Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 18(2): 366–399.
DOI: 10.3934/cam.2026014
Received: 30 July 2025
Revised: 05 January 2026
Accepted: 26 January 2026
Published: 21 April 2026

Research article

Effect of parallel magnetic field on the compressible Kelvin-Helmholtz problem

Binqiang Xie* **and Jiakai Ma**

School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou, 510006, China

* **Correspondence:** Email: xbqmath@gdut.edu.cn.

Abstract: In this paper, we present an analysis of the KH instability in 2D magnetohydrodynamic (MHD) flows, providing rigorous confirmation that a parallel magnetic field can have a destabilizing effect on this instability. When the Mach number $M := \frac{\dot{v}_1^+}{c}$ lies strictly between $M_{\text{low}} := \sqrt{1 - \sqrt{\frac{1-\beta}{1+\beta}} + \epsilon_0}$ and $M_{\text{upp}} := \sqrt{1 + \sqrt{\frac{1-\beta}{1+\beta}} - \epsilon_0}$, where $\beta := \frac{c_A^2}{c^2}$ and $\epsilon_0 > 0$ is a small but fixed constant, we prove the linear and nonlinear ill-posedness of the KH problem for compressible MHD flows.

Keywords: free surface; Kelvin-Helmholtz instability; two-dimensional MHD flows; parallel magnetic field
Mathematics Subject Classification: 76W05, 35Q35, 35D05, 76X05

1. Introduction

In this paper, we provide rigorous confirmation that a parallel magnetic field does not always stabilize the compressible system under consideration. More precisely, we consider the following compressible inviscid MHD equations in the full plane $\mathbb{R}^2$ for $t \geq 0$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u - H \otimes H) + \nabla(p + \dfrac{|H|^2}{2}) = 0, \\ \partial_t H - \nabla \times (u \times H) = 0, \end{cases} \tag{1.1}$$

where ρ, $u = (u_1, u_2)$, p, and $H = (H_1, H_2)$ denote the fluid density, velocity, pressure, and magnetic field, respectively. We assume that p is a C^∞ function of ρ on $(0, \infty)$ satisfying $p'(\rho) > 0$ for all $\rho > 0$. For clarity, we adopt a polytropic pressure law $p(\rho) = A\rho^\gamma$ with $A > 0$, $\gamma \geq 1$, and without loss of generality, we set $A = 1$. The speed of sound $c(\rho)$ is defined by

$$c(\rho) = \sqrt{p'(\rho)}, \qquad \rho > 0. \tag{1.2}$$

The system (1.1) is supplemented with the divergence constraint

$$\mathrm{div} H = 0. \tag{1.3}$$

This condition is preserved for all $t > 0$ if it holds initially.

Let $U(t, x_1, x_2) = (\rho, u, H)(t, x_1, x_2)$ be a solution of (1.1) that is smooth on each side of a moving interface $\Gamma(t) := \{x_2 = f(t, x_1)\}$. The front f is part of the unknowns, i.e., it is a free boundary, and x_1 denotes the tangential coordinate. The interface $\Gamma(t)$ separates $\mathbb{R}^2$ into an upper domain $\Omega^+(t) := \{x_2 > f(t, x_1)\}$ and a lower domain $\Omega^-(t) := \{x_2 < f(t, x_1)\}$. We denote

$$U = \begin{cases} U^+(t, x_1, x_2), & \text{in } \Omega^+(t), \\ U^-(t, x_1, x_2), & \text{in } \Omega^-(t), \end{cases} \tag{1.4}$$

where $U^\pm = (\rho^\pm, u^\pm, H^\pm)$. To be weak solutions of (1.1), such piecewise smooth functions must satisfy the Rankine-Hugoniot conditions on $\Gamma(t)$:

$$\begin{cases} [j] = 0, \\ j[u_n] + (1 + (\partial_2 f)^2)[q] = 0, \quad j[u_\tau] = H_n[H_\tau], \\ j[H_\tau/\rho] = H_n[u_\tau], \\ [H_n] = 0, \end{cases} \tag{1.5}$$

where $j = \rho(\partial_t f - u \cdot n)$ is the mass transfer flux across the discontinuity surface, $q = p + \frac{|H|^2}{2}$ is the total pressure, $n = (-\partial_{x_1} f, 1)$ is a normal vector to $\Gamma(t)$, and $\tau = (1, \partial_1 f)$ is a tangential vector to $\Gamma(t)$ .We also introduce the notation $H_n = H \cdot n$, $u_n = u \cdot n$, $H_\tau = H \cdot \tau$,and $u_\tau = u \cdot \tau$. Here, $[\phi] = \phi^+ - \phi^-$ denotes the jump of the function ϕ across the hypersurface Γ.

Since we are interested in the KH instability for 2D MHD flows, there is no mass transfer across the interface $\Gamma(t)$:

$$j^\pm = 0 \text{ on } \Gamma(t). \tag{1.6}$$

Now in view of (1.5) and (1.6), we obtain

$$[q] = 0, H_n^\pm = 0 \text{ on } \Gamma(t). \tag{1.7}$$

Therefore, for the 2D MHD KH instability, the Rankine-Hugoniot conditions reduce to the boundary conditions

$$\partial_t f = u^+ \cdot n = u^- \cdot n, \ \ p^+ + \frac{|H^+|^2}{2} = p^- + \frac{|H^-|^2}{2}, \ \ H^+ \cdot n = H^- \cdot n = 0 \text{ on } \Gamma(t). \tag{1.8}$$

The system (1.1) is supplemented with the initial data

$$\rho^\pm(0, x) = \rho_0^\pm(x), \ u^\pm(0, x) = u_0^\pm(x), \ H^\pm(0, x) = H_0^\pm(x) \text{ in } \Omega^\pm(0). \tag{1.9}$$

Since $p'(\rho) > 0$, the function $p = p(\rho)$ is invertible, and we write $\rho = \rho(p)$. We define $\sigma(p) = \log(\rho(p))$ and take σ as a new unknown. In terms of (σ, u, H), the system (1.1) can be written equivalently as

$$\begin{cases} \partial_t \sigma + (u \cdot \nabla)\, \sigma + \nabla \cdot u = 0, \\ \partial_t u + (u \cdot \nabla)\, u + c^2(\sigma) \nabla \sigma + \frac{1}{e^\sigma} \nabla \frac{|H|^2}{2} = \frac{1}{e^\sigma} (H \cdot \nabla)\, H, \\ \partial_t H + (u \cdot \nabla)\, H = (H \cdot \nabla)\, u - H \nabla \cdot u, \end{cases} \tag{1.10}$$

where the speed of sound, now viewed as a function of σ, is given by $c^2(\sigma) = \gamma e^{\sigma(\gamma-1)}$.

The jump conditions (1.8) can be rewritten as

$$e^{\sigma^+\gamma} + \frac{|H^+|^2}{2} = e^{\sigma^-\gamma} + \frac{|H^-|^2}{2}, u^+ \cdot n = u^- \cdot n,\ H^+ \cdot n = H^- \cdot n = 0 \ \text{ on } \ \Gamma(t). \tag{1.11}$$

1.1. Rectilinear solution

It is easy to see that system (1.1)–(1.7) admits rectilinear solutions $\dot{U} = (\dot{f}, \dot{\rho}^\pm, \dot{u}^\pm, \dot{H}^\pm)$ with the flat interface $\{x_2 = 0\}$ for all $t \geq 0$. Then $\Omega^+ = \mathbb{R} \times (0, \infty)$ and $\Omega^- = \mathbb{R} \times (-\infty, 0)$ for all $t \geq 0$. More precisely, the front is flat, i.e., $\dot{f} = 0$. To ensure that the constant densities $\dot{\rho}^\pm$ satisfy the jump condition (1.7), we impose

$$\dot{\rho}^+ = \dot{\rho}^- := \dot{\rho}, \tag{1.12}$$

where $\dot{\rho}$ is a positive constant. The upper fluid moves horizontally with constant velocity, and the lower fluid moves with the same speed in the opposite direction. That is, the constant velocity field $\dot{u}^\pm$ takes the form:

$$\dot{u} = \begin{cases} (\dot{u}_1^+, 0) & x_2 \geq 0, \\ (\dot{u}_1^-, 0) & x_2 < 0, \end{cases} \tag{1.13}$$

where the constants $\dot{u}_1^+$ and $\dot{u}_1^-$ satisfy

$$\dot{u}_1^+ = -\dot{u}_1^-. \tag{1.14}$$

As in the work of Yu and Wang [1], the constant transverse magnetic field $\dot{H}$ is of the form:

$$\dot{H} = \begin{cases} (\dot{H}_1^+, 0) & x_2 \geq 0, \\ (\dot{H}_1^-, 0) & x_2 < 0, \end{cases} \tag{1.15}$$

where the constants $\dot{H}_1^+, \dot{H}_1^-$ satisfy

$$|\dot{H}_1^+| = |\dot{H}_1^-|. \tag{1.16}$$

1.2. The new formulations

Our analysis in this paper relies on reformulating the problem in new coordinates. We first define the fixed domains $\Omega^\pm$ as

$$\begin{aligned} \Omega^+ &:= \left\{x \in \mathbb{R}^2 : x_2 > 0\right\}, \\ \Omega^- &:= \left\{x \in \mathbb{R}^2 : x_2 < 0\right\}. \end{aligned} \tag{1.17}$$

We also define the fixed boundary Γ by

$$\Gamma := \left\{x \in \mathbb{R}^2 : x_2 = 0\right\}.$$

To transform the free boundary problem to the fixed domains $\Omega^\pm$, we introduce a change of variables that map the original domains back to the fixed ones:

$$(t, x) \mapsto (t, x_1, x_2 + \psi(t, x)).$$

We construct ψ by multiplying the front f by a smooth cut-off function in x_2:

$$\psi(t, x) = \chi(\frac{x_2}{3(1 + a)}) f(t, x_1), \tag{1.18}$$

where $a := \|f_0\|_{L^\infty(\mathbb{R})}$ and $f_0(x_1) := f(x_1, 0)$. Here, $\chi \in C_c^\infty(\mathbb{R})$ is a smooth cut-off function satisfying $0 \le \chi \le 1$, with $\chi(x_2) = 1$ for $|x_2| \le 1$, $\chi(x_2) = 0$ for $|x_2| \ge 3$, and $|\partial_2\chi(x_2)| \le 1$ for all $x_2 \in \mathbb{R}$, where $\partial_j := \partial/\partial x_j$. We also assume

$$\|f_0\|_{L^\infty(\mathbb{R})} \le 1. \tag{1.19}$$

Moreover, the following properties hold:

$$\begin{aligned} &\psi(x_1, 0, t) = f(x_1, t),\\ &\partial_2\psi(x_1, 0, t) = 0,\\ &|\partial_2\psi| \le \frac{1}{3(1+a)}|f|. \end{aligned} \tag{1.20}$$

The change of variables that transform the free boundary problem (1.10)–(1.11) to the fixed domains $\Omega^\pm$ is given in the following lemma.

Lemma 1.1. *Define the function* Ψ *by*

$$\Psi(t, x) := (x_1, x_2 + \psi(t, x)), \quad (t, x) \in [0, T] \times \Omega. \tag{1.21}$$

Then there exists $T > 0$ *such that for each* $t \in [0, T]$*, the map* $\Psi(t, \cdot) : (x_1, x_2) \mapsto (x_1, x_2 + \psi(t, x))$ *is a diffeomorphism from* $\Omega^\pm$ *onto* $\Omega^\pm(t)$*. In particular,* $\Psi|_{t=0} : \bar{\Omega}^\pm \to \bar{\Omega}^\pm(0)$ *is a diffeomorphism.*

Proof. Since $\|f_0\|_{L^\infty(\mathbb{R})} \le 1$, we can choose $T > 0$ sufficiently small so that $\sup_{[0,T]} \|f\|_{L^\infty} < 2$. Consequently, the free interface remains a graph on $[0, T]$, and

$$\partial_2\Psi_2(t, x) = 1 + \partial_2\psi(t, x) \ge 1 - \frac{1}{3} \times 2 = \frac{1}{3},$$

which ensures that for each $t \in [0, T]$, the map $\Psi(t, \cdot) : \Omega \to \Omega(t)$ is a diffeomorphism. □

We introduce the matrix

$$A = [D\Psi]^{-1} = \begin{pmatrix} 1 & 0 \\ -\partial_1\psi/J & 1/J \end{pmatrix},$$

where $J = \det[D\Psi] = 1 + \partial_2\psi$. Using the change of variables from Lemma 1.1, we transform the free boundary problem (1.10)–(1.11) into a problem on the fixed domains $\Omega^\pm$. Let us set

$$\begin{aligned} &v^\pm(t, x) := u^\pm(t, \Psi(t, x)), B^\pm(t, x) := H^\pm(t, \Psi(t, x)),\\ &q^\pm(t, x) := p^\pm(t, \Psi(t, x)), \quad \varrho^\pm(t, x) := \rho^\pm(t, \Psi(t, x)),\\ &\quad h^\pm(t, x) := \sigma^\pm(t, \Psi(t, x)). \end{aligned} \tag{1.22}$$

In the rest of this paper, an equation written on Ω is understood to hold on both Ω^+ and Ω^-. For convenience, we consolidate the notation by writing v, B, q, ϱ, h to refer to $v^\pm$, $B^\pm$, $q^\pm$,$\varrho^\pm$, $h^\pm$ except when necessary to distinguish the two.

We introduce the notation:

$$\nabla^\psi = A^T\nabla, \Delta^\psi = A^T\nabla \cdot A^T\nabla. \tag{1.23}$$

Then the system (1.10) and boundary conditions (1.11) can be reformulated as:

$$\begin{cases} \partial_t h + (\tilde{v}\cdot\nabla)\, h + \nabla^{\psi}\cdot v = 0, & \text{in } \Omega, \\ \partial_t v + (\tilde{v}\cdot\nabla)\, v + c^2\nabla^{\psi} h + \frac{1}{\varrho}\nabla^{\psi}\frac{|B|^2}{2} = \frac{1}{\varrho}(\tilde{B}\cdot\nabla)B, & \text{in } \Omega, \\ \partial_t B + (\tilde{v}\cdot\nabla)\, B + B\nabla^{\psi}\cdot v = (\tilde{B}\cdot\nabla)v, & \text{in } \Omega, \\ \nabla^{\psi} B = 0, & \text{in } \Omega, \\ \partial_t f = v\cdot n, \quad [e^{\gamma h} + \frac{|B|^2}{2}] = 0, \quad B\cdot n = 0, & \text{on } \Gamma, \end{cases} \tag{1.24}$$

where we define

$$\tilde{v} := Av - (0, \partial_t\psi/J) = (v_1, (v\cdot n - \partial_t\psi)/J)\,,$$
$$\tilde{B} := AB = (B_1, B\cdot n/J)\,.$$

The initial data are required to satisfy

$$e^{\gamma h_0^+} + \frac{|B_0^+|^2}{2} = e^{\gamma h_0^-} + \frac{|B_0^-|^2}{2}, v_0^+\cdot n_0 = v_0^-\cdot n_0, B_0^{\pm}\cdot n_0 = 0 \text{ on } \Gamma. \tag{1.25}$$

Notice that on the boundary Γ , we have

$$J = 1, \qquad \tilde{v}_2 = 0, \qquad \tilde{B}_2 = 0. \tag{1.26}$$

Since we are concerned with the KH instability in MHD flows, the instability manifests primarily at the interface. To this end, we derive the evolution equation for the front f on the fixed boundary Γ. First, using the momentum equation in (1.24), we obtain on Γ:

$$\begin{aligned} \partial_t^2 f =& \partial_t v^+\cdot n + v^+\cdot\partial_t n \\ =& -((\tilde{v}^+\cdot\nabla)\, v^+ - \frac{1}{\varrho^+}\left(\tilde{B}^+\cdot\nabla\right)B^+ + c^2\nabla^{\psi}h^+ + \nabla^{\psi}\frac{|B^+|^2}{2})\cdot n - v^+\cdot(\partial_1\partial_t f, 0) \\ =& -v_1^+\partial_1 v^+\cdot n + \frac{1}{\varrho^+}B_1^+\partial_1 B^+\cdot n - c^2A^T\nabla h^+\cdot n - v_1^+\partial_1\partial_t f \\ =& v_1^+\partial_1 n\cdot v^+ - v_1^+\partial_1\partial_t f - \frac{1}{\varrho^+}B_1^+\partial_1 n\cdot B^+ + c^2\nabla^{\psi}h^+\cdot n - \nabla^{\psi}\frac{|B^+|^2}{2}\cdot n - v_1^+\partial_1\partial_t f \\ =& -2v_1^+\partial_1\partial_t f - c^2\nabla^{\psi}h^+\cdot n - \nabla^{\psi}\frac{|B^+|^2}{2}\cdot n - (v_1^+)^2\partial_1^2 f + \frac{1}{\varrho^+}(B_1^+)^2\partial_1^2 f \quad \text{on} \quad \Gamma. \end{aligned} \tag{1.27}$$

Similarly, we obtain an evolution equation from the lower side:

$$\partial_t^2 f = -2v_1^-\partial_1\partial_t f - c^2\nabla^{\psi}h^-\cdot n - \nabla^{\psi}\frac{|B^-|^2}{2}\cdot n - (v_1^-)^2\partial_1^2 f + \frac{1}{\varrho^-}(B_1^-)^2\partial_1^2 f \quad \text{on } \Gamma. \tag{1.28}$$

Adding (1.27) and (1.28) yields the evolution equation for the front:

$$\begin{aligned} &\partial_t^2 f + (v_1^+ + v_1^-)\partial_1\partial_t f + \frac{1}{2}((c^+)^2\nabla^{\psi}h^+\cdot n + (c^-)^2\nabla^{\psi}h^-\cdot n) \\ &+ \frac{1}{2}(\nabla^{\psi}\frac{|B^+|^2}{2}\cdot n + \nabla^{\psi}\frac{|B^-|^2}{2}\cdot n) + \frac{1}{2}((v_1^+)^2 + (v_1^-)^2)\partial_1^2 f \\ &- \frac{1}{2}(\frac{1}{\varrho^+}(B_1^+)^2 + \frac{1}{\varrho^-}(B_1^-)^2)\partial_1^2 f = 0 \quad \text{on } \Gamma. \end{aligned} \tag{1.29}$$

Applying $\partial_t + \tilde{v} \cdot \nabla$ to the first equation in (1.24) and $\nabla^\psi \cdot$ to the second yields

$$\begin{cases} (\partial_t + \tilde{v} \cdot \nabla)^2 h + (\partial_t + \tilde{v} \cdot \nabla) \nabla^\psi \cdot v = 0, \\ \nabla^\psi \cdot ((\partial_t + \tilde{v} \cdot \nabla) v) + \nabla^\psi \cdot (c^2 \nabla^\psi h) + \nabla^\psi \cdot (\frac{1}{\varrho} \nabla^\psi \frac{|B|^2}{2}) = \nabla^\psi \cdot \left(\frac{1}{\varrho} (\tilde{B} \cdot \nabla) B \right). \end{cases} \tag{1.30}$$

Next, subtracting the two equations in (1.30) yields a wave-type equation for h:

$$(\partial_t + \tilde{v} \cdot \nabla)^2 h - c^2 \Delta^\psi h - \frac{1}{\varrho} \Delta^\psi \frac{|B|^2}{2} = \mathcal{F}, \tag{1.31}$$

where $\mathcal{F}$ denotes a collection of lower-order terms given explicitly by

$$\mathcal{F} = -[\partial_t + \tilde{v} \cdot \nabla, \nabla^\psi] v + \nabla^\psi c^2 \cdot \nabla^\psi h + \nabla^\psi \frac{1}{\varrho} \cdot \nabla^\psi \frac{|B|^2}{2} - \nabla^\psi (\frac{1}{\varrho} \tilde{B}) \cdot \nabla B.$$

In contrast to the Euler case, the MHD wave equation (1.31) involves both h and B. To relate h and B, we apply $\partial_t + \tilde{v} \cdot \nabla$ to the third equation in (1.24) and $\tilde{B} \cdot \nabla$ to the second one, which yields

$$\begin{aligned} &(\partial_t + \tilde{v} \cdot \nabla)^2 B - B(\partial_t + \tilde{v} \cdot \nabla)^2 h + c^2 (\tilde{B} \cdot \nabla) \nabla^\psi h + \frac{1}{\varrho} (\tilde{B} \cdot \nabla) \nabla^\psi \frac{|B|^2}{2} - \frac{1}{\varrho} (\tilde{B} \cdot \nabla)^2 B \\ &= \mathcal{G}, \end{aligned} \tag{1.32}$$

where $\mathcal{G}$ denotes the lower-order terms:

$$\mathcal{G} = (\partial_t + \tilde{v} \cdot \nabla) B \nabla^\psi \cdot v + [\partial_t + \tilde{v} \cdot \nabla, \tilde{B} \cdot \nabla] u + \tilde{B} \cdot \nabla c^2 \nabla^\psi h - \tilde{B} \cdot \nabla \frac{1}{\varrho} \tilde{B} \cdot \nabla B + \tilde{B} \cdot \nabla \frac{1}{\varrho} \nabla^\psi \frac{|B|^2}{2}.$$

From the boundary conditions (1.8) , we obtain

$$[e^{\gamma h} + \frac{|B|^2}{2}] = 0 \quad \text{on } \Gamma(t), \tag{1.33}$$

where $[h] := h^+|_\Gamma - h^-|_\Gamma$ denotes the jump across $\Gamma(t)$.

To close the system for h, we supplement it with the jump of the normal derivative of the total pressure. Subtracting (1.27) and (1.28) gives

$$[c^2 \nabla^\psi h \cdot n + \nabla^\psi \frac{|B|^2}{2} \cdot n] = [-2 v_1 \partial_1 \partial_t f - (v_1)^2 \partial_1^2 f + \frac{1}{\varrho} (B_1)^2 \partial_1^2 f] \quad \text{on } \Gamma. \tag{1.34}$$

Combing (1.34), (1.33), (1.32), and (1.31) yields the following closed system for h:

$$\begin{cases} (\partial_t + \tilde{v} \cdot \nabla)^2 h - c^2 \Delta^\psi h - \frac{1}{\varrho} \Delta^\psi \frac{|B|^2}{2} = \mathcal{F} & \text{in } \Omega, \\ (\partial_t + \tilde{v} \cdot \nabla)^2 B - B(\partial_t + \tilde{v} \cdot \nabla)^2 h + c^2 (\tilde{B} \cdot \nabla) \nabla^\psi h & \\ + \frac{1}{\varrho} (\tilde{B} \cdot \nabla) \nabla^\psi \frac{|B|^2}{2} - \frac{1}{\varrho} (\tilde{B} \cdot \nabla)^2 B = \mathcal{G}, & \text{in } \Omega, \\ [e^{\gamma h} + \frac{|B|^2}{2}] = 0 & \text{on } \Gamma, \\ [c^2 \nabla^\psi h \cdot n + \nabla^\psi \frac{|B|^2}{2} \cdot n] = [-2 v_1 \partial_1 \partial_t f - (v_1)^2 \partial_1^2 f + \frac{1}{\varrho} (B_1)^2 \partial_1^2 f] & \text{on } \Gamma. \end{cases} \tag{1.35}$$

1.4. History result

In Chandrasekhar's monograph [2], the stability problem of superposed fluids is classified into two types. The first is the Rayleigh-Taylor instability. There is a substantial body of mathematical work on the Rayleigh–Taylor instability; see, e.g., [3–7]. Ebin [8] established the instability of the Rayleigh-Taylor problem for the incompressible Euler equations, while Guo and Tice [4] proved the analogous result for the compressible inviscid case. Moreover, the Rayleigh-Taylor instability was established for viscous compressible fluids in [5] and for the inhomogeneous Euler equations in [6]. The second type occurs when layers of stratified, heterogeneous fluids are in relative horizontal motion. This paper is concerned with the latter.

The stability of two fluids in relative motion has attracted wide interest among researchers from various fields. This instability is known as the KH instability, first studied by Hermann von Helmholtz [9] and William Thomson (Lord Kelvin) [10]. The KH instability is central to understanding a variety of space and astrophysical phenomena involving sheared plasma flows, including the stability of the magnetopause [11–13], interactions between solar wind streams of different velocities [14], and the dynamics of cometary tails [15].

Much progress has been made on the well-posedness of the KH problem for ideal fluids. This configuration is also known in the literature as 'vortex sheets', since the vorticity is concentrated as a δ-function supported on the velocity discontinuity along the sheet. In pioneering works, Coulombel and Secchi [16, 17] proved the nonlinear stability of vortex sheets for ideal compressible flows using microlocal analysis and the Nash-Moser iteration technique. Morando, Trebeschi, and Wang [18, 19] later extended this result to two-dimensional, ideal, non-isentropic, compressible flows. For vortex sheets in elastic fluids, Chen, Hu, and Wang [20, 21] proved that strong elasticity can suppress the KH instability in the 2D case. Furthermore, Chen, Huang, Wang, and Yuan [22] established the stabilizing effect of elasticity on 3D compressible vortex sheets.

Progress has also been made on the ill-posedness of the KH problem for ideal fluids. For incompressible Euler flows, Ebin [8] proved the linear and nonlinear ill-posedness of the KH problem. Recently, we proved the linear and nonlinear ill-posedness of the KH problem for incompressible MHD fluids [23] under conditions that violate the Syrovatskij stability condition. For compressible Euler flows, normal mode analysis [24, 25] showed that the linear KH instability is suppressed when the Mach number satisfies $M := \dot{v}_1^+/c > \sqrt{2}$, while the solutions become strongly unstable when $M < \sqrt{2}$. In [26], we proved that the nonlinear KH problem for compressible Euler fluids exhibits the same ill-posedness as its linearized counterpart [24, 25] under the condition $\epsilon_0 \leq M < \sqrt{2}$, where $\epsilon_0 > 0$ is a small fixed constant. Recently, we also showed that weak elasticity has a destabilizing effect on the KH instability [27].

The linear theory of Chandrasekhar analyzes the KH instability in incompressible MHD fluids [2]. It is known that the instability sets in when the relative velocity between the layers exceeds twice the local Alfvén speed, provided the magnetic field is aligned with the direction of maximum velocity shear. Under the Syrovatskij stability condition, Sun, Wang, and Zhang [28] proved the nonlinear stability of the incompressible MHD KH problem. In contrast, for a magnetic field parallel to the flow, we proved in [23] that the incompressible MHD KH problem is both linearly and nonlinearly ill-posed when the Syrovatskij condition is violated, using the framework introduced by Ebin [8]. For transverse magnetic fields, Chandrasekhar observed that, in the linear regime, the development of the KH instability in the streaming direction is unaffected by the transverse field. We further showed that the nonlinear MHD KH problem exhibits the same ill-posed behavior as its linearized counterpart in this configuration [29].

In the context of compressible MHD flows, Wang and Yu [1] demonstrated that, for 2D flows, a parallel magnetic field stabilizes the KH instability when the Mach number satisfies either $M < \sqrt{1-\sqrt{\frac{1-\beta}{1+\beta}}}$ or $M > \sqrt{1+\sqrt{\frac{1-\beta}{1+\beta}}}$. For 3D compressible MHD flows, Trakhinin [30, 31] and Chen and Wang [32] independently employed distinct symmetrization techniques to establish the linear and nonlinear stability of compressible vortex sheets, highlighting the stabilizing role of magnetic fields in such configurations. Conversely, in regimes where the magnetic field is sufficiently weak,we show in this paper that a parallel magnetic field can have a destabilizing effect on the KH instability. Similarly, we proved that a transverse magnetic field also has a destabilizing effect [33]. The mechanism of the transverse field in ideal MHD flows operates via its contribution to the pressure, thereby modifying the magnetosonic wave speed.

In this paper, through an eigenvalue analysis of the linearized KH problem for ideal compressible MHD flows, we establish that, in the high-frequency limit, the perturbed interface exhibits instantaneous growth when the Mach number satisfies

$$\sqrt{1-\sqrt{\frac{1-\beta}{1+\beta}}} < M <= \sqrt{1+\sqrt{\frac{1-\beta}{1+\beta}}}.$$

Motivated by the works [4, 26, 34], we further prove that the nonlinear system (1.1)–(1.7) is ill-posed for Mach numbers in the range $M_{\text{low}} \leq M \leq M_{\text{upp}}$.

2. Main result

2.1. Definitions and terminology

Before stating the main result, we introduce some notation that will be used throughout the paper. $C > 0$ denotes a generic universal constant that may depend on the parameters of the problem but not on the data; its value may change from line to line. Such constants are referred to as universal and are allowed to vary from one inequality to another. We write $a \lesssim b$ to indicate that $a \leq Cb$ for some universal constant $C > 0$, and $a \gtrsim b$ is understood analogously.

Since we consider two disjoint fluid layers, for a function ψ defined on Ω, we write ψ^+ for its restriction to Ω^+ and ψ^- for its restriction to Ω^-. For any $j \in \mathbb{R}$, we define the piecewise Sobolev space by

$$H^j(\Omega) := \{\psi | \psi^+ \in H^j(\Omega^+), \psi^- \in H^j(\Omega^-)\}, \tag{2.1}$$

endowed with the norm $\|\psi\|^2_{H^j} = \|\psi^+\|^2_{H^j(\Omega^+)} + \|\psi^-\|^2_{H^j(\Omega^-)}$. The norm on $H^j(\Omega^\pm)$ is defined by

$$\begin{aligned} \|\psi\|^2_{H^j(\Omega^\pm)} : &= \sum_{s=0}^{j} \int_{\mathbb{R}\times I_\pm} (1+\eta^2)^{j-s} |\partial_2^s \hat{\psi}_\pm(\eta, x_2)|^2 d\eta dx_2 \\ &= \sum_{s=0}^{j} \int_{\mathbb{R}} (1+\eta^2)^{j-s} \|\partial_2^s \hat{\psi}_\pm(\eta, \cdot)\|^2_{L^2(I_\pm)} d\eta, \end{aligned} \tag{2.2}$$

where $I_- = (-\infty, 0)$, and $I_+ = (0, \infty)$, and $\hat{\psi}$ denotes the Fourier transform of ψ defined by

$$\hat{\psi}(\eta) = \int_{\mathbb{R}} \psi e^{-ix_1\eta} dx_1. \tag{2.3}$$

For a function ψ defined on $\Gamma \simeq \mathbb{R}$, we define the usual Sobolev space $H^j(\mathbb{R}) \simeq H^j(\Gamma)$ as the set of functions $\psi = \psi(x_1)$ for which the following norm is finite:

$$\|\psi\|^2_{H^j(\Gamma)} := \int_{\mathbb{R}} (1+\eta^2)^j |\hat{\psi}(\eta)|^2 d\eta. \tag{2.4}$$

To shorten notation, for $j \geq 0$, we define

$$\|(f,h,v,B)(t)\|_{H^j} = \|f(t)\|_{H^j(\Gamma)} + \|h(t)\|_{H^j(\Omega)} + \|v(t)\|_{H^j(\Omega)} + \|B(t)\|_{H^j(\Omega)}. \tag{2.5}$$

2.2. Main result

This paper is devoted to proving the ill-posedness of the KH problem for the MHD system under the following condition:

$$M_{low} := \sqrt{1 - \sqrt{\frac{1-\beta}{1+\beta}}} + \epsilon_0 \leq M := \frac{|\dot{v}_1^+|}{c} \leq M_{upp} := \sqrt{1 + \sqrt{\frac{1-\beta}{1+\beta}}} - \epsilon_0, \tag{2.6}$$

in which $\beta := \frac{c_A^2}{c^2}$ and ϵ_0 is a small but fixed number.

Definition 2.1. *We say that the perturbed problem* (6.8) *has property $EE(k)$ for some $k \geq 3$ if there exist $\delta, t_0, C > 0$ and a function $F : [0,\delta) \to \mathbb{R}^+$ satisfying $F(z) \leq Cz$ for $z \in [0,\delta)$ such that for any initial data $(\tilde{f}_0, \tilde{h}_0, \tilde{v}_0)$ satisfying*

$$\|\tilde{f}_0, \tilde{h}_0, \tilde{v}_0, \tilde{B}_0\|_{H^k} < \delta, \tag{2.7}$$

there exists a unique solution $(\tilde{f}, \tilde{h}, \tilde{v}, \tilde{B})$ of (6.8) *with initial data $(\tilde{f}, \tilde{h}, \tilde{v}, \tilde{B})|_{t=0} = (\tilde{f}_0, \tilde{h}_0, \tilde{v}_0, \tilde{B}_0)$, and the estimate*

$$\sup_{0\leq t\leq t_0} \|(\tilde{f}, \tilde{h}, \tilde{v}, \tilde{B})(t)\|_{H^3} \leq F(\|(\tilde{f}_0, \tilde{h}_0, \tilde{v}_0, \tilde{B}_0)\|_{H^k}). \tag{2.8}$$

Theorem 2.2. *Suppose that the initial data satisfies the constraint condition* (1.19) *and* (1.25).*We further assume the rectilinear solution* (3.1)-(3.3) *satisfies the instability condition* (2.6). *Then the perturbed problem* (6.8) *does not have property $EE(k)$ in the sense of Definition 2.1.*

Remark 2.1. *We construct a growing normal mode solution for the front f when $\sqrt{1 - \sqrt{\frac{1-\beta}{1+\beta}}} < M := \frac{|\dot{v}_1^+|}{c} < \sqrt{1 + \sqrt{\frac{1-\beta}{1+\beta}}}$. For the linear and nonlinear problems, however, we can only prove the ill-posedness of the solutions (h,v) of the KH problem for ideal compressible flows when $M_{\rm low} \leq M \leq M_{\rm upp}$ due to technical reasons.*

Remark 2.2. *Since $\Psi(t,\cdot) : (x_1,x_2) \mapsto (x_1, x_2 + \psi(t,x))$ is a diffeomorphism, the ill-posedness of system* (3.8) *in the flattened coordinates implies the ill-posedness of the solution to the original system* (1.1).

Remark 2.3. *By taking $\dot{B}_1 = 0$, the instability condition* (2.6) *reduces to*

$$\epsilon_0 \leq M := \frac{|\dot{v}_1^+|}{c} < \sqrt{2} - \epsilon_0, \tag{2.9}$$

which is precisely the instability condition for the KH problem in compressible Euler fluids (see [26]).

Remark 2.4. *On page 851 of [35], Pu and Kivelson examined a specific nightside configuration where $B_1 = B_2 = B$, $k_t \parallel U \parallel B$, and $\rho_1 = \rho_2$. For a velocity U satisfying $U_c < U < U_u$ with*

$$U_{u,c}^2 = 4\beta(1 \pm \sqrt{(\beta - 1)/(\beta + 1)}),$$

they demonstrated that the interface is unstable and will excite KH waves. We note that these two critical values coincide with those obtained in the present paper.

2.3. Strategy of the proof and organization of the paper

First, the motion of the free surface $\Gamma(t)$ and the domains $\Omega^\pm(t)$ introduces several mathematical difficulties. To overcome this, we switch to a flattened coordinate system (1.24) in which the upper and lower fluid domains are fixed in time as $\Omega^+ = \mathbb{R} \times (0, \infty)$ and $\Omega^- = \mathbb{R} \times (-\infty, 0)$, respectively. Since we are concerned with the KH instability, the instability arises from the discontinuity of the tangential velocity across the interface. Using the momentum equation (the second equation in (1.24)) together with the kinematic boundary condition, we derive a second-order evolution equation for the front in the flattened coordinates:

$$\begin{aligned}
&\partial_t^2 f + (v_1^+ + v_1^-)\partial_1\partial_t f + \frac{1}{2}((c^+)^2\nabla^\psi h^+ \cdot n + (c^-)^2\nabla^\psi h^- \cdot n) \\
&+ \frac{1}{2}(\nabla^\psi \frac{|B^+|^2}{2} \cdot n + \nabla^\psi \frac{|B^-|^2}{2} \cdot n) + \frac{1}{2}((v_1^+)^2 + (v_1^-)^2)\partial_1^2 f \\
&- \frac{1}{2}(\frac{1}{\varrho^+}(B_1^+)^2 + \frac{1}{\varrho^-}(B_1^-)^2)\partial_1^2 f = 0 \quad \text{on } \Gamma.
\end{aligned}$$

From the above equation,we see that it is necessary to establish a relation between the pressure h and the front f, a feature that distinguishes the compressible case from the KH instability in incompressible MHD flows. To overcome this problem, we apply $\partial_t + \tilde{v} \cdot \nabla$ to the first equation in (1.34) and $\nabla^\psi \cdot$ to the second one, obtaining (1.30). Subtracting the two equations in (1.30) yields the nonlinear wave equation (1.31) for the pressure h. In contrast to the Euler case, the wave equation (1.31) in MHD flows involves both h and B. Hence, a relation between h and B is required. To obtain such a relation, we apply $\partial_t + \tilde{v} \cdot \nabla$ to the third equation in (1.24) and $\tilde{B} \cdot \nabla$ to the second one, which leads to (1.32) together with the boundary conditions (1.33)–(1.34), both expressed in terms of the front f.

Next, we study the linearized KH problem for compressible MHD equations, reformulated in the flattened coordinates and linearized around the rectilinear solution. A key ingredient of our proof is to derive an explicit expression relating the pressure h to the front f. Since we are investigating the ill-posedness of the linearized KH problem for compressible MHD flows, it is natural to seek normal mode solutions, i.e., we assume that the solution grows exponentially in time as $e^{\tau(\eta)t}$ with $\tau(\eta) \in \mathbb{C}$, $\operatorname{Re}\tau(\eta) > 0$, where $\eta \in \mathbb{R}$ is the horizontal spatial frequency. Taking the horizontal Fourier transform in x_1 reduces (3.10)–(3.12) to a system of Ordinary Differential Equations(ODEs) (3.13)–(3.15). From the second equation in (3.15), we deduce the relation between the magnetic field component $\hat{b}_1$ and the pressure $\hat{m}$:

$$\hat{b}_1 = \frac{\dot{B}_1[(\tau + i\dot{v}_1\eta)^2 + c^2\eta^2]}{(\tau + i\dot{v}_1\eta)^2}\hat{m}. \tag{2.10}$$

Substituting (3.16) into (3.15) yields

$$\begin{cases}((\tau + i\dot{v}_1\eta)^2 + C_B^2\eta^2)\hat{m} - C_B^2\partial_2^2\hat{m} = 0, & in\ \Omega \quad , \\ [C_B^2\hat{m}] = 0, & on\ \Gamma \quad , \\ [C_B^2\partial_2\hat{m}] = -4i\dot{v}_1^+\eta\tau\hat{g}, & on\ \Gamma \quad , \end{cases} \tag{2.11}$$

where

$$C_B^2 = c^2 + c_A^2\frac{(\tau + i\dot{v}_1^+\eta)^2 + c^2\eta^2}{(\tau + i\dot{v}_1^+\eta)^2\eta^2} \ \ and \ \ c_A^2 = \frac{|\dot{B}_1|^2}{\dot{\rho}}. \tag{2.12}$$

Here we remark that C_B is called the magneto-acoustic speed and c_A is called the Alfven speed. Thus, we first solve this ODE system for $\hat{m}$ and then substitute the resulting expression (3.20) into the front equation (3.19) to obtain the symbol equation (3.23) for the front in Fourier space. By analyzing the roots of the symbol of $\hat{g}$, we obtain the instability condition for the linearized KH problem in ideal compressible MHD fluids; this is stated in Lemma 4.2.

Another key idea of this paper is to identify an elliptic structure in the front equation (3.7) when

$$c^2 - c^2\sqrt{\frac{c^2 - c_A^2}{c^2 + c_A^2}} < (\dot{v}_1^+)^2 < c^2 + c^2\sqrt{\frac{c^2 - c_A^2}{c^2 + c_A^2}}.$$

We then construct a sequence of solutions of the form

$$f_n(t, x_1) = e^{\tau t}g_n(x_1) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\tau(\eta)t}\chi_n(\eta)e^{i\eta x_1}\, d\eta,$$

which satisfies (5.12). From this, we see that the linearized front equation (5.12) is qualitatively unstable at high frequencies. As $\eta \to \infty$, solutions of (5.12) with higher frequency grow faster over the same time interval, which provides a mechanism for the KH instability. We remark that the linearized front equation (5.12) for the KH instability in compressible Euler flows is, in some sense, similar to the Birkhoff-Rott equation, which was shown to be of elliptic type in [36, 37]. Using the explicit expressions for the velocity and density (or pressure) in terms of the front in Fourier space, together with the solution sequence constructed above, we show that the solution of the linearized KH problem for compressible MHD flows is not uniformly continuously dependent on the initial data when the Mach number satisfies $M_{\text{low}} \leq M \leq M_{\text{upp}}$.

In general, it is not trivial to pass from linear instability to nonlinear instability in conservative partial differential equations. Inspired by the work on the instability of the Rayleigh-Taylor problem for compressible Euler equations [4], we prove the ill-posedness of the nonlinear system via a rescaling argument that connects the nonlinear problem to its linearized counterpart. However, the tangential discontinuity in the boundary condition arising from the KH instability requires a more careful analysis and estimation of the boundary terms.

The rest of the paper is organized as follows. In Section 3, we reformulate the system in the flattened coordinates and derive the wave equation for the pressure. In Section 4, we analyze the roots of the symbol of the linearized system for the 2D ideal compressible MHD KH problem, which helps us identify the instability condition that the rectilinear solutions must satisfy. In Sections 5 and 6, we prove the linear and nonlinear ill-posedness, respectively, of the compressible KH problem with a parallel magnetic field.

3. The Linearized equations in new coordinates

In this section, we consider the linearized system in the flattened coordinates and construct a growing normal mode solution. Taking the Fourier transform of the linearized system yields a second-order ODE for $\hat{g}$.

3.1. Construction of a growing solution of the linearized system

One easily verifies that the particular solution in Euler coordinates remains a particular solution in the flattened coordinates, with

$$\dot{v}^{\pm} = \dot{u}^{\pm} = \begin{cases} (\dot{v}_1^{+}, 0) & x_2 \geq 0, \\ (\dot{v}_1^{-}, 0) & x_2 < 0, \end{cases} \tag{3.1}$$

and

$$\dot{\varrho}^{+} = \dot{\varrho}^{-} := \dot{\varrho}, \tag{3.2}$$

and

$$\dot{B} = \dot{H} = \begin{cases} (\dot{B}_1^{+}, 0) & x_2 \geq 0, \\ (\dot{B}_1^{-}, 0) & x_2 < 0. \end{cases} \tag{3.3}$$

Remark 3.1. *From now on and throughout this paper, we use the new notation $(\dot{f}, \dot{\varrho}^{\pm}, \dot{v}^{\pm}, \dot{B}^{\pm})$ to denote the rectilinear solution $(\dot{f}, \dot{\rho}^{\pm}, \dot{u}^{\pm}, \dot{H}^{\pm})$, which is in fact the same constant quantity. Here, we use the new notation to match the notation in the new coordinates.*

We now consider the constant-coefficient linearized equations obtained by linearizing (1.24), (1.29) and (1.35) around the following configuration: the constant velocity $\dot{v}^{\pm} = (\dot{v}_1^{\pm}, 0)$ along the x_1-direction, the constant parallel magnetic field $\dot{B}^{\pm} = (\dot{B}_1^{\pm}, 0)$ also along the x_1-direction, constant pressure $\dot{h}^{+} = \dot{h}^{-}$, flat front $\Gamma = \{x_2 = 0\}$,and the normal vector $n = (0, 1)$. Moreover, by a Galilean transformation and a suitable change of scale, the rectilinear solution can be normalized so that

$$\dot{v}_1^{+} + \dot{v}_1^{-} = 0, \ |\dot{B}_1^{+}| = |\dot{B}_1^{-}|. \tag{3.4}$$

Therefore, we have the following linearized equations:

$$\begin{cases} \partial_t \tilde{h} + \dot{v}_1 \partial_1 \tilde{h} + \nabla \cdot \tilde{v} = 0, & \text{in } \Omega, \\ \partial_t \tilde{v} + \dot{v}_1 \partial_1 \tilde{v} + c^2 \nabla h + \frac{\dot{B}_1}{\dot{\varrho}} \nabla \tilde{B}_1 = \frac{\dot{B}_1}{\dot{\varrho}} \partial_1 \tilde{B}, & \text{in } \Omega, \\ \partial_t \tilde{B} + \dot{v}_1 \partial_1 \tilde{B} + \dot{B} \nabla \cdot \tilde{v} = \dot{B}_1 \partial_1 \tilde{v}, & \text{in } \Omega, \\ \nabla \cdot \tilde{B} = 0, & \text{in } \Omega, \\ \partial_t \tilde{f} = \tilde{v}_2 - \dot{v}_1 \partial_1 \tilde{f} & \text{on } \Gamma, \end{cases} \tag{3.5}$$

where $c^2 := c^2(\dot{\varrho})$. In fact, let $v = \dot{v} + \tilde{v}$ and $n = e_2 + \tilde{n}$, so we linearize the origin boundary condition $[v \cdot n] = 0$ as follows:

$$[(\dot{v} + \tilde{v}) \cdot (e_2 + \tilde{n})] = [\tilde{v} \cdot e_2] + [\dot{v} \cdot \tilde{n}] + [\tilde{v} \cdot \tilde{n}] = 0,$$

where $\tilde{n} = (-\partial_1 \tilde{f}, 0)$. Obviously, the third term is a nonlinear term. Then the linear boundary condition is written as

$$[\tilde{v} \cdot e_2] = -[\dot{v} \cdot \tilde{n}] = 2\dot{v}_1^{+} \partial_1 \tilde{f}.$$

Similarly, we deduce that

$$\tilde{B}_2 - \dot{B}_1\partial_1\tilde{f} = 0.$$

Thus, the jump conditions on the boundary can be linearized as follows:

$$[c^2\tilde{h} + \dot{\varrho}^{-1}\dot{B}_1\tilde{B}_1] = 0,\ [\tilde{v}_2] = 2\dot{v}_1^+\partial_1\tilde{f},\ \tilde{B}_2 - \dot{B}_1\partial_1\tilde{f} = 0 \quad \text{on } \Gamma. \tag{3.6}$$

We also get a linearized equation for the front $\tilde{f}$

$$\partial_t^2\tilde{f} + (\dot{v}_1^+)^2\partial_1^2\tilde{f} + \frac{1}{2}c^2(\partial_2\tilde{h}^+ + \partial_2\tilde{h}^-) + \frac{1}{2}(\frac{\dot{B}_1^+}{\dot{\varrho}}\partial_2\tilde{B}_1^+ + \frac{\dot{B}_1^-}{\dot{\varrho}}\partial_2\tilde{B}_1^-) = \frac{(\dot{B}_1^+)^2}{\dot{\varrho}}\partial_1^2\tilde{f} \quad \text{on } \Gamma, \tag{3.7}$$

and a linearized system for the pressure $\tilde{h}$

$$\begin{cases} (\partial_t + \dot{v}_1\partial_1)^2\tilde{h} - c^2\Delta\tilde{h} - \dfrac{\dot{B}_1}{\dot{\rho}}\Delta\tilde{B}_1 = 0 & \text{in } \Omega, \\ (\partial_t + \dot{v}_1\partial_1)^2\tilde{B}_1 - \dot{B}_1(\partial_t + \dot{v}_1\partial_1)^2\tilde{h} + c^2\dot{B}_1\partial_1^2\tilde{h} = 0 & \text{in } \Omega, \\ [c^2\tilde{h} + \dot{\varrho}^{-1}\dot{B}_1\tilde{B}_1] = 0 & \text{on } \Gamma, \\ [c^2\partial_2\tilde{h} + \dot{\varrho}^{-1}\dot{B}_1\partial_2\tilde{B}_1] = -4\dot{v}_1^+\partial_t\partial_1\tilde{f} & \text{on } \Gamma. \end{cases} \tag{3.8}$$

Then, before going on to the analysis, we are going to drop the tilde from all unknowns of the linearized problem by $\tilde{f}, \tilde{h}, \tilde{v}, \tilde{B}$ in order to avoid overloading notation.

We want to construct a solution to the linear system (3.5)-(3.8) that has a growing H^k norm for any k. To begin with, we assume the solution is in the following normal mode form:

$$\begin{aligned} &h(t, x_1, x_2) = e^{\tau t}m(x_1, x_2), v(t, x_1, x_2) = e^{\tau t}w(x_1, x_2), \\ &B(t, x_1, x_2) = e^{\tau t}b(x_1, x_2), f(t, x_1) = e^{\tau t}g(x_1). \end{aligned} \tag{3.9}$$

Here we assume that $\tau = \gamma + i\delta \in \mathbb{C}\backslash\{0\}$ is the same above and below the interface. A solution with $\Re(\tau) > 0$ corresponds to a growing mode. Plugging the ansatz (3.9) into (3.5)-(3.8), we have

$$\begin{cases} \tau m + \dot{v}_1\partial_1 m + \partial_1 w_1 + \partial_2 w_2 = 0 & \text{in } \Omega, \\ \tau w_1 + \dot{v}_1\partial_1 w_1 + c^2\partial_1 m = 0 & \text{in } \Omega, \\ \tau w_2 + \dot{v}_1\partial_1 w_2 + c^2\partial_2 m + \frac{\dot{B}_1}{\dot{\varrho}}\partial_2 b_1 = \frac{\dot{B}_1}{\dot{\varrho}}\partial_1 b_2 & \text{in } \Omega, \\ \tau b_1 + \dot{v}_1\partial_1 b_1 + \dot{B}_1\partial_2 w_2 = 0 & \text{in } \Omega, \\ \tau b_2 + \dot{v}_1\partial_1 b_2 = \dot{B}_1\partial_1 w_2 & \text{in } \Omega, \\ \partial_1 b_1 + \partial_2 b_2 = 0 & \text{in } \Omega, \\ \tau g = w_2 - \dot{v}_1\partial_1 g & \text{on } \Gamma, \\ [c^2 m + \dot{\varrho}^{-1}\dot{B}_1 b_1] = 0,\ [w_2] = 2\dot{v}_1^+\partial_1 g,\ b_2 = \dot{B}_1\partial_1 g & \text{on } \Gamma, \end{cases} \tag{3.10}$$

$$\tau^2 g + (\dot{v}_1^+)^2\partial_1^2 g + \frac{1}{2}c^2(\partial_2 m^+ + \partial_2 m^-) + \frac{1}{2}(\frac{\dot{B}_1^+}{\dot{\varrho}}\partial_2 b_1^+ + \frac{\dot{B}_1^-}{\dot{\varrho}}\partial_2 b_1^-) = \frac{(\dot{B}_1^+)^2}{\dot{\varrho}}\partial_1^2 g \quad \text{on } \Gamma, \tag{3.11}$$

and

$$\begin{cases}(\tau+\dot v_1\partial_1)^2m-c^2(\partial_1^2+\partial_2^2)m-\dfrac{\dot B_1}{\dot\varrho}(\partial_1^2+\partial_2^2)b_1=0 & \text{in } \Omega,\\ (\tau+\dot v_1\partial_1)^2b_1-\dot B_1(\tau+\dot v_1\partial_1)^2m+c^2\dot B_1\partial_1^2m=0 & \text{in } \Omega,\\ [\dot\rho c^2m+\dot\rho^{-1}\dot B_1b_1]=0 & \text{on } \Gamma,\\ [c^2\partial_2m+\dot\rho^{-1}\dot B_1\partial_2b_1]=-4\dot v_1^+\tau\partial_1g & \text{on } \Gamma.\end{cases} \tag{3.12}$$

3.2. *The formula for $\partial_2\hat m^++\partial_2\hat m^-$ on Γ*

By taking the Fourier transform of problem (3.11) and (3.12), we deduce a formula for $\partial_2\hat m^++\partial_2\hat m^-$ on Γ, then, substituting this formula into (3.12), it follows that a second-order wave-type equation for the front g is satisfied. More precisely, we define the Fourier transform of m, w, b, and g as follows:

$$\hat m=\int_{-\infty}^{\infty}e^{-i\eta x_1}m\,dx_1,\ \hat w=\int_{-\infty}^{\infty}e^{-i\eta x_1}w\,dx_1,$$

and

$$\hat b=\int_{-\infty}^{\infty}e^{-i\eta x_1}b\,dx_1,\ \hat g=\int_{-\infty}^{\infty}e^{-i\eta x_1}g\,dx_1,$$

in which the dual variable of the space variable x_1 is denoted by η.

Taking the Fourier transform of the equations (3.10)-(3.12), we derive the following equations:

$$\begin{cases}(\tau+i\dot v_1\eta)\hat m+i\eta\hat w_1+\partial_2\hat w_2=0 & \text{in } \Omega,\\ (\tau+i\dot v_1\eta)\hat w_1+c^2i\eta\hat m=0 & \text{in } \Omega,\\ (\tau+i\dot v_1\eta)\hat w_2+c^2\partial_2\hat m+\frac{\dot B_1}{\dot\varrho}\partial_2\hat b_1=\frac{\dot B_1}{\dot\varrho}i\eta\hat b_2 & \text{in } \Omega,\\ (\tau+i\dot v_1\eta)\hat b_1+\dot B_1\partial_2w_2=0 & \text{in } \Omega,\\ (\tau+i\dot v_1\eta)\hat b_2=\dot B_1i\eta\hat w_2 & \text{in } \Omega,\\ i\eta\hat b_1+\partial_2\hat b_2=0 & \text{in } \Omega,\\ \tau\hat g=\hat w_2-i\dot v_1\eta\hat g & \text{on } \Gamma,\\ \quad [c^2\hat m+\dot\varrho^{-1}\dot B_1\hat b_1]=0,\ [\hat w_2]=2i\dot v_1^+\eta\hat g,\ \hat b_2=i\dot B_1\eta\hat g & \text{on } \Gamma,\end{cases} \tag{3.13}$$

and

$$\tau^2\hat g-(\dot v_1^+)^2\eta^2\hat g+\frac12(c^2(\partial_2\hat m^++\partial_2\hat m^-)+\frac12(\frac{\dot B_1^+}{\dot\varrho}\partial_2b_1^++\frac{\dot B_1^+}{\dot\varrho}\partial_2b_1^-)+\frac{(\dot B_1^+)^2}{\dot\varrho}\eta^2g=0 \quad \text{on } \Gamma, \tag{3.14}$$

and

$$\begin{cases}(\tau+i\dot v_1\eta)^2+c^2k^2\hat m+\dfrac{\dot B_1}{\dot\varrho}\eta^2\hat b_2-c^2\partial_2^2\hat m-\partial_2^2\hat b_1=0 & \text{in } \Omega,\\ (\tau+i\dot v_1\eta)^2\hat b_1-\dot B_1(\tau+i\dot v_1\eta)^2\hat m-c^2\dot B_1\eta^2m=0 & \text{in } \Omega,\\ [c^2\hat m+\dot\varrho^{-1}\dot B_1\hat b_1]=0 & \text{on } \Gamma,\\ [c^2\partial_2\hat m+\dot\varrho^{-1}\dot B_1\partial_2\hat b_1]=-4i\dot v_1^+\eta\tau\hat g & \text{on } \Gamma.\end{cases} \tag{3.15}$$

From the second equation in (3.15), we can deduce the relationship between the magnetic $\hat{b}_1$ and the pressure $\hat{m}$:

$$\hat{b}_1 = \frac{\dot{B}_1[(\tau + i\dot{v}_1\eta)^2 + c^2\eta^2]}{(\tau + i\dot{v}_1\eta)^2}\hat{m}, \tag{3.16}$$

then we substitute this relationship (3.16) into (3.15) to get

$$\begin{cases} ((\tau + i\dot{v}_1\eta)^2 + C_B^2\eta^2)\hat{m} - C_B^2\partial_2^2\hat{m} = 0, & \text{in } \Omega, \\ [C_B^2\hat{m}] = 0, & \text{on } \Gamma, \\ [C_B^2\partial_2\hat{m}] = -4i\dot{v}_1^+\eta\tau\hat{g}, & \text{on } \Gamma, \end{cases} \tag{3.17}$$

where

$$(C_B^\pm)^2 = c^2 + c_A^2\frac{(\tau \pm i\dot{v}_1^+\eta)^2 + c^2\eta^2}{(\tau \pm i\dot{v}_1^+\eta)^2}, \quad c_A^2 = \frac{|\dot{B}_1|^2}{\dot{\rho}}. \tag{3.18}$$

Here, we remark that C_B is called the magneto-acoustic speed, and c_A is called the Alfven speed and we also substitute this relationship (3.16) into (3.14) to get

$$\tau^2\hat{g} - (\dot{v}_1^+)^2\eta^2\hat{g} + \frac{(C_B^+)^2}{2}\partial_2\hat{m}^+ + \frac{(C_B^-)^2}{2}\partial_2\hat{m}^- + c_A^2\eta^2\hat{g} = 0, \quad \text{on } \Gamma. \tag{3.19}$$

Solving the system (3.17), we obtain

$$\hat{m} = \begin{cases} \dfrac{4i\dot{v}_1^+\tau\eta\hat{g}}{(C_B^+)^2(\mu^+ + \mu^-)}e^{-\mu^+x_2} & x_2 \geq 0, \\ \dfrac{4i\dot{v}_1^+\tau\eta\hat{g}}{(C_B^-)^2(\mu^+ + \mu^-)}e^{\mu^-x_2} & x_2 < 0, \end{cases} \tag{3.20}$$

where $\mu^\pm = \sqrt{\frac{(\tau\pm i\dot{v}_1^+\eta)^2}{(C_B^\pm)^2} + \eta^2}$ are the roots of the equation

$$(C_B^\pm)^2s^2 + (\tau \pm i\dot{v}_1^+\eta)^2 - (C_B^\pm)^2\eta^2 = 0. \tag{3.21}$$

Here, we notice that $\mathcal{R}\mu^\pm > 0$ since $\mathcal{R}\tau > 0$.

By direct computation, we deduce

$$\frac{(C_B^+)^2}{2}\partial_2\hat{m}^+ + \frac{(C_B^-)^2}{2}\partial_2\hat{m}^- = -4i\dot{v}_1^+\eta\tau\hat{f}\frac{\mu^+ - \mu^-}{\mu^+ + \mu^-} \text{ on } \Gamma. \tag{3.22}$$

Finaly,we substitute (3.22) into (3.19) to obtain a second-order equation for $\hat{g}$:

$$(\tau^2 - (\dot{v}_1^+)^2\eta^2 - 2i\dot{v}_1^+\eta\tau\frac{\mu^+ - \mu^-}{\mu^+ + \mu^-} + c_A^2\eta^2)\hat{g} = 0 \text{ on } \Gamma, \tag{3.23}$$

where the symbol of (3.23) is defined as follows:

$$\Sigma = \tau^2 - (\dot{v}_1^+)^2\eta^2 - 2i\dot{v}_1^+\eta\tau\frac{\mu^+ - \mu^-}{\mu^+ + \mu^-} + c_A^2\eta^2. \tag{3.24}$$

4. The analysis of the symbol (3.24)

In order to further analyze the equation (3.23), we define a set of frequencies

$$\Xi = \{(\tau, \eta) \in \mathbb{C} \times \mathbb{R} : \Re\tau > 0\}. \tag{4.1}$$

Since it is known that $\Re\mu^{\pm} > 0$ when $\Re\tau > 0$, it follows that $\Re(\mu^{+} + \mu^{-}) > 0$, and thus $\mu^{+} + \mu^{-} \neq 0$ in all such points. To study the symbol Σ, we also need to know whether the difference $\mu^{+} - \mu^{-}$ vanishes.

Lemma 4.1. *Let $(\tau, \eta) \in \Xi$. Then $\mu^{+} = \mu^{-}$ if and only if $(\tau, \eta) = (\tau, 0)$.*

Proof. From (3.21), it implies that $(\mu^{+})^2 = (\mu^{-})^2$ if and only if $\eta = 0$ or $\tau = 0$. Since $(\tau, \eta) \in \Xi$, only the $\eta = 0$ case needs study. When $\eta = 0$, it follows that $\mu^{+} = \mu^{-} = \tau / \sqrt{c^2 + c_A^2}$. □

Now we will discuss the roots of the symbol (3.24) in the instability case.

Lemma 4.2. *Let $\Sigma(\tau, \eta)$ be the symbol defined in* (3.24)*. For $(\tau, \eta) \in \Xi$, when $c > c_A$, if $c^2 - c^2\sqrt{\frac{c^2 - c_A^2}{c^2 + c_A^2}} < (\dot{v}_1^{+})^2 < c^2 + c^2\sqrt{\frac{c^2 - c_A^2}{c^2 + c_A^2}}$, then $\Sigma(\tau, \eta) = 0$ if and only if*

$$\tau = +V_3\eta,$$

where $V_3^2 = \left(c^2(c^2 + 4(\dot{v}_1^{+})^2) - 2\frac{c^4 c_A^2}{c^2 + c_A^2}\right)^{\frac{1}{2}} - (\dot{v}_1^{+})^2 - c^2 > 0.$

Proof. To begin with, we introduce a quantity

$$\theta := \frac{\mu^{+} - \mu^{-}}{\mu^{+} + \mu^{-}}. \tag{4.2}$$

From this, we can deduce that

$$\frac{\mu^{+}}{1 + \theta} = \frac{\mu^{-}}{1 - \theta}. \tag{4.3}$$

Since our goal is to find the root of $\Sigma = 0$, from the definition of Σ (3.24), it follows that $\theta = \frac{\tau^2 - (\dot{v}_1^{+})^2\eta^2 + c_A^2\eta^2}{2i\dot{v}_1^{+}\eta\tau}$, therefore we have

$$1 + \theta = \frac{2i\dot{v}_1^{+}\eta\tau + (\tau^2 - (\dot{v}_1^{+})^2\eta^2 + c_A^2\eta^2)}{2i\dot{v}_1^{+}\eta\tau}, 1 - \theta = \frac{2i\dot{v}_1^{+}\eta\tau - (\tau^2 - (\dot{v}_1^{+})^2\eta^2 + c_A^2\eta^2)}{2i\dot{v}_1^{+}\eta\tau}. \tag{4.4}$$

Substituting (4.4) into (4.3) leads to a characteristic equation

$$\frac{\mu^{+}}{(\tau + i\dot{v}_1^{+}\eta)^2 + c_A^2\eta^2} = -\frac{\mu^{-}}{(\tau - i\dot{v}_1^{+}\eta)^2 + c_A^2\eta^2}. \tag{4.5}$$

In order to find the root of $\Sigma = 0$, we also introduce new variables

$$V = \frac{\tau}{\eta}, \ \tilde{\mu}^{\pm} = \frac{\mu^{\pm}}{\eta}. \tag{4.6}$$

Plugging these new variables (4.6) into the characteristic equation (4.5), we have

$$\tilde{\mu}^+((V - i\dot{v}_1^+)^2 + c_A^2) + \tilde{\mu}^-((V + i\dot{v}_1^+)^2 + c_A^2) = 0. \tag{4.7}$$

By the formula of the roots $\mu^\pm$ (3.21), it follows that

$$(\tilde{\mu}^+)^2 = \frac{[(V + i\dot{v}_1^+)^2 + c^2][(V + i\dot{v}_1^+)^2 + c_A^2]}{(c^2 + c_A^2)(V + i\dot{v}_1^+)^2 + c^2 c_A^2} \tag{4.8}$$

and

$$(\tilde{\mu}^-)^2 = \frac{[(V - i\dot{v}_1^+)^2 + c^2][(V - i\dot{v}_1^+)^2 + c_A^2]}{(c^2 + c_A^2)(V - i\dot{v}_1^+)^2 + c^2 c_A^2}. \tag{4.9}$$

Squaring the equation (4.6), we get

$$[\tilde{\mu}^+((V - i\dot{v}_1^+)^2 + c_A^2)]^2 - [\tilde{\mu}^-((V + i\dot{v}_1^+)^2 + c_A^2)]^2 = 0, \tag{4.10}$$

then substituting (4.8) and (4.9) into (4.10) leads to the following equation for V^2:

$$V[V^4 + 2(c^2 + (\dot{v}_1^+)^2)V^2 + (\dot{v}_1^+)^2((\dot{v}_1^+)^2 - 2c^2) + 2\frac{c^4 c_A^2}{c^2 + c_A^2}] = 0. \tag{4.11}$$

The roots of this equation are

$$V_1 = 0, \tag{4.12}$$

$$V_2^2 = -c^2 - (\dot{v}_1^+)^2 - [c^2(c^2 + 4(\dot{v}_1^+)^2) - 2\frac{c^4 c_A^2}{c^2 + c_A^2}]^{\frac{1}{2}}, \tag{4.13}$$

or

$$V_3^2 = -c^2 - (\dot{v}_1^+)^2 + [c^2(c^2 + 4(\dot{v}_1^+)^2) - 2\frac{c^4 c_A^2}{c^2 + c_A^2}]^{\frac{1}{2}}. \tag{4.14}$$

Now we verify whether these possible root are the real roots of $\Sigma(\tau, \eta) = 0$.

i) Since the first root $V_1 = 0$, from the definition (4.6), we deduce that $\tau = 0$. However, since we seek a growing mode, this contradicts with the assumption $\Re(\tau) > 0$. Therefore, $V_1 = 0$ is not a root of $\Sigma(\tau, \eta) = 0$.

ii) We claim that the points $(\tau, \eta) \in \Xi$ with $\tau = \pm V_2 \eta$ are also not roots of $\Sigma(\tau, \eta) = 0$. Without loss of generality, we can assume that V_2 is positive. From (4.13), we deduce

$$V_2 = iY_2, \ Y_2 \geq \dot{v}_1^+ + c. \tag{4.15}$$

And from this, we deduce $Y_2 \pm \dot{v}_1^+ > c$. Thus, it follows that $(Y_2 \pm \dot{v}_1^+)^2 > c^2 > c_A^2$. In accordance with the equation (4.8) and (4.9), we deduce that $\tilde{\mu}^+ = i\frac{[(Y_2+\dot{v}_1^+)^2 - c^2][(Y_2+\dot{v}_1^+)^2 - c_A^2]}{(c^2+c_A^2)(Y_2+\dot{v}_1^+)^2 - c^2 c_A^2}$ *and* $\tilde{\mu}^- = i\frac{[(Y_2-\dot{v}_1^+)^2 - c^2][(Y_2-\dot{v}_1^+)^2 - c_A^2]}{(c^2+c_A^2)(Y_2-\dot{v}_1^+)^2 - c^2 c_A^2}$, from which we know that $\tilde{\mu}^+$ and $\tilde{\mu}^-$ have the same sign. Also, we have $(V - i\dot{v}_1^+)^2 + c_A^2 < 0$ and $(V + i\dot{v}_1^+)^2 + c_A^2 < 0$. From these relationships, we deduce that (4.7) is not satisfied. Similarly, we can show that $(\tau, \eta) \in \Sigma$ with $\tau = -V_2 \eta$ is not the root of $\mu^+ \mu^- = \eta^2$. On the other hand, from (4.15), we know that $\tau = iY_2 \eta$ is an imaginary root, thus it implies that $\Re \tau = 0$ and $(\pm V_2 \eta, \eta) \notin \Xi$.

iii) We discuss whether the root V_3 is a root of $\Sigma(\tau,\eta)=0$. Under this constraint, $c^2 - c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}} < (\dot{v}_1^+)^2 < c^2 + c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}}$. We get $V_3^2 > 0$, thus it follows that $\tau = \pm V_3\eta$ are real. The point $(-V_3\eta,\eta) \notin \Xi$, thus we omit this point, and we only study the root $\tau = +V_3\eta$. Using a fact that square roots of the complex number $a + ib$ are

$$\pm\{\sqrt{\frac{r+a}{2}} + isgn(b)\sqrt{\frac{r-a}{2}}\},\ r = |a+ib|, \tag{4.16}$$

in our case, we compute

$$\mu^+ = \sqrt{\frac{r+a}{2}} + i\sqrt{\frac{r-a}{2}}, \mu^- = \sqrt{\frac{r+a}{2}} - i\sqrt{\frac{r-a}{2}}, \tag{4.17}$$

where

$$a = \frac{(c^2+c_A^2)(V_3^2-(\dot{v}_1^+)^2)^2(V_3^2-(\dot{v}_1^+)^2) + c^2c_A^2(V_3^4+(\dot{v}_1^+)^4-6V_3^2(\dot{v}_1^+)^2)}{(c^2+c_A^2)^2(V_3^2+(\dot{v}_1^+)^2)^2 + 2c^2c_A^2(c^2+c_A^2)(V_3^2-(\dot{v}_1^+)^2) + c^4c_A^4}\eta^2, \tag{4.18}$$

and

$$b = \frac{2V_3\dot{v}_1^+((c^2+c_A^2)(V_3^2-(\dot{v}_1^+)^2)^2 + 2c^2c_A^2(V_3^2-(\dot{v}_1^+)^2))}{(c^2+c_A^2)^2(V_3^2+(\dot{v}_1^+)^2)^2 + 2c^2c_A^2(c^2+c_A^2)(V_3^2-(\dot{v}_1^+)^2) + c^4c_A^4}\eta^2, \tag{4.19}$$

so that (4.7) is satisfied. Therefore we deduce that in the case of $c^2 - c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}} < (\dot{v}_1^+)^2 < c^2 + c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}}$, the root of the symbol Σ is the point $(+V_3\eta,\eta)$. In summary, we can get a root (τ,η) with $\Re\tau > 0$, which is an unstable solution. □

5. Ill-posedness of solutions for the linear problem

5.1. Uniqueness for the linearized equations (3.5)

To begin with, we prove a uniqueness result for the linearized equations (3.5).

Lemma 5.1. *Let* (f,h,v,B) *be a solution to the linearized equations* (3.5) *with* $(f,h,v,B)|_{t=0} = 0$. *Then* $(f,h,v,B) \equiv 0$.

Proof. Taking the standard inner product of the first, second, and third equations in (3.5) with c^2h^+, v^+ and integrating over Ω^+, we obtain

$$\frac{1}{2}\partial_t\int_{\Omega^+} c^2|h^+|^2 + \frac{1}{2}\int_{\Omega^+}\dot{v}_1^+\partial_1(c^2|h^+|^2) + \int_{\Omega^+} c^2h^+\mathrm{div}v^+ = 0 \tag{5.1}$$

and

$$\frac{1}{2}\partial_t\int_{\Omega^+}|v^+|^2 + \frac{1}{2}\int_{\Omega^+}\dot{v}_1^+\partial_1(|v^+|^2) + \int_{\Omega^+} c^2\nabla h^+\cdot v^+ + \int_{\Omega^+}\frac{\dot{B}_1}{\dot{\varrho}}\nabla B_1\cdot v^+ = \int_{\Omega^+}\frac{\dot{B}_1^+}{\dot{\varrho}}\partial_1 B\cdot v^+. \tag{5.2}$$

Meanwhile, we take the inner product of the third equation in (3.5) with $\frac{B^+}{\dot\varrho}$ and integrate over Ω^+ to arrive at

$$\frac{1}{2}\partial_t\int_{\Omega^+}\frac{|B^+|^2}{\dot\varrho}+\frac{1}{2}\int_{\Omega^+}\frac{\dot v_1^+}{\dot\varrho}\partial_1(|B^+|^2)+\int_{\Omega^+}\frac{\dot B_1^+}{\dot\varrho}B_1^+\nabla\cdot v^+=\int_{\Omega^+}\frac{\dot B_1^+}{\dot\varrho}\partial_1 v^+\cdot B^+. \tag{5.3}$$

After integrating by parts, the second terms on the left-hand side of (5.1) and (5.2) vanish, and also the sum of the terms on the right-hand side of (5.2) and (5.7) vanish, thus adding (5.1), (5.2) and (5.7) and integrating by parts, we get

$$\frac{1}{2}\partial_t\int_{\Omega^+}(c^2|h^+|^2+|v^+|^2+\frac{|B^+|^2}{\dot\varrho})=c^2\int_\Gamma h^+v^+\cdot e_2+\frac{\dot B_1^+}{\dot\varrho}\int_\Gamma B_1^+v^+\cdot e_2. \tag{5.4}$$

A similar result holds on Ω_- with the opposite sign on the right-hand side:

$$\frac{1}{2}\partial_t\int_{\Omega^-}(c^2|h^-|^2+|v^-|^2+\frac{|B^-|^2}{\dot\varrho})=-c^2\int_\Gamma h^-v^-\cdot e_2-\frac{\dot B_1^-}{\dot\varrho}\int_\Gamma B_1^-v^-\cdot e_2. \tag{5.5}$$

Adding (5.4) and (5.5) implies

$$\begin{aligned}\frac{1}{2}\partial_t\int_{\Omega}(c^2|h|^2+|v|^2+\frac{|B|^2}{\dot\varrho})&=\int_\Gamma[(c^2h+\dot\varrho^{-1}\dot B_1B_1)v\cdot e_2]\\&=2\int_\Gamma(c^2h+\dot\varrho^{-1}\dot B_1B_1)\dot v_1^+\partial_1 f.\end{aligned} \tag{5.6}$$

Also multiplying the fifth equation in (3.5) by f, we have

$$\frac{1}{2}\partial_t\int_\Gamma|f|^2=\int_\Gamma v_2 f. \tag{5.7}$$

Adding (5.6) and (5.7) and using the *Hölder* inequality yields

$$\begin{aligned}&\frac{1}{2}\partial_t\int_{\Omega}(c^2|h|^2+|v|^2+\frac{|B|^2}{\dot\varrho})+\frac{1}{2}\partial_t\int_\Gamma|f|^2\\&=2\int_\Gamma(c^2h+\dot\varrho^{-1}\dot B_1B_1)\dot v_1^+\partial_1 f+\int_\Gamma v_2 f\\&\le 2c^2\dot v_1^+\|h\|_{L^2(\Gamma)}\|\partial_1 f\|_{L^2(\Gamma)}+\|B_1\|_{L^2(\Gamma)}\|\partial_1 f\|_{L^2(\Gamma)}+\|v_2\|_{L^2(\Gamma)}\|f\|_{L^2(\Gamma)}:=J.\end{aligned} \tag{5.8}$$

To avoid the loss of derivatives, we suppose that the solutions are band-limited at radius $R>0$, i.e., that

$$\underset{x_2\in\mathbb{R}}{\cup} supp(|\hat f(\cdot)|+|\hat h(\cdot,x_2)|+|\hat v(\cdot,x_2)|+|\hat B(\cdot,x_2)|)\subset B(0,R).$$

We also introduce an anisotropic trace estimate in Lemma B.1 ([38]):

$$\|\phi\|^2_{L^2(\Gamma)}\le C(\|\dot v\cdot\nabla\phi\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)}+\|\phi\|^2_{L^2(\Omega)}), \tag{5.9}$$

where $\dot v=(\dot v_1^+,0)$ with $\dot v_1^+=U_{low}+c\epsilon_0>0$. Now we estimate J as follows:

$$\begin{aligned}J&\lesssim(\|\dot v_1^+\partial_1 h\|_{L^2(\Omega)}\|h\|_{L^2(\Omega)}+\|h\|^2_{L^2(\Omega)})^{\frac12}\|\eta\hat f\|_{L^2(\Gamma)}\\&+(\|\dot v_1^+\partial_1 B_1\|_{L^2(\Omega)}\|B_1\|_{L^2(\Omega)}+\|B_1\|^2_{L^2(\Omega)})^{\frac12}\|\eta\hat f\|_{L^2(\Gamma)}\\&+(\|\dot v_1^+\partial_1 v_2\|_{L^2(\Omega)}\|v_2\|_{L^2(\Omega)}+\|v_2\|^2_{L^2(\Omega)})^{\frac12}\|f\|_{L^2(\Gamma)}\\&\lesssim((\dot v_1^+R+1)R^2\|h\|^2_{L^2(\Omega)})^{\frac12}\|f\|_{L^2(\Gamma)}+((\dot v_1^+R+1)R^2\|B_1\|^2_{L^2(\Omega)})^{\frac12}\|f\|_{L^2(\Gamma)}\\&+((\dot v_1^+R+1)\|v_2\|^2_{L^2(\Omega)})^{\frac12}\|f\|_{L^2(\Gamma)}.\end{aligned} \tag{5.10}$$

Finally, plugging (5.10) into (5.8) and taking use of *Grönwall′s* inequality, for arbitrary R, we have

$$\|f\|^2_{L^2(\Gamma)} + \|h\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|B\|^2_{L^2(\Omega)} \leq C(\|f_0\|^2_{L^2(\Gamma)} + \|h_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega)} + \|B\|^2_{L^2(\Omega)}). \tag{5.11}$$

From this, we infer that if $(f_0, h_0, v_0, B_0) = 0$, then it follows that $(f, h, v, B) \equiv 0$.

□

5.2. Discontinuous dependence on the initial data

In accordance with Lemma 3.2 and (4.14), if $c^2 - c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}} < (\dot{v}_1^+)^2 < c^2 + c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}}$, we deduce that V_3^2 is positive, and it follows that $\tau = V_3\eta$ is real and positive. Also, we can infer that the equation (3.7) reduces to the following form:

$$\partial_t^2 f + \lambda\partial_1^2 f = 0, \tag{5.12}$$

where λ must be positive in the case of $c^2 - c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}} < (\dot{v}_1^+)^2 < c^2 + c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}}$. In fact, plugging $f = e^{\tau t}g$ into equation (5.12), we get $\tau^2 g + \lambda\partial_1^2 g = 0$. Then we take the Fourier transform of this identity with respect the x_1 variable, and we have

$$(\tau^2 - \lambda\eta^2)\hat{g} = 0, \tag{5.13}$$

which yields $\lambda = \frac{\tau^2}{\eta^2}$. From Lemma 3.2, we know that $V_3^2 = \frac{\tau^2}{\eta^2} > 0$ in the case of $c^2 - c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}} < (\dot{v}_1^+)^2 < c^2 + c^2\sqrt{\frac{c^2-c_A^2}{c^2+c_A^2}}$. Thus, we have $\lambda = V_3^2 > 0$. Therefore, (5.12) is an elliptic equation.

We are now in a position to prove ill-posedness for this linear problem in the following lemma.

Lemma 5.2. *In the case of* $M_{low} := \sqrt{1 - \sqrt{\frac{1-\beta}{1+\beta}}} + \epsilon_0 \leq M := \frac{\dot{v}_1^+}{c} < M_{upp} := \sqrt{1 + \sqrt{\frac{1-\beta}{1+\beta}}} - \epsilon_0$, $\beta := \frac{c_A^2}{c^2}$, *the linear problem* (3.5) *with the correesponding jump boundary conditions* (3.6) *is ill-posed in the sense of Hadamard in* $H^k(\Omega)$ *for every k. More precisely, for any k,* $j \in \mathbb{N}$ *with* $j \geq k$, *and for any* $T_0 > 0$ *and* $\alpha > 0$, *there exists a sequence* $\{(f_n, h_n, v_n, B_n)\}_{n=1}^\infty$ *to* (3.5), *satisfying jump boundary conditions* (3.6), *so that*

$$\|(f_n(0), h_n(0), v_n(0), B_n(0))\|_{H^j} \lesssim \frac{1}{n} \tag{5.14}$$

but

$$\|(f_n(t), h_n(t), v_n(t), B_n(t))\|_{H^k} \geq \alpha, \ for\ all\ t \geq T_0. \tag{5.15}$$

Proof. For any $j \in \mathbb{N}$, we let $\chi_n(\eta) \in C_c^\infty(\mathbb{R})$ be a real-valued function so that $supp(\chi_n) \subset B(0, n + 1)\backslash B(0, n)$ and

$$\int_{\mathbb{R}} (1 + |\eta|^2)^{j+1}|\chi_n(\eta)|^2 d\eta = \frac{1}{\bar{C}_j^{\,2} n^2}. \tag{5.16}$$

We define

$$f_n(t, x_1) = e^{\tau t}g_n(x_1) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{V_3\eta t}\chi_n(\eta)e^{i\eta x_1}d\eta, \tag{5.17}$$

which solves (5.12). Here, we make use of $\tau = V_3\eta$ according with Lemma 3.2. Meanwhile we can see that $\hat{g}_n = \chi_n(\eta)$. From this, we can see that the linearized front equation is qualitatively more unstable for large frequencies η. Since $\eta \to \infty$, the solutions f_n of (5.12) with a higher frequency grow faster in time, which provides a mechanism for KH instability. By the choice of χ_n and the Plancherel theorem, we have the estimate

$$\begin{aligned} \|f_n(t=0)\|_{H^j(\Gamma)} &= \|g_n\|_{H^j(\Gamma)} \\ &= (\int_{\mathbb{R}} (1+|\eta|^2)^j |\chi_n(\eta)|^2 d\eta)^{1/2} \lesssim \frac{1}{n}. \end{aligned} \tag{5.18}$$

Meanwhile for $n+1 \geq \eta \geq n$ and $t \geq T_0$, we get

$$\begin{aligned} \|f_n(t)\|^2_{H^k(\Gamma)} &\geq e^{2V_3 n T_0} \int_{\mathbb{R}} (1+|\eta|^2)^k |\chi_n(\eta)|^2 d\eta \\ &\geq \frac{e^{2V_3 n T_0}}{(1+(n+1)^2)^{j-k+1}} \int_{\mathbb{R}} (1+\eta^2)^{j+1} |\chi_n(\eta)|^2 d\eta. \end{aligned} \tag{5.19}$$

Let n be sufficiently large so that

$$\frac{e^{2V_3 n T_0}}{(1+(n+1)^2)^{j-k+1}} \geq \alpha^2 \bar{C}_j^{\,2} n^2, \tag{5.20}$$

thus we may estimate

$$\|f_n(t)\|_{H^k(\Gamma)} \geq \alpha. \tag{5.21}$$

From the computation in Section 2.2, we know that

$$\hat{m}_n(\eta, x_2) = \begin{cases} \dfrac{4i\dot{v}_1^+ \eta\tau}{(C_B^+)^2(\mu^+ + \mu^-)} \hat{g}_n(\eta) e^{-\mu^+ x_2} & x_2 \geq 0, \\ \dfrac{4i\dot{v}_1^+ \eta\tau}{(C_B^-)^2(\mu^+ + \mu^-)} \hat{g}_n(\eta) e^{\mu^- x_2} & x_2 < 0. \end{cases} \tag{5.22}$$

Since $\tau = V_3\eta > 0$ and $\eta > 0$, from Lemma 3.1, we know that $\mu^+ - \mu^- \neq 0$, then (5.22) can be rewritten as

$$\hat{m}_n(\eta, x_2) = \begin{cases} C_1(M)(\mu^+ - \mu^-)\hat{g}_n(\eta) e^{-\mu^+ x_2} & x_2 \geq 0, \\ C_2(M)(\mu^+ - \mu^-)\hat{g}_n(\eta) e^{\mu^- x_2} & x_2 < 0, \end{cases} \tag{5.23}$$

where $C_1(M)$ and. Since $C_2(M)$ are constants that depend on the Mach number M. Here, we note that $\mu^\pm$ only depend on η, we get $\tau = X_1\eta$, it implies that $\mu(\tau, \eta) = \mu(X_1\eta, \eta)$.

We have the following fact:

$$|\partial_2^s e^{-\mu^+ x_2}|^2 = |\mu^+|^{2s} |e^{-\mu^+ x_2}|^2 = |\mu^+|^{2s} e^{-2(\Re\mu^+)x_2} = |\mu^+|^{2s} e^{-2\sqrt{\frac{r+a}{2}}x_2}, \tag{5.24}$$

and the same equality holds for μ^-.

By the Plancherel theorem and (5.22), we have

$$\begin{aligned}
\|h_n(t,x_1,x_2)\|^2_{H^k(\Omega)} &= \|e^{\tau t} m_n(x_1,x_2)\|^2_{H^k(\Omega)} \\
&\geq (C_1(M))^2 \int_{\mathbb{R}} (1+\eta^2)^k |\mu^+ - \mu^-|^2 |e^{\tau t}\hat{g}_n(\eta)|^2 \int_0^\infty |e^{-\mu^+ x_2}|^2 dx_2 d\eta \\
&+ (C_2(M))^2 \int_{\mathbb{R}} (1+\eta^2)^k |\mu^+ - \mu^-|^2 |e^{\tau t}\hat{g}_n(\eta)|^2 \int_{-\infty}^0 |e^{\mu^- x_2}|^2 dx_2 d\eta \\
&\geq (C_1(M))^2 \int_{\mathbb{R}} (1+\eta^2)^k \left|\frac{\mu^+ - \mu^-}{\mu^+}\right|^2 \left|\frac{\mu^+}{\mu^+ + \mu^-}\right| |\mu^+| e^{2V_3\eta t} |\chi_n(\eta)|^2 d\eta \\
&+ (C_2(M))^2 \int_{\mathbb{R}} (1+\eta^2)^k \left|\frac{\mu^+ - \mu^-}{\mu^-}\right|^2 \left|\frac{\mu^-}{\mu^+ + \mu^-}\right| |\mu^-| e^{2V_3\eta t} |\chi_n(\eta)|^2 d\eta,
\end{aligned} \tag{5.25}$$

then recalling the formula (4.16), we deduce that

$$\left|\frac{\mu^+ - \mu^-}{\mu^+}\right|^2 = \frac{|2i\sqrt{\frac{r-a}{2}}|^2}{|\sqrt{\frac{r+a}{2}} + i\sqrt{\frac{r-a}{2}}|^2} = 2\frac{r-a}{r} = 2(1-\frac{a}{r}). \tag{5.26}$$

Making use of $M_{low} \leq M \leq M_{upp}$, we estimate $\frac{a}{r}$ as follows:

$$-1 < -1 + C_3(\epsilon_0) \leq \frac{a}{r} \leq 1 - C_4(\epsilon_0) < 1, \tag{5.27}$$

where $C_3(\epsilon_0)$ and $C_4(\epsilon_0)$ are very small constants that depend on the parameter ϵ_0. Meanwhile the lower bound is reached when M takes the value of M_{upp} in the above inequality.
Also, we compute

$$\begin{aligned}
|\mu^\pm| &= |\sqrt{\frac{r+a}{2}} \pm i\sqrt{\frac{r-a}{2}}| = \sqrt{r}, \\
|\mu^+ + \mu^-| &= |2\sqrt{\frac{r+a}{2}}| = \sqrt{2(r+a)}.
\end{aligned} \tag{5.28}$$

Finally, combining with (5.26), (5.27), and (5.28) implies that

$$2C_4(\epsilon_0) \leq \left|\frac{\mu^+ - \mu^-}{\mu^\pm}\right|^2 < 4. \tag{5.29}$$

Here we remark that $M_{low} \leq M$ must be satisfied, and ϵ_0 is a small but fixed number. Because if the Mach number M tends to $\sqrt{1-\sqrt{\frac{1-\beta}{1+\beta}}}$, this lower bound tends to zero.

Meanwhile, we have to estimate the terms $|\frac{\mu^\pm}{\mu^+ + \mu^-}|$. Noting the fact (5.28) and (5.27), we have

$$\left|\frac{\mu^\pm}{\mu^+ + \mu^-}\right| = \frac{|\mu^\pm|}{|\mu^+ + \mu^-|} = \frac{1}{\sqrt{2}\sqrt{1+\frac{a}{r}}}. \tag{5.30}$$

Since $\sqrt{1-\sqrt{\frac{1-\beta}{1+\beta}}} < M \leq M_{upp}$, the above equality has the following uniform lower bound and upper bound:

$$\frac{1}{2} < \left|\frac{\mu^\pm}{\mu^+ + \mu^-}\right| \leq \frac{1}{\sqrt{2}\sqrt{C_3(\epsilon_0)}}. \tag{5.31}$$

Noting that when ϵ_0 goes to zero, the uniform upper bound will not be obtained.

Therefore, employing (5.29), (5.23), (5.28), and (5.24), we estimate $\|h_n(0)\|_{H^k(\Omega)}$ as follows:

$$
\begin{aligned}
&\|h_n(t=0,x_1,x_2)\|^2_{H^j(\Omega)} = \|m_n(x_1,x_2)\|^2_{H^j(\Omega)}\\
&\leq \sum_{s=0}^{j}(C_1(M))^2\int_{\mathbb{R}}(1+\eta^2)^{j-s}|\mu^+-\mu^-|^2|\hat{g}_n(\eta)|^2\int_0^{\infty}|\partial_2^s e^{-\mu^+x_2}|^2dx_2d\eta\\
&+\sum_{s=0}^{j}(C_2(M))^2\int_{\mathbb{R}}(1+\eta^2)^{j-s}|\mu^+-\mu^-|^2|\hat{g}_n(\eta)|^2\int_{-\infty}^{0}|\partial_2^s e^{\mu^-x_2}|^2dx_2d\eta\\
&\leq \sum_{s=0}^{j}(C_1(M))^2\int_{\mathbb{R}}(1+\eta^2)^{j-s}\left|\frac{\mu^+-\mu^-}{\mu^+}\right|^2\left|\frac{\mu^+}{\mu^++\mu^-}\right||\mu^+|^{2s+1}|\chi_n(\eta)|^2d\eta\\
&+\sum_{s=0}^{j}(C_2(M))^2\int_{\mathbb{R}}(1+\eta^2)^{j-s}\left|\frac{\mu^+-\mu^-}{\mu^-}\right|^2\left|\frac{\mu^-}{\mu^++\mu^-}\right||\mu^-|^{2s+1}|\chi_n(\eta)|^2d\eta\\
&\lesssim \int_{\mathbb{R}}(1+\eta^2)^{j+1}|\chi_n(\eta)|^2d\eta \lesssim \frac{1}{n^2}.
\end{aligned}
\tag{5.32}
$$

At the same time, noting that $supp(\chi_n)\subset B(0,n+1)\backslash B(0,n)$, then for $\eta\geq n\geq 1$ and $t\geq T_0$, we may estimate (5.25) as follows:

$$\|h_n(t)\|^2_{H^k(\Omega)} \geq \tilde{C}(M,\epsilon)\frac{e^{2V_3nT_0}}{(1+(n+1)^2)^{j-k+1}}\int_{\mathbb{R}}(1+\eta^2)^{j+1}|\chi_n(\eta)|^2d\eta. \tag{5.33}$$

Let n be sufficiently large so that

$$\tilde{C}(M,\epsilon)\frac{e^{2V_3nT_0}}{(1+(n+1)^2)^{j-k+1}} \geq \alpha^2n^2\bar{C}_j^2. \tag{5.34}$$

Hence, we may estimate

$$\|h_n(t)\|_{H^k(\Omega)} \geq \alpha. \tag{5.35}$$

Next, we substitute (3.16) into the second equation and the fourth equation in (3.13) and arrive at

$$(\tau+i\dot{v}_1\eta)\hat{w}_1 + c^2 i\eta\hat{m} = 0, \tag{5.36}$$

and

$$(\tau+i\dot{v}_1\eta)\hat{w}_2 + C_B^2\partial_2\hat{m} = -c_A^2\frac{\eta^2}{(\tau+i\dot{v}_1\eta)}\hat{w}_2. \tag{5.37}$$

Since $\tau+i\dot{v}_1\eta\neq 0$, we directly compute to find

$$\hat{v}_1 = c^2\frac{i\eta}{\tau+i\dot{v}_1^+\eta}\hat{h}. \tag{5.38}$$

Also from (3.16), we have

$$\hat{v}_2 = C_B^2\frac{(\tau+i\dot{v}_1^+\eta)\mu^+}{(\tau+i\dot{v}_1^+\eta)^2+c_A^2\eta^2}\hat{h}, \tag{5.39}$$

then we may estimate $|\frac{i\eta}{\tau+i\dot{v}_1^+\eta}|^2$ and $|\frac{(\tau+i\dot{v}_1^+\eta)\mu^+}{(\tau+i\dot{v}_1^+\eta)^2+c_A^2\eta^2}|^2$ as follows:

$$\begin{aligned} |\frac{i\eta}{\tau+i\dot{v}_1^+\eta}|^2 &= \frac{1}{V_3^2+(\dot{v}_1^+)^2} = \frac{1}{\left(c^2(c^2+4(\dot{v}_1^+)^2)-2\frac{c^4c_A^2}{c^2+c_A^2}\right)^{\frac{1}{2}}-c^2} \\ &= \frac{1}{c^2(\sqrt{4M^2+\frac{1-\beta}{1+\beta}}-1)}. \end{aligned} \tag{5.40}$$

Since $1-\sqrt{\frac{1-\beta}{1+\beta}} < M^2 < 1+\sqrt{\frac{1-\beta}{1+\beta}}$, we know that

$$\frac{1}{c^2(1+\sqrt{\frac{1-\beta}{1+\beta}})} < |\frac{i\eta}{\tau+i\dot{v}_1^+\eta}|^2 < \frac{1}{c^2(1-\sqrt{\frac{1-\beta}{1+\beta}})}. \tag{5.41}$$

In accordance with (4.6) , (4.14) and $\sqrt{1-\sqrt{\frac{1-\beta}{1+\beta}}} < M < \sqrt{1+\sqrt{\frac{1-\beta}{1+\beta}}}$ then imply that

$$C_5(\beta) < |\frac{(\tau+i\dot{v}_1^+\eta)\mu^+}{(\tau+i\dot{v}_1^+\eta)^2+c_A^2\eta^2}|^2 \le C_6(\beta). \tag{5.42}$$

Therefore, employing (5.41), (5.42), (5.38) and (5.39), we deduce

$$\|(v_{n,1},v_{n,2})(t=0)\|_{H^j(\Omega)} \lesssim \frac{1}{n}, \tag{5.43}$$

whereas for $\eta \ge n$ and $t \ge T_0$, we deduce

$$\|(v_{n,1},v_{n,2})(t)\|_{H^k(\Omega)} \ge \alpha. \tag{5.44}$$

Since $\tau+i\dot{v}_1\eta \neq 0$, from the fifth equation in (3.13), we know that

$$\hat{b}_2 = \dot{B}_1\frac{i\eta}{\tau+i\dot{v}_1^+\eta}\hat{v}_2. \tag{5.45}$$

Also from (3.16), we have

$$\hat{b}_1 = \frac{\dot{B}_1[(\tau+i\dot{v}_1\eta)^2+c^2\eta^2]}{(\tau+i\dot{v}_1\eta)^2}\hat{h}. \tag{5.46}$$

As with the previous computation, it is easier to show that the upper and lower bounds of these two quantities $\dot{B}_1\frac{i\eta}{\tau+i\dot{v}_1^+\eta}$ and $\frac{\dot{B}_1[(\tau+i\dot{v}_1\eta)^2+c^2\eta^2]}{(\tau+i\dot{v}_1\eta)^2}$ are controlled by two constants, therefore we deduce that

$$\|(b_{n,1},b_{n,2})(t=0)\|_{H^j(\Omega)} \lesssim \frac{1}{n},\ \|(b_{n,1},b_{n,2})(t)\|_{H^k(\Omega)} \ge \alpha. \tag{5.47}$$

Collecting the estimates (5.18), (5.32), (5.43), and (5.47) gives

$$\|f_n(0)\|_{H^j(\Omega)} + \|h_n(0)\|_{H^j(\Gamma)} + \|v_n(0)\|_{H^j(\Omega)} + \|B_n(0)\|_{H^k(\Omega)} \lesssim \frac{1}{n}, \tag{5.48}$$

but the estimates (5.21), (5.35), (5.44), and (5.47) yield

$$\|f_n(t)\|_{H^k(\Gamma)} + \|h_n(t)\|_{H^k(\Omega)} + \|v_n(t)\|_{H^k(\Omega)} + \|B_n(t)\|_{H^k(\Omega)} \ge \alpha,\ for\ all\ t \ge T_0. \tag{5.49}$$

□

6. Ill-posedness for the nonlinear problem

Now we will prove nonlinear ill-posedness for the nonlinear problem (1.24). To begin with, we rewrite the nonlinear system (1.24) in a perturbation formulation around the steady state. Let

$$\begin{aligned} f &= 0 + \tilde{f},\ v = \dot{v} + \tilde{v},\ h = \dot{h} + \tilde{h}, \\ \Psi &= Id + \tilde{\Psi}, \varrho = \dot{\varrho} + \tilde{\varrho},\ \psi = 0 + \tilde{\psi}, \\ B &= \dot{B} + \tilde{B},\ n = e_2 + \tilde{n},\ A = I - K, \end{aligned} \tag{6.1}$$

where

$$K = \sum_{n=1}^{\infty} (-1)^{n-1} (D\tilde{\Psi})^n. \tag{6.2}$$

We can rewrite the term $\breve{v}$ as follows:

$$\begin{aligned} \breve{v} &= (I-K)(\dot{v}+\tilde{v}) - (0, \frac{\partial_t \tilde{\psi}}{1+\partial_2\tilde{\psi}}) \\ &= \dot{v} + \tilde{v} - K(\dot{v}+\tilde{v}) - (0, \frac{\partial_t \tilde{\psi}}{1+\partial_2\tilde{\psi}}) := \dot{v} + M, \end{aligned} \tag{6.3}$$

in which the M is defined as follows:

$$M = \tilde{v} - K(\dot{v}+\tilde{v}) - (0, \frac{\partial_t \tilde{\psi}}{1+\partial_2\tilde{\psi}}). \tag{6.4}$$

Similarly, we rewrite the term $\breve{B}$ as follows:

$$\begin{aligned} \breve{B} &= (I-K)(\dot{B}+\tilde{B}) \\ &= \dot{B} + \tilde{B} - K(\dot{B}+\tilde{B}) := \dot{B} + N, \end{aligned} \tag{6.5}$$

in which the N is defined as follows:

$$N = \tilde{B} - K(\dot{B}+\tilde{B}). \tag{6.6}$$

To linearize the term $c^2(h) = c^2(\dot{h}+\tilde{h})$, we employ the Taylor formula to get

$$c^2(\dot{h}+\tilde{h}) = c^2(\dot{h}) + \mathcal{R}, \tag{6.7}$$

where the reminder term is defined by

$$\mathcal{R} = (c^2)'(\dot{h} + (1-\alpha)\tilde{h})\tilde{h},\ 0 < \alpha < 1. \tag{6.8}$$

Similarly, we also have

$$e^h = e^{\dot{h}+\tilde{h}} = 1 + \tilde{h} + \mathcal{R}', \tag{6.9}$$

where the reminder term is defined by

$$\mathcal{R}' = e^{\dot{h}+(1-\alpha)\tilde{h}}\tilde{h},\ 0 < \alpha < 1. \tag{6.10}$$

For the term $v \cdot n$ and $B \cdot n$, we can rewrite it as

$$v \cdot n = (\dot{v} + \tilde{v}) \cdot (e_2 + \tilde{n}) = \tilde{v}_2 - \dot{v}_1 \partial_1 \tilde{f} + \tilde{v} \cdot \tilde{n}, \tag{6.11}$$

and

$$B \cdot n = (\dot{B} + \tilde{B}) \cdot (e_2 + \tilde{n}) = \tilde{B}_2 - \dot{B}_1 \partial_1 \tilde{f} + \tilde{B} \cdot \tilde{n}. \tag{6.12}$$

Then the nonlinear system (1.24) can be rewritten as a system of the perturbation terms $(\tilde{h}, \tilde{v}, \tilde{f}, \tilde{B})$ as follows:

$$\begin{cases} \partial_t \tilde{h} + (\dot{v} \cdot \nabla)\tilde{h} + \nabla \cdot \tilde{v} = -(M \cdot \nabla)\tilde{h} + K^T \nabla \cdot \tilde{v} & \text{in } \Omega, \\ \dot{\varrho}(\partial_t \tilde{v} + (\dot{v} \cdot \nabla)\tilde{v}) + \dot{\varrho} c^2(\dot{h}) \nabla \tilde{h} + \dot{B} \nabla \tilde{B} - (\dot{B} \cdot \nabla)\tilde{B} = -\dot{\varrho}(M \cdot \nabla)\tilde{v} & \\ -\tilde{\varrho}(\partial_t \tilde{v} + ((\dot{v} + M) \cdot \nabla)\tilde{v}) - \dot{\varrho} c^2(\dot{h}) K^T \nabla \tilde{h} - \tilde{\varrho}(c^2(\dot{h}) + \mathcal{R})(\nabla \tilde{h} - K^T \nabla \tilde{h}) & \\ -\dot{\varrho}\mathcal{R}(\nabla \tilde{h} - K^T \nabla \tilde{h}) + \dot{B} K^T \nabla \tilde{B} - \tilde{B}(\nabla \tilde{B} - K^T \nabla \tilde{B}) + (N \cdot \nabla)\tilde{B} & \text{in } \Omega, \\ \partial_t \tilde{B} + (\dot{v} \cdot \nabla)\tilde{B} + \dot{B} \nabla \cdot \tilde{v} - (\dot{B} \cdot \nabla)\tilde{v} = -(M \cdot \nabla)\tilde{B} & \\ -\dot{B} K^T \nabla \cdot \tilde{v} + \tilde{B}(\nabla \cdot \tilde{v} - K^T \nabla \cdot \tilde{v}) + (N \cdot \nabla)\tilde{v} & \text{on } \Gamma. \\ \partial_t \tilde{f} + \dot{v}_1 \partial_1 \tilde{f} - \tilde{v}_2 = \tilde{v} \cdot \tilde{n} & \text{on } \Gamma. \end{cases} \tag{6.13}$$

The jump conditions take new form in terms of $\tilde{h}, \tilde{v}, \tilde{f}, \tilde{B}$

$$\begin{cases} (\tilde{v}^+ - \tilde{v}^-) \cdot e_2 + (\dot{v}^+ - \dot{v}^-) \cdot \tilde{n} = -(\tilde{v}^+ - \tilde{v}^-) \cdot \tilde{n} & \text{on } \Gamma, \\ \tilde{B}^+ \cdot e_2 + \dot{B}^+ \cdot \tilde{n} + \tilde{B}^+ \cdot \tilde{n} = 0,\ \tilde{B}^- \cdot e_2 + \dot{B}^- \cdot \tilde{n} + \tilde{B}^- \cdot \tilde{n} = 0 & \text{on } \Gamma, \\ c^2(\dot{h}^+)\tilde{h}^+ + c^2(\dot{h}^+)\mathcal{R}' + \mathcal{R}(1 + \tilde{h}^+ + \mathcal{R}') + \frac{\gamma}{2\dot{\rho}}(\dot{B}^+ \tilde{B}^+ + \frac{|\tilde{B}^+|^2}{2}) & \\ = c^2(\dot{h}^-)\tilde{h}^- + c^2(\dot{h}^-)\mathcal{R}' + \mathcal{R}(1 + \tilde{h}^- + \mathcal{R}'^{,-}) + \frac{\gamma}{2\dot{\rho}}(\dot{B}^- \tilde{B}^- + \frac{|\tilde{B}^-|^2}{2}) & \text{on } \Gamma. \end{cases} \tag{6.14}$$

Proof of Theorem 2.2 Now we are ready to prove the main Theorem 1.3. We prove it by the method of contradiction. Suppose that the perturbed problem (6.13) has property $EE(k)$ for some $k \geq 3$. Let $\delta, t_0, C > 0$ be the constants provided by Definition 1.2. Fix $n \in \mathbb{N}$ so that $n > C$. Applying Lemma 4.2 with this $n, T_0 = t_0/2, k \geq 3$, and $\alpha = 2$, we can find f^L, h^L, v^L, B^L solving (3.5) so that

$$\|(f_0^L, h_0^L, v_0^L, B_0^L)\|_{H^k} \lesssim \frac{1}{n}, \tag{6.15}$$

but

$$\|(f^L(t), h^L(t), v^L(t), B^L(t))\|_{H^3} \geq 2 \quad \text{for } t \geq t_0/2. \tag{6.16}$$

We define $\tilde{f}_0^\varepsilon = f_0^\varepsilon - \dot{f} := \varepsilon f_0^L$, $\tilde{h}_0^\varepsilon = h_0^\varepsilon - \dot{h} := \varepsilon h_0^L$, $\tilde{v}_0^\varepsilon = v_0^\varepsilon - \dot{v} := \varepsilon v_0^L$, and $\tilde{B}_0^\varepsilon = B_0^\varepsilon - \dot{B} := \varepsilon B_0^L$. Then for $\varepsilon < \delta n$, we have $\|(\tilde{f}_0^\varepsilon, \tilde{h}_0^\varepsilon, \tilde{v}_0^\varepsilon, \tilde{B}_0^\varepsilon)\|_{H^k} < \delta$, so according to Definition 1.2, there exist $\left(\tilde{f}^\varepsilon := f^\varepsilon - \dot{f}, \tilde{h}^\varepsilon := h^\varepsilon - \dot{h}, \tilde{v}^\varepsilon := v^\varepsilon - \dot{v}, \tilde{B}^\varepsilon := B^\varepsilon - \dot{B}\right) \in L^\infty\left([0, t_0]\,; H^3(\Omega)\right)$ that solve (6.13)-(6.14) with $(\tilde{f}_0^\varepsilon, \tilde{h}_0^\varepsilon, \tilde{v}_0^\varepsilon, \tilde{B}_0^\varepsilon)$ as initial data and that satisfy the inequality

$$\begin{aligned} \sup_{0 \leq t \leq t_0} \left\| \left(\tilde{f}^\varepsilon, \tilde{h}^\varepsilon, \tilde{v}^\varepsilon, \tilde{B}^\varepsilon\right)(t) \right\|_{H^3} &\leq C\left(\left\| \left(f_0^\varepsilon, h_0^\varepsilon, v_0^\varepsilon, \tilde{B}_0^\varepsilon\right) \right\|_{H^k} \right) \\ &\leq C\varepsilon \frac{1}{n} < \varepsilon. \end{aligned} \tag{6.17}$$

Now define the rescaled functions $\bar{f}^\varepsilon = \tilde{f}^\varepsilon/\varepsilon, \bar{h}^\varepsilon = \tilde{h}^\varepsilon/\varepsilon, \bar{v}^\varepsilon = \tilde{v}^\varepsilon/\varepsilon, and \bar{B}^\varepsilon = \tilde{B}^\varepsilon/\varepsilon$; rescaling (6.17) then shows that

$$\sup_{0\le t\le t_0} \left\|(\bar{f}^\varepsilon, \bar{h}^\varepsilon, \bar{v}^\varepsilon, \bar{B}^\varepsilon)(t)\right\|_{H^3} < 1. \tag{6.18}$$

By construction, we know that $(\bar{f}_0^\varepsilon, \bar{h}_0^\varepsilon, \bar{v}_0^\varepsilon, \bar{B}_0^\varepsilon) = (f_0^L, h_0^L, v_0^L, B_0^L)$. We are going to show that the rescaled functions $(\bar{f}^\varepsilon, \bar{h}^\varepsilon, \bar{v}^\varepsilon, \bar{B}^\varepsilon)$ converge as $\varepsilon \to 0$ to the solutions (f^L, h^L, v^L, B^L) of the linearized equations (3.5) and the boundary conditions (3.6).

Now we are going to reformulate (6.13)-(6.14) in terms of rescaled functions $(\bar{f}^\varepsilon, \bar{h}^\varepsilon, \bar{v}^\varepsilon, \bar{B}^\varepsilon)$ and show some convergence results. The fourth equation in (6.13) can be rewritten in terms of rescaled function $(\bar{f}^\varepsilon, \bar{h}^\varepsilon, \bar{v}^\varepsilon, \bar{B}^\varepsilon)$ as follows:

$$\partial_t \bar{f}^\varepsilon + \dot{v}_1 \partial_1 \bar{f}^\varepsilon - \bar{v}_2^\varepsilon = \varepsilon \bar{v}^\varepsilon \cdot n^\varepsilon, \tag{6.19}$$

where the rescaled normal vector $n^\varepsilon = \frac{(-\varepsilon\partial_1 \bar{f}^\varepsilon, 0)}{\varepsilon} = (-\partial_1 \bar{f}^\varepsilon, 0)$ is well defined and uniformly bounded in $L^\infty\left([0, t_0]\,; H^2(\Gamma)\right)$ since

$$\|n^\varepsilon\|_{H^2(\Gamma)} \le \|\bar{f}^\varepsilon\|_{H^3(\Gamma)} < 1. \tag{6.20}$$

Hence, by (6.18) and (6.20), we obtain

$$\lim_{\varepsilon\to 0} \sup_{0\le t\le t_0} \left\|\partial_t \bar{f}^\varepsilon + \dot{v}_1 \partial_1 \bar{f}^\varepsilon - \bar{v}_2^\varepsilon\right\|_{H^2} = 0, \tag{6.21}$$

and

$$\sup_{0\le t\le t_0} \left\|\partial_t \bar{f}^\varepsilon(t)\right\|_{H^2} \le \dot{v}_1 \sup_{0\le t\le t_0} \left\|\partial_1 \bar{f}^\varepsilon(t)\right\|_{H^2} + \sup_{0\le t\le t_0} \left\|\bar{v}_2^\varepsilon\right\|_{H^2} \le C. \tag{6.22}$$

Expanding the first equation in (6.13) implies that

$$\partial_t \bar{h}^\varepsilon + (\dot{v}\cdot\nabla)\bar{h}^\varepsilon + \nabla\cdot\bar{v}^\varepsilon = -\varepsilon(M^\varepsilon\cdot\nabla)\bar{h}^\varepsilon + \varepsilon(K^\varepsilon)^T\nabla\cdot\bar{v}^\varepsilon, \tag{6.23}$$

where we define M^ε as follows:

$$M^\varepsilon = \bar{v}^\varepsilon - K^\varepsilon(\dot{v} + \varepsilon\bar{v}^\varepsilon) - (0, \frac{\partial_t \psi^\varepsilon}{1 + \varepsilon\partial_2\psi^\varepsilon}), \text{ where } \psi^\varepsilon = \theta \bar{f}^\varepsilon. \tag{6.24}$$

In order to estimate the bound of M^ε, we first estimate the bound of K^ε. We assume that ε is sufficiently small so that $\varepsilon < 1/(2C_1)$, where $C_1 > 0$ is the best constant in the inequality $\|UV\|_{H^2} \le C_1\|U\|_{H^2}\|V\|_{H^2}$ for 2×2 matrix-valued functions U, V. This assumption guarantees that $K^\varepsilon := (I - (I + \varepsilon\nabla\Psi^\varepsilon)^{-1})/\varepsilon$ is well defined and uniformly bounded in $L^\infty\left([0, t_0]\,; H^2(\Omega)\right)$ since

$$\begin{aligned}
\|K^\varepsilon\|_{H^2} &= \left\|\sum_{n=1}^{\infty}(-\varepsilon)^{n-1}(\nabla\Psi^\varepsilon)^n\right\|_{H^2} \le \sum_{n=1}^{\infty}\varepsilon^{n-1}\|(\nabla\Psi^\varepsilon)^n\|_{H^2} \\
&\le \sum_{n=1}^{\infty}(\varepsilon C_1)^{n-1}\|\nabla\Psi^\varepsilon\|_{H^2}^n \le \sum_{n=1}^{\infty}\frac{1}{2^{n-1}}\|\psi^\varepsilon\|_{H^3}^n \\
&\le \sum_{n=1}^{\infty}\frac{1}{2^{n-1}}\left\|\bar{f}^\varepsilon\right\|_{H^3}^n < \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = 2,
\end{aligned} \tag{6.25}$$

whereas we show that

$$\begin{aligned}\|M^\varepsilon\|_{H^2} &\le \|\bar{v}^\varepsilon\|_{H^2} + \dot{v}\|K^\varepsilon\|_{H^2}\| + \varepsilon C_1\|K^\varepsilon\|_{H^2}\|\bar{v}^\varepsilon\|_{H^2} + \|\partial_t\psi^\varepsilon\|_{H^2} \\ &\le \|\bar{v}^\varepsilon\|_{H^2} + \dot{v}\|K^\varepsilon\|_{H^2}\| + \varepsilon C_1\|K^\varepsilon\|_{H^2}\|\bar{v}^\varepsilon\|_{H^2} + \|\partial_t\bar{f}^\varepsilon\|_{H^2} \\ &\le C.\end{aligned} \tag{6.26}$$

Therefore,by employing (6.18), (6.25), and (6.26), we get by (6.23)

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le t_0}\left\|\partial_t\bar{h}^\varepsilon + (\dot{v}\cdot\nabla)\bar{h}^\varepsilon + \nabla\cdot\bar{v}^\varepsilon\right\|_{H^2} = 0, \tag{6.27}$$

and

$$\sup_{0\le t\le t_0}\left\|\partial_t\bar{h}^\varepsilon(t)\right\|_{H^2} < C. \tag{6.28}$$

Expanding the second equation in (6.13), we find that

$$\begin{aligned}&\dot{\varrho}(\partial_t\bar{v}^\varepsilon + (\dot{v}\cdot\nabla)\bar{v}^\varepsilon) + \dot{\varrho}c^2(\dot{h})\nabla\bar{h}^\varepsilon + \dot{B}\nabla\bar{B}^\varepsilon - (\dot{B}\cdot\nabla)\bar{B}^\varepsilon = -\dot{\varrho}\varepsilon(M^\varepsilon\cdot\nabla)\bar{v}^\varepsilon \\ &- \varepsilon\bar{\varrho}^\varepsilon(\partial_t\bar{v}^\varepsilon + ((\dot{v}+\varepsilon M^\varepsilon)\cdot\nabla)\bar{v}^\varepsilon) - \varepsilon\dot{\varrho}c^2(\dot{h})(K^\varepsilon)^T\nabla\bar{h}^\varepsilon - \varepsilon\dot{\varrho}\mathcal{R}^\varepsilon(\nabla\bar{h}^\varepsilon - \varepsilon(K^\varepsilon)^T\nabla\bar{h}^\varepsilon) \\ &- \varepsilon\bar{\varrho}^\varepsilon(c^2(\dot{h}) + \varepsilon\mathcal{R}^\varepsilon)(\nabla\bar{h}^\varepsilon - \varepsilon(K^\varepsilon)^T\nabla\bar{h}^\varepsilon) + \varepsilon\dot{B}(K^\varepsilon)^T\nabla\bar{B}^\varepsilon \\ &- \varepsilon\bar{B}^\varepsilon(\nabla\bar{B}^\varepsilon - \varepsilon(K^\varepsilon)^T\nabla\bar{B}^\varepsilon) + \varepsilon(N^\varepsilon\cdot\nabla)\bar{B}^\varepsilon,\end{aligned} \tag{6.29}$$

where we define N^ε as follows:

$$N^\varepsilon = \bar{B}^\varepsilon - K^\varepsilon(\dot{B} + \varepsilon\bar{B}^\varepsilon). \tag{6.30}$$

Making full use of (6.18) and (6.25), we show that

$$\begin{aligned}\|N^\varepsilon\|_{H^2} &\le \|\bar{B}^\varepsilon\|_{H^2} + \dot{B}\|K^\varepsilon\|_{H^2}\| + \varepsilon\|K^\varepsilon\|_{H^2}\|\bar{B}^\varepsilon\|_{H^2} \\ &\le C.\end{aligned} \tag{6.31}$$

We also define the normalized remainder function by

$$\mathcal{R}^\varepsilon(x,t) = \frac{(c^2)'(\bar{h} + (1-\alpha)\varepsilon\bar{h}^\varepsilon)\varepsilon\bar{h}^\varepsilon}{\varepsilon} = (c^2)'(\bar{h} + (1-\alpha)\varepsilon\bar{h}^\varepsilon)\bar{h}^\varepsilon. \tag{6.32}$$

It is easier to show that $\bar{h} + (1-\alpha)\varepsilon\bar{h}^\varepsilon$ is bounded above by a positive constant. We take use of (6.18), which implies

$$\sup_{0\le t\le t_0}\|\mathcal{R}^\varepsilon(x,t)\|_{H^3} \le C. \tag{6.33}$$

We also need to define $\bar{\varrho}^\varepsilon$ as follows:

$$\bar{\varrho}^\varepsilon = \frac{\tilde{\varrho}}{\varepsilon} = \frac{\dot{\varrho}(e^{\varepsilon\bar{h}^\varepsilon} - 1)}{\varepsilon} = \dot{\varrho}(\bar{h}^\varepsilon + \varepsilon\mathcal{R}'^{,\varepsilon}), \tag{6.34}$$

where $\mathcal{R}'^{,\varepsilon} := \frac{\mathcal{R}'}{\varepsilon}$ is well-defined and uniformly bounded in $L^\infty\left([0,t_0]\,;H^3(\Omega)\right)$ since

$$\|\mathcal{R}'^{,\varepsilon}\|_{H^2} = \|e^{\dot{h}+(1-\alpha)\varepsilon\bar{h}^\varepsilon}\bar{h}^\varepsilon\|_{H^3} \le C. \tag{6.35}$$

We take use of (6.34) and (6.18), which imply

$$\sup_{0\le t\le t_0} \|\bar{\varrho}^\varepsilon\|_{H^3} \le C. \tag{6.36}$$

Therefore, from (6.31), (6.33), (6.35) , (6.36), (6.24), and (6.25), we deduce by (6.29) that

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le t_0} \left\| \dot{\varrho}(\partial_t \bar{v}^\varepsilon + (\dot{v}\cdot\nabla)\bar{v}^\varepsilon) + \dot{\varrho}c^2(\dot{h})\nabla\bar{h}^\varepsilon + \dot{B}\nabla\bar{B}^\varepsilon - (\dot{B}\cdot\nabla)\bar{B}^\varepsilon \right\|_{H^2} = 0, \tag{6.37}$$

and

$$\sup_{0\le t\le t_0} \|\partial_t \bar{v}^\varepsilon(t)\|_{H^2} < C. \tag{6.38}$$

Finally, we expand the third equation in (6.13), and we find that

$$\begin{aligned} &\partial_t \bar{B}^\varepsilon + (\dot{v}\cdot\nabla)\bar{B}^\varepsilon + \dot{B}\nabla\cdot\bar{v}^\varepsilon - (\dot{B}\cdot\nabla)\bar{v}^\varepsilon = -\varepsilon(M^\varepsilon\cdot\nabla)\bar{B}^\varepsilon \\ &- \varepsilon\dot{B}(K^\varepsilon)^T\nabla\cdot\bar{v}^\varepsilon + \varepsilon\bar{B}^\varepsilon(\nabla\cdot\bar{v}^\varepsilon - \varepsilon(K^\varepsilon)^T\nabla\cdot\bar{v}^\varepsilon) + \varepsilon(N^\varepsilon\cdot\nabla)\bar{v}^\varepsilon. \end{aligned} \tag{6.39}$$

In accordance with (6.25), (6.26), and (6.31), we deduce that

$$\sup_{0\le t\le t_0} \left\| -\varepsilon(M^\varepsilon\cdot\nabla)\bar{B}^\varepsilon - \varepsilon\dot{B}(K^\varepsilon)^T\nabla\cdot\bar{v}^\varepsilon + \varepsilon\bar{B}^\varepsilon(\nabla\cdot\bar{v}^\varepsilon - \varepsilon(K^\varepsilon)^T\nabla\cdot\bar{v}^\varepsilon) + \varepsilon(N^\varepsilon\cdot\nabla)\bar{v}^\varepsilon \right\|_{H^3} \le C. \tag{6.40}$$

Therefore, from (6.39) and (6.40), we deduce that

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le t_0} \left\| \partial_t \bar{B}^\varepsilon + (\dot{v}\cdot\nabla)\bar{B}^\varepsilon + \dot{B}\nabla\cdot\bar{v}^\varepsilon - (\dot{B}\cdot\nabla)\bar{v}^\varepsilon \right\|_{H^2} = 0, \tag{6.41}$$

and

$$\sup_{0\le t\le t_0} \left\| \partial_t \bar{B}^\varepsilon(t) \right\|_{H^2} < C. \tag{6.42}$$

Next, we deal with some convergence results for the jump conditions. For the first equation in (6.14), we rewrite the normal vector n as follows:

$$n = e_2 + \tilde{n}^\varepsilon := e_2 + \varepsilon n^\varepsilon,\ n^\varepsilon = (-\partial_1 \bar{f}^\varepsilon, 0).$$

Noting that $\dot{v}^+\cdot e_2 = 0$ and $\dot{B}^+\cdot e_2 = 0$, we may rewrite the first equation and second equation in (6.14) as

$$(\dot{v}^+ + \varepsilon\bar{v}^{+,\varepsilon} - \dot{v}^- - \varepsilon\bar{v}^{-,\varepsilon})\cdot(e_2 + \varepsilon n^\varepsilon) = 0, \tag{6.43}$$

and

$$(\dot{B}^+ + \varepsilon\bar{B}^{+,\varepsilon})\cdot(e_2 + \varepsilon n^\varepsilon) = 0,\ (\dot{B}^- + \varepsilon\bar{B}^{-,\varepsilon})\cdot(e_2 + \varepsilon n^\varepsilon) = 0. \tag{6.44}$$

Since $\sup_{0\le t\le t_0} \|n^\varepsilon(t)\|_{L^\infty} \le \|n^\varepsilon\|_{H^3(\Gamma)} < 1$ is bounded uniformly, we find that

$$\sup_{0\le t\le t_0} \left\| e_2\cdot(\bar{v}^{+,\varepsilon}(t) - \bar{v}^{-,\varepsilon}(t) + (\dot{v}^+ - \dot{v}^-)\cdot n^\varepsilon) \right\|_{L^\infty} \to 0 \quad \text{as } \varepsilon\to 0, \tag{6.45}$$

$$\sup_{0\le t\le t_0} \|e_2\cdot\bar{B}^{+,\varepsilon} + \dot{B}^+\cdot n^\varepsilon\|_{L^\infty} \to 0 \quad \text{as } \varepsilon\to 0, \tag{6.46}$$

and

$$\sup_{0\le t\le t_0} \|e_2\cdot \bar{B}^{-,\varepsilon} + \dot{B}^- \cdot n^\varepsilon\|_{L^\infty} \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.47}$$

Therefore, we have

$$[\bar{v}^\varepsilon \cdot e_2] = 2\dot{v}_1^+ \partial_1 f^\varepsilon \text{ on } \Gamma, \tag{6.48}$$

and

$$\bar{B}^{+,\varepsilon}\cdot e_2 = \dot{B}_1^+\partial_1 f^\varepsilon,\ \ \bar{B}^{-,\varepsilon}\cdot e_2 = \dot{B}_1^-\partial_1 f^\varepsilon \text{ on } \Gamma. \tag{6.49}$$

We expand the third equation in (6.14) as follows:

$$\begin{aligned} &c^2(\dot{h}^+)\bar{h}^{+,\varepsilon} + c^2(\dot{h}^+)\mathcal{R}^{+,\prime,\varepsilon} + \mathcal{R}^\varepsilon(1+\varepsilon\bar{h}^{+,\varepsilon} + \varepsilon\mathcal{R}^{+,\prime,\varepsilon}) + \frac{\gamma}{2\dot{\rho}}(\dot{B}^+\bar{B}^{+,\varepsilon} + \varepsilon\frac{|\bar{B}^{+,\varepsilon}|^2}{2}) \\ &= c^2(\dot{h}^-)\bar{h}^{-,\varepsilon} + c^2(\dot{h}^-)\mathcal{R}^{-,\prime,\varepsilon} + \mathcal{R}^\varepsilon(1+\varepsilon\bar{h}^{-,\varepsilon} + \varepsilon\mathcal{R}^{-,\prime,\varepsilon}) + \frac{\gamma}{2\dot{\rho}}(\dot{B}^-\bar{B}^{-,\varepsilon} + \varepsilon\frac{|\bar{B}^{-,\varepsilon}|^2}{2}) \text{ on } \Gamma. \end{aligned} \tag{6.50}$$

Since $\dot{h}^+ = \dot{h}^-$, we may eliminate these two terms from equation (6.50) and divide both sides by ε to get

$$\bar{h}^{+,\varepsilon} = \bar{h}^{-,\varepsilon} \text{ on } \Gamma. \tag{6.51}$$

According to the bound (6.18) and sequential weak-* compactness, we have that, up to the extraction of a subsequence (which we still denote using only ε)

$$(\bar{f}^\varepsilon, \bar{h}^\varepsilon, \bar{v}^\varepsilon, \bar{B}^\varepsilon) \overset{*}{\rightharpoonup} (f^\star, h^\star, v^\star, B^\star) \quad \text{weakly } -* \text{ in } L^\infty\left([0,t_0]\,;H^3(\Omega)\right). \tag{6.52}$$

By lower semicontinuity, we know that

$$\sup_{0\le t\le t_0}\left\|(f^\star, h^\star, v^\star, B^\star)(t)\right\|_{H^3} \le 1. \tag{6.53}$$

According to (6.22), (6.28), and (6.38), we get

$$\limsup_{\varepsilon\to 0}\sup_{0\le t\le t_0}\left\|(\partial_t\bar{f}^\varepsilon, \partial_t\bar{h}^\varepsilon, \partial_t\bar{v}^\varepsilon, \partial_t\bar{B}^\varepsilon)(t)\right\|_{H^2} < \infty. \tag{6.54}$$

By the Lions-Aubin lemma in [39], we then have that the sequence $\{(f^\varepsilon, h^\varepsilon, v^\varepsilon, B^\varepsilon)\}$ is strongly precompact in the space $L^\infty\left([0,t_0]\,;H^{8/3}(\Omega)\right)$, so

$$(\bar{f}^\varepsilon, \bar{h}^\varepsilon, \bar{v}^\varepsilon, \bar{B}^\varepsilon) \to (f^\star, h^\star, v^\star, B^\star) \quad \text{strongly in } L^\infty\left([0,t_0]\,;H^{8/3}(\Omega)\right). \tag{6.55}$$

This strong convergence, together with (6.21), (6.28), and (6.38), implies that

$$(\partial_t\bar{f}^\varepsilon, \partial_t\bar{h}^\varepsilon, \partial_t\bar{v}^\varepsilon, \partial_t\bar{B}^\varepsilon) \to (\partial_t f^\star, \partial_t h^\star, \partial_t v^\star, \partial_t B^\star) \text{ strongly in } L^\infty\left([0,t_0]\,;H^{5/3}(\Omega)\right). \tag{6.56}$$

The indexes $\frac{8}{3}$ and $\frac{5}{3}$ are sufficiently large to give $L^\infty([0,t_0];L^\infty)$ convergence of $\{(f^\varepsilon, h^\varepsilon, v^\varepsilon, B^\varepsilon)\}$, thus we have

$$\begin{cases} \partial_t h^\star + (\dot{v}\cdot\nabla)h^\star + \nabla\cdot v^\star = 0, & \text{in } \Omega, \\ \dot{\varrho}(\partial_t v^\star + (\dot{v}\cdot\nabla)v^\star + c^2\nabla h^\star) + \dot{B}\nabla\bar{B}^\star = (\dot{B}\cdot\nabla)\bar{B}^\star, & \text{in } \Omega, \\ \partial_t B^\star + (\dot{v}\cdot\nabla)B^\star + \dot{B}\nabla\cdot v^\star = (\dot{B}\cdot\nabla)v^\star, & \text{in } \Omega, \\ \partial_t f^\star + \bar{v}_1\partial_1 f^\star - v_2^\star = 0, & \text{on } \Gamma, \end{cases} \tag{6.57}$$

and

$$(v^{+,\star}-v^{-,\star})\cdot e_2 = 2\dot{v}_1^{+}\partial_1 f^{\star},\quad \text{on}\ \Gamma.$$
$$B^{+,\star}\cdot e_2 = \dot{B}_1^{+}\partial_1 f^{\star},\ B^{-,\star}\cdot e_2 = \dot{B}_1^{-}\partial_1 f^{\star},\quad \text{on}\ \Gamma, \tag{6.58}$$
$$h^{+,\star} = h^{-,\star},\quad \text{on}\ \Gamma.$$

We also pass to the limit in the initial conditions $(\bar{f}_0^{\varepsilon}, \bar{h}_0^{\varepsilon}, \bar{v}_0^{\varepsilon}, \bar{B}_0^{\varepsilon}) = (f_0^L, h_0^L, v_0^L, B_0^L)$ to obtain

$$(f_0^{\star}, v_0^{\star}, h_0^{\star}, B_0^{\star}) = (f_0^L, h_0^L, v_0^L, B_0^L).$$

Now we can see that $(f^{\star}, v^{\star}, h^{\star}, B^{\star})(t)$ are solutions to (3.2) and boundary conditions (3.3) with the same initial data. In accordance with the uniqueness result in Lemma 4.1, we have

$$(f^{\star}, v^{\star}, h^{\star}, B^{\star})(t) = (f^L, v^L, h^L, B^L)(t). \tag{6.59}$$

Therefore, we combine inequalities (6.53) with (6.16) to get

$$2 = \alpha < \sup_{0\le t\le t_0} \left\|(f^{\star}, h^{\star}, v^{\star}, B^{\star})(t)\right\|_{H^3} \le 1, \tag{6.60}$$

which is a contradiction. Therefore, the proof of Theorem 1.3 is completed.

Acknowledgements

The work of Binqiang Xie is supported by NSF-China under grant number 11901207, 12326355, Guangzhou Science and Technology Project 2023A04J1315 and the National R&D Key Program of China (No.2021YFA1002900).

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

The first author conducted the research and contributed the ideas and framework of the article, while the second author was responsible for the calculations and the writing of the paper. All authors have read and approved the final version of the manuscript.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. Y. G. Wang, F. Yu, Stabilization effect of magnetic fields on two-dimensional compressible current-vortex sheets, *Arch. Ration. Mech. Anal.*, **208** (2013), 341–389. https://doi.org/10.1007/s00205-012-0601-9

2. S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability,* Oxford University Press, New York, 1961. https://doi.org/10.1063/1.3058072
3. A. Castro, D. Cordoba, C. Fefferman, F. Gancedo, M. Lopez-Fernandez, Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves, *Ann. of Math.*, **175** (2012), 909–948. https://doi.org/10.4007/annals.2012.175.2.9
4. Y. Guo, I. Tice, Compressible, inviscid Rayleigh–Taylor instability, *Indiana Univ. Math. J.*, **60** (2011), 677–712. https://doi.org/10.1512/iumj.2011.60.4193
5. Y. Guo, I. Tice, Linear Rayleigh–Taylor instability for viscous, compressible fluids, *SIAM J. Math. Anal.*, **42** (2011), 1688–1720. https://doi.org/10.1137/090777438
6. H. J. Hwang, Y. Guo, On the dynamical Rayleigh–Taylor instability, *Arch. Ration. Mech. Anal.*, **167** (2003), 235–253. https://doi.org/10.1007/s00205-003-0243-z
7. F. Jiang, S. Jiang, W. Zhan, Instability of the abstract rayleigh-taylor problem and applications, *Math. Models Methods Appl. Sci.*, **30** (2020), 2299–2388. https://doi.org/10.1142/S021820252050044X
8. D. G. Ebin, Ill-posedness of the Rayleigh-Taylor and Helmholtz problems for incompressible fluids, *Comm. Partial Differential Equations*, **13** (1988), 1265–1295. https://doi.org/10.1080/03605308808820576
9. H. L. F. Helmholtz, On the discontinuous movements of fluids, *Sitz.ber. Preuss. Akad.Wiss. Berl. Philos.-Hist. Kl.*, **23** (1868), 215–228.
10. W. Thomson, Hydrokinetic solutions and observations, *Philos. Mag.*, **42** (1871), 362–377.
11. H. Hasegawa, M. Fujimoto, T. D. Phan, H. Reme, A. Balogh, M. W. Dunlop, et al., Transport of solar wind into earth's magnetosphere through rolled-up kelvin–helmholtz vortices, *Nature*, **430** (2004), 755–758. https://doi.org/10.1038/nature02799
12. J. W. Dungey, Electrodynamics of the outer atmosphere, In: *Proceedings of the Ionosphere,* The Physical Society of London, London, 1955, 255–265.
13. E. N. Parker, Dynamics of the interplanetary gas and magnetic fields, *Astrophys. J.*, **128** (1958), 664–678. https://doi.org/10.1086/146579
14. P. A. Sturrock, R. E. Hartle, Two-fluid model of the solar wind, *Phys. Rev. Lett.*, **16** (1966), 628–636. https://doi.org/10.1103/physrevlett.16.628
15. A. I. Ershkovich, Solar wind interaction with the tail of comet kohoutek, *Planet. Space Sci.*, **24** (1976), 287–290. https://doi.org/10.1016/0032-0633(76)90025-8
16. J. F. Coulombel, P. Secchi, The stability of compressible vortex sheets in two space dimensions, *Indiana Univ. Math. J.*, **53** (2004), 941–1012.
17. J. F. Coulombel, P. Secchi, Nonlinear compressible vortex sheets in two space dimensions, *Ann. Sci. cole Norm. Supr.*, **41** (2008), 85–139. https://doi.org/10.24033/asens.2064
18. A. Morando, P. Trebeschi, Two-dimensional vortex sheets for the nonisentropic euler equations: linear stability, *J. Hyperbolic Differ. Equ.*, **5** (2008), 487–518. https://doi.org/10.1142/S021989160800157X
19. A. Morando, P. Trebeschi, T. Wang, Two-dimensional vortex sheets for the nonisentropic euler equations: nonlinear stability, *J. Differential Equations*, **266** (2019), 5397–5430. https://doi.org/10.1016/j.jde.2018.10.029

20. R. M. Chen, J. Hu, D. Wang, Linear stability of compressible vortex sheets in 2d elastodynamics: variable coefficients, *Math. Ann.*, **376** (2020), 863–912. https://doi.org/10.1007/s00208-018-01798-w
21. R. M. Chen, J. Hu, D. Wang, Linear stability of compressible vortex sheets in two-dimensional elastodynamics, *Adv. Math.*, **311** (2017), 18–60. https://doi.org/10.1016/j.aim.2017.02.014
22. R. M. Chen, F. M. Huang, D. Wang, D. F. Yuan, Stabilization effect of elasticity on three-dimensional compressible vortex sheets, *J. Math. Pures Appl.*, **172** (2023), 105–138. https://doi.org/10.1016/j.matpur.2023.01.005
23. B. Q. Xie, B. Zhao, D. W. Huang, Ill-posedness of the Kelvin-Helmholtz problem for incompressible MHD fluids, Preprint.
24. J. W. Miles, On the disturbed motion of a plane vortex sheet, *J. Fluid Mech.*, **4** (1958), 538–552. https://doi.org/10.1017/S0022112058000653
25. J. A. Fejer, J. W. Miles, On the stability of a plane vortex sheet with respect to three-dimensional disturbances, *J. Fluid Mech.*, **15** (1963), 335–336. https://doi.org/10.1017/S002211206300029X
26. B. Q. Xie, B. Zhao, Kelvin-Helmholtz instability for compressible Euler fluids, *Math. Methods Appl. Sci.*, accepted. https://doi.org/10.1002/mma.70669
27. B. Q. Xie, B. L. Guo, B. Zhao, Effect of weak elasticity on the Kelvin-Helmholtz instability, *J.Math. Fluid Mech.*, **28** (2026), 20. https://doi.org/10.1007/s00021-025-00983-7
28. Y. Z. Sun, W. Wang, Z. F. Zhang, Nonlinear stability of the current-vortex sheet to the incompressible mhd equations, *Commun. Pure Appl. Math.*, **71** (2018), 356–403. https://doi.org/10.1002/cpa.21710
29. B. Q. Xie, B. L. Guo, B. Zhao, Ill-posedness of incompressible Kelvin-Helmholtz problem with transverse magnetic field, *J. Math. Phys.*, **66** (2025), 121506. https://doi.org/10.1063/5.0268159
30. Y. Trakhinin, Existence of compressible current-vortex sheets: variable coefficients linear analysis, *Arch. Ration. Mech. Anal.*, **177** (2005), 331–366. https://doi.org/10.1007/s00205-005-0364-7
31. Y. Trakhinin, The existence of current-vortex sheets in ideal compressible magnetohydrodynamics, *Arch. Ration. Mech. Anal.*, **191** (2009), 245–310. https://doi.org/10.1007/s00205-008-0124-6
32. G. Q. Chen, Y. G. Wang, Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics, *Arch. Ration. Mech. Anal.*, **187** (2008), 369–408. https://doi.org/10.1007/s00205-007-0070-8
33. B. Q. Xie, B. Zhao, Effect of transverse magnetic field on the compressible Kelvin-Helmholtz problem, *J. Geom. Anal.*, **36** (2026), 7. https://doi.org/10.1007/s12220-025-02247-2
34. A. Morando, P. Secchi, P. Trebeschi, On the evolution equation of compressible vortex sheets, *Math. Nachr.*, **293** (2020), 945–969. https://doi.org/10.1002/mana.201800162
35. Z. Y. Pu, M. G. Kivelson, Kelvin-helmholtz instability at the magnetopause solution for compressible plasmas, *J. Geophysical Res.*, **88** (1983), 841–853. https://doi.org/10.1029/ja088ia02p00841
36. S. J. Wu, Mathematical analysis of vortex sheets, *Comm. Pure. Appl. Math*, **59** (2006), 1065–1206. https://doi.org/10.1002/cpa.20110
37. G. Lebeau, Régularité du problème de Kelvin-Helmholtz pour l'équation d'euler 2d, *ESAIM: Control, Optimisation and Calculus of Variations*, **8** (2002), 801–825. https://doi.org/10.1051/cocv:2002052

38. Y. J. Wang, Z. P. Xin, Existence of multi-dimensional contact discontinuities for the ideal compressible magnetohydrodynamics, *Comm. Pure Appl. Math.*, **77** (2024), 583–629. https://doi.org/10.1002/cpa.22148

39. J. Simon, Comact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.*, **146** (1987), 65–96.

AIMS Press