$

Moreover, from the C^1 regularity estimate, we have

$$|v_\lambda(x)| = |u(\lambda x) - u(0)| \le C\lambda \le 1,$$

and then

$$\left(v_\lambda(0) - \inf_{B_1} v_\lambda\right) = -\inf_{B_1} v_\lambda = -\inf_{B_\lambda} u \le \frac{\delta_*}{2}$$

by choosing λ small enough if necessary. Therefore, from Lemma 2.3, we obtain

$$\operatorname*{osc}_{B_{1/2}} v_\lambda(x) \le \epsilon_* = 2^{-(1+s_a')}.$$

Now, we aim to infer by inductive reasoning that

$$\operatorname*{osc}_{B_{2^{-k}}} v_\lambda(x) \le \epsilon_*^k = 2^{-k(1+s_a')}. \tag{3.10}$$

Suppose that the above conclusion holds true for $k \in \mathbb{N}$. We denote $v_{k+1}(x)$ by

$$v_{k+1}(x) := 2^{k(1+s'_a)} v_\lambda \left(2^{-k} x\right) = 2^{k(1+s'_a)} u \left(2^{-k} \lambda x\right).$$

Then, it is easy to check that $v_{k+1}(0) = 0$, $|v_{k+1}(x)| \leq 1$ for any $x \in B_1$, and $v_{k+1}(x)$ satisfies

$$-\mathrm{div}\, (a_k(|\nabla v_{k+1}|)\nabla v_{k+1}) = f_k \quad \text{in } B_1,$$

where

$$a_k(t) = \frac{a\left(\frac{2^{-ks'_a}}{\lambda} t\right)}{a\left(\frac{2^{-ks'_a}}{\lambda}\right)}$$

and

$$f_k = \frac{2^{k(s'_a-1)} \lambda^2 f(2^{-k} \lambda x)}{a\left(\frac{2^{-ks'_a}}{\lambda}\right)}.$$

Because from (3.9), we can estimate

$$\begin{aligned} \left(v_{k+1}(0) - \inf_{B_1} v_{k+1}(x)\right) &= -2^{k(1+s'_a)} \inf_{B_1} v_\lambda \left(2^{-k} x\right) \\ &\leq -2^{k(1+s'_a)} \inf_{B_{2^{-k}}} \varphi_\lambda \\ &\leq \frac{\delta_*}{2} \end{aligned}$$

and

$$|f_k| \leq C 2^{k(s'_a + s'_a s_a - 1)} \lambda^{2+i_a} \leq \frac{\delta_*}{2},$$

we apply Lemma 2.3 again to deduce that

$$\operatorname*{osc}_{B_{1/2}} v_{k+1}(x) = 2^{k(1+s'_a)} \operatorname*{osc}_{B_{2^{-(k+1)}}} v_\lambda(x) \leq 2^{-(1+s'_a)},$$

which implies that (3.10) is true for any positive integer $k \in \mathbb{N}$. Let any $r \in (0, 1/2)$ and then $2^{-(k+1)} < r < 2^{-k}$ for some $k \in \mathbb{N}$. Thus, we use (3.10) to prove that

$$\operatorname*{osc}_{B_{\lambda r}} u = \operatorname*{osc}_{B_r} v_\lambda \leq \operatorname*{osc}_{B_{2^{-k}}} v_\lambda \leq 2^{-k(1+s'_a)} \leq C r^{1+s'_a} \leq C(\lambda r)^{1+s'_a},$$

which finishes our proof by a standard finite covering argument. □

Use of Generative-AI tools declaration

The author declares that he has not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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