Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 18(2): 400–405.
DOI: 10.3934/cam.2026015
Received: 16 December 2025
Revised: 23 March 2026
Accepted: 21 April 2026
Published: 28 April 2026

Correction

On the necessity of the connectedness condition of Ω in : "Nontrivial solutions for the Laplace equation with a nonlinear Goldstein–Wentzell boundary condition"

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Abstract: This note corrects an inaccuracy in the paper "Nontrivial solutions for the Laplace equation with a nonlinear Goldstein–Wentzell boundary condition" by the author, which appeared in Commun. Anal. Mech. **15** (2023), no. 4, 811–830. In particular, we point out that the bounded open set Ω must be connected for some of the main results in the paper to hold. To motivate this claim, we give an example of a disconnected open set Ω combined with a partition (Γ_0, Γ_1) of its boundary, for which the potential–well depth associated to the problem vanishes. When this occurs, the framework developed in the paper breaks down. We also show that, conversely, when Ω is connected, this phenomenon does not show up and all assertions in the paper are correct.

Keywords: Laplace equation; Laplace–Beltrami operator; existence and multiplicity for nontrivial solutions; Wentzell boundary conditions; Ventcel boundary conditions; Mountain Pass Theorem
Mathematics Subject Classification: 35D30, 35J05, 35J20, 25J25, 35J61, 35J67

1. The content of the paper [1]

The aim of this note is to correct an inaccuracy in the paper [1]. To keep the discussion as self–contained as possible, we first recall the problem studied in [1] and summarize its main results.

We are dealing with the doubly elliptic problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ -\Delta_\Gamma u + \partial_\nu u = |u|^{p-2}u & \text{on } \Gamma_1, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with C^1 boundary $\Gamma = \partial\Omega$ (see [2]). Moreover, we assume that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_1 being nonempty and relatively open on Γ (or equivalently $\overline{\Gamma_0} = \Gamma_0$).

We denote by $\mathcal{H}^{N-1}$ the Hausdorff measure, see [3], Chapter 1, pp. 15–16. By the Area Theorem, the restriction of $\mathcal{H}^{N-1}$ to measurable subsets of Γ coincides with the $(N-1)$ – dimensional area on the manifold Γ, see [4], Theorem 9.27, p. 253, and [5], Chapter 2, p. 19. We also assume that $\mathcal{H}^{N-1}(\overline{\Gamma}_0 \cap \overline{\Gamma}_1) = 0$ and $\mathcal{H}^{N-1}(\Gamma_0) > 0$.

Moreover, in (1.1), we respectively denote by Δ and Δ_Γ the Laplace and the Laplace–Beltrami operators, while ν stands for the outward normal to Ω. The nonlinearity $|u|^{p-2}u$ is superlinear and subcritical with respect to the Sobolev embedding $H^1(\Gamma) \hookrightarrow L^p(\Gamma)$, that is,

$$2 < p < r, \qquad \text{where} \quad r = \begin{cases} \frac{2(N-1)}{N-3} & \text{if } N \geq 4, \\ \infty & \text{if } N = 2, 3. \end{cases} \tag{1.2}$$

We introduce the space

$$H^1 = \left\{(u, v) \in H^1(\Omega) \times H^1(\Gamma) : v = u_{|\Gamma}, v = 0 \ \text{ on } \Gamma_0\right\}.$$

The linear and bijective operator $(u, u_{|\Gamma}) \mapsto u$ from H^1 onto the space

$$H^1_{\Gamma_0}(\Omega, \Gamma) = \{u \in H^1(\Omega) : u_{|\Gamma} \in H^1(\Gamma);\ u_{|\Gamma} = 0 \ \text{ on } \Gamma_0\}, \tag{1.3}$$

allows us to identify H^1 with $H^1_{\Gamma_0}(\Omega, \Gamma)$, which we shall do throughout. Hence, we shall write, without further mention, $u \in H^1$ for functions defined in Ω. Moreover, when working with $u \in H^1$, we will often omit the trace notation $u_{|\Gamma}$ when it is clear from the context. Lebesgue spaces and integrals on the boundary will be the ones which are defined in relation with $\mathcal{H}^{N-1}$, the notation $d\mathcal{H}^{N-1}$ being dropped in integrals, so $\int_{\Gamma_1} |u|^2 = \int_{\Gamma_1} |u|^2 \, d\mathcal{H}^{N-1}$.

The space H^1 is naturally equipped with the topology inherited from the product, which is induced, see Lemma 1 in [6], by the norm $|\!|\!|\cdot|\!|\!|_{H^1}$ given by

$$|\!|\!|u|\!|\!|^2_{H^1} = \int_\Omega |\nabla u|_2^2 + \int_{\Gamma_1} |\nabla_\Gamma u|^2_{2,\Gamma_1} + \int_{\Gamma_1} |u|^2,$$

where ∇_Γ denotes the Riemannian gradient on Γ, $|\cdot|_\Gamma = (\cdot, \cdot)_\Gamma^{1/2}$, and $(\cdot, \cdot)_\Gamma$ denotes the Riemannian scalar product on the tangent bundle of Γ.

The following result, which is going to be discussed in the sequel, is a key one in [1].

Lemma 1 ([1, Lemma 1]). *Let $\mathcal{H}^{N-1}(\Gamma_0) > 0$. Then, setting for all $u, v \in H^1$*

$$(u, v)_{H^1} = \int_\Omega \nabla u \nabla v + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma, \quad \textit{and} \quad \|\cdot\|_{H^1} = (\cdot, \cdot)^{1/2}_{H^1}, \tag{1.4}$$

$\|\cdot\|_{H^1}$ defines on H^1 a norm equivalent to $|\!|\!|\cdot|\!|\!|_{H^1}$.

We now introduce the potential energy functional $I : H^1 \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_{\Gamma_1} |\nabla_\Gamma u|^2_\Gamma - \frac{1}{p} \int_{\Gamma_1} |u|^p. \tag{1.5}$$

By assumption (1.2) and standard results one has $I \in C^1(H^1) = C^1(H^1; \mathbb{R})$. By[1, Lemma 3], critical points of I coincide with weak solutions of (1.1). They are defined as $u \in H^1$ such that

$$\int_\Omega \nabla u \nabla \phi + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma \phi)_\Gamma - \int_{\Gamma_1} |u|^{p-2} u \phi = 0 \quad \text{for all } \phi \in H^1.$$

The potential–well depth d associated with the functional I, and hence with problem (1.1), is defined by

$$d = \inf_{u \in H^1, u_{|\Gamma} \not\equiv 0} \sup_{\lambda > 0} I(\lambda u). \tag{1.6}$$

The positivity of d is asserted in [1]. This assertion, which is an essential part of the following main result of [1], will be discussed in the sequel.

Theorem 1 ([1, Theorem 1]). *When* (1.2) *holds, problem* (1.1) *has at least a couple* $(u, -u)$ *of antipodal weak solutions in* H^1 *such that* $I(u) = I(-u) = d > 0$.

Moreover d coincides with the Mountain Pass level of the functional I, that is,

$$d = \inf_{\sigma \in \Sigma} \max_{t \in [0,1]} I(\sigma(t)), \quad \textit{where} \quad \Sigma = \{\sigma \in C([0,1]; H^1) : \sigma(0) = 0, I(\sigma(1)) < 0\}.$$

By using Lemma 1 one introduces the positive constant

$$B = \sup_{u \in H^1 \setminus \{0\}} \frac{\left(\int_{\Gamma_1} |u|^p\right)^{1/p}}{\left(\int_\Omega |\nabla u|^2 + \int_{\Gamma_1} |\nabla_\Gamma u|_\Gamma^2\right)^{1/2}}, \tag{1.7}$$

and state the second main result in [1].

Theorem 2 ([1, Theorem 2]). *When* (1.2) *holds, setting*

$$\lambda_1 = B^{-p/(p-2)} \qquad \textit{and} \quad \lambda_2 = B^{-2/(p-2)}, \tag{1.8}$$

one has

$$d = \left(\tfrac{1}{2} - \tfrac{1}{p}\right) \lambda_1^2 = \left(\tfrac{1}{2} - \tfrac{1}{p}\right) \lambda_2^p. \tag{1.9}$$

Moreover, if u is a weak solution of (1.1) *with* $I(u) = d$*, we also have*

$$\int_\Omega |\nabla u|^2 + \int_{\Gamma_1} |\nabla_\Gamma u|_\Gamma^2 = \lambda_1^2 \qquad \textit{and} \qquad \int_{\Gamma_1} |u|^p = \lambda_2^p.$$

Finally, weak solutions at the energy level d are least energy non–trivial solutions of (1.1)*, that is, for any nontrivial weak solutions u of* (1.1) *one has* $I(u) \geq d$.

The third main result in [1] reads as follows.

Theorem 3 ([1, Theorem 3]). *When* (1.2) *holds there is a sequence* $(u_n)_n$ *of nontrivial weak solutions of* (1.1) *such that* $I(u_n) \to \infty$ *as* $n \to \infty$.

2. Corrigenda

The author recently pointed out that Ω must be connected for some of the results in the previous section to hold. More specifically, when Ω is disconnected, Theorems 1 and 2 may be wrong, together with Lemma 1, also depending on the location of the part Γ_0 of the boundary where the Dirichlet homogeneous condition is posed. As long as Theorem 3 is concerned, its proof is correct only when Ω is connected, but we have no evidence that its statement fails to hold when Ω is disconnected.

To prove our claim concerning Theorems 1 and 2 and Lemma 1, let us consider the case when $\Omega = \Omega_1 \cup \Omega_2$, with Ω_1 and Ω_2 open and nonempty, $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, and $\Gamma_0 = \partial\Omega_2$, $\Gamma_1 = \partial\Omega_1$, so Γ_0 and Γ_1 are nonempty as well.

We introduce the characteristic function χ_{Ω_1} of Ω_1, as a function defined in Ω, up to a.e. equivalence. It has a representative, with vanishing gradient, in $C^\infty(\overline{\Omega})$, hence also in $H^1(\Omega)$. Moreover, trivially $\chi_{\Omega_1|\Gamma} \equiv 0$ on Γ_0 and $\chi_{\Omega_1|\Gamma} \equiv 1$ on Γ_1. Consequently we have $\nabla\chi_{\Omega_1} \equiv 0$ in Ω and $\nabla_\Gamma\chi_{\Omega_1} \equiv 0$ on Γ_1. By (1.3) and the identification between H^1 and $H^1_{\Gamma_0}(\Omega, \Gamma)$ made above, we have $\chi_{\Omega_1} \in H^1$ and

$$\|\chi_{\Omega_1}\|^2_{H^1} = (\chi_{\Omega_1}, \chi_{\Omega_1})_{H^1} = 0. \tag{2.1}$$

Since $\chi_{\Omega_1} \not\equiv 0$ and (2.1) holds, in this case the form $(\cdot,\cdot)_{H^1}$ defined in H^1 is not even positive definite, so $\|\cdot\|_{H^1}$ is not a norm on H^1 and Lemma 1 is wrong.

Moreover, in this case, the potential–well depth d defined in (1.6) vanishes. To prove this assertion, we point out that, as done in the proof of [1, Lemma 6], an easy calculation shows that, for any $u \in H^1$ with $u_{|\Gamma} \not\equiv 0$, one has

$$\sup_{\lambda>0} I(\lambda u) = \max_{\lambda>0} I(\lambda u) = \left(\frac{1}{2} - \frac{1}{p}\right)\left[\frac{\|u\|_{H^1}}{\left(\int_{\Gamma_1} |u|^p\right)^{1/p}}\right]^{\frac{2p}{p-2}} \geq 0. \tag{2.2}$$

By (2.1) and (2.2), when selecting $u = \chi_{\Omega_1}$, one has $\sup_{\lambda>0} I(\lambda u) = 0$. Hence, the infimum in (1.6) is attained, as a minimum, when $u = \chi_{\Omega_1}$, so $d = 0$ in this case. Consequently, a part of the statement of Theorem 1 is wrong. Moreover, in this case the functional I does not even possesses the geometric properties which characterize Mountain Pass type critical points, since $\alpha, \rho > 0$ such that

$$I(u) \geq \alpha \qquad \text{when} \quad \|u\|_{H^1} = \rho, \tag{2.3}$$

can not exist. Indeed, when selecting $u = \chi_{\Omega_1}$, one has $I(\lambda u) = -\frac{1}{p}\lambda^p[\mathcal{H}^{N-1}(\Gamma_1)]^p < 0$ for all $\lambda > 0$, making (2.3) impossible. Hence, the whole of Theorem 1 is wrong.

As to Theorem 2, it can be stated only when the constant B defined in (1.7) is well defined. But, when taking $u = \chi_{\Omega_1}$ in (1.7), the denominator in the fraction vanishes, by (2.1), so the fraction and the supremum, are ill defined in this case, making the numbers λ_1 and λ_2 in (1.8) ill–defined as well. So, also Theorem 2 is wrong in this case.

Let us now explain at which point the proofs of Theorems 1–3 and of Lemma 1 fail and how they are correct when Ω is connected. The proofs of Theorems 1–3 use Lemma 1, and its failure is responsible for the incorrectness of Theorems 1 and 2. Hence, we just have to analyze the proof of Lemma 1.

A careful examination of this proof shows that it is correct only when Ω is connected, as we are going to explain. By [3], Chapter 2, Theorem 2.6.16, p. 75, one derives that, when $\mathcal{H}^{N-1}(\Gamma_0) > 0$, the capacity $B_{1,2}(\Gamma_0)$ is positive. One then applies [3], Chapter 4, Corollary 4.5.2, p. 195, to conclude that the following Poincaré-type inequality holds: there is a positive constant $c_1 = c_1(\Omega, \Gamma_0)$ such that

$$\|u\|_2 \leq c_1\|\nabla u\|_2 \tag{2.4}$$

for all $u \in H^1(\Omega)$ such that $u_{|\Gamma} = 0$ on Γ_0. Unfortunately, the author did not notice, before the publication of [1], that in the reference [3], Chapter 4, p. 195, Corollary 4.5.2 is obtained as a consequence of Theorem 4.5.1, where Ω is assumed to be connected.

Therefore, the assumption that "Ω is a bounded open subset of $\mathbb{R}^N$" must be replaced in paper [1] by "Ω is a bounded domain (that is, a connected open subset) of $\mathbb{R}^N$". This assumption appears in two instances, in Abstract and at the beginning of Section 1 of [1]. Under this new assumption, all the results in paper [1] hold.

3. Final remarks

As explained in [1], the doubly elliptic problem (1.1) is motivated by a series of papers by the author concerning wave equations with hyperbolic dynamical boundary conditions, boundary damping, and source terms. The prototype of this kind of problems is the evolutionary boundary value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times\Gamma_0, \\ u_{tt} + \partial_\nu u - \Delta_\Gamma u + |u_t|^{m-2}u_t = |u|^{p-2}u & \text{on } (0,\infty)\times\Gamma_1, \end{cases} \tag{3.1}$$

where $u = u(t,x)$, $t \geq 0$, $x \in \Omega$, and Δ, Δ_Γ stand for the operators defined above, when they are taken with respect to the space variable.

The initial–value problem associated with (3.1) was introduced in [7] and then studied, as a particular case, in [8, 9] and [10]. We refer to [8, 11] for the physical derivation of the problem, describing the vibrations of a membrane with a part of the boundary carrying a linear density of kinetic energy.

When dealing with problem (3.1), one also assumes that

$$m > 1, \qquad p \leq 1 + r/\overline{m}', \quad \text{where} \quad \overline{m} := \max\{2, m\}, \tag{3.2}$$

the last assumption being related to the correct definition of weak solutions of (3.1), and with well–posedness issues.

As proved in [1, Lemma 2], when (1.2) and (3.2) hold, weak stationary solutions of (3.1) coincide with weak solutions of (1.1). Hence, when applying Theorems 1–3 to weak stationary solutions of (3.1), one also has to assume that Ω is connected.

In conclusion, we would like to point out that, while in the present note we corrected the paper [1] in the simplest possible way, that is, by taking Ω connected, other corrections are possible. For example, one can strengthen the condition $\mathcal{H}^{N-1}(\Gamma_0) > 0$ as follows: one assumes that the intersection between Γ_0 and the boundary of each connected component of Ω has positive $\mathcal{H}^{N-1}$ measure. Since Ω has finitely many connected components, one then trivially gets (2.4) also in this case.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author thanks the anonymous referees for many helpful comments.

This work has been funded by the European Union - NextGenerationEU within the framework of PNRR Mission 4 - Component 2 - Investment 1.1 under the Italian Ministry of University and Research (MUR) programme "PRIN 2022" - 2022BCFHN2 - Advanced theoretical aspects in PDEs and their applications - CUP: J53D23003700006.

Conflict of interest

The author declares there is no conflict of interest.

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