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*Research article*

## Stability and exponential decay for the 2D incompressible MHD system with fractional horizontal dissipation

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**Abstract:** This paper focuses on two-dimensional (2D) incompressible magnetohydrodynamic (MHD) equations with only fractional horizontal dissipation  $\Lambda_1^{2\alpha}u$  and  $\Lambda_1^{2\beta}b$  in the spatial domain  $\Omega = \mathbb{T} \times \mathbb{R}$  (where  $\mathbb{T} = [0, 1]$  is a periodic box, and  $\mathbb{R}$  is the whole line). For this system, Feng, Wang, and Wu assessed the global stability of perturbations near the steady solution given by a background magnetic field  $A = (A_1, A_2)$  with  $A_1, A_2 \in \mathbb{R}$  and established nonlinear stability in the Sobolev space  $H^3(\Omega)$ . In this paper, we consider the stability problem in the lower regularity space  $H^2(\Omega)$ . By applying the Hölder's inequality with various anisotropic fractional exponents and some special anisotropic interpolation inequalities, we obtain the global stability of the system when  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$  in  $H^2(\Omega)$ . In addition, we prove that the oscillation part  $(\tilde{u}, \tilde{b})$  of the solution in  $H^1(\Omega)$  decays to zero exponentially in time.

**Keywords:** magnetohydrodynamic equations; fractional horizontal dissipation; anisotropic interpolation inequalities; global stability; exponential decay

**Mathematics Subject Classification:** 35B35, 35B40, 76E25

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### 1. Introduction

Magnetohydrodynamic (MHD) equations describe the fundamental physical laws governing the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. MHD equations have been widely used in the study of astrophysics, geophysics, and cosmological phenomena (see, e.g., [1–3]). Particularly, under some specific physical conditions, MHD equations can model so-called anomalous diffusion or attenuation phenomena by incorporating fractional Laplacian operators (see, e.g., [4, 5]).

MHD equations involving fractional dissipation are also mathematically significant. These equations not only allow us to investigate a family of equations simultaneously but also provide a comprehensive understanding of how the behavior of solutions changes with the dissipation power. Two-dimensional (2D) incompressible MHD equations with fractional dissipation can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu(-\Delta)^\alpha u + \nabla P = B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B + \eta(-\Delta)^\beta B = B \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \end{cases} \quad (1.1)$$

where  $u$  represents the velocity field,  $B$  represents the magnetic field, and  $P$  denotes the pressure. The positive constants  $\nu$  and  $\eta$  are the viscosity and the magnetic damping coefficients, respectively. The fractional partial operator  $(-\Delta)^\alpha$  is defined by the Fourier transform

$$\widehat{(-\Delta)^\alpha f(\xi)} = |\xi|^{2\alpha} \hat{f}(\xi).$$

Here, we also write  $\Lambda^{2\alpha} = (-\Delta)^\alpha$  for any  $\alpha \in \mathbb{R}$ . In particular,  $\Lambda^{2\alpha}$  with  $\alpha = 0$  becomes the identity operator.

Due to their physical applications and mathematical importance, the MHD equations with fractional dissipation given by (1.1) have recently attracted considerable interest. So far, substantial progress has been achieved in the analysis of global regularity (see, e.g., [6–10]) and well-posedness (see, e.g., [11–14]); further related results can be found in [15–19]. However, there are relatively few studies on the stability problem of System (1.1). In fact, studying the global stability of System (1.1) is significant for understanding the behavior of solutions during long-term evolution and how the system responds to small perturbations. In recent decades, increasing attention has been devoted to the stability problem of MHD equations, particularly for systems with partial dissipation (see, e.g., [20–24]). Specifically, Feng, Wang, and Wu [22] analyzed the stability problem for the 2D incompressible MHD system with fractional horizontal dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P + \nu \Lambda_1^{2\alpha} u = B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B + \eta \Lambda_1^{2\beta} B = B \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \end{cases} \quad (1.2)$$

where the spatial domain is defined as

$$\Omega = \mathbb{T} \times \mathbb{R}$$

with  $\mathbb{T} = [0, 1]$  being a 1D periodic box and  $\mathbb{R}$  being the whole line. When  $\alpha, \beta > 0$ , the authors in [10] established the nonlinear stability in the Sobolev space  $H^3(\Omega)$  for perturbations near any fixed magnetic field  $B^{(0)} = A = (A_1, A_2)$  with  $A_1, A_2 \in \mathbb{R}$ . Furthermore, in the case of  $\alpha = 1$  and  $\beta = 0$ , Feng, Hafeez, Wu, and Regmi [21] obtained the stability of the solution  $(u, b)$  in  $H^3(\Omega)$  when  $B$  is close to the equilibrium state  $B^{(0)} = (1, 0)$ .

Based on [21] and [22], a natural question is whether the global stability holds in  $H^2(\Omega)$  with lower regularity when the horizontal dissipative powers  $\alpha$  and  $\beta$  satisfy  $0 < \alpha, \beta \leq 1$ . The answer is affirmative. In this paper, we investigate the stability problem of System (1.2) near the equilibrium state

$$u^{(0)} = (0, 0), \quad B^{(0)} = A = (A_1, A_2),$$

where  $A_1$  and  $A_2$  are arbitrarily fixed real numbers. Let  $(u, b)$  be the perturbation near the steady state  $(u^{(0)}, B^{(0)})$  with  $b = B - A$ . The system governing the perturbation is given as follows:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P + \nu \Lambda_1^{2\alpha} u = b \cdot \nabla b + A \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b + \eta \Lambda_1^{2\beta} b = b \cdot \nabla u + A \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \\ u(x, 0) = u_0(x), b(x, 0) = b_0(x). \end{cases} \quad (1.3)$$

In order to understand the desired stability, we utilize the properties of the spatial domain  $\Omega$ , which allows a physical quantity to be decomposed into its horizontal average and the corresponding oscillation part. Precisely, for a function  $f = f(x_1, x_2)$  integrable in  $x_1$  on  $\mathbb{T}$ , we define the horizontal average

$$\bar{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2) dx_1 \quad (1.4)$$

and write

$$f = \bar{f} + \tilde{f}. \quad (1.5)$$

This decomposition and the oscillation part possess several desirable mathematical properties, such as that  $\bar{f}$  and  $\tilde{f}$  are orthogonal in  $L^2(\Omega)$ , and that the oscillation part  $\tilde{f}$  satisfies a very general strong Poincaré type inequality. Detailed statements and proofs are given in Lemmas 2.1 and 2.2.

To estimate the nonlinear terms in the spatial domain  $\Omega$ , Feng, Wang, and Wu [22] used the following inequality:

$$\left| \int_{\Omega} fgh dx \right| \leq C \|f\|_{L_{x_2}^2(\mathbb{R})L_{x_1}^\infty(\mathbb{T})} \|g\|_{L_{x_2}^\infty(\mathbb{R})L_{x_1}^2(\mathbb{T})} \|h\|_{L_{x_2}^2(\mathbb{R})L_{x_1}^2(\mathbb{T})}. \quad (1.6)$$

Applying the inequality (1.6) and some anisotropic inequalities, the nonlinear stability for System (1.3) in  $H^3(\Omega)$  is established. Compared with [22], we focus on the stability in the framework of  $H^2(\Omega)$ . However, using the existing methods in [22], the stability problem in the lower regularity space  $H^2(\Omega)$  cannot be resolved. Therefore, some new approaches are needed. In the following, we will briefly describe the main differences between our methods and those in [22].

Unlike inequality (1.6), we utilize Hölder's inequality with various anisotropic fractional exponents in this paper. For example, when  $0 < \alpha < 1$ , we use the following inequality:

$$\left| \int_{\Omega} fgh dx \right| \leq C \|f\|_{L_{x_2}^\infty(\mathbb{R})L_{x_1}^{\frac{2}{\alpha}}(\mathbb{T})} \|g\|_{L_{x_2}^2(\mathbb{R})L_{x_1}^2(\mathbb{T})} \|h\|_{L_{x_2}^2(\mathbb{R})L_{x_1}^{\frac{2}{1-\alpha}}(\mathbb{T})}. \quad (1.7)$$

Here,  $\|f\|_{L_{x_2}^\infty(\mathbb{R})L_{x_1}^{2/\alpha}(\mathbb{T})}$  represents the  $L^{2/\alpha}$ -norm in the  $x_1$ -variable over  $\mathbb{T}$ , followed by the  $L^\infty$ -norm in  $x_2$  over  $\mathbb{R}$ . To derive the upper bounds of the terms on the right-hand side of inequality (1.7), we use some elementary 1D interpolation inequalities (see Lemma 2.3 for details), Minkowski's inequality, and the Sobolev embedding  $W^{(1-\eta)/2,2}(\mathbb{T}) \hookrightarrow L^{2/\eta}(\mathbb{T})$  for any fixed  $\eta \in (0, 1]$ . These tools allow us to obtain, for  $\epsilon > 0$ ,  $\eta \in (0, 1]$  and  $\gamma \in [0, 1]$ ,

$$\|f\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\eta}}(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1-\eta}{2}} f\|_{L^2(\Omega)} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1-\eta}{2}} \Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} \right),$$

and

$$\|h\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\gamma}}(\Omega)} \leq C \|h\|_{L^2(\Omega)}^{\frac{1}{2}} \left( \|h\|_{L^2(\Omega)} + \|\Lambda_1^\gamma h\|_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

The detailed proofs can be found in Lemma 2.4.

Applying Hölder's inequality with various anisotropic fractional exponents and the inequalities in Lemmas 2.3 and 2.4, combined with a very general strong Poincaré-type inequality, we establish the global stability for System (1.3) when  $0 < \alpha, \beta \leq 1$  in  $H^2(\Omega)$ .

**Theorem 1.1.** *Let  $\Omega = \mathbb{T} \times \mathbb{R}$ , with  $\mathbb{T} = [0, 1]$  being a 1D periodic box and  $\mathbb{R}$  being the whole line. Let  $\nu, \eta > 0$ ,  $\alpha \in (0, 1]$ , and  $\beta \in [0, 1]$ . Consider (1.3) with the initial data  $u_0, b_0 \in H^2(\Omega)$ , and  $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ . Then, there exists a constant  $\varepsilon_0 := \varepsilon_0(\nu, \eta) > 0$  such that if  $\varepsilon \leq \varepsilon_0$  and*

$$\|u_0\|_{H^2} + \|b_0\|_{H^2} \leq \varepsilon, \quad (1.8)$$

then (1.3) has a unique global solution that remains uniformly bounded,

$$\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + 2\nu \int_0^t \|\Lambda_1^\alpha u(\tau)\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\Lambda_1^\beta b(\tau)\|_{H^2}^2 d\tau \leq C_1 \varepsilon^2, \quad (1.9)$$

for some constant  $C_1$  and for all  $t > 0$ .

The proof of Theorem 1.1 is not trivial. A natural starting point is to bound  $\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2$  via energy estimates. The framework of the proof for the  $H^2$ -bound is the bootstrapping argument (see, e.g., [25]). The process starts with the setup of a suitable energy functional. More precisely, we set

$$E(t) := \sup_{0 \leq \tau \leq t} (\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2) + 2\nu \int_0^t \|\Lambda_1^\alpha u(\tau)\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\Lambda_1^\beta b(\tau)\|_{H^2}^2 d\tau.$$

We primarily focus on proving that for some constant  $C > 0$  and for all  $t > 0$ ,

$$E(t) \leq E(0) + CE(t)^{\frac{3}{2}}. \quad (1.10)$$

The bootstrapping argument then implies that when

$$\|u_0\|_{H^2} + \|b_0\|_{H^2} \leq \varepsilon,$$

there exists a constant  $C_1 > 0$ , and for any  $t > 0$ ,

$$E(t) \leq C_1 \varepsilon^2.$$

We thus derive the global uniform  $H^2$ -bound for  $(u, b)$  and stability.

The second main result indicates that the oscillation component  $(\tilde{u}, \tilde{b})$  decays to zero exponentially in time in the  $H^1$ -norm.

**Theorem 1.2.** *Let  $u_0, b_0 \in H^2(\Omega)$  with  $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ . Assume that  $(u_0, b_0)$  satisfies (1.8) for sufficiently small  $\varepsilon > 0$ . Let  $(u, b)$  be the corresponding solution to (1.3) with  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$ . Then, the  $H^1$ -norm of the oscillation part  $(\tilde{u}, \tilde{b})$  decays exponentially in time,*

$$\|\tilde{u}(t)\|_{H^1} + \|\tilde{b}(t)\|_{H^1} \leq C (\|u_0\|_{H^1} + \|b_0\|_{H^1}) e^{-C_2 t}, \quad (1.11)$$

for some constant  $C_2 > 0$  and for all  $t > 0$ .

**Remark 1.1.** For  $\alpha, \beta \in (0, 1]$ , Theorems 1.1 and 1.2 address the stability and decay of System (1.3) near the origin  $u^{(0)} = (0, 0)$ ,  $B^{(0)} = (0, 0)$ , which corresponds to the stability and decay of System (1.2) about the steady state  $u^{(0)} = (0, 0)$ ,  $B^{(0)} = A = (A_1, A_2)$ .

When  $\beta = 0$ , and  $\alpha \in (0, 1]$  with  $(A_1, A_2) \neq (0, 0)$ , the state  $u^{(0)} = (0, 0)$ ,  $B^{(0)} = A = (A_1, A_2)$  is no longer a steady state of System (1.2). In this case, the conclusions of Theorems 1.1 and 1.2 are only suitable for System (1.3) (i.e., the following system (1.12)) near the origin  $u^{(0)} = (0, 0)$ ,  $B^{(0)} = (0, 0)$ .

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P + \nu \Lambda_1^{2\alpha} u = b \cdot \nabla b + A \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b + \eta b = b \cdot \nabla u + A \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \\ u(x, 0) = u_0(x), b(x, 0) = b_0(x). \end{cases} \quad (1.12)$$

The structure of this paper is as follows. In Section 2, we present some properties of the orthogonal decomposition, prove a very general strong Poincaré-type inequality for the oscillation part, and derive several anisotropic inequalities with fractional derivatives. Section 3 is devoted to the proof of Theorem 1.1, and Section 4 proves Theorem 1.2.

## 2. Preliminaries

In this section, we first give some fundamental properties of the horizontal average and corresponding oscillatory part. Then, several anisotropic inequalities with fractional derivatives are presented, which are crucial in the proofs of Theorems 1.1 and 1.2.

**Lemma 2.1.** Assume the 2D function  $f$  defined on  $\Omega = \mathbb{T} \times \mathbb{R}$  is sufficiently regular. Let  $\bar{f}$  and  $\widetilde{f}$  be defined as in (1.4) and (1.5). Then,

$$\widetilde{\bar{f}} = 0, \quad \overline{\partial_1 f} = \partial_1 \bar{f} = 0, \quad \overline{\partial_2 f} = \partial_2 \bar{f}, \quad \widetilde{\partial_2 f} = \partial_2 \widetilde{f}. \quad (2.1)$$

If  $f$  is a divergence-free vector field (i.e.,  $\nabla \cdot f = 0$ ), then both the average component  $\bar{f}$  and the oscillatory component  $\widetilde{f}$  inherit the divergence-free property,

$$\nabla \cdot \bar{f} = 0 \quad \text{and} \quad \nabla \cdot \widetilde{f} = 0. \quad (2.2)$$

Let  $\Omega = \mathbb{T} \times \mathbb{R}$ . For any  $f \in L^2(\Omega)$ , we have

$$(\bar{f}, \widetilde{f}) := \int_{\Omega} \bar{f} \widetilde{f} dx = 0, \quad \|f\|_{L^2(\Omega)}^2 = \|\bar{f}\|_{L^2(\Omega)}^2 + \|\widetilde{f}\|_{L^2(\Omega)}^2. \quad (2.3)$$

*Proof of Lemma 2.1.* It is easy to show (2.1) by the definitions of  $\bar{f}$  and  $\widetilde{f}$ . If  $\nabla \cdot f = 0$ , then

$$0 = \overline{\partial_1 f + \partial_2 f} = \partial_1 \bar{f} + \partial_2 \bar{f} = \nabla \cdot \bar{f} = \nabla \cdot f - \nabla \cdot \widetilde{f} = -\nabla \cdot \widetilde{f}.$$

For (2.3), according to the definitions of  $\bar{f}$  and  $\widetilde{f}$ ,

$$(\bar{f}, \widetilde{f}) = \int_{\Omega} \bar{f} \widetilde{f} dx = \int_{\mathbb{R}} \bar{f} \left( \int_{\mathbb{T}} \widetilde{f}(x_1, x_2) dx_1 \right) dx_2 = 0.$$

This completes the proof of Lemma 2.1. □

The following lemma assesses that the oscillation part  $\tilde{f}$  obeys a very general strong Poincaré-type inequality.

**Lemma 2.2.** *Let  $\Omega = \mathbb{T} \times \mathbb{R}$ , and let  $\tilde{f}$  be the oscillation part of  $f$ . Then, for any  $\gamma \geq 0$ ,*

$$\|\tilde{f}\|_{L^2(\Omega)} \leq \|\Lambda_1^\gamma \tilde{f}\|_{L^2(\Omega)} = \|\Lambda_1^\gamma f\|_{L^2(\Omega)}. \quad (2.4)$$

*Proof of Lemma 2.2.* Invoking the Fourier transform, we have

$$\begin{aligned} \|\tilde{f}\|_{L^2(\Omega)}^2 &= \sum_{n \in \mathbb{Z}, n \neq 0} \int_{\mathbb{R}} |\hat{f}(n, \xi)|^2 d\xi \\ &\leq \sum_{n \in \mathbb{Z}, n \neq 0} \int_{\mathbb{R}} |n|^\gamma |\hat{f}(n, \xi)|^2 d\xi \\ &= \|\Lambda_1^\gamma \tilde{f}\|_{L^2(\Omega)}^2. \end{aligned}$$

In addition,

$$\sum_{n \in \mathbb{Z}, n \neq 0} \int_{\mathbb{R}} |n|^\gamma |\hat{f}(n, \xi)|^2 d\xi = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |n|^\gamma |\hat{f}(n, \xi)|^2 d\xi = \|\Lambda_1^\gamma f\|_{L^2(\Omega)}^2,$$

where  $\hat{f}(n, \xi)$  is the Fourier transform of  $f$  in  $\Omega$ , and

$$\hat{f}(n, \xi) = \int_{\mathbb{R}} \int_{\mathbb{T}} f(x, y) e^{i(nx + \xi y)} dx dy.$$

This completes the proof of Lemma 2.2. □

Next, we present some basic interpolation inequalities for one-dimensional functions on the whole line  $\mathbb{R}$  and bounded domains (including periodic domains), respectively.

**Lemma 2.3.** ([26]) *For any 1D function  $f = f(x)$ ,*

(1) *suppose that  $f(x) \in H^{(1/2)+\epsilon}(\mathbb{R})$  with  $\epsilon > 0$ , which gives*

$$\|f\|_{L^\infty(\mathbb{R})} \leq C \left( \|f\|_{L^2(\mathbb{R})} + \|\Lambda^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbb{R})} \right); \quad (2.5)$$

(2) *suppose that  $f(x) \in H^{(1/2)+\epsilon}(\mathbb{T})$  with  $\epsilon > 0$ , which gives*

$$\|f\|_{L^\infty(\mathbb{T})} \leq C \left( \|f\|_{L^2(\mathbb{T})} + \|\Lambda^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbb{T})} \right); \quad (2.6)$$

(3) *suppose that  $f(x) \in H^\alpha(\mathbb{T})$  for any  $\alpha \in [0, 1]$ , which gives*

$$\|f\|_{L^{\frac{2}{1-\alpha}}(\mathbb{T})} \leq C \left( \|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \|\Lambda^\alpha f\|_{L^2(\mathbb{T})}^{\frac{1}{2}} + \|f\|_{L^2(\mathbb{T})} \right). \quad (2.7)$$

As a consequence of (2.5)–(2.7), we obtain the anisotropic inequalities with fractional derivatives in the following lemma.

**Lemma 2.4.** Let  $\Omega = \mathbb{T} \times \mathbb{R}$ . For any 2D function  $f = f(x_1, x_2)$  on  $\Omega$ ,

(1) suppose that  $f \in H^\gamma(\Omega)$  with  $\gamma \in [0, 1]$ , which yields

$$\|f\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\gamma}}(\Omega)} \leq C \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \left( \|f\|_{L^2(\Omega)} + \|\Lambda_1^\gamma f\|_{L^2(\Omega)} \right)^{\frac{1}{2}}; \quad (2.8)$$

(2) suppose that  $f \in H^{1+2\epsilon}(\Omega)$  with  $\epsilon > 0$ , which yields

$$\|f\|_{L^\infty(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} \right); \quad (2.9)$$

(3) suppose that  $f \in H^{1+\epsilon}(\Omega)$  with  $\epsilon > 0$  sufficiently small. For any fixed  $\eta \in (0, 1]$ ,

$$\begin{aligned} \|f\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\eta}}(\Omega)} &\leq C \left( \|f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1-\eta}{2}} f\|_{L^2(\Omega)} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1-\eta}{2}} \Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} \right) \\ &\leq C \|f\|_{H^1(\Omega)}. \end{aligned} \quad (2.10)$$

*Proof of Lemma 2.4.* We first prove (2.8). It follows from (2.7) that

$$\begin{aligned} \|f\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\gamma}}(\Omega)} &= \left\| \|f\|_{L_{x_1}^{\frac{2}{1-\gamma}}(\mathbb{T})} \right\|_{L_{x_2}^2(\mathbb{R})} \\ &\leq C \left\| \|f\|_{L_{x_1}^2(\mathbb{T})}^{\frac{1}{2}} \|\Lambda^\gamma f\|_{L_{x_1}^2(\mathbb{T})}^{\frac{1}{2}} + \|f\|_{L_{x_1}^2(\mathbb{T})} \right\|_{L_{x_2}^2(\mathbb{R})} \\ &\leq C \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\Lambda^\gamma f\|_{L^2(\Omega)}^{\frac{1}{2}} + \|f\|_{L^2(\Omega)} \\ &\leq C \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \left( \|\Lambda^\gamma f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right)^{\frac{1}{2}}. \end{aligned}$$

To prove (2.9), we resort to Minkowski's inequality, (2.5), and (2.6),

$$\begin{aligned} \|f\|_{L^\infty(\Omega)} &= \left\| \|f\|_{L_{x_2}^\infty(\mathbb{R})} \right\|_{L_{x_1}^\infty(\mathbb{T})} \\ &\leq C \left\| \|f\|_{L_{x_2}^2(\mathbb{R})} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_2}^2(\mathbb{R})} \right\|_{L_{x_1}^\infty(\mathbb{T})} \\ &\leq C \left( \left\| \|f\|_{L_{x_1}^\infty(\mathbb{T})} \right\|_{L_{x_2}^2(\mathbb{R})} + \left\| \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^\infty(\mathbb{T})} \right\|_{L_{x_2}^2(\mathbb{R})} \right) \\ &\leq C \left\| \|f\|_{L_{x_1}^\infty(\mathbb{T})} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^2(\mathbb{T})} \right\|_{L_{x_2}^2(\mathbb{R})} \\ &\quad + C \left\| \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^2(\mathbb{T})} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^2(\mathbb{T})} \right\|_{L_{x_2}^2(\mathbb{R})} \\ &\leq C \left( \|f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} \right). \end{aligned}$$

For (2.10), applying (2.5), Minkowski's inequality and the Sobolev embedding  $W^{(1-\eta)/2,2}(\mathbb{T}) \hookrightarrow L^{2/\eta}(\mathbb{T})$

for any fixed  $\eta \in (0, 1]$ , we have

$$\begin{aligned}
 \|f\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\eta}}(\Omega)} &\leq C \left\| \|f\|_{L_{x_2}^\infty(\mathbb{R})} \right\|_{L_{x_1}^{\frac{2}{\eta}}(\mathbb{T})} \\
 &\leq C \left\| \|f\|_{L_{x_2}^2(\mathbb{R})} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_2}^2(\mathbb{R})} \right\|_{L_{x_1}^{\frac{2}{\eta}}(\mathbb{T})} \\
 &\leq C \left( \left\| \|f\|_{L_{x_1}^{\frac{2}{\eta}}(\mathbb{T})} \right\|_{L_{x_2}^2(\mathbb{R})} + \left\| \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^{\frac{2}{\eta}}(\mathbb{T})} \right\|_{L_{x_2}^2(\mathbb{R})} \right) \\
 &\leq C \left( \|f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1-\eta}{2}} f\|_{L^2(\Omega)} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} + \|\Lambda_1^{\frac{1-\eta}{2}} \Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\Omega)} \right) \\
 &\leq C \|f\|_{H^1(\Omega)}.
 \end{aligned}$$

This completes the proof of Lemma 2.4.  $\square$

### 3. Proof of Theorem 1.1

This section proves Theorem 1.1. Our main efforts are devoted to establishing (1.10).

*Proof of Theorem 1.1.* To establish the global existence and stability of the solutions to (1.3), we first prove the local existence result, which can be achieved by the standard contraction mapping argument combined with a local-in-time a priori bound. The application of the contraction mapping argument is standard and can be found in [27]. Here, we focus on deriving the local a priori bound. Taking the inner product in  $H^2$  of  $(u, b)$  in (1.3), we find

$$\begin{aligned}
 &\frac{d}{dt} \left( \|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 \right) + 2\nu \|\Lambda_1^\alpha u\|_{H^2}^2 + 2\eta \|\Lambda_1^\beta b\|_{H^2}^2 \\
 &= -2 \int_{\Omega} \nabla u \cdot \nabla \omega \cdot \nabla \omega dx + 2 \int_{\Omega} [\nabla b + (\nabla b)^\top] \cdot \nabla j \cdot \nabla \omega dx - 2 \int_{\Omega} \nabla u \cdot \nabla j \cdot \nabla j dx \\
 &\quad + 2 \int_{\Omega} Q \cdot j dx + 2 \int_{\Omega} \nabla Q \cdot \nabla j dx \\
 &=: I_1 + I_2 + I_3 + I_4 + I_5,
 \end{aligned} \tag{3.1}$$

where  $\|(u, b)\|_{H^2}^2 = \|(u, b)\|_{L^2}^2 + \|(u, b)\|_{H^1}^2 + \|(u, b)\|_{H^2}^2$ . Here, in order to estimate the  $\dot{H}^1$ -norm and  $\dot{H}^2$ -norm, we employ the vorticity equations associated with the system (1.3),

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \nu \Lambda_1^{2\alpha} \omega = b \cdot \nabla j + A \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j + \eta \Lambda_1^{2\beta} j = b \cdot \nabla \omega + Q + A \cdot \nabla \omega, \end{cases} \tag{3.2}$$

with

$$Q = 2\partial_1 b_1 (\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1 (\partial_2 b_1 + \partial_1 b_2),$$

where the  $\omega = \nabla \times u$  is transported by the velocity field, and the  $j = \nabla \times b$  indicates the current density. By Hölder's, Minkowski's, Young's, and Sobolev inequalities, (2.8), and (2.9), we have for sufficiently

small  $\epsilon > 0$  and  $\alpha \in (0, 1]$ ,

$$\begin{aligned} |I_1| &\leq C \|\nabla u\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\alpha}}} \|\nabla \omega\|_{L_{x_2}^2 L_{x_1}^2} \|\nabla \omega\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \\ &\leq C \|\nabla u\|_{H^1} \|\nabla \omega\|_{L^2} \|\nabla \omega\|_{L^2}^{\frac{1}{2}} \left( \|\nabla \omega\|_{L^2} + \|\Lambda_1^\alpha \nabla \omega\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{H^2} \|\nabla \omega\|_{L^2} \|\nabla \omega\|_{L^2}^{\frac{1}{2}} \left( \|\nabla \omega\|_{L^2} + \|\Lambda_1^\alpha \nabla \omega\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq \epsilon \|\Lambda_1^\alpha u\|_{H^2}^2 + C \left( \|u\|_{H^2}^3 + \|u\|_{H^2}^{\frac{10}{3}} \right). \end{aligned}$$

For  $I_2$ , when  $\alpha \in (0, 1]$ ,

$$\begin{aligned} |I_2| &\leq C \|\nabla b\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\alpha}}} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} \|\nabla \omega\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \\ &\leq C \|\nabla b\|_{H^1} \|\nabla j\|_{L^2} \|\nabla \omega\|_{L^2}^{\frac{1}{2}} \left( \|\nabla \omega\|_{L^2} + \|\Lambda_1^\alpha \nabla \omega\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|b\|_{H^2} \|\nabla j\|_{L^2} \|\nabla \omega\|_{L^2}^{\frac{1}{2}} \left( \|\nabla \omega\|_{L^2} + \|\Lambda_1^\alpha \nabla \omega\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq \epsilon \|\Lambda_1^\alpha u\|_{H^2}^2 + C \left( \|(u, b)\|_{H^2}^3 + \|(u, b)\|_{H^2}^{\frac{10}{3}} \right). \end{aligned}$$

For the term  $I_3$ , it follows for  $\beta \in (0, 1]$  that

$$\begin{aligned} |I_3| &\leq C \|\nabla u\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\beta}}} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\beta}}} \\ &\leq C \|\nabla u\|_{H^1} \|\nabla j\|_{L^2} \|\nabla j\|_{L^2}^{\frac{1}{2}} \left( \|\nabla j\|_{L^2} + \|\Lambda_1^\beta \nabla j\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{H^2} \|\nabla j\|_{L^2} \|\nabla j\|_{L^2}^{\frac{1}{2}} \left( \|\nabla j\|_{L^2} + \|\Lambda_1^\beta \nabla j\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq \epsilon \|\Lambda_1^\beta u\|_{H^2}^2 + C \left( \|(u, b)\|_{H^2}^3 + \|(u, b)\|_{H^2}^{\frac{10}{3}} \right), \end{aligned}$$

and when  $\beta = 0$ , it follows from (2.9) that

$$\begin{aligned} |I_3| &\leq C \|\nabla u\|_{L_{x_2}^\infty L_{x_1}^\infty} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \left( \|\nabla u\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \nabla u\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \nabla u\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \nabla u\|_{L^2} \right) \|b\|_{H^2}^2. \end{aligned}$$

To proceed the estimate of  $I_3$ , we consider two cases:  $\alpha \in (0, 1/2]$  and  $\alpha \in (1/2, 1]$ . If  $\alpha \in (0, 1/2]$ , choosing  $\epsilon$  small enough, we have

$$\begin{aligned} |I_3| &\leq C \left( \|u\|_{H^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon-\alpha} \Lambda_2^{\frac{1}{2}+\epsilon} \Lambda_1^\alpha \nabla u\|_{L^2} \right) \|b\|_{H^2}^2 \\ &\leq C \left( \|u\|_{H^2} + \|\Lambda_1^\alpha u\|_{H^2} \right) \|b\|_{H^2}^2. \end{aligned} \tag{3.3}$$

When  $\alpha \in (1/2, 1]$ , choosing  $\epsilon = \alpha - 1/2$ , we obtain

$$\|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \nabla u\|_{L^2} = \|\Lambda_1^\alpha \Lambda_2^\alpha \nabla u\|_{L^2} \leq C \|\Lambda_1^\alpha u\|_{H^2}.$$

Thus, for  $\beta = 0$  and  $\alpha \in (0, 1]$ , the following estimate holds:

$$\begin{aligned} |I_3| &\leq C\left(\|u\|_{H^2} + \|\Lambda_1^\alpha u\|_{H^2}\right)\|b\|_{H^2}^2 \\ &\leq \varepsilon\|\Lambda_1^\alpha u\|_{H^2}^2 + C\left(\|(u, b)\|_{H^2}^3 + \|b\|_{H^2}^4\right). \end{aligned}$$

For  $I_4$ , by Hölder's, Young's, and Sobolev inequalities, we have for  $\beta \in (0, 1]$  that

$$\begin{aligned} |I_4| &= \left| 2 \int_{\Omega} [2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)] \cdot j dx \right| \\ &\leq C\|\nabla b\|_{L_{x_2}^\infty L_{x_1}^2} \|\nabla u\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{\beta}}} \|j\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\beta}}} \\ &\leq C\left(\|\nabla b\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\varepsilon} \nabla b\|_{L^2}\right) \|\Lambda_1^{\frac{1-\beta}{2}} \nabla u\|_{L^2} \|j\|_{L^2}^{\frac{1}{2}} \left(\|j\|_{L^2} + \|\Lambda_1^\beta j\|_{L^2}\right)^{\frac{1}{2}} \\ &\leq C\|b\|_{H^2} \|u\|_{H^2} \|j\|_{L^2}^{\frac{1}{2}} \left(\|j\|_{L^2} + \|\Lambda_1^\beta j\|_{L^2}\right)^{\frac{1}{2}} \\ &\leq \varepsilon\|\Lambda_1^\beta b\|_{H^2}^2 + C\left(\|(u, b)\|_{H^2}^3 + \|(u, b)\|_{H^2}^{\frac{10}{3}}\right), \end{aligned}$$

and in the case of  $\beta = 0$ , it follows that

$$\begin{aligned} |I_4| &\leq C\|\nabla b\|_{L_{x_2}^\infty L_{x_1}^2} \|\nabla u\|_{L_{x_2}^2 L_{x_1}^\infty} \|j\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C\left(\|\nabla b\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\varepsilon} \nabla b\|_{L^2}\right) \left(\|\nabla u\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\varepsilon} \nabla u\|_{L^2}\right) \|b\|_{H^2} \\ &\leq C\|u\|_{H^2} \|b\|_{H^2}^2 \leq \varepsilon\|b\|_{H^2}^2 + C\|(u, b)\|_{H^2}^4. \end{aligned}$$

For the last term  $I_5$  in (3.1), we first deal with the special case  $\beta = 0$  and  $\alpha \in (0, 1]$ ,

$$\begin{aligned} |I_5| &= \left| 2 \int_{\Omega} \nabla [2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)] \cdot \nabla j dx \right| \\ &\leq C\|\nabla^2 b\|_{L_{x_2}^2 L_{x_1}^2} \|\nabla u\|_{L_{x_2}^\infty L_{x_1}^\infty} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} + C\|\nabla b\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\alpha}}} \|\nabla^2 u\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C\left(\|\nabla u\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\varepsilon} \nabla u\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\varepsilon} \nabla u\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\varepsilon} \Lambda_2^{\frac{1}{2}+\varepsilon} \nabla u\|_{L^2}\right) \|\nabla^2 b\|_{L^2} \|\nabla j\|_{L^2} \\ &\quad + C\|\nabla b\|_{H^1} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \left(\|\nabla^2 u\|_{L^2} + \|\Lambda_1^\alpha \nabla^2 u\|_{L^2}\right)^{\frac{1}{2}} \|\nabla j\|_{L^2} \\ &\leq C\left(\|u\|_{H^2} + \|\Lambda_1^\alpha u\|_{H^2}\right) \|b\|_{H^2}^2 \\ &\leq \varepsilon\|\Lambda_1^\alpha u\|_{H^2}^2 + C\left(\|(u, b)\|_{H^2}^3 + \|b\|_{H^2}^4\right). \end{aligned}$$

In the case of  $\alpha, \beta \in (0, 1]$ ,  $I_5$  can be bounded as

$$\begin{aligned} |I_5| &\leq C\|\nabla^2 b\|_{L_{x_2}^2 L_{x_1}^2} \|\nabla u\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\beta}}} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\beta}}} + C\|\nabla b\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\alpha}}} \|\nabla^2 u\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C\|\nabla^2 b\|_{L^2} \|\nabla u\|_{H^1} \|\nabla j\|_{L^2}^{\frac{1}{2}} \left(\|\nabla j\|_{L^2} + \|\Lambda_1^\beta \nabla j\|_{L^2}\right)^{\frac{1}{2}} \\ &\quad + C\|\nabla b\|_{H^1} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \left(\|\nabla^2 u\|_{L^2} + \|\Lambda_1^\alpha \nabla^2 u\|_{L^2}\right)^{\frac{1}{2}} \|\nabla j\|_{L^2} \\ &\leq C\left(\|b\|_{H^2} + \|\Lambda_1^\beta b\|_{H^2}\right) \|(u, b)\|_{H^2}^2 + C\left(\|u\|_{H^2} + \|\Lambda_1^\alpha u\|_{H^2}\right) \|b\|_{H^2}^2 \\ &\leq \varepsilon\left(\|\Lambda_1^\alpha u, \Lambda_1^\beta b\|_{H^2}\right)^2 + C\left(\|(u, b)\|_{H^2}^3 + \|(u, b)\|_{H^2}^4\right). \end{aligned}$$

Inserting these estimates above into (3.1) yields a differential inequality that establishes the local upper bound for  $\|(u, b)\|_{H^2}$ .

To prove the global existence and the stability result, we need to obtain the uniform-in-time  $H^2$  estimate. Due to the equivalence of  $\|(u, b)\|_{H^2}$  with  $\|(u, b)\|_{L^2} + \|(u, b)\|_{\dot{H}^2}$ , it suffices to estimate the  $L^2$  and homogeneous  $\dot{H}^2$ -norm of  $(u, b)$ . It is easy to see that, due to  $\nabla \cdot u = \nabla \cdot b = 0$ , we have

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda_1^\alpha u(\tau)\|_{L^2}^2 d\tau + 2\eta \int_0^t \|\Lambda_1^\beta b(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \quad (3.4)$$

To estimate the  $\dot{H}^2$ -norm, we resort to the vorticity equations (3.2). Taking the inner product of  $(\nabla\omega, \nabla j)$  with the gradient of (3.2), due to  $\nabla \cdot u = \nabla \cdot b = 0$  and after integrating by parts, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u(t)\|_{\dot{H}^2}^2 + \|b(t)\|_{\dot{H}^2}^2) + \nu \|\Lambda_1^\alpha u\|_{\dot{H}^2}^2 + \eta \|\Lambda_1^\beta b\|_{\dot{H}^2}^2 \\ &= - \int_{\Omega} \nabla u \cdot \nabla \omega \cdot \nabla \omega dx + \int_{\Omega} [\nabla b + (\nabla b)^\top] \cdot \nabla j \cdot \nabla \omega dx - \int_{\Omega} \nabla u \cdot \nabla j \cdot \nabla j dx + \int_{\Omega} \nabla Q \cdot \nabla j dx \\ &=: M + N + J + K. \end{aligned} \quad (3.5)$$

For the first term  $M$ , we break it into four terms as

$$\begin{aligned} M &= - \int_{\Omega} \partial_1 u_1 (\partial_1 \omega)^2 dx - \int_{\Omega} \partial_1 u_2 \partial_1 \omega \partial_2 \omega dx \\ &\quad - \int_{\Omega} \partial_2 u_1 \partial_1 \omega \partial_2 \omega dx - \int_{\Omega} \partial_2 u_2 (\partial_2 \omega)^2 dx \\ &=: M_1 + M_2 + M_3 + M_4. \end{aligned}$$

$M_1$  and  $M_2$  can be bounded directly. By Lemma 2.1,  $\partial_1 \bar{u} = 0$ , and  $\partial_1 u = \partial_1 \bar{u}$ . Applying Hölder's inequality, Lemma 2.2, and Lemma 2.4, we have for  $\alpha \in (0, 1]$  that

$$\begin{aligned} |M_1| &= \left| - \int_{\Omega} \partial_1 \bar{u}_1 \partial_1 \bar{\omega} \partial_1 \bar{\omega} dx \right| \\ &\leq C \|\partial_1 \bar{u}_1\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\alpha}}} \|\partial_1 \bar{\omega}\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_1 \bar{\omega}\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \\ &\leq C \|\partial_1 \bar{u}_1\|_{H^1} \|\partial_1 \bar{\omega}\|_{L^2} \|\partial_1 \bar{\omega}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \bar{\omega}\|_{L^2} + \|\Lambda_1^\alpha \partial_1 \bar{\omega}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^\alpha u\|_{H^2}^2. \end{aligned}$$

Similarly, for  $M_2$ , when  $\alpha \in (0, 1]$ , we have

$$\begin{aligned} |M_2| &= \left| - \int_{\Omega} \partial_1 \bar{u}_2 \partial_1 \bar{\omega} \partial_2 \bar{\omega} dx \right| \\ &\leq C \|\partial_1 \bar{u}_2\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\alpha}}} \|\partial_1 \bar{\omega}\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \|\partial_2 \bar{\omega}\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \|\partial_1 \bar{u}_2\|_{H^1} \|\partial_1 \bar{\omega}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \bar{\omega}\|_{L^2} + \|\Lambda_1^\alpha \partial_1 \bar{\omega}\|_{L^2} \right)^{\frac{1}{2}} \|\partial_2 \bar{\omega}\|_{L^2} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^\alpha u\|_{H^2}^2. \end{aligned}$$

The estimate of  $M_3$  is slightly more delicate.

$$\begin{aligned} M_3 &= - \int_{\Omega} \partial_2 u_1 \partial_1 \omega \partial_2 \omega dx = - \int_{\Omega} \partial_2 (\bar{u}_1 + \tilde{u}_1) \partial_1 \bar{\omega} \partial_2 (\bar{\omega} + \tilde{\omega}) dx \\ &= - \int_{\Omega} \partial_2 \bar{u}_1 \partial_1 \bar{\omega} \partial_2 \bar{\omega} dx - \int_{\Omega} \partial_2 \bar{u}_1 \partial_1 \tilde{\omega} \partial_2 \tilde{\omega} dx - \int_{\Omega} \partial_2 \tilde{u}_1 \partial_1 \bar{\omega} \partial_2 \omega dx \\ &=: M_{31} + M_{32} + M_{33}. \end{aligned}$$

The first term  $M_{31}$  is clearly zero:

$$M_{31} = - \int_{\Omega} \partial_2 \bar{u}_1 \partial_1 \bar{\omega} \partial_2 \bar{\omega} dx = - \int_{\mathbb{R}} \partial_2 \bar{u}_1 \partial_2 \bar{\omega} \left( \int_{\mathbb{T}} \partial_1 \bar{\omega} dx_1 \right) dx_2 = 0.$$

The terms  $M_{32}$  and  $M_{33}$  can be bounded in a similar manner to  $M_1$ . When  $\alpha \in (0, 1]$ ,

$$\begin{aligned} |M_{32}| &\leq C \|\partial_2 \bar{u}_1\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\alpha}}} \|\partial_1 \bar{\omega}\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_2 \bar{\omega}\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \left( \|\partial_2 \bar{u}_1\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \partial_2 \bar{u}_1\|_{L^2} \right) \|\partial_1 \bar{\omega}\|_{L^2} \|\partial_2 \bar{\omega}\|_{L^2} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^{\alpha} u\|_{H^2}^2, \end{aligned}$$

and

$$\begin{aligned} |M_{33}| &\leq C \|\partial_2 \tilde{u}_1\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\alpha}}} \|\partial_1 \tilde{\omega}\|_{L_{x_2}^2 L_{x_1}^{1-\alpha}} \|\partial_2 \omega\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \|\partial_2 \tilde{u}_1\|_{H^1} \|\partial_1 \tilde{\omega}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \tilde{\omega}\|_{L^2} + \|\Lambda_1^{\alpha} \partial_1 \tilde{\omega}\|_{L^2} \right)^{\frac{1}{2}} \|\partial_2 \omega\|_{L^2} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^{\alpha} u\|_{H^2}^2. \end{aligned}$$

For  $M_4$ , when  $\alpha \in (0, 1]$ , it follows that

$$\begin{aligned} |M_4| &= \left| \int_{\Omega} \partial_1 \tilde{u}_1 (\partial_2 \tilde{\omega} + \partial_2 \bar{\omega})^2 dx \right| \\ &= \left| 2 \int_{\Omega} \partial_1 \tilde{u}_1 \partial_2 \tilde{\omega} \partial_2 \bar{\omega} dx + \int_{\Omega} \partial_1 \tilde{u}_1 (\partial_2 \tilde{\omega})^2 dx \right| \\ &\leq C \|\partial_1 \tilde{u}_1\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\alpha}}} \left( \|\partial_2 \tilde{\omega}\|_{L_{x_2}^2 L_{x_1}^2} + \|\partial_2 \bar{\omega}\|_{L_{x_2}^2 L_{x_1}^2} \right) \|\partial_2 \tilde{\omega}\|_{L_{x_2}^2 L_{x_1}^{1-\alpha}} \\ &\leq C \|\partial_1 \tilde{u}_1\|_{H^1} \|u\|_{H^2} \|\partial_2 \tilde{\omega}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_2 \tilde{\omega}\|_{L^2} + \|\Lambda_1^{\alpha} \partial_2 \tilde{\omega}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^{\alpha} u\|_{H^2}^2. \end{aligned}$$

Thus, for  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$ , we have

$$|M| \leq C \|u\|_{H^2} \|\Lambda_1^{\alpha} u\|_{H^2}^2. \quad (3.6)$$

For the term  $N$  in (3.5), when  $\beta = 0$  and  $\alpha \in (0, 1]$ ,

$$\begin{aligned} |N| &= \left| \int_{\Omega} [\nabla b + (\nabla b)^{\top}] \cdot \nabla j \cdot \nabla \omega dx \right| \\ &\leq C \|\nabla b\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\alpha}}} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} \|\nabla \omega\|_{L_{x_2}^2 L_{x_1}^{1-\alpha}} \\ &\leq C \|\nabla b\|_{H^1} \|\nabla j\|_{L^2} \|\nabla \omega\|_{L^2}^{\frac{1}{2}} \left( \|\nabla \omega\|_{L^2} + \|\Lambda_1^{\alpha} \nabla \omega\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|b\|_{H^2}^2 \left( \|u\|_{H^2} + \|\Lambda_1^{\alpha} u\|_{H^2} \right). \end{aligned}$$

In the case of  $\alpha, \beta \in (0, 1]$ , we can decompose  $N$  as

$$\begin{aligned} N &= 2 \int_{\Omega} \partial_1 b_1 \partial_1 j \partial_1 \omega dx + \int_{\Omega} \partial_1 b_2 \partial_2 j \partial_1 \omega dx + \int_{\Omega} \partial_2 b_1 \partial_1 j \partial_2 \omega dx \\ &\quad + 2 \int_{\Omega} \partial_2 b_2 \partial_2 j \partial_2 \omega dx + \int_{\Omega} \partial_1 b_2 \partial_1 j \partial_2 \omega dx + \int_{\Omega} \partial_2 b_1 \partial_2 j \partial_1 \omega dx \\ &=: N_1 + N_2 + N_3 + N_4 + N_5 + N_6. \end{aligned}$$

According to Hölder's inequality, Lemma 2.1, Lemma 2.2, and Lemma 2.4, for  $N_1$ , we have for  $\alpha, \beta \in (0, 1]$  that

$$\begin{aligned} |N_1| &= \left| 2 \int_{\Omega} \partial_1 \tilde{b}_1 \partial_1 \tilde{j} \partial_1 \tilde{\omega} dx \right| \\ &\leq C \|\partial_1 \tilde{b}_1\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\alpha}}} \|\partial_1 \tilde{j}\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_1 \tilde{\omega}\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \\ &\leq C \|\partial_1 \tilde{b}_1\|_{H^1} \|\partial_1 \tilde{j}\|_{L^2} \|\partial_1 \tilde{\omega}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \tilde{\omega}\|_{L^2} + \|\Lambda_1^{\alpha} \partial_1 \tilde{\omega}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|b\|_{H^2} \|\Lambda_1^{\beta} b\|_{H^2} \|\Lambda_1^{\alpha} u\|_{H^2}. \end{aligned}$$

Similarly, for  $N_2$ , it follows for  $\alpha, \beta \in (0, 1]$  that

$$\begin{aligned} |N_2| &= \left| \int_{\Omega} \partial_1 \tilde{b}_2 \partial_2 j \partial_1 \tilde{\omega} dx \right| \\ &\leq C \|\partial_1 \tilde{b}_2\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\alpha}}} \|\partial_2 j\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_1 \tilde{\omega}\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \\ &\leq C \|\partial_1 \tilde{b}_2\|_{H^1} \|\partial_2 j\|_{L^2} \|\partial_1 \tilde{\omega}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \tilde{\omega}\|_{L^2} + \|\Lambda_1^{\alpha} \partial_1 \tilde{\omega}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|\Lambda_1^{\beta} b\|_{H^2} \|b\|_{H^2} \|\Lambda_1^{\alpha} u\|_{H^2}. \end{aligned}$$

For  $N_3$ , it can be split into three terms,

$$\begin{aligned} N_3 &= \int_{\Omega} \partial_2 b_1 \partial_1 j \partial_2 \omega dx = \int_{\Omega} \partial_2 (\tilde{b}_1 + \bar{b}_1) \partial_1 \tilde{j} \partial_2 (\tilde{\omega} + \bar{\omega}) dx \\ &= \int_{\Omega} \partial_2 \tilde{b}_1 \partial_1 \tilde{j} \partial_2 \tilde{\omega} dx + \int_{\Omega} \partial_2 \bar{b}_1 \partial_1 \tilde{j} \partial_2 \tilde{\omega} dx + \int_{\Omega} \partial_2 \tilde{b}_1 \partial_1 \tilde{j} \partial_2 \bar{\omega} dx \\ &=: N_{31} + N_{32} + N_{33}. \end{aligned}$$

Obviously, the first term  $N_{31}$  is equal to zero; for  $N_{32}$ , we estimate for  $\alpha, \beta \in (0, 1]$  that

$$\begin{aligned} |N_{32}| &\leq C \|\partial_2 \tilde{b}_1\|_{L_{x_2}^{\infty}} \|\partial_1 \tilde{j}\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_2 \tilde{\omega}\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \left( \|\partial_2 \tilde{b}_1\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \partial_2 \tilde{b}_1\|_{L^2} \right) \|\partial_1 \tilde{j}\|_{L^2} \|\partial_2 \tilde{\omega}\|_{L^2} \\ &\leq C \|b\|_{H^2} \|\Lambda_1^{\beta} b\|_{H^2} \|\Lambda_1^{\alpha} u\|_{H^2}. \end{aligned}$$

For  $N_{33}$ , when  $\alpha, \beta \in (0, 1]$ , we have

$$\begin{aligned} |N_{33}| &\leq C \|\partial_2 \tilde{b}_1\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\alpha}}} \|\partial_1 \tilde{j}\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \|\partial_2 \omega\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \|\partial_2 \tilde{b}_1\|_{H^1} \|\partial_1 \tilde{j}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \tilde{j}\|_{L^2} + \|\Lambda_1^{\alpha} \partial_1 \tilde{j}\|_{L^2} \right)^{\frac{1}{2}} \|\partial_2 \omega\|_{L^2} \\ &\leq C \|\Lambda_1^{\beta} b\|_{H^2}^2 \|u\|_{H^2}. \end{aligned}$$

For  $N_4$ , we continue to decompose it into three terms,

$$\begin{aligned} N_4 &= -2 \int_{\Omega} \partial_1 \bar{b}_1 \partial_2 (\bar{j} + \tilde{j}) \partial_2 (\bar{\omega} + \tilde{\omega}) dx \\ &= -2 \int_{\Omega} \partial_1 \bar{b}_1 \partial_2 \bar{j} \partial_2 \bar{\omega} dx - 2 \int_{\Omega} \partial_1 \bar{b}_1 \partial_2 \tilde{j} \partial_2 \tilde{\omega} dx - 2 \int_{\Omega} \partial_1 \bar{b}_1 \partial_2 \tilde{j} \partial_2 \bar{\omega} dx \\ &=: N_{41} + N_{42} + N_{43}. \end{aligned}$$

Clearly,  $N_{41} = 0$ . For  $N_{42}$ , when  $\alpha, \beta \in (0, 1]$ ,

$$\begin{aligned} |N_{42}| &\leq C \|\partial_1 \bar{b}_1\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\alpha}}} \|\partial_2 \bar{j}\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_2 \bar{\omega}\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \\ &\leq C \|\partial_1 \bar{b}_1\|_{H^1} \|\partial_2 \bar{j}\|_{L^2} \|\partial_2 \bar{\omega}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_2 \bar{\omega}\|_{L^2} + \|\Lambda_1^{\alpha} \partial_2 \bar{\omega}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|\Lambda_1^{\beta} b\|_{H^2} \|b\|_{H^2} \|\Lambda_1^{\alpha} u\|_{H^2}. \end{aligned}$$

The term  $N_{43}$  can be similarly bounded for  $\alpha, \beta \in (0, 1]$ ,

$$\begin{aligned} |N_{43}| &\leq C \|\partial_1 \bar{b}_1\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\beta}}} \|\partial_2 \tilde{j}\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\beta}}} \|\partial_2 \omega\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \|\partial_1 \bar{b}_1\|_{H^1} \|\partial_2 \tilde{j}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_2 \tilde{j}\|_{L^2} + \|\Lambda_1^{\beta} \partial_2 \tilde{j}\|_{L^2} \right)^{\frac{1}{2}} \|\partial_2 \omega\|_{L^2} \\ &\leq C \|\Lambda_1^{\beta} b\|_{H^2}^2 \|u\|_{H^2}. \end{aligned}$$

The estimates for  $N_5$  and  $N_6$  follow from those for  $N_2$  and  $N_3$ , respectively. Thus, for  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$ , we have

$$|N| \leq C \|(u, b)\|_{H^2} \left( \|\Lambda_1^{\alpha} u\|_{H^2}^2 + \|\Lambda_1^{\beta} b\|_{H^2}^2 \right). \quad (3.7)$$

As for  $J$ , similar to  $N$ , it follows for  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$  that

$$|J| \leq C \|(u, b)\|_{H^2} \left( \|\Lambda_1^{\alpha} u\|_{H^2}^2 + \|\Lambda_1^{\beta} b\|_{H^2}^2 \right). \quad (3.8)$$

For the last term  $K$  in (3.5), we further write it into eight terms,

$$\begin{aligned} K &= \int_{\Omega} \nabla [2\partial_1 b_1 (\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1 (\partial_2 b_1 + \partial_1 b_2)] \cdot \nabla j dx \\ &= 2 \int_{\Omega} \nabla [\partial_1 b_1 (\partial_2 u_1 + \partial_1 u_2)] \cdot \nabla j dx - 2 \int_{\Omega} \nabla [\partial_1 u_1 (\partial_2 b_1 + \partial_1 b_2)] \cdot \nabla j dx \\ &= 2 \int_{\Omega} \partial_1 \nabla b_1 \partial_2 u_1 \cdot \nabla j dx + 2 \int_{\Omega} \partial_1 b_1 \partial_2 \nabla u_1 \cdot \nabla j dx + 2 \int_{\Omega} \partial_1 \nabla b_1 \partial_1 u_2 \cdot \nabla j dx \\ &\quad + 2 \int_{\Omega} \partial_1 b_1 \partial_1 \nabla u_2 \cdot \nabla j dx - 2 \int_{\Omega} \partial_1 \nabla u_1 \partial_2 b_1 \cdot \nabla j dx - 2 \int_{\Omega} \partial_1 u_1 \partial_2 \nabla b_1 \cdot \nabla j dx \\ &\quad - 2 \int_{\Omega} \partial_1 \nabla u_1 \partial_1 b_2 \cdot \nabla j dx - 2 \int_{\Omega} \partial_1 u_1 \partial_1 \nabla b_2 \cdot \nabla j dx \\ &=: K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8. \end{aligned}$$

Similarly, we estimate the case  $\beta = 0$  and  $\alpha \in (0, 1]$ . According to Hölder's inequality and Lemma 2.4, we have

$$\begin{aligned} & |K_1 + K_3 + K_6 + K_8| \\ & \leq C \|\nabla^2 b\|_{L^2_{x_2} L^2_{x_1}} \|\nabla u\|_{L^\infty_{x_2} L^\infty_{x_1}} \|\nabla j\|_{L^2_{x_2} L^2_{x_1}} \\ & \leq C \left( \|\nabla u\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \nabla u\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \nabla u\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \nabla u\|_{L^2} \right) \|b\|_{H^2}^2 \\ & \leq C \left( \|u\|_{H^2} + \|\Lambda_1^\alpha u\|_{H^2} \right) \|b\|_{H^2}^2, \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} & |K_2 + K_4 + K_5 + K_7| \\ & \leq C \|\nabla b\|_{L^\infty_{x_2} L^{\frac{2}{\alpha}}_{x_1}} \|\nabla^2 u\|_{L^2_{x_2} L^{\frac{2}{1-\alpha}}_{x_1}} \|\nabla j\|_{L^2_{x_2} L^2_{x_1}} \\ & \leq C \|\nabla b\|_{H^1} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \left( \|\nabla^2 u\|_{L^2} + \|\Lambda_1^\alpha \nabla^2 u\|_{L^2} \right)^{\frac{1}{2}} \|b\|_{H^2} \\ & \leq C \left( \|u\|_{H^2} + \|\Lambda_1^\alpha u\|_{H^2} \right) \|b\|_{H^2}^2. \end{aligned} \quad (3.10)$$

Next, we consider the case  $\alpha, \beta \in (0, 1]$ . For  $K_1$ , it can be divided into two pieces,

$$\begin{aligned} K_1 &= 2 \int_{\Omega} \partial_1 \nabla b_1 \partial_2 u_1 \cdot \nabla j dx \\ &= 2 \int_{\Omega} \partial_1 \partial_1 b_1 \partial_2 u_1 \partial_1 j dx + 2 \int_{\Omega} \partial_1 \partial_2 b_1 \partial_2 u_1 \partial_2 j dx \\ &=: K_{11} + K_{12}. \end{aligned}$$

For  $K_{11}$ , one has for  $\beta \in (0, 1]$  that

$$\begin{aligned} |K_{11}| &= \left| 2 \int_{\Omega} \partial_1 \partial_1 \tilde{b}_1 \partial_2 u_1 \partial_1 \tilde{j} dx \right| \\ &\leq C \|\partial_1 \partial_1 \tilde{b}_1\|_{L^2_{x_2} L^2_{x_1}} \|\partial_2 u_1\|_{L^\infty_{x_2} L^{\frac{2}{\beta}}_{x_1}} \|\partial_1 \tilde{j}\|_{L^2_{x_2} L^{1-\beta}_{x_1}} \\ &\leq C \|\partial_1 \partial_1 \tilde{b}_1\|_{L^2} \|\partial_2 u_1\|_{H^1} \|\partial_1 \tilde{j}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \tilde{j}\|_{L^2} + \|\Lambda_1^\beta \partial_1 \tilde{j}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^\beta b\|_{H^2}^2. \end{aligned}$$

For  $K_{12}$ , we can decompose it as

$$\begin{aligned} K_{12} &= 2 \int_{\Omega} \partial_1 \partial_2 \tilde{b}_1 \partial_2 (\bar{u}_1 + \tilde{u}_1) \partial_2 (\bar{j} + \tilde{j}) dx \\ &= 2 \int_{\Omega} \partial_1 \partial_2 \tilde{b}_1 \partial_2 \bar{u}_1 \partial_2 \bar{j} dx + 2 \int_{\Omega} \partial_1 \partial_2 \tilde{b}_1 \partial_2 \tilde{u}_1 \partial_2 \tilde{j} dx + 2 \int_{\Omega} \partial_1 \partial_2 \tilde{b}_1 \partial_2 \bar{u}_1 \partial_2 \tilde{j} dx \\ &=: K_{121} + K_{122} + K_{123}. \end{aligned}$$

Clearly,  $K_{121} = 0$ . For  $K_{122}$ , when  $\beta \in (0, 1]$ ,

$$\begin{aligned} |K_{122}| &\leq C \|\partial_1 \partial_2 \tilde{b}_1\|_{L^2_{x_2} L^2_{x_1}} \|\partial_2 \tilde{u}_1\|_{L^\infty_{x_2}} \|\partial_2 \tilde{j}\|_{L^2_{x_2} L^2_{x_1}} \\ &\leq C \|\partial_1 \partial_2 \tilde{b}_1\|_{L^2} \left( \|\partial_2 \tilde{u}_1\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \partial_2 \tilde{u}_1\|_{L^2} \right) \|\partial_2 \tilde{j}\|_{L^2} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^\beta b\|_{H^2}^2. \end{aligned}$$

A similar bound holds for  $K_{123}$  when  $\alpha, \beta \in (0, 1]$ .

$$\begin{aligned} |K_{123}| &\leq C \|\partial_1 \partial_2 \tilde{b}_1\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\beta}}} \|\partial_2 \tilde{u}_1\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\beta}}} \|\partial_2 j\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \|\partial_1 \partial_2 \tilde{b}_1\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \partial_2 \tilde{b}_1\|_{L^2} + \|\Lambda_1^\beta \partial_1 \partial_2 \tilde{b}_1\|_{L^2} \right)^{\frac{1}{2}} \|\partial_2 \tilde{u}_1\|_{H^1} \|\partial_2 j\|_{L^2} \\ &\leq C \|b\|_{H^2} \|\Lambda_1^\alpha u\|_{H^2} \|\Lambda_1^\beta b\|_{H^2}. \end{aligned}$$

Thus, for  $\alpha, \beta \in (0, 1]$ , we have

$$|K_1| \leq C \|(u, b)\|_{H^2} \left( \|\Lambda_1^\alpha u\|_{H^2}^2 + \|\Lambda_1^\beta b\|_{H^2}^2 \right). \quad (3.11)$$

Similar to  $K_1$ , for  $\alpha, \beta \in (0, 1]$ ,  $K_2$  can be bounded by

$$|K_2| \leq C \|(u, b)\|_{H^2} \left( \|\Lambda_1^\alpha u\|_{H^2}^2 + \|\Lambda_1^\beta b\|_{H^2}^2 \right). \quad (3.12)$$

Next, we estimate  $K_3$ . By Hölder's inequality, Lemma 2.1, Lemma 2.2, and Lemma 2.4, we get for  $\alpha, \beta \in (0, 1]$  that

$$\begin{aligned} |K_3| &= \left| 2 \int_{\Omega} \partial_1 \nabla \tilde{b}_1 \partial_1 \tilde{u}_2 \cdot \nabla j dx \right| \\ &\leq C \|\partial_1 \nabla \tilde{b}_1\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\beta}}} \|\partial_1 \tilde{u}_2\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\beta}}} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \|\partial_1 \nabla \tilde{b}_1\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \nabla \tilde{b}_1\|_{L^2} + \|\Lambda_1^\beta \partial_1 \nabla \tilde{b}_1\|_{L^2} \right)^{\frac{1}{2}} \|\partial_1 \tilde{u}_2\|_{H^1} \|\nabla j\|_{L^2} \\ &\leq C \|b\|_{H^2} \|\Lambda_1^\alpha u\|_{H^2} \|\Lambda_1^\beta b\|_{H^2}. \end{aligned} \quad (3.13)$$

For  $K_4$ , when  $\alpha, \beta \in (0, 1]$ ,

$$\begin{aligned} |K_4| &= \left| 2 \int_{\Omega} \partial_1 \tilde{b}_1 \partial_1 \nabla \tilde{u}_2 \cdot \nabla j dx \right| \\ &\leq C \|\partial_1 \tilde{b}_1\|_{L_{x_2}^\infty L_{x_1}^{\frac{2}{\alpha}}} \|\partial_1 \nabla \tilde{u}_2\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \|\nabla j\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \|\partial_1 \tilde{b}_1\|_{H^1} \|\partial_1 \nabla \tilde{u}_2\|_{L^2}^{\frac{1}{2}} \left( \|\partial_1 \nabla \tilde{u}_2\|_{L^2} + \|\Lambda_1^\alpha \partial_1 \nabla \tilde{u}_2\|_{L^2} \right)^{\frac{1}{2}} \|\nabla j\|_{L^2} \\ &\leq C \|b\|_{H^2} \|\Lambda_1^\alpha u\|_{H^2} \|\Lambda_1^\beta b\|_{H^2}. \end{aligned} \quad (3.14)$$

The estimates for  $K_5$  and  $K_6$  are similar to those of  $K_1$ . The terms  $K_7$  and  $K_8$  can be bounded in the same manner as  $K_4$  and  $K_3$ , respectively. Thus, for  $\alpha, \beta \in (0, 1]$ , we have

$$|K_5 + K_6 + K_7 + K_8| \leq C \|(u, b)\|_{H^2} \left( \|\Lambda_1^\alpha u\|_{H^2}^2 + \|\Lambda_1^\beta b\|_{H^2}^2 \right). \quad (3.15)$$

Collecting the bounds from (3.9)–(3.15), we have for  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$  that

$$|K| \leq C \|(u, b)\|_{H^2} \left( \|\Lambda_1^\alpha u\|_{H^2}^2 + \|\Lambda_1^\beta b\|_{H^2}^2 \right). \quad (3.16)$$

Combining the upper bounds in (3.6)–(3.8) and (3.16), we get

$$|M| + |N| + |J| + |K| \leq C \|(u, b)\|_{H^2} \left( \|\Lambda_1^\alpha u\|_{H^2}^2 + \|\Lambda_1^\beta b\|_{H^2}^2 \right). \quad (3.17)$$

Then, we can obtain from (3.5) that

$$\begin{aligned}
 & \|u(t)\|_{\dot{H}^2}^2 + \|b(t)\|_{\dot{H}^2}^2 + 2\nu\|\Lambda_1^\alpha u\|_{\dot{H}^2}^2 + 2\eta\|\Lambda_1^\beta b\|_{\dot{H}^2}^2 \\
 & \leq E(0) + C \int_0^t (\|u\|_{H^2} + \|b\|_{H^2}) \left( \|\Lambda_1^\alpha u\|_{H^2}^2 + \|\Lambda_1^\beta b\|_{H^2}^2 \right) d\tau \\
 & \leq E(0) + C \sup_{0 \leq \tau \leq t} (\|u(\tau)\|_{H^2} + \|b(\tau)\|_{H^2}) \int_0^t \left( \|\Lambda_1^\alpha u(\tau)\|_{H^2}^2 + \|\Lambda_1^\beta b(\tau)\|_{H^2}^2 \right) d\tau.
 \end{aligned} \tag{3.18}$$

Adding (3.18) to (3.4), we have

$$E(t) \leq E(0) + CE(t)^{\frac{3}{2}}.$$

By the bootstrapping argument, if  $E(0)$  is sufficiently small (i.e.,  $E(0) < \varepsilon^2$  for some  $\varepsilon > 0$ ), then there exists a constant  $C_1 > 0$ , and for any  $t > 0$ ,

$$E(t) \leq C_1 \varepsilon^2.$$

This completes the proof of Theorem 1.1. □

#### 4. Proof of Theorem 1.2

This section proves Theorem 1.2. We consider the equations of  $(\bar{u}, \bar{b})$  and apply the properties of the orthogonal decomposition and various anisotropic inequalities.

*Proof of Theorem 1.2.* We first write the equation of  $(\bar{u}, \bar{b})$ . According to Lemma 2.1, we have  $\partial_1 \bar{u} = 0$  and

$$\overline{u \cdot \nabla \bar{u}} = \overline{u_1 \partial_1 \bar{u}} + \overline{u_2 \partial_2 \bar{u}} = \bar{u}_2 \partial_2 \bar{u}.$$

Using the divergence-free condition  $\nabla \cdot u = 0$  and Lemma 2.1, we have

$$\partial_1 \bar{u}_1 + \partial_2 \bar{u}_2 = 0 \Rightarrow \bar{u}_2 = C,$$

where  $C$  is a constant. Furthermore,

$$\|\bar{u}_2\|_{L^2} \leq \|u\|_{L^2} < \infty.$$

Consequently, we deduce that

$$\bar{u}_2 = 0 \quad \text{and} \quad \overline{u \cdot \nabla \bar{u}} = 0. \tag{4.1}$$

Similarly,

$$\bar{b}_2 = 0 \quad \text{and} \quad \overline{u \cdot \nabla \bar{b}} = 0. \tag{4.2}$$

Taking the average of (1.3) and using (4.1) and (4.2), we have

$$\begin{cases} \partial_t \bar{u} + \overline{u \cdot \nabla \bar{u}} + \begin{pmatrix} 0 \\ \partial_2 \bar{P} \end{pmatrix} + \nu \Lambda_1^{2\alpha} \bar{u} = \overline{b \cdot \nabla \bar{u}} + \overline{A \cdot \nabla \bar{b}}, \\ \partial_t \bar{b} + \overline{u \cdot \nabla \bar{b}} + \eta \Lambda_1^{2\beta} \bar{b} = \overline{b \cdot \nabla \bar{u}} + \overline{A \cdot \nabla u}. \end{cases} \tag{4.3}$$

Taking the difference between (1.3) and (4.3), we find

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u} \cdot \widetilde{\nabla} \widetilde{u} + u_2 \partial_2 \widetilde{u} + \nabla \widetilde{P} + \nu \Lambda_1^{2\alpha} \widetilde{u} = \widetilde{b} \cdot \widetilde{\nabla} \widetilde{b} + b_2 \partial_2 \widetilde{b} + \widetilde{A} \cdot \widetilde{\nabla} \widetilde{b}, \\ \partial_t \widetilde{b} + \widetilde{u} \cdot \widetilde{\nabla} \widetilde{b} + u_2 \partial_2 \widetilde{b} + \eta \Lambda_1^{2\beta} \widetilde{b} = \widetilde{b} \cdot \widetilde{\nabla} \widetilde{u} + b_2 \partial_2 \widetilde{u} + \widetilde{A} \cdot \widetilde{\nabla} \widetilde{u}. \end{cases} \quad (4.4)$$

Taking the inner product of  $(\widetilde{u}, \widetilde{b})$  with (4.4) yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2 \right) + \nu \|\Lambda_1^\alpha \widetilde{u}\|_{L^2}^2 + \eta \|\Lambda_1^\beta \widetilde{b}\|_{L^2}^2 \\ &= - \int_{\Omega} \widetilde{u} \cdot \widetilde{\nabla} \widetilde{u} \cdot \widetilde{u} dx - \int_{\Omega} u_2 \partial_2 \widetilde{u} \cdot \widetilde{u} dx + \int_{\Omega} \widetilde{b} \cdot \widetilde{\nabla} \widetilde{b} \cdot \widetilde{u} dx + \int_{\Omega} b_2 \partial_2 \widetilde{b} \cdot \widetilde{u} dx \\ & \quad - \int_{\Omega} \widetilde{u} \cdot \widetilde{\nabla} \widetilde{b} \cdot \widetilde{b} dx - \int_{\Omega} u_2 \partial_2 \widetilde{b} \cdot \widetilde{b} dx + \int_{\Omega} \widetilde{b} \cdot \widetilde{\nabla} \widetilde{u} \cdot \widetilde{b} dx + \int_{\Omega} b_2 \partial_2 \widetilde{u} \cdot \widetilde{b} dx \\ &=: Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8. \end{aligned} \quad (4.5)$$

For  $Q_1$ , by Lemma 2.1 and divergence-free conditions, we have

$$Q_1 = - \int_{\Omega} \widetilde{u} \cdot \widetilde{\nabla} \widetilde{u} \cdot \widetilde{u} dx = - \int_{\Omega} u \cdot \nabla \widetilde{u} \cdot \widetilde{u} dx + \int_{\Omega} \overline{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx = 0.$$

For  $Q_2$ , according to (4.1), Hölder's inequality, Lemma 2.2, and Lemma 2.4, we get for  $\alpha \in (0, 1]$  that

$$|Q_2| = \left| - \int_{\Omega} \widetilde{u}_2 \partial_2 \widetilde{u} \cdot \widetilde{u} dx \right| \leq C \|\widetilde{u}_2\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_2 \widetilde{u}\|_{L_{x_2}^\infty} \|\widetilde{u}\|_{L_{x_2}^2 L_{x_1}^2} \leq C \|u\|_{H^2} \|\Lambda_1^\alpha \widetilde{u}\|_{L^2}^2.$$

For  $Q_3$  and  $Q_7$ , by Lemma 2.1 and divergence-free conditions,

$$\begin{aligned} Q_3 + Q_7 &= \int_{\Omega} \widetilde{b} \cdot \widetilde{\nabla} \widetilde{b} \cdot \widetilde{u} dx + \int_{\Omega} \widetilde{b} \cdot \widetilde{\nabla} \widetilde{u} \cdot \widetilde{b} dx \\ &= \int_{\Omega} \widetilde{b} \cdot \widetilde{\nabla} \widetilde{b} \cdot \widetilde{u} dx - \int_{\Omega} \overline{\widetilde{b} \cdot \widetilde{\nabla} \widetilde{b}} \cdot \widetilde{u} dx + \int_{\Omega} \widetilde{b} \cdot \nabla \widetilde{u} \cdot \widetilde{b} dx - \int_{\Omega} \overline{\widetilde{b} \cdot \nabla \widetilde{u}} \cdot \widetilde{b} dx \\ &= 0. \end{aligned}$$

In the case of  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$ ,  $Q_4$  can be bounded by

$$|Q_4| = \left| \int_{\Omega} \widetilde{b}_2 \partial_2 \widetilde{b} \cdot \widetilde{u} dx \right| \leq C \|\widetilde{b}_2\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_2 \widetilde{b}\|_{L_{x_2}^\infty} \|\widetilde{u}\|_{L_{x_2}^2 L_{x_1}^2} \leq C \|\Lambda_1^\beta \widetilde{b}\|_{L^2} \|b\|_{H^2} \|\Lambda_1^\alpha \widetilde{u}\|_{L^2}.$$

Similar to  $Q_1$ , we have  $Q_5 = 0$ . For  $Q_6$ , when  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$ ,

$$|Q_6| = \left| - \int_{\Omega} \widetilde{u}_2 \partial_2 \widetilde{b} \cdot \widetilde{b} dx \right| \leq C \|\widetilde{u}_2\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_2 \widetilde{b}\|_{L_{x_2}^\infty} \|\widetilde{b}\|_{L_{x_2}^2 L_{x_1}^2} \leq C \|\Lambda_1^\alpha \widetilde{u}\|_{L^2} \|b\|_{H^2} \|\Lambda_1^\beta \widetilde{b}\|_{L^2}.$$

For  $Q_8$ , when  $\beta \in [0, 1]$ , we have

$$|Q_8| = \left| \int_{\Omega} \widetilde{b}_2 \partial_2 \widetilde{u} \cdot \widetilde{b} dx \right| \leq C \|\widetilde{b}_2\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_2 \widetilde{u}\|_{L_{x_2}^\infty} \|\widetilde{b}\|_{L_{x_2}^2 L_{x_1}^2} \leq C \|u\|_{H^2} \|\Lambda_1^\beta \widetilde{b}\|_{L^2}^2.$$

Inserting the estimates for  $Q_1$  through  $Q_8$  into (4.5), we have

$$\frac{d}{dt} \left( \|\bar{u}\|_{L^2}^2 + \|\bar{b}\|_{L^2}^2 \right) + (2\nu - C\|(u, b)\|_{H^2}) \|\Lambda_1^\alpha \bar{u}\|_{L^2}^2 + (2\eta - C\|(u, b)\|_{H^2}) \|\Lambda_1^\beta \bar{b}\|_{L^2}^2 \leq 0.$$

By Theorem 1.1, if  $\varepsilon > 0$  is sufficiently small, and  $\|u_0\|_{H^2} + \|b_0\|_{H^2} \leq \varepsilon$ , then  $\|(u, b)\|_{H^2} \leq C\varepsilon$ , and

$$2\nu - C\|(u, b)\|_{H^2} \geq \nu, \quad 2\eta - C\|(u, b)\|_{H^2} \geq \eta.$$

Then, we get

$$\frac{d}{dt} \left( \|\bar{u}\|_{L^2}^2 + \|\bar{b}\|_{L^2}^2 \right) + \nu \|\Lambda_1^\alpha \bar{u}\|_{L^2}^2 + \eta \|\Lambda_1^\beta \bar{b}\|_{L^2}^2 \leq 0.$$

Invoking the Poincaré inequality in Lemma 2.2 and applying Grönwall's lemma, we obtain

$$\|\bar{u}\|_{L^2}^2 + \|\bar{b}\|_{L^2}^2 \leq (\|u_0\|_{L^2} + \|b_0\|_{L^2}) e^{-C_3 t}, \quad (4.6)$$

where  $C_3 = C_3(\nu, \eta) > 0$ .

Next, we turn to the exponential decay of  $\|(\nabla \bar{u}(t), \nabla \bar{b}(t))\|_{L^2}$ . Applying  $\nabla$  to (4.4) yields

$$\begin{cases} \partial_t \nabla \bar{u} + \nabla(\bar{u} \cdot \nabla \bar{u}) + \nabla(u_2 \partial_2 \bar{u}) + \nabla \nabla \bar{P} + \nu \Lambda_1^{2\alpha} \nabla \bar{u} = \nabla(\bar{b} \cdot \nabla \bar{b}) + \nabla(b_2 \partial_2 \bar{b}) + \nabla(\bar{A} \cdot \nabla \bar{b}), \\ \partial_t \nabla \bar{b} + \nabla(\bar{u} \cdot \nabla \bar{b}) + \nabla(u_2 \partial_2 \bar{b}) + \eta \Lambda_1^{2\beta} \nabla \bar{b} = \nabla(\bar{b} \cdot \nabla \bar{u}) + \nabla(b_2 \partial_2 \bar{u}) + \nabla(\bar{A} \cdot \nabla \bar{u}). \end{cases} \quad (4.7)$$

Taking the  $L^2$  inner product of System (4.7) with  $(\nabla \bar{u}, \nabla \bar{b})$ , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\nabla \bar{u}\|_{L^2}^2 + \|\nabla \bar{b}\|_{L^2}^2 \right) + \nu \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2}^2 + \eta \|\Lambda_1^\beta \nabla \bar{b}\|_{L^2}^2 \\ &= - \int_{\Omega} \nabla(\bar{u} \cdot \nabla \bar{u}) \cdot \nabla \bar{u} dx - \int_{\Omega} \nabla(u_2 \partial_2 \bar{u}) \cdot \nabla \bar{u} dx + \int_{\Omega} \nabla(\bar{b} \cdot \nabla \bar{b}) \cdot \nabla \bar{u} dx \\ & \quad + \int_{\Omega} \nabla(b_2 \partial_2 \bar{b}) \cdot \nabla \bar{u} dx - \int_{\Omega} \nabla(\bar{u} \cdot \nabla \bar{b}) \cdot \nabla \bar{b} dx - \int_{\Omega} \nabla(u_2 \partial_2 \bar{b}) \cdot \nabla \bar{b} dx \\ & \quad + \int_{\Omega} \nabla(\bar{b} \cdot \nabla \bar{u}) \cdot \nabla \bar{b} dx + \int_{\Omega} \nabla(b_2 \partial_2 \bar{u}) \cdot \nabla \bar{b} dx \\ &=: R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8. \end{aligned} \quad (4.8)$$

For  $R_1$ , when  $\alpha \in (0, 1]$ , it follows that

$$\begin{aligned} |R_1| &= \left| - \int_{\Omega} \nabla(\bar{u} \cdot \nabla \bar{u}) \cdot \nabla \bar{u} dx + \int_{\Omega} \nabla(\bar{u} \cdot \nabla \bar{u}) \cdot \nabla \bar{u} dx \right| \\ &= \left| - \int_{\Omega} \nabla \bar{u} \cdot \nabla \bar{u} \cdot \nabla \bar{u} dx \right| \\ &\leq C \|\nabla \bar{u}\|_{L_{x_2}^{\infty} L_{x_1}^{\frac{2}{\alpha}}} \|\nabla \bar{u}\|_{L_{x_2}^2 L_{x_1}^2} \|\nabla \bar{u}\|_{L_{x_2}^2 L_{x_1}^{\frac{2}{1-\alpha}}} \\ &\leq C \|\nabla \bar{u}\|_{H^1} \|\nabla \bar{u}\|_{L^2} \|\nabla \bar{u}\|_{L^2}^{\frac{1}{2}} \left( \|\nabla \bar{u}\|_{L^2} + \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2}^2. \end{aligned}$$

For  $R_2$ , we can decompose it as

$$\begin{aligned} R_2 &= - \int_{\Omega} \partial_1 u_2 \partial_2 \bar{u} \cdot \partial_1 \bar{u} dx - \int_{\Omega} u_2 \partial_1 \partial_2 \bar{u} \cdot \partial_1 \bar{u} dx \\ &\quad - \int_{\Omega} \partial_2 u_2 \partial_2 \bar{u} \cdot \partial_2 \bar{u} dx - \int_{\Omega} u_2 \partial_2 \partial_2 \bar{u} \cdot \partial_2 \bar{u} dx \\ &=: R_{21} + R_{22} + R_{23} + R_{24}. \end{aligned}$$

In the case of  $\alpha \in (0, 1]$ ,  $R_{21}$  can be bounded by

$$|R_{21}| = \left| - \int_{\Omega} \partial_1 \bar{u}_2 \partial_2 \bar{u} \cdot \partial_1 \bar{u} dx \right| \leq C \|\partial_1 \bar{u}_2\|_{L^2_{x_2} L^2_{x_1}} \|\partial_2 \bar{u}\|_{L^\infty_{x_2}} \|\partial_1 \bar{u}\|_{L^2_{x_2} L^2_{x_1}} \leq C \|u\|_{H^2} \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2}^2.$$

According to the definition of  $\bar{u}$ , we have  $\partial_1 \bar{u} = 0$ . Then, we obtain

$$R_{22} = - \int_{\Omega} u_2 \partial_2 \partial_1 \bar{u} \cdot \partial_1 \bar{u} dx = 0.$$

For  $R_{23}$ , when  $\alpha \in (0, 1]$ , we have

$$|R_{23}| = \left| \int_{\Omega} \partial_1 \bar{u}_1 \partial_2 \bar{u} \cdot \partial_2 \bar{u} dx \right| \leq C \|\partial_1 \bar{u}_1\|_{L^2_{x_2} L^2_{x_1}} \|\partial_2 \bar{u}\|_{L^\infty_{x_2}} \|\partial_2 \bar{u}\|_{L^2_{x_2} L^2_{x_1}} \leq C \|u\|_{H^2} \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2}^2.$$

From (4.1), we know that  $\bar{u}_2 = 0$ ; then, for  $R_{24}$ , when  $\alpha \in (0, 1]$ ,

$$\begin{aligned} |R_{24}| &= \left| - \int_{\Omega} \bar{u}_2 \partial_2 \partial_2 \bar{u} \cdot \partial_2 \bar{u} dx \right| \\ &\leq C \|\bar{u}_2\|_{L^\infty_{x_2} L^{\frac{2}{\alpha}}_{x_1}} \|\partial_2 \partial_2 \bar{u}\|_{L^2_{x_2} L^2_{x_1}} \|\partial_2 \bar{u}\|_{L^2_{x_2} L^{1-\frac{2}{\alpha}}_{x_1}} \\ &\leq C \|\bar{u}_2\|_{H^1} \|\partial_2 \partial_2 \bar{u}\|_{L^2} \|\partial_2 \bar{u}\|_{L^2}^{\frac{1}{2}} \left( \|\partial_2 \bar{u}\|_{L^2} + \|\Lambda_1^\alpha \partial_2 \bar{u}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2}^2. \end{aligned}$$

Thus, when  $\alpha \in (0, 1]$ , we have

$$|R_2| \leq C \|u\|_{H^2} \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2}^2.$$

Regarding  $R_3$  and  $R_7$ , we have

$$\begin{aligned} R_3 + R_7 &= \int_{\Omega} \nabla(b \cdot \nabla \bar{b}) \cdot \nabla \bar{u} dx - \int_{\Omega} \nabla(\overline{b \cdot \nabla \bar{b}}) \cdot \nabla \bar{u} dx + \int_{\Omega} \nabla(b \cdot \nabla \bar{u}) \cdot \nabla \bar{b} dx \\ &\quad - \int_{\Omega} \nabla(\overline{b \cdot \nabla \bar{u}}) \cdot \nabla \bar{b} dx \\ &= \int_{\Omega} \nabla b \cdot \nabla \bar{b} \cdot \nabla \bar{u} dx + \int_{\Omega} b \cdot \nabla^2 \bar{b} \cdot \nabla \bar{u} dx + \int_{\Omega} \nabla b \cdot \nabla \bar{u} \cdot \nabla \bar{b} dx \\ &\quad + \int_{\Omega} b \cdot \nabla^2 \bar{u} \cdot \nabla \bar{b} dx \\ &= \int_{\Omega} \nabla b \cdot \nabla \bar{b} \cdot \nabla \bar{u} dx + \int_{\Omega} \nabla b \cdot \nabla \bar{u} \cdot \nabla \bar{b} dx. \end{aligned}$$

Then, we get for  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$  that

$$\begin{aligned} |R_3 + R_7| &\leq C \|\nabla b\|_{L^\infty_{x_2} L^{\frac{2}{1-\alpha}}_{x_1}} \|\nabla \tilde{b}\|_{L^2_{x_2} L^2_{x_1}} \|\nabla \tilde{u}\|_{L^2_{x_2} L^{\frac{2}{1-\alpha}}_{x_1}} \\ &\leq C \|\nabla b\|_{H^1} \|\nabla \tilde{b}\|_{L^2} \|\nabla \tilde{u}\|_{L^2}^{\frac{1}{2}} \left( \|\nabla \tilde{u}\|_{L^2} + \|\Lambda_1^\alpha \nabla \tilde{u}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|b\|_{H^2} \|\Lambda_1^\alpha \nabla \tilde{u}\|_{L^2} \|\Lambda_1^\beta \nabla \tilde{b}\|_{L^2}. \end{aligned}$$

Similar to  $R_2$ , for  $R_4$ , when  $\alpha \in (0, 1]$ , and  $\beta \in [0, 1]$ , we have

$$|R_4| \leq C \|b\|_{H^2} \left( \|\Lambda_1^\alpha \nabla \tilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \nabla \tilde{b}\|_{L^2}^2 \right).$$

In the following, we estimate  $R_5$ . In the case of  $\beta = 0$  and  $\alpha \in (0, 1]$ ,

$$\begin{aligned} |R_5| &= \left| - \int_{\Omega} \nabla(u \cdot \nabla \tilde{b}) \cdot \nabla \tilde{b} dx + \int_{\Omega} \nabla(\overline{u \cdot \nabla \tilde{b}}) \cdot \nabla \tilde{b} dx \right| \\ &= \left| - \int_{\Omega} \nabla \tilde{u} \cdot \nabla \tilde{b} \cdot \nabla \tilde{b} dx - \int_{\Omega} \nabla \tilde{u} \cdot \nabla \tilde{b} \cdot \nabla \tilde{b} dx \right| \\ &\leq C \|\nabla \tilde{u}\|_{L^2_{x_2} L^{\frac{2}{1-\alpha}}_{x_1}} \|\nabla \tilde{b}\|_{L^\infty_{x_2} L^{\frac{2}{1-\alpha}}_{x_1}} \|\nabla \tilde{b}\|_{L^2_{x_2} L^2_{x_1}} + C \|\nabla \tilde{u}\|_{L^\infty_{x_2}} \|\nabla \tilde{b}\|_{L^2} \|\nabla \tilde{b}\|_{L^2} \\ &\leq C \|\nabla \tilde{u}\|_{L^2}^{\frac{1}{2}} \left( \|\nabla \tilde{u}\|_{L^2} + \|\Lambda_1^\alpha \nabla \tilde{u}\|_{L^2} \right)^{\frac{1}{2}} \|\nabla \tilde{b}\|_{H^1} \|\nabla \tilde{b}\|_{L^2} + C \|\nabla \tilde{u}\|_{L^\infty_{x_2}} \|\nabla \tilde{b}\|_{L^2}^2 \\ &\leq C \|\Lambda_1^\alpha \nabla \tilde{u}\|_{L^2} \|b\|_{H^2} \|\nabla \tilde{b}\|_{L^2} + C \|u\|_{H^2} \|\nabla \tilde{b}\|_{L^2}^2. \end{aligned}$$

When  $\alpha, \beta \in (0, 1]$ ,  $R_5$  can be bounded by

$$\begin{aligned} |R_5| &\leq C \|\nabla u\|_{L^\infty_{x_2} L^{\frac{2}{1-\beta}}_{x_1}} \|\nabla \tilde{b}\|_{L^2_{x_2} L^2_{x_1}} \|\nabla \tilde{b}\|_{L^2_{x_2} L^{\frac{2}{1-\beta}}_{x_1}} \\ &\leq C \|\nabla u\|_{H^1} \|\nabla \tilde{b}\|_{L^2} \|\nabla \tilde{b}\|_{L^2}^{\frac{1}{2}} \left( \|\nabla \tilde{b}\|_{L^2} + \|\Lambda_1^\beta \nabla \tilde{b}\|_{L^2} \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{H^2} \|\Lambda_1^\beta \nabla \tilde{b}\|_{L^2}^2. \end{aligned}$$

Accordingly, we have for  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$  that

$$|R_5| \leq C \|(u, b)\|_{H^2} \left( \|\Lambda_1^\alpha \nabla \tilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \nabla \tilde{b}\|_{L^2}^2 \right).$$

For  $R_6$ , when  $\beta = 0$ , and  $\alpha \in (0, 1]$ , thanks to  $\bar{u}_2 = 0$ , we have

$$\begin{aligned} |R_6| &= \left| - \int_{\Omega} \bar{u}_2 \partial_2 \nabla \tilde{b} \cdot \nabla \tilde{b} dx - \int_{\Omega} \nabla \bar{u}_2 \partial_2 \tilde{b} \cdot \nabla \tilde{b} dx \right| \\ &\leq C \|\bar{u}_2\|_{L^\infty_{x_2} L^\infty_{x_1}} \|\partial_2 \nabla \tilde{b}\|_{L^2_{x_2} L^2_{x_1}} \|\nabla \tilde{b}\|_{L^2_{x_2} L^2_{x_1}} + C \|\nabla \bar{u}_2\|_{L^2_{x_2} L^2_{x_1}} \|\partial_2 \tilde{b}\|_{L^\infty_{x_2}} \|\nabla \tilde{b}\|_{L^2_{x_2} L^2_{x_1}} \\ &\leq C \left( \|\bar{u}_2\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \bar{u}_2\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \bar{u}_2\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \bar{u}_2\|_{L^2} \right) \|b\|_{H^2} \|\nabla \tilde{b}\|_{L^2} \\ &\quad + C \|\nabla \bar{u}_2\|_{L^2} \left( \|\partial_2 \tilde{b}\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \partial_2 \tilde{b}\|_{L^2} \right) \|\nabla \tilde{b}\|_{L^2} \\ &\leq C \|\Lambda_1^\alpha \nabla \tilde{u}\|_{L^2} \|b\|_{H^2} \|\nabla \tilde{b}\|_{L^2}. \end{aligned}$$

When  $\alpha, \beta \in (0, 1]$ , it follows that

$$|R_6| \leq C \|b\|_{H^2} \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2} \|\Lambda_1^\beta \nabla \bar{b}\|_{L^2}.$$

For the last term  $R_8$ , we similarly start by discussing the case  $\beta = 0$ . By the definition of the horizontal average,

$$\begin{aligned} |R_8| &= \left| \int_{\Omega} \nabla \bar{b}_2 \partial_2 \bar{u} \cdot \nabla \bar{b} dx + \int_{\Omega} \bar{b}_2 \partial_2 \nabla \bar{u} \cdot \nabla \bar{b} dx \right| \\ &\leq C \|\nabla \bar{b}_2\|_{L_{x_2}^2 L_{x_1}^2} \|\partial_2 \bar{u}\|_{L_{x_2}^\infty} \|\nabla \bar{b}\|_{L_{x_2}^2 L_{x_1}^2} + C \|\bar{b}_2\|_{L_{x_2}^\infty L_{x_1}^2} \|\partial_2 \nabla \bar{u}\|_{L_{x_2}^2} \|\nabla \bar{b}\|_{L_{x_2}^2 L_{x_1}^2} \\ &\leq C \|\nabla \bar{b}_2\|_{L^2} \left( \|\partial_2 \bar{u}\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \partial_2 \bar{u}\|_{L^2} \right) \|\nabla \bar{b}\|_{L^2} + C \left( \|\bar{b}_2\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \bar{b}_2\|_{L^2} \right) \|u\|_{H^2} \|\nabla \bar{b}\|_{L^2} \\ &\leq C \|u\|_{H^2} \|\nabla \bar{b}\|_{L^2}^2. \end{aligned}$$

On the other hand, when  $\alpha, \beta \in (0, 1]$ , similar to  $R_2$ , we have

$$|R_8| \leq C \|u\|_{H^2} \|\Lambda_1^\beta \nabla \bar{b}\|_{L^2}^2.$$

Substituting the estimates for  $R_1$  to  $R_8$  into (4.8), we conclude that

$$\begin{aligned} \frac{d}{dt} \left( \|\nabla \bar{u}\|_{L^2}^2 + \|\nabla \bar{b}\|_{L^2}^2 \right) &+ (2\nu - C\|(u, b)\|_{H^2}) \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2}^2 \\ &+ (2\eta - C\|(u, b)\|_{H^2}) \|\Lambda_1^\beta \nabla \bar{b}\|_{L^2}^2 \leq 0. \end{aligned}$$

Choosing  $\varepsilon > 0$  is sufficiently small. According to Theorem 1.1, when  $\|u_0\|_{H^2} + \|b_0\|_{H^2} \leq \varepsilon$ , we have  $\|(u(t), b(t))\|_{H^2} \leq C\varepsilon$  and

$$2\nu - C\|(u, b)\|_{H^2} \geq \nu, \quad 2\eta - C\|(u, b)\|_{H^2} \geq \eta.$$

Thus,

$$\frac{d}{dt} \left( \|\nabla \bar{u}\|_{L^2}^2 + \|\nabla \bar{b}\|_{L^2}^2 \right) + \nu \|\Lambda_1^\alpha \nabla \bar{u}\|_{L^2}^2 + \eta \|\Lambda_1^\beta \nabla \bar{b}\|_{L^2}^2 \leq 0.$$

Using Grönwall's lemma, we have

$$\|\nabla \bar{u}\|_{L^2}^2 + \|\nabla \bar{b}\|_{L^2}^2 \leq (\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2) e^{-C_4 t}, \quad (4.9)$$

where  $C_4 = C_4(\nu, \eta) > 0$ .

Combining the estimates in (4.6) and (4.9), we obtain the desired decay result. This completes the proof of Theorem 1.2.  $\square$

### Author contributions

Mingxi Zhang: writing-original draft, writing-review & editing, formal analysis; Ling Gao: methodology, writing-review & editing; Wenjing Yang: conceptualization, supervision, writing-review & editing.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflict of interest.

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