Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 18(2): 406–418.
DOI: 10.3934/cam.2026016
Received: 04 September 2025
Revised: 08 December 2025
Accepted: 22 January 2026
Published: 07 May 2026

Research article

Robustness of regularity and convergence of Galerkin approximations for 3D generalized incompressible Navier-Stokes equations

Ning Duan, Hao Pan and Xiaopeng Zhao*

College of Sciences, Northeastern University, Shenyang 110004, China

* **Correspondence:** Email: zhaoxiaopeng@mail.neu.edu.cn; Tel: +862483683382; Fax: +862483676883.

Abstract: In this paper, we first consider the robustness of strong solutions for 3D generalized Navier-Stokes equations, i.e., we show the set of all initial conditions $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ that give rise to a strong solution of the generalized Navier-Stokes equations on the time interval $[0, T]$ is open in $H^1(\mathbb{R}^3)$. Moreover, we prove that the Galerkin approximations of a strong solution of the 3D generalized Navier-Stokes equations converge strongly to u in $L^\infty(0, T; H^1_{0,div}(\mathbb{T}^3))$ and $L^2(0, T; H^{\kappa+1}(\mathbb{T}^3) \bigcap H^1_{0,div}(\mathbb{T}^3))$.

Keywords: robustness; convergence; Galerkin approximations; strong solution; generalized Navier-Stokes equations
Mathematics Subject Classification: 35B65, 76D05

1. Introduction

Let $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, and consider the Cauchy problem of 3D generalized incompressible Navier-Stokes equations:

$$\begin{cases} u_t + u \cdot \nabla u + (-\Delta)^\kappa u + \nabla \pi = 0, \\ \nabla \cdot u = 0, & x \in \Omega, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity field of the fluid and $\pi \in \mathbb{R}$ is the pressure. The definitions of the fractional Laplacian operator are different between $\mathbb{R}^3$ and $\mathbb{T}^3$:

- If $\Omega = \mathbb{R}^3$, the fractional Laplacian operator $(-\Delta)^\kappa$ is defined through the Fourier transform [1], namely,

$$\widehat{(-\Delta)^\kappa f}(\xi) = \widehat{\Lambda^\kappa f}(\xi) = |\xi|^{2\delta} \widehat{f}(\xi),$$

  and $\widehat{f}$ is the Fourier transform of f. Sometimes we write $\Lambda = (-\Delta)^{\frac{1}{2}}$ for notational convenience.

- For the case $\Omega = \mathbb{T}^3$, we use the Fourier decomposition to define the fractional Laplacian $(-\Delta)^\kappa$ (see [2,3]). For any $u \in L^2_{per}(\Omega)$, define

$$u(x) = \sum_{k_1,k_2,k_3\in\mathbb{Z}} \hat{u}_{k_1k_2k_3} e^{ik_1x_1+ik_2x_2+ik_3x_3},$$

where $i^2 = -1$. The Fourier coefficients are given by

$$\hat{u}_{k_1k_2k_3} = \langle u, e^{ik_1x_1+ik_2x_2+ik_3x_3}\rangle = \frac{1}{(2\pi)^3}\int_\Omega u e^{ik_1x_1+ik_2x_2+ik_3x_3} dx.$$

Hence, the fractional Laplacian can be defined by

$$(-\Delta)^\kappa u := \Lambda^{2\kappa} u := \sum_{k_1,k_2,k_3\in\mathbb{Z}} (k_1^2 + k_2^2 + k_3^2)^\kappa \hat{u}_{k_1k_2k_3} e^{ik_1x_1+ik_2x_2+ik_3x_3}.$$

We remark that if $\kappa = 1$, system (1.1) reduces to be Navier-Stokes equations, which have drawn much attention. Especially, the results of the global well-posedness of the initial value problem for Navier-Stokes equations in various classical function spaces were obtained due to the absence of global well-posedness for large initial data. Fujita and Kato [4] obtained the global well-posedness for small initial data and the local well-posedness for large initial data in $H^s(\mathbb{R}^n)$ with $s \geq \frac{n}{2} - 1$. Many other interesting and improved results have been established in $L^n(\mathbb{R}^n)$ by Kato [5], in the Besov space by Cannone [6] and Planchon [7], and in the larger bounded mean oscillation (BMO) space by Koch and Tataru [8]. Furthermore, Lei and Lin [9] proved global well-posedness results in the Wieneer space χ^{-1}. For more results on the Navier-Stokes equations, we refer the reader to [10–15] and the references therein.

It was Lions [16] who first investigated the generalized incompressible Navier-Stokes equations, and established the existence and uniqueness of a global regular solution under the condition that $\kappa \geq \frac{5}{4}$ (see also [17] and [18]; note that the magnetohydrodynamics (MHD) equations reduce to the Navier-Stokes equations when the magnetic field $b = 0$). In cases where $\kappa < \frac{5}{4}$, several studies have aimed at deriving regularity criteria or establishing partial regularity for the 3D generalized Navier-Stokes system; see, for example, [19,20] and [18]. Furthermore, a number of works have explored the global existence of strong or smooth solutions for these equations with small initial data in various function spaces, such as Sobolev spaces [21,22], pseudomeasure spaces [23], and the Lei-Lin space [24]. Results concerning the asymptotic behavior and regularity criteria space of solutions to the generalized Navier-Stokes equations can be found in [25,26] and the references therein. In the past two years, some classical results on the generalized Navier-Stokes equations have also emerged. For example, for the stationary generalized Navier-Stokes equations in dimensions four and five, Liu and Zuo [27] proved the existence of partial regular weak solutions and the Liouville-type theorem; Chamorro and Mansais [28] studied the mild solutions for the forced, incompressible generalized Navier-Stokes equations via a fixed-point argument, which relies on suitable estimates for the initial data, the nonlinearity, and the external forces. Besides, Wu [29] proved the energy equality for distributional solutions to the generalized Navier-Stokes equations. For more results on the generalized Navier-Stokes equations, we refer the reader to [30–33] and the references cited therein.

We note that in [14,34], Robinson and collaborators investigated the robustness of strong solutions for two types of second-order dissipative equations in $\mathbb{R}^3$. However, for fractional dissipative equations, the stability of solutions under perturbations has not yet been addressed. In this paper, adopting an approach similar to that in [14, 34], we first study the robustness of strong solutions to system (1.1)–a Cauchy problem involving fractional dissipation. Specifically, we demonstrate that on a finite time interval $[0,T]$, the smoothness of solutions to the generalized Navier-Stokes equations remains stable under small perturbations in the initial data with respect to the H^1-norm. More precisely, we establish the following theorem:

Theorem 1.1. *Let $\Omega = \mathbb{R}^3$. Assume that $\frac{3}{4} < \kappa < \frac{3}{2}$, $u_0 \in H^1(\mathbb{R}^3)$, and $\nabla \cdot u_0 = 0$, which gives rise to a strong solution $u(x,t)$ of system (1.1) on the time interval $[0,T]$. Moreover, if $v_0 \in H^1(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$, and*

$$\|\nabla v_0 - \nabla u_0\|_{L^2} \le R,$$

then v_0 also gives rise to a strong solution of system (1.1) on $[0,T]$. Here, the number R satisfies

$$R(u) = \left(\frac{4\kappa - 3}{2\kappa C T}\right)^{1-\frac{3}{4\kappa}} \exp\left[-c\int_0^t \left(\|\nabla u\|_{L^2}^{\frac{4\kappa}{4\kappa-3}} + \|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3-2\kappa}{\kappa}}\|\nabla u\|_{L^2}^{4-\frac{3}{\kappa}}\right)ds\right], \tag{1.2}$$

where C is an absolute constant.

The second goal of this paper is to study the convergence of Galerkin approximations of a strong solution u of the 3D generalized Navier-Stokes equations in $\mathbb{T}^3$. In order to consider this problem, we introduce the space $C_{0,div}^{\infty}(\mathbb{T}^3)$ as the space of divergence free vector fields in $(C_0^{\infty}(\mathbb{T}^3))^3$, and define $L_{0,div}^2(\mathbb{T}^3)$ and $H_{0,div}^1(\mathbb{T}^3)$ as the closure of $C_{0,div}^{\infty}(\mathbb{T}^3)$ with respect to the L^2 and H_0^1 norms, respectively. Define the Stokes operator $A : H_{0,div}^1(\mathbb{T}^3) \to L_{0,div}^2(\mathbb{R}^3)$ such that

$$(Au, \xi) = (\nabla u, \nabla \xi), \quad \forall \xi \in H_{0,div}^1(\mathbb{T}^3),$$

with domain $D(A) = H_{0,div}^1(\mathbb{T}^3) \bigcap H^2(\mathbb{T}^3)$ (see, e.g., [35, Chapter III]). Assume that P_n is the projection onto the space spanned by the first n eigenvectors of the Stokes operator A, and we consider

$$\begin{cases} \partial_t u_n + A^{\kappa} u_n + P_n (u_n \cdot \nabla u_n) = 0, \\ u_n(0) = P_n u_0, \end{cases} \tag{1.3}$$

where $\kappa \in [1, \frac{3}{2})$. We will prove that the solution of problem (1.3) converges strongly to the solution u of problem (1.1) in $L^{\infty}(0,T; H_{0,div}^1(\mathbb{T}^3))$ and $L^2(0,T; H^{\kappa+1}(\mathbb{T}^3) \bigcap H_{0,div}^1(\mathbb{T}^3))$.

It should be noted that establishing strong convergence in $L^2(0,T; H^1(\mathbb{T}^3))$ for Galerkin approximations of weak solutions remains challenging, even when the solution u itself belongs to $L^2(0,T; H^1(\mathbb{T}^3))$. In contrast, the situation improves significantly for strong solutions. Specifically, we demonstrate that every strong solution of the generalized incompressible Navier-Stokes equations in $\mathbb{T}^3$ can be obtained as the limit of its corresponding Galerkin approximations. The following theorem formalizes this result:

Theorem 1.2. *Let $\Omega = \mathbb{T}^3$. Assume that $\frac{3}{4} < \kappa < \frac{3}{2}$ and $u(x,t)$ is the strong solution of problem (1.1) on the time interval $[0,T]$ corresponding to $u_0 \in H^1(\mathbb{T}^3)$. Suppose u_n is the Galerkin approximations corresponding to the same initial condition u_0, i.e., u_n satisfies (1.3), then u_n converges strongly to u in $L^{\infty}(0,T; H_{0,div}^1(\mathbb{T}^3))$ and $L^2(0,T; H^{\kappa+1}(\mathbb{T}^3) \bigcap H_{0,div}^1(\mathbb{T}^3))$.*

Remark 1.3. *The main purpose of this section is to consider two problems: the robustness of strong solutions and the Galerkin approximations of strong solutions for generalized Navier-Stokes equations. It should be noted that in [14], the authors studied the 3D incompressible Navier-Stokes equations and derived corresponding conclusions. By comparison, our main results can be regarded as an extension of Robinson et al. [14] to fractional-order equations. In other words, the main conclusions of Robinson et al. [14] can be seen as a special case of our results when* $\kappa = 1$.

The remainder of this paper is structured as follows. In Section 2, we introduce some preliminary results and lemmas. In Section 3, we present the proof of Theorem 1.1, which addresses the robustness of regularity. Subsequently, Section 4 is devoted to analyzing the convergence of Galerkin approximations.

2. Preliminaries

This section presents several essential inequalities and lemmas that will be instrumental in proving the main results.

The following Gagliardo-Nirenberg inequality was proved in [36]:

Lemma 2.1 ([36])**.** *Let* $\Omega = \mathbb{R}^3$ *or* $\mathbb{T}^3$. *Suppose that* $0 \le m, \alpha \le l$, *then*

$$\|\nabla^\alpha f\|_{L^p(\Omega)} \le C\|\nabla^m f\|^{1-\theta}_{L^q(\Omega)}\|\nabla^l f\|^{\theta}_{L^r(\Omega)}, \tag{2.1}$$

where $\theta \in [0, 1]$ *and*

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta. \tag{2.2}$$

Here, when $p = \infty$, *we require that* $0 < \theta < 1$.

The following lemma will be useful for our subsequent analysis.

Lemma 2.2 ([14])**.** *Let* f *be a non-negative function and* X *a function satisfying the differential inequality*

$$\frac{dX}{dt} \le \alpha X^\beta + f(t), \quad X(0) = a,$$

where $a > 0$ *and* $\alpha, \beta > 0$. *Moreover, assume that* Y *is the solution of*

$$\frac{dY}{dt} = \alpha Y^\beta, \quad Y(0) = b + \int_0^T f(s)ds.$$

Prove that

(i) $a < b \Rightarrow X(t) < Y(t)$ *and*
(ii) $a \le b \Rightarrow X(t) \le Y(t)$

on the interval of existence of Y.

By using Lemma 2.2, we can easily obtain the following result:

Lemma 2.3 ([14]). *Assume the two parameters α, $\beta > 0$. Let $f_n \geq 0$ be a sequence of non-negative functions on $[0, T]$. Suppose that (y_n) is a sequence of non-negative functions on $(0, T)$ that satisfy*

$$\frac{dy_n}{dt}(t) \leq \alpha y_n^{\beta}(t) + f_n(t).$$

If

$$\eta_n := y_n(0) + \int_0^T f_n(s)ds \to 0 \quad \text{as } n \to \infty,$$

then $y_n \to 0$ uniformly on $[0, T]$ as $n \to \infty$.

3. Proof of Theorem 1.1

Assume that u and v are strong solutions for system (1.1) with the initial data u_0 and v_0, respectively. Moreover, set

$$\omega(t) = u(t) - v(t).$$

The main purpose of this section is to show that the H^1-norm of ω cannot blow up on $[0, T]$ if v_0 is sufficiently close to u_0.

It is easy to see that if v loses regularity on $[0, T]$, then there must exists some time $T^* \leq T$ such that v is smooth on $(0, T^*)$ and $\|\nabla v\|_{L^2} \to \infty$ as $t \to T^*$. Clearly, if this happens, then one also has $\|\nabla\omega\|_{L^2} \to \infty$ as $t \to T^*$. Hence, we suppose that there exists such a T^* and derive a contradiction.

We begin by deriving a differential inequality for $|\nabla\omega|$ that depends solely on ω and u, with no explicit dependence on v. Since u and v are two strong solutions for system (1.1) on $(0, T^*)$, we have

$$(u_t, \Delta\omega) + ((-\Delta)^{\kappa} u, \Delta\omega) + (u \cdot \nabla u, \Delta\omega) = 0,$$

and

$$-(v_t, \Delta\omega) - ((-\Delta)^{\kappa} v, \Delta\omega) - (v \cdot \nabla v, \Delta\omega) = 0.$$

Note that $\omega = u - v$. Therefore, combining the above two equalities together gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 + \|\Lambda^{\kappa+1}\omega\|_{L^2}^2 \leq |(u \cdot \nabla\omega, \Delta\omega)| + |(\omega \cdot \nabla u, \Delta\omega)| + |(\omega \cdot \nabla\omega, \Delta\omega)|. \tag{3.1}$$

By applying Hölder's inequality, Lebesgue interpolation, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, and Young's inequality, we can estimate the three terms on the right-hand side of (3.1) as follows:

$$\begin{aligned}
|(u \cdot \nabla\omega, \Delta\omega)| \leq& \|u\|_{L^6}\|\Delta\omega\|_{L^{\frac{6}{5-2\kappa}}}\|\nabla\omega\|_{L^{\frac{3}{\kappa}}} \\
\leq& C\|\nabla u\|_{L^2}\|\Lambda^{\kappa+1}\omega\|_{L^2}\|\Lambda^{\frac{5}{2}-\kappa}\omega\|_{L^2} \\
\leq& C\|\nabla u\|_{L^2}\|\Lambda^{\kappa+1}\omega\|_{L^2}\|\Lambda^{\kappa+1}\omega\|_{L^2}^{\frac{3}{2\kappa}-1}\|\nabla\omega\|_{L^2}^{2-\frac{3}{2\kappa}} \\
=& C\|\Lambda^{\kappa+1}\omega\|_{L^2}^{\frac{3}{2\kappa}}\|\nabla u\|_{L^2}\|\nabla\omega\|_{L^2}^{2-\frac{3}{2\kappa}} \\
\leq& \frac{1}{3}\|\Lambda^{\kappa+1}\omega\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{\frac{4\kappa}{4\kappa-3}}\|\nabla\omega\|_{L^2}^2,
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
|(\omega\cdot\nabla u,\Delta\omega)| \leq & \|\omega\|_{L^6}\|\Delta\omega\|_{L^{\frac{6}{5-2\kappa}}}\|\nabla u\|_{L^{\frac{3}{\kappa}}} \\
\leq & C\|\nabla\omega\|_{L^2}\|\Lambda^{\kappa+1}\omega\|_{L^2}\|\Lambda^{\frac{5}{2}-\kappa}u\|_{L^2} \\
\leq & C\|\nabla\omega\|_{L^2}\|\Lambda^{\kappa+1}\omega\|_{L^2}\|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3}{2\kappa}-1}\|\nabla u\|_{L^2}^{2-\frac{3}{2\kappa}} \\
\leq & \frac{1}{3}\|\Lambda^{\kappa+1}\omega\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^2\|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3}{\kappa}-2}\|\nabla u\|_{L^2}^{4-\frac{3}{\kappa}},
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
|(\omega\cdot\nabla\omega,\Delta\omega)| \leq & \|\omega\|_{L^6}\|\Delta\omega\|_{L^{\frac{6}{5-2\kappa}}}\|\nabla\omega\|_{L^{\frac{3}{\kappa}}} \\
\leq & C\|\nabla\omega\|_{L^2}\|\Lambda^{\kappa+1}\omega\|_{L^2}\|\Lambda^{\frac{5}{2}-\kappa}\omega\|_{L^2} \\
\leq & C\|\nabla\omega\|_{L^2}\|\Lambda^{\kappa+1}\omega\|_{L^2}\|\Lambda^{\kappa+1}\omega\|_{L^2}^{\frac{3}{2\kappa}-1}\|\nabla\omega\|_{L^2}^{2-\frac{3}{2\kappa}} \\
= & C\|\Lambda^{\kappa+1}\omega\|_{L^2}^{\frac{3}{2\kappa}}\|\nabla\omega\|_{L^2}^{3-\frac{3}{2\kappa}} \\
\leq & \frac{1}{3}\|\Lambda^{\kappa+1}\omega\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^{\frac{12\kappa-6}{4\kappa-3}}.
\end{aligned}
\tag{3.4}
$$

Plugging (3.2)–(3.4) into (3.1) gives

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 \leq & C\|\nabla u\|_{L^2}^{\frac{4\kappa}{4\kappa-3}}\|\nabla\omega\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^2\|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3}{\kappa}-2}\|\nabla u\|_{L^2}^{4-\frac{3}{\kappa}} + C\|\nabla\omega\|_{L^2}^{\frac{12\kappa-6}{4\kappa-3}} \\
\leq & C\|\nabla\omega\|_{L^2}^{\frac{12\kappa-6}{4\kappa-3}} + C\|\nabla\omega\|_{L^2}^2(\|\nabla u\|_{L^2}^{\frac{4\kappa}{4\kappa-3}} + \|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3}{\kappa}-2}\|\nabla u\|_{L^2}^{4-\frac{3}{\kappa}}),
\end{aligned}
$$

which can be rewritten as

$$
\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 - \alpha(t)\|\nabla\omega\|_{L^2}^2 \leq C\|\nabla\omega\|_{L^2}^{\frac{12\kappa-6}{4\kappa-3}},
\tag{3.5}
$$

where

$$
\alpha(t) = \|\nabla u\|_{L^2}^{\frac{4\kappa}{4\kappa-3}} + \|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3}{\kappa}-2}\|\nabla u\|_{L^2}^{4-\frac{3}{\kappa}}.
\tag{3.6}
$$

Next, multiplying (3.5) by $e^{-\int_0^t\alpha(s)ds}$, we deduce that

$$
\begin{aligned}
\frac{d}{dt}\left(e^{-\int_0^t\alpha(s)ds}\|\nabla\omega\|_{L^2}^2\right) \leq & Ce^{-\int_0^t\alpha(s)ds}\|\nabla\omega\|_{L^2}^{\frac{12\kappa-6}{4\kappa-3}} \\
= & Ce^{\frac{2\kappa}{4\kappa-3}\int_0^t\alpha(s)ds}e^{-\frac{6\kappa-3}{4\kappa-3}\int_0^t\alpha(s)ds}\|\nabla\omega\|_{L^2}^{\frac{12\kappa-6}{4\kappa-3}} \\
\leq & Ce^{\frac{2\kappa}{4\kappa-3}\int_0^t\alpha(s)ds}\left(e^{-\int_0^t\alpha(s)ds}\|\nabla\omega\|_{L^2}^2\right)^{\frac{6\kappa-3}{4\kappa-3}},
\end{aligned}
$$

that is,

$$
\frac{d}{dt}\left(e^{-\int_0^t\alpha(s)ds}\|\nabla\omega\|_{L^2}^2\right) \leq \beta\left(e^{-\int_0^t\alpha(s)ds}\|\nabla\omega\|_{L^2}^2\right)^{\frac{6\kappa-3}{4\kappa-3}},
\tag{3.7}
$$

where α is given in (3.6) and β satisfies

$$
\beta = C\exp\left[\frac{2\kappa}{4\kappa-3}\int_0^t\alpha(s)ds\right].
\tag{3.8}
$$

Define $X(t) = e^{-\int_0^t\alpha(s)ds}\|\nabla\omega\|_{L^2}^2$. On the basis of the above two steps, we derive that on $(0,T^*)$, the function $X(t)$ satisfies

$$
\dot{X} \leq \beta X^{\frac{6\kappa-3}{4\kappa-3}}, \quad X(0) = \|\nabla\omega(0)\|_{L^2}^2,
$$

where β is given in (3.8). Hence, $X(t) \leq Y(t)$, where Y is the solution of the following ordinary differential equation:

$$\dot{Y} = \beta Y^{\frac{6\kappa-3}{4\kappa-3}}, \quad Y(0) = \|\nabla\omega(0)\|_{L^2}^2,$$

as long as the solution Y exists. Then, by direct computation of Y, one arrives at

$$e^{-\int_0^t \alpha(s)ds}\|\nabla\omega(t)\|_{L^2}^2 \leq \frac{Y(0)}{\left(1 - \frac{2\kappa}{4\kappa-3}\beta t Y^{\frac{2\kappa}{4\kappa-3}}(0)\right)^{\frac{4\kappa-3}{2\kappa}}},$$

as long as

$$Y(0) < \left(\frac{2\kappa}{4\kappa-3}\beta t\right)^{-\frac{4\kappa-3}{2\kappa}}. \tag{3.9}$$

Hence, if we know that $Y(0) < \left(\frac{2\kappa}{4\kappa-3}\beta(t)T\right)^{-\frac{4\kappa-3}{2\kappa}}$, then clearly for each $t \in (0, T^*)$, where $T^* \leq T$, we have (3.9) together with

$$\begin{aligned}\|\nabla\omega(t)\|_{L^2}^2 \leq& e^{\int_0^t \alpha(s)ds}\frac{Y(0)}{\left(1 - \frac{2\kappa}{4\kappa-3}\beta t Y^{\frac{2\kappa}{4\kappa-3}}(0)\right)^{\frac{4\kappa-3}{2\kappa}}}\\ \leq& e^{\int_0^T \alpha(s)ds}\frac{Y(0)}{\left(1 - \frac{2\kappa}{4\kappa-3}\beta T Y^{\frac{2\kappa}{4\kappa-3}}(0)\right)^{\frac{4\kappa-3}{2\kappa}}} := C_T.\end{aligned}$$

Rewritting the condition (3.9) in our original variables, we deduce that if

$$\|\nabla\omega(0)\|_{L^2} < \left(\frac{4\kappa-3}{2\kappa c T}\right)^{1-\frac{3}{4\kappa}} \exp\left[-c\int_0^t \left(\|\nabla u\|_{L^2}^{\frac{4\kappa}{4\kappa-3}} + \|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3-2\kappa}{\kappa}}\|\nabla u\|_{L^2}^{4-\frac{3}{\kappa}}\right)ds\right],$$

then $\|\nabla\omega(t)\|_{L^2}^2 \leq C_T$ for all $t \in [0, T^*)$, so $\|\nabla\omega(T^*)\|_{L^2}^2 \leq C_T$, In consequence, ω does not blow up at any time $T^* \leq T$, hence we complete the proof.

4. Proof of Theorem 1.2

Assume that $u(x, t)$ is the strong solution of problem (1.1) on the time interval $[0, T]$ corresponding to $u_0 \in H^1(\mathbb{T}^3)$. Then, we have

$$u \in L^\infty(0, T; H^1_{0,div}(\mathbb{T}^3)) \bigcap L^2(0, T; H^{\kappa+1}(\mathbb{T}^3) \bigcap H^1_{0,div}(\mathbb{T}^3)). \tag{4.1}$$

Using (4.1) and the fact that $H^{\kappa+1}(\mathbb{T}^3) \to H^{\frac{5}{2}-\kappa}(\mathbb{T}^3)$ ($\frac{3}{4} \leq \kappa < \frac{3}{2}$), we arrive at

$$\begin{aligned}\int_0^T \|u\cdot\nabla u\|_{L^{\frac{6}{1+2\kappa}}}^2 dt \leq& C\int_0^T \|u\|_{L^6}^2\|\nabla u\|_{L^{\frac{3}{\kappa}}}^2 dt \leq C\int_0^T \|u\|_{L^6}^2\|\Lambda^{\frac{5}{2}-\kappa}u\|_{L^2}^2 dt\\ \leq& C\|u\|_{L^\infty(0,T;H^1)}\|\Lambda^{\frac{5}{2}-\kappa}u\|_{L^2(0,T;L^2)}^2\\ \leq& C\|u\|_{L^\infty(0,T;H^1)}\|u\|_{L^2(0,T;H^{\kappa+1})}^2 \leq C,\end{aligned} \tag{4.2}$$

which implies that $u\cdot\nabla u \in L^2(0, T; L^{\frac{6}{1+2\kappa}}(\mathbb{T}^3))$. Moreover, suppose that u_n is the sequence of Galerkin approximations of u. We will show that the H^1 norm of the difference

$$\omega_n = u - u_n,$$

tends uniformly to 0 on $[0, T]$ as n tends to infinity.

We first derive a differential inequality for $\|\nabla\omega_n\|_{L^2}$ that depends on ω_n and the strong solution u, but not explicitly on u_n. Note that $u(t)$ is a strong solution to system (1.1) on $(0, T)$. Hence,

$$(u_t, A\omega_n) + (A^{\kappa+1}u, A^{\kappa+1}\omega_n) + (\mathbb{P}[u\cdot\nabla u], A\omega_n) = 0, \tag{4.3}$$

where $\mathbb{P}$ is the Leray projector. Introduce Q_n via the identity $Id = P_n + Q_n$, then the above equation can be rewritten as

$$(u_t, A\omega_n) + (A^{\kappa+1}u, A^{\kappa+1}\omega_n) + (P_n[u\cdot\nabla u], A\omega_n) = -(Q_n\mathbb{P}[u\cdot\nabla u], A\omega_n). \tag{4.4}$$

Taking the inner product of $(1.3)_1$ with $A\omega_n$, it yields that

$$(\partial_t u_n, A\omega_n) + (A^{\kappa+1}u_n, A^{\kappa+1}\omega_n) + (P_n[u_n\cdot\nabla u_n], A\omega_n) = 0. \tag{4.5}$$

Combining (4.4) and (4.5) together gives

$$\begin{aligned}&(\partial_t\omega_n, A\omega_n) + (A^{\kappa+1}\omega_n, A^{\kappa+1}\omega_n)\\ =&(P_n[u_n\cdot\nabla u_n - u\cdot\nabla u], A\omega_n) - (Q_n\mathbb{P}[u\cdot\nabla u], A\omega_n),\end{aligned}$$

that is,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\omega_n\|_{L^2}^2 + \|\Lambda^{\kappa+1}\omega_n\|_{L^2}^2 \le&|(u\cdot\nabla\omega_n, P_nA\omega)| + |(\omega_n\cdot\nabla u, P_nA\omega)|\\ &+ |(\omega_n\cdot\nabla\omega_n, P_nA\omega)| + |(Q_n\mathbb{P}[u\cdot\nabla u], A\omega_n)|.\end{aligned} \tag{4.6}$$

By using Hölder's inequality, Lebesgue's interpolation, Sobolev's embedding $H^1(\mathbb{T}^3) \to L^6(\mathbb{T}^3)$, and Young's inequality, we can bound the three terms on the right hand side of (4.6) as

$$\begin{aligned}|(u\cdot\nabla\omega_n, P_nA\omega_n)| \le&\|u\|_{L^6}\|\Delta\omega_n\|_{L^{\frac{6}{5-2\kappa}}}\|\nabla\omega_n\|_{L^{\frac{3}{\kappa}}}\\ \le&C\|\nabla u\|_{L^2}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}\|\Lambda^{\frac{5}{2}-\kappa}\omega_n\|_{L^2}\\ \le&C\|\nabla u\|_{L^2}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}^{\frac{3}{2\kappa}-1}\|\nabla\omega_n\|_{L^2}^{2-\frac{3}{2\kappa}}\\ \le&\frac{1}{8}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{\frac{4\kappa}{4\kappa-3}}\|\nabla\omega_n\|_{L^2}^2,\end{aligned} \tag{4.7}$$

$$\begin{aligned}|(\omega_n\cdot\nabla u, P_nA\omega_n)| \le&\|\omega_n\|_{L^6}\|\Delta\omega_n\|_{L^{\frac{6}{5-2\kappa}}}\|\nabla u\|_{L^{\frac{3}{\kappa}}}\\ \le&C\|\nabla\omega_n\|_{L^2}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}\|\Lambda^{\frac{5}{2}-\kappa}u\|_{L^2}\\ \le&C\|\nabla\omega_n\|_{L^2}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}\|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3}{2\kappa}-1}\|\nabla u\|_{L^2}^{2-\frac{3}{2\kappa}}\\ \le&\frac{1}{8}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}^2 + C\|\nabla\omega_n\|_{L^2}^2\|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3}{\kappa}-2}\|\nabla u\|_{L^2}^{4-\frac{3}{\kappa}},\end{aligned} \tag{4.8}$$

$$\begin{aligned}|(\omega_n\cdot\nabla\omega_n, P_nA\omega_n)| \le&\|\omega_n\|_{L^6}\|\Delta\omega_n\|_{L^{\frac{6}{5-2\kappa}}}\|\nabla\omega_n\|_{L^{\frac{3}{\kappa}}}\\ \le&C\|\nabla\omega_n\|_{L^2}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}\|\Lambda^{\frac{5}{2}-\kappa}\omega_n\|_{L^2}\\ \le&C\|\nabla\omega_n\|_{L^2}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}^{\frac{3}{2\kappa}-1}\|\nabla\omega_n\|_{L^2}^{2-\frac{3}{2\kappa}}\\ \le&\frac{1}{8}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}^2 + C\|\nabla\omega_n\|_{L^2}^{\frac{12\kappa-6}{4\kappa-3}},\end{aligned} \tag{4.9}$$

and

$$
\begin{aligned}
|(Q_n\mathbb{P}[u\cdot\nabla u], A\omega_n)| \le& \|A\omega_n\|_{L^{\frac{6}{5-2\kappa}}} \|Q_n\mathbb{P}[u\cdot\nabla u]\|_{\frac{6}{1+2\kappa}} \\
\le& \frac{1}{8}\|\Lambda^{\kappa+1}\omega_n\|_{L^2}^2 + C\|Q_n\mathbb{P}[u\cdot\nabla u]\|_{\frac{6}{1+2\kappa}}^2 .
\end{aligned} \tag{4.10}
$$

Plugging (4.7)–(4.10) into (4.6) gives

$$
\frac{d}{dt}\|\nabla\omega_n\|_{L^2}^2 + \|A^{\kappa+1}\omega_n\|_{L^2}^2 \le \alpha(t)\|\nabla\omega_n\|_{L^2}^2 + C\|\nabla\omega_n\|_{L^2}^{\frac{12\kappa-6}{4\kappa-3}} + f_n(t), \tag{4.11}
$$

where

$$
\alpha(t) = C\left(\|\nabla u\|_{L^2}^{\frac{4\kappa}{4\kappa-3}} + \|\Lambda^{\kappa+1}u\|_{L^2}^{\frac{3}{\kappa}-2}\|\nabla u\|_{L^2}^{4-\frac{3}{\kappa}}\right),
$$

and

$$
f_n(t) = C\|Q_n\mathbb{P}[u\cdot\nabla u]\|_{L^{\frac{6}{1+2\kappa}}}^2 .
$$

Next, an integrating factor is applied to simplify equation (4.11). The detailed steps are omitted here, as they closely follow the reasoning provided in Section 3. The result is that

$$
\frac{d}{dt}y_n(t) \le \beta y_n^{\frac{6\kappa-3}{4\kappa-3}}(t) + f_n(t), \tag{4.12}
$$

where

$$
\beta = C\exp\left[\frac{2\kappa}{4\kappa-3}\int_0^t \alpha(s)ds\right],
$$

and

$$
y_n(t) = \|\nabla\omega_n(t)\|_{L^2}^2 \exp\left(-\int_0^t \alpha(s)ds\right).
$$

Hence, set

$$
\eta_n = \|\nabla\omega_n(0)\|_{L^2}^2 + \int_0^T \|Q_n\mathbb{P}[u\cdot\nabla u]\|_{\frac{6}{1+2\kappa}}^2 ds. \tag{4.13}
$$

What we need to do is to show that $\eta_n \to 0$ as $n \to \infty$. In fact, P_n is the projection onto the space spanned by the first n eigenvectors of of the Stokes operator A and Q_n in its orthogonal complement. Denoting these eigenfunctions by $\{\zeta_j\}_{j=1}^{\infty}$, and their corresponding eigenvalues by $0 < \lambda_1 \le \lambda_2 \le \cdots$, we obtain

$$
P_n u = \sum_{j=1}^{n}(u,\zeta_j)\zeta_j, \quad \text{and} \quad Q_n u = \int_{j=n+1}^{\infty}(u,\zeta_j)\zeta_j.
$$

Hence, if $u \in H^1_{0,div}(\mathbb{T}^3)$, then

$$
\|Q_n u\|_{H^1_{0,div}(\mathbb{T}^3)}^2 = \sum_{j=n+1}^{\infty}\lambda_j|(u,\zeta_j)|^2 \le \sum_{j=1}^{\infty}\lambda_j|(u,\zeta_j)|^2 = \|u\|_{H^1_{0,div}(\mathbb{T}^3)}^2,
$$

and clearly, $Q_n u \to 0$ in $H^1_{0,div}(\mathbb{T}^3)$ as $n \to \infty$ (see [37]). Based on the above properties of Q_n, we immediately obtain

$$
\|\nabla\omega_n(0)\|_{L^2}^2 = \|\nabla u(0) - \nabla P_n u(0)\|_{L^2}^2 \le \|u_0 - P_n u_0\|_{H^1}^2 \to 0. \tag{4.14}
$$

On the other hand, to show the convergence of the integral term, observe that since

$$\|\mathbb{P}[u\cdot\nabla u]\|^2_{L^{\frac{6}{1+2\kappa}}} \le C\|u\|_{L^6}\|\nabla u\|_{L^{\frac{3}{\kappa}}} \le C\|\nabla u\|_{L^2}\|\nabla u\|_{L^{\frac{3}{2}-\kappa}} \le C\|\nabla u\|_{L^2}\|\nabla u\|_{H^\kappa},$$

and the regularity result (4.1) guarantees that $u \in L^2(0,T;H^{\kappa+1}(\mathbb{T}^3)\bigcap H^1_{0,div}(\mathbb{T}^3))$, it follows that

$$\mathbb{P}(u\cdot\nabla u) \in H^1(\mathbb{T}^3) \quad \text{for a.e. } s\in[0,T].$$

One therefore knows that $\|Q_n\mathbb{P}[u\cdot\nabla u]\|^2_{L^{\frac{6}{1+2\kappa}}}$ convergens pointwise to 0 for a.e. $s\in[0,T]$, while it is clear that

$$\|Q_n\mathbb{P}[u\cdot\nabla u]\|^2_{L^{\frac{6}{1+2\kappa}}} \le \|\mathbb{P}(u\cdot\nabla u)\|^2_{L^{\frac{6}{1+2\kappa}}} \quad \text{for a.e. } s\in[0,T].$$

The right-hand side is an element of $L^1(0,T)$ (based on (4.2)). It follows from the Lebesgue dominated convergence theorem that

$$\int_0^T \|Q_n\mathbb{P}[u\cdot\nabla u]\|^2_{L^{\frac{6}{1+2\kappa}}}\,ds \to 0 \quad \text{as } n\to\infty. \tag{4.15}$$

Plugging (4.14) and (4.15) into (4.13), we obtain $\eta_n \to 0$ as required. Then, from Lemma 2.3, we deduce that

$$\|\nabla\omega_n\|^2_{L^2} \to 0 \quad \text{uniformly on } (0,T).$$

Thus, $u_n \to u$ in $L^\infty(0,T;H^1_{0,div}(\mathbb{T}^3))\bigcap L^2(0,T;H^{\kappa+1}(\mathbb{T}^3)\bigcap H^1_{0,div}(\mathbb{T}^3))$. This complete the proof.

5. Conclusions

This work establishes two key results for the 3D generalized Navier-Stokes equations. First, the set of initial data in $H^1(\mathbb{R}^3)$ yielding a strong solution on $[0,T]$ is open, proving structural stability under H^1 perturbations. Second, Galerkin approximations converge strongly to the solution in $L^\infty(0,T;H^1_{0,\mathrm{div}}(\mathbb{T}^3))$ and $L^2(0,T;H^{\kappa+1}(\mathbb{T}^3)\cap H^1_{0,\mathrm{div}}(\mathbb{T}^3))$. The robustness result ensures that small H^1 errors in initial conditions (e.g., from measurement or discretization) do not prevent strong solution existence, enhancing reliability in physical modeling. The convergence proof rigorously validates spectral and finite element methods, which is crucial for high-fidelity simulations such as direct numerical simulation. Future research may extend robustness to weaker (e.g., L^2) norms, generalize the results to bounded domains or higher dimensions, analyze coupled systems (e.g., MHD), and incorporate stochastic initial perturbations for uncertainty quantification.

Author contributions

Ning Duan: Writing-original draft; Hao Pan: Formal analysis; Xiaopeng Zhao: Writing-review & editing, methodology.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970. https://doi.org/10.1112/blms/5.1.121
2. M. Ainsworth, Z. Mao, Analysis and approximation of a fractional Cahn-Hilliard equation, *SIAM J. Numer. Anal.*, **55** (2017), 1689–1718. https://doi.org/10.1137/16M1075302
3. X. Zhao, Long time behavior of solutions to 3D generalized MHD equations, *Forum Math.*, **32** (2020), 977–993. https://doi.org/10.1515/forum-2019-0155
4. H. Fujita, T. Kato, On the Navier-Stokes initial value problem I, *Arch. Ration. Mech. Anal.*, **16** (1964), 269–315. https://doi.org/10.1007/BF00276188
5. T. Kato, Strong L^p-solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions, *Math. Z.*, **187** (1984), 471–480. https://doi.org/10.1007/BF01174182
6. M. Cannone, *Ondelettes, Paraproduits et Navier-Stokes*, Arts et Sciences, Diderot editeur, 1995.
7. F. Planchon, Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **13** (1996), 319–336. https://doi.org/10.1016/S0294-1449(16)30107-X
8. H. Koch, D. Tataru, Well-posedness for the Navier-Stokes equations, *Adv. Math.*, **157** (2001), 22–35. https://doi.org/10.1006/aima.2000.1937
9. Z. Lei, F. Lin, Global mild solutions of Navier-Stokes equations, *Comm. Pure Appl. Math.*, **64** (2011), 1297–1304. https://doi.org/10.1002/cpa.20361
10. R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland Publishing Company, Amsterdam, 1979. https://doi.org/10.1115/1.3424338
11. J.-Y. Chemin, I. Gallagher, Well-posedness and stability results for the Navier-Stokes equations in $\mathbb{R}^3$, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **26** (2009), 599–624. https://doi.org/10.1016/j.anihpc.2007.05.008
12. G. Seregin, *Lecture Notes on Regularity Theory for The Navier-Stokes Equation*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. https://doi.org/10.1142/9314
13. D. Song, X. Zhao, Large-time behavior of cylindrically symmetric Navier-Stokes equations with temperature-dependent viscosity and heat conductivity, *Commun. Anal. Mech.*, **16** (2024), 599–632. https://doi.org/10.3934/cam.2024028
14. J. C. Robinson, J. L. Rodrigo, W. Sadowski, *The Three-Dimensional Navier-Stokes Equations*, In: *Cambridge Studies in Advanced Mathematics,* Cambridge University Press, Cambridge, 2016. https://doi.org/10.1017/CBO9781139095143
15. S. Wang, Global well-posedness and viscosity vanishing limit of a new initial-boundary value problem on two/three-dimensional incompressible Navier-Stokes equations and/or Boussinesq equations, *Commun. Anal. Mech.*, **17** (2025), 582–605. https://doi.org/ 10.3934/cam.2025023

16. J.-L. Lions, *uelques méthodes de résolution des problémes aux limites non linéaires, Dunod, Gauthier-Villars*, Paris, 1969.

17. J. Wu, Generalized MHD equations, *J. Differential Equations*, **195** (2003), 284–312. https://doi.org/10.1016/j.jde.2003.07.007

18. Y. Zhou, Regularity criteria for the generalized viscous MHD equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **24** (2007), 491–505. https://doi.org/10.1016/j.anihpc.2006.03.014

19. J. Fan, Y. Fukumoto, Y. Zhou, Logarithmically improved regularity criteria for the generalized Navier-Stokes and related equations, *Kinet. Relat. Models*, **6** (2013), 545–556. https://doi.org/10.3934/krm.2013.6.545

20. J. Wu, Regularity criteria for the generalized MHD equations, *Comm. Partial Differential Equations*, **33** (2008), 285–306. https://doi.org/10.1080/03605300701382530

21. N. Duan, Well-posedness and decay of solutions for three-dimensional generalized Navier-Stokes equations, *Comput. Math. Appl.*, **76** (2018), 1026–1033. https://doi.org/10.1016/j.camwa.2018.05.038

22. X. Zhao, Y. Zhou, Well-posedness and decay of solutions to 3D generalized Navier-Stokes equations, *Discret. Cont. Dyn. Syst. Ser. B*, **26** (2021), 795–813. https://doi.org/10.3934/dcdsb.2020142

23. Q. Liu, J. Zhao, S. Cui, Existence and regularizing rate estimates of solutions to a generalized magneto-hydrodynamic system in pseudomeasure spaces, *Ann. Mat. Pura Appl.*, **191** (2012), 293–309. https://doi.org/10.1007/s10231-010-0184-8

24. Z. Ye, Global well-posedness and decay results to 3D generalized viscous magnetohydrodynamic equations, *Ann. Mat. Pura Appl.*, **195** (2016), 1111–1121. https://doi.org/10.1007/s10231-015-0507-x

25. Q. Jiu, H. Yu, Decay of solutions to the three-dimensional generalized Navier-Stokes equations, *Asymptotic Anal.*, **94** (2015), 105–124. https://doi.org/10.3233/ASY-151307

26. N. T. Le, L. T. Tinh, On the three dimensional generalized Navier-Stokes equations with damping, *Frac. Calc. Appl. Anal.*, **28** (2025), 1923–1967. https://doi.org/10.1007/s13540-025-00421-5

27. Q. Liu, Z. Zuo, Partially regular weak solutions and Liouville-type theorem to the stationary fractional Navier-Stokes equations in dimensions four and five, *J. Differential Equations*, **421** (2025), 291–335. https://doi.org/10.1016/j.jde.2024.11.057

28. D. Chamorro, M. Mansais, Some general external forces and critical mild solutions for the fractional Navier-Stokes equations, *J. Elliptic Parabolic Equ.*, **11** (2025), 1071–1100. https://doi.org/10.1007/s41808-025-00365-0

29. F. Wu, A note on energy equality for the fractional Navier-Stokes equations, *Proc. Royal Soc. Edinburg. Sec. A Math.*, **154** (2024), 201–208. https://doi.org/ 10.1017/prm.2023.3

30. O. Jarrin, Asymptotic behavior in time of a generalized Navier-Stokes-alpha model, *Disc. Cont. Dyn. Syst. Ser. B*, **30** (2025), 1669–1709. https://doi.org/10.3934/dcdsb.2024155

31. J. L. Wu, Regularity criteria for the 3D generalized Navier-Stokes equations with nonlinear damping term, *AIMS Math.*, **9** (2025), 16250–16259. https://doi.org/10.3934/math.2024786

32. X. Xi, Y. Zhou, M. Hou, Well-posedness of mild solutions for the fractional Navier-Stokes equations in Besov spaces, *Qualitative Theory Dyn. Syst.*, **23** (2024), 15. https://doi.org/10.1007/s12346-023-00867-z

33. D. W. Boutros, J. D. Gibbon, Phase transitions in the fractional three-dimensional Navier-Stokes equations, *Nonlinearity*, **37** (2024), 045010. https://doi.org/10.1088/1361-6544/ad25be

34. K. W. Hajduk, J. C. Robinsion, W. Sadowski, Robustness of regularity for the 3D convective Brinkman-Forchheimer equations, *J. Math. Anal. Appl.*, **500** (2021), 125058. https://doi.org/10.1016/j.jmaa.2021.125058

35. H. Sohr, *The Navier-Stokes Equations: An Elementary Functional Analytic Approach*, Birkhäuser Advanced Texts, Springer, Basel, 2012.

36. L. Nirenberg, On elliptic partial differential equations, *Ann. Sc. Norm. Super. Pisa CI. Sci.*, **13** (1959), 115–162.

37. S. I. Chernysehnko, P. Constantin, J. C. Robinson, E. S. Titi, A posteriori regularity of the three-dimensional Navier-Stokes equations from numerical computations, *J. Math. Phys.*, **48** (2006), 1–15. https://doi.org/10.1063/1.2372512