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Research article

Approximate controllability of Hilfer fractional integro-differential neutral dynamical system with infinite delay

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Abstract: This paper investigates the approximate controllability of Hilfer integro-differential neutral dynamical systems with infinite delay governed by almost sectorial operators. The mild solution of the proposed system is derived using the Laplace transform technique. Subsequently, the approximate controllability of the system is established by the Bohnenblust–Karlin - fixed point theorem. Furthermore, the controllability of the associated neutral system is analyzed in detail. Finally, a numerical example is presented to demonstrate the validity and applicability of the theoretical results and our system explained via a filter system.

Keywords: approximate controllability; Hilfer fractional derivative; almost sectorial operators; multivalued maps
Mathematics Subject Classification: 26A33, 34A08, 47D09

1. Introduction

Fractional calculus has gained significant attention in recent years due to its ability to more accurately model real-world processes that exhibit memory and hereditary characteristics. Unlike classical integer-order differential equations, fractional calculus offers a more flexible and nuanced framework for describing complex dynamics observed in various systems and materials. Its applications span a wide range of disciplines, including biochemistry, fluid dynamics, biomathematics, electrical circuits, control theory, viscoelasticity, and electrochemistry. By incorporating fractional derivatives and integrals into mathematical models, researchers can better capture the nonlocal and non-Markovian behaviors that are intrinsic to many natural and engineered systems.

Numerous studies have explored the implications and applications of fractional systems. Some notable references include the works of [1–5]; and others cited in the references. These studies delve into specific applications of fractional calculus, elucidating its significance in diverse fields. Research paper works [6–9], contribute to the ongoing investigation into the existence and properties of solutions for fractional differential equations and inclusions. These studies play a crucial role in advancing our understanding of fractional calculus and its practical implications across various disciplines. Fractional calculus offers a powerful framework for modeling intricate systems with memory and hereditary properties, with applications ranging from fluid dynamics to control theory. The ongoing research in this field continues to uncover new insights and applications, driving innovation in science and technology.

In control theory, controllability is a basic and broadly applicable term that is essential to the study and design of control systems. It describes the capacity to use appropriate control inputs to guide a system from any beginning condition to any desired final state within a certain amount of time. This characteristic holds true for both finite-dimensional and infinite-dimensional systems and is crucial in many different fields. Research on controllability is still quite active, with continuous efforts to expand controllability principles to new application and solve difficult control issues. Researchers aim to solve difficult control challenges and enhance the performance of engineered systems in a variety of domains by expanding our knowledge of controllability and creating novel control techniques. In [10] the authors investigated the approximate controllability of fractional differential systems for the order $r \in (1,2)$. In [11], the authors studied the controllability of an integro-differential system. In [12], the authors completed their study on the existence theory with the help of the Bohnenblust–Karlin's fixed-point theorem. In [13], studied the Hyers–Ulam stability, exponential stability, and relative controllability of the fractional differential equations.

Hilfer [14] presented another form of fractional derivatives containing the Riemann–Liouville derivative as well as the Caputo fractional derivative. Nowadays, Hilfer fractional differential equations have an important role in research. Recently, researchers are used some notions such as the nondense domain, almost sectorial operator, numerous fixed-point and a measure of noncompactness to study the Hilfer fractional dynamical systems with different delays. Many authors have recently expressed a strong interest in this field, which has prompted on the work in [15, 16]. In [17, 18], the researchers turned to a Hilfer fractional dynamical system based on the conclusion of the Schauder's fixed point theorem along with the almost sectorial operators. Later, in [19–21], the authors completed their studies on existence and controllability results in the Hilfer fractional differential equations by using the almost sectorial operators along with the fixed point theorem method.

H. A. Ahmed [22–24] investigated the controllability of impulsive neutral stochastic differential equations with fractional Brownian motion; focused the nolocal controllability of Sobolev-type fractional and the approximate controllability of Atangana–Baleanu fractional with Poisson jumps. In [25], studied the existence of the solution; graphical and numerical approach of Hilfer fractional derivatives. Recently, [26] studied controllability of the impulsive Hilfer fractional integro-differential equation of order $1 > r < 2$ and numerical study of the system. In [27, 28], the authors focused mathematical modeling and optimal control strategies of fractional diffential equations. In [29], studied the exponential behavior and optimal controllability of fractional stochastic differential equations. In [30] focused approximate controllability of the Hilfer fractional stochastic differential system with nonlocal conditions

To state the main purpose of our study, it is that no work has been published on the existence and approximate controllability Hilfer-type neutral integro-differential dynamic systems with nonlocal conditions and infinite delay based on the almost sectorial operators. Our essential goals in the article are to examine the approximate controllability aforementioned dynamic Hilfer-type neutral system almost sectorial operators. A new neutral dynamic system of Integro-differential inclusions of the Hilfer type, given by

$$D_{0^+}^{\eta,\zeta}\mathfrak{y}(\mathtt{z}) \in \mathtt{A}\mathfrak{y}(\mathtt{z}) + F\Big(\mathtt{z}, \mathfrak{y}_{\mathtt{z}}, \int_0^{\mathtt{z}} e(\mathtt{z}, s, \mathfrak{y}_s)ds\Big) + \mathtt{B}v(\mathtt{z}), \qquad \mathtt{z} \in \mathcal{I}' = (0, b], \tag{1.1}$$

$$I_{0^+}^{(1-\eta)(1-\zeta)}\mathfrak{y}(0) = \mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \dots, u_{\mathtt{z}_n}) \in L^2(D, \mathfrak{D}_h), \ \mathtt{z} \in (-\infty, 0] \tag{1.2}$$

where $D_{0^+}^{\eta,\zeta}$ is the Hilfer fractional derivative of order $\eta,\ 0 < \eta < 1$ and type $\zeta,\ 0 \le \zeta \le 1, \{\mathtt{T}(\mathtt{z}),\ \mathtt{z} \ge 0\}$ is the analytic semigroup on Y, generated from the almost sectorial operator and denoted by $\mathtt{A}$. Here $\mathfrak{y}(\cdot), v(\cdot)$ are the state and control function of the systems respectively. Take $\mathtt{B}$ as a linear bounded operator from the Banach space U into the Banach space Y. $\mathcal{I} = [0, b],\ F : \mathcal{I} \times \mathfrak{D}_h \times Y \to 2^Y \backslash \{0\}$ is a nonempty, bounded, closed convex multivalued map, and also, $e : \mathcal{I} \times \mathcal{I} \times \mathfrak{D}_h \to Y$ and $0 < \mathtt{z}_1 < \mathtt{z}_2 < \cdots < \mathtt{z}_n \le b$, $\xi : \mathfrak{D}_h^n \to \mathfrak{D}_h$ are two appropriate functions so that $\mathfrak{D}_h$ introduces a phase space. The $\mathfrak{y}_{\mathtt{z}} : (-\infty, 0] \to Y$ so that $\mathfrak{y}_{\mathtt{z}}(s) = \mathfrak{y}(\mathtt{z} + s)$, is to the phase space $\mathfrak{D}_h$.

More precisely, the contribution of this study is that we prove that the Hilfer fractional differential inclusions of the kinds (1.1) – (1.2) and (5.1) – (5.2) become approximately controllable under the basic and fundamental operators; in particular, the equivalent linear system is approximately controllable. In Section 2, fractional calculus, semigroup, multivalued mappings, and almost sectorial operators are all discussed. In Section 3, we can find the mild solution of the system. The system's approximate controllability was first established in Section 4 of the article. We continue our research in Section 5 to extend the results to approximately controllable neutral system with nonlocal condition. In order to clarify our key points, we give an example in Section 6 and some conclusions.

2. Preliminaries

Take $\mathcal{X} = \{\mathfrak{y} \in C(\mathcal{I}', Y) : \lim_{\mathtt{z}\to 0} \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\mathfrak{y}(\mathtt{z}) \text{ is finite}\}$. It will be a Banach space with $\|\cdot\|_{\mathcal{X}}$ so that $\|\mathfrak{y}\|_{\mathcal{X}} = \sup_{\mathtt{z}\in\mathcal{I}'}\{\mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\|\mathfrak{y}(\mathtt{z})\|\}$.

Definition 2.1. *[5] The fractional integral of order η for the function $F : [b, \infty) \to \mathbb{R}$, the lower limit b is*

$$I_{b^+}^{\eta}F(\mathtt{z}) = \frac{1}{\Gamma(\eta)}\int_b^{\mathtt{z}} \frac{F(\varsigma)}{(\mathtt{z} - \varsigma)^{1-\eta}}d\varsigma \quad \mathtt{z} > 0;\ \eta \in \mathbb{R}^+.$$

Definition 2.2. *[14] The Hilfer fractional derivative of order $0 < \eta < 1$ and type $\zeta \in [0, 1]$ for the function $F : [b, +\infty) \to \mathbb{R}$; is*

$$D_{b^+}^{\eta,\zeta}F(\mathtt{z}) = [I_{b^+}^{(1-\eta)\zeta}D(I_{b^+}^{(1-\eta)(1-\zeta)}F)](\mathtt{z}).$$

From [31–34], define the abstract phase space $\mathfrak{D}_w$. Let $w : (-\infty, 0] \to \mathbb{R}$ be continuous along with the condition $l = \int_{-\infty}^0 w(\mathtt{z})d\mathtt{z} < +\infty$. Now, for each $n > 0$, define

$$\mathfrak{D} = \Big\{\delta : [-n, 0] \to Y \text{ such that } \delta \text{ is bounded and measurable}\Big\},$$

and define

$$\|\delta\|_{[-n,0]} = \sup_{\tau\in[-n,0]} \|\delta(\tau)\|, \text{ for all } \delta \in \mathfrak{D}.$$

Now, one can define

$$\mathfrak{D}_w = \left\{\delta : (-\infty, 0] \to Y, \text{ such that } \forall\, n > 0,\ \delta|_{[-n,0]} \in \mathfrak{D} \text{ and } \int_{-\infty}^{0} w(\tau)\|\delta\|_{[\tau,0]} d\tau < +\infty\right\}.$$

If $\mathfrak{D}_w$

$$\|\delta\|_{\mathfrak{D}_w} = \int_{-\infty}^{0} w(\tau)\|\delta\|_{[\tau,0]} d\tau, \forall\, \delta \in \mathfrak{D}_w,$$

then; $(\mathfrak{D}_w, \|\cdot\|)$ is a Banach space.

Now, we define the space

$$\mathfrak{D}'_w = \left\{\mathfrak{y} : (-\infty, b] \to Y \text{ such that } \mathfrak{y}|_{\mathcal{I}} \in \mathfrak{C},\ \mathfrak{y}_0 = \xi \in \mathfrak{D}_w\right\}.$$

We define the seminorm $\|\cdot\|_b$ in $\mathfrak{D}'_w$ as

$$\|\mathfrak{y}\|_b = \|\mathfrak{y}_0\|_{\mathfrak{D}_w} + \sup\{\|\mathfrak{y}(\tau)\| : \tau \in [0, b]\},\ \mathfrak{y} \in \mathfrak{D}'_w.$$

Lemma 2.3. *[31] If $\mathfrak{y} \in \mathfrak{D}'_w$, then for $\mathtt{z} \in \mathcal{I}$, $\mathfrak{y}_\mathtt{z} \in \mathfrak{D}_w$. Moreover,*

$$l|\mathfrak{y}(\mathtt{z})| \le \|\mathfrak{y}_\mathtt{z}\|_{\mathfrak{D}_w} \le \|\mathfrak{y}_0\|_{\mathfrak{D}_w} + l \sup_{r\in[0,\mathtt{z}]} |\mathfrak{y}(r)|,$$

where $l = \int_{-\infty}^{0} w(\mathtt{z})d\mathtt{z} < \infty$.

Let $\mathtt{A}$ be a linear operator as $Y \to Y$; sets $D(\mathtt{A})$ and $\sigma(\mathtt{A})$ are the domain and sprectrum of $\mathtt{A}$, respectively, so that the resolvent of $\mathtt{A}$ is given by $\rho(\mathtt{A}) = \mathbb{C} - \sigma(\mathtt{A})$. Moreover, the open sector is defined by

$$S^0_\delta = \{\theta \in \mathbb{C}\backslash\{0\} : |\arg\theta| < \delta\},$$

for $0 < \delta < \pi$ and its closure is given by

$$S_\delta = \{\theta \in \mathbb{C}\backslash\{0\} : |\arg\theta| \le \delta\} \cup \{0\}.$$

Definition 2.4. *[35] Let $0 < \vartheta < 1$; $0 < \varphi < \frac{\pi}{2}$. We define $\Theta^{-\vartheta}_\varphi$ the family of all closed linear operators, $S_\varphi = \{\theta \in \mathbb{C}\backslash\{0\}$ with $|arg\ \theta| \le \varphi\}$ the sector and $\mathtt{A} : D(\mathtt{A}) \subset Y \to Y$ to be such that*

(i) $\sigma(\mathtt{A}) \subseteq S_\varphi$;

(ii) $\mathbb{M}_\delta$ *(is a constant) such that*

$$\left\|(\theta I - \mathtt{A})^{-1}\right\| \le \mathbb{M}_\delta|\mathtt{z}|^{-\vartheta}, \textit{ for every } \quad \varphi < \delta < \pi,\ \theta \in \mathbb{C}\backslash S_\delta.$$

Then $\mathtt{A} \in \Theta^{-\vartheta}_\varphi$ is called an almost sectorial operator on Y.

Let $\{T(\mathtt{z})\}_{\mathtt{z}\geq 0}$ be the semigroup related to $\mathtt{A}$, given by

$$T(\mathtt{z}) = e^{\mathtt{z}\theta}(\mathtt{A}) = \frac{1}{2\pi i}\int_{\Gamma_\mu} e^{-\mathtt{z}\theta}R(\theta;\mathtt{A})d\theta,\ \mathtt{z}\in S_{\frac{\pi}{2}-\varphi},$$

where the contour integral

$$\Gamma_\mu = \{\mathbb{R}^+ e^{i\mu}\} \cup \{\mathbb{R}^+ e^{-i\mu}\},$$

is oriented counter-clockwise such that $\varphi < \mu < \frac{\pi}{2} - |\arg \mathtt{z}|$, be an analytic semigroup of the growth order $1-\vartheta$.

Consider the operator families $\{\mathcal{S}_\eta(\mathtt{z})\}_{\mathtt{z}\in S_{\frac{\pi}{2}-\varphi}}$, $\{\mathtt{Q}_\eta(\mathtt{z})\}_{\mathtt{z}\in S_{\frac{\pi}{2}-\varphi}}$ defined as follows:

$$\mathcal{S}_\eta(\mathtt{z}) = \int_0^\infty W_\eta(\xi)T(\mathtt{z}^\eta\xi)d\xi,$$

$$\mathtt{Q}_\eta(\mathtt{z}) = \int_0^\infty \eta\xi W_\eta(\xi)T(\mathtt{z}^\eta\xi)d\xi.$$

The Wright-type function $W_\eta(\beta)$ is formulated as,

$$W_\eta(\beta) = \sum_{k\in\mathbb{N}} \frac{(-\beta)^{k-1}}{\Gamma(1-\eta k)(k-1)!},\ \beta\in\mathbb{C}. \tag{2.1}$$

Let $-1 < \iota < \infty,\ p > 0$. Then

(1) $W_\eta(\beta) \geq 0, \quad \beta > 0;$

(2) $\displaystyle\int_0^\infty \beta^\iota W_\eta(\beta)d\beta = \frac{\Gamma(1+\iota)}{\Gamma(1+\eta\iota)},$ for $\beta \geq 0;$

(3) $\displaystyle\int_0^\infty \frac{\eta}{\beta^{(\eta+1)}}e^{-p\beta}W_\eta(\frac{1}{\beta^\eta})d\beta = e^{-p^\eta}.$

Definition 2.5. *[36] Let F be the multivalued map to be an upper semi-continuous on Y if, for each $\mathfrak{y}_0 \in Y$, the set $F(\mathfrak{y}_0)$ is a nonempty; closed subset of Y, and if for each open set $\mathcal{U}$ of Y containing $F(\mathfrak{y}_0)$, there exists an open neighborhood $\mathcal{V}$ of $\mathfrak{y}_0$, such that $F(\mathcal{V}) \subseteq \mathcal{U}$.*

Definition 2.6. *[36] F is called completely continuous if $F(C)$ is relatively compact for each bounded subset C of Y. If a multivalued map F is completely continuous with nonempty compact values, then F is upper semi-continuous if and only if F has a closed graph i.e., $\mathfrak{y}_m \to \mathfrak{y}_0$, $z_m \to z_0$, $z_m \in F(\mathfrak{y}_m)$ imply $z_0 \in F(\mathfrak{y}_0)$.*

For more details about multivalued maps, refer to [37].

3. Mild solution

The structure of the mild solution is discussed in this section.

Lemma 3.1. *The Hilfer fractional differential system* (1.1) – (1.2) *is equivalent to the integral equation*

$$\mathfrak{y}(z) = \frac{\mathfrak{y}(0) + \xi(\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_n)}{\Gamma(\zeta(1-\eta)+\eta)}\mathtt{z}^{(1-\eta)(\zeta-1)} + \frac{1}{\Gamma(\eta)}\int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}$$
$$\times\Big[\mathtt{A}\mathfrak{y}_\varsigma + F\Big(\varsigma, \mathfrak{y}_\varsigma, \int_0^\varsigma e(\mathtt{z}, s, \mathfrak{y}_s)ds\Big) + \mathtt{B}v(\varsigma)\Big]d\varsigma. \tag{3.1}$$

Proof. First, we apply $I_{0^+}^{\eta}$ on (1.1) as,

$$I_{0^+}^{\eta}(D_{0^+}^{\eta,\zeta}\mathfrak{y}(\mathtt{z})) = I_{0^+}^{\eta}(\mathtt{A}\mathfrak{y}(\mathtt{z}) + F\Big(\mathtt{z},\mathfrak{y}_{\mathtt{z}},\int_0^{\mathtt{z}} e(\mathtt{z},s,\mathfrak{y}_s)ds\Big) + \mathtt{B}\nu(\mathtt{z})).$$

Using properties of the fractional integral equation [5, 38], we get

$$\begin{aligned}\mathfrak{y}(\mathtt{z}) &= \frac{I_{0^+}^{(1-\eta)(1-\zeta)}\mathfrak{y}(0)}{\Gamma(\eta+\zeta(1-\eta))} + I_{0^+}^{\eta}(\mathtt{A}\mathfrak{y}(\mathtt{z}) + F\Big(\mathtt{z},\mathfrak{y}_{\mathtt{z}},\int_0^{\mathtt{z}} e(\mathtt{z},s,\mathfrak{y}_s)ds\Big) + \mathtt{B}\nu(\mathtt{z}))\\ &= \frac{\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)}{\Gamma(\zeta(1-\eta)+\eta)}\mathtt{z}^{(1-\eta)(\zeta-1)} + \frac{1}{\Gamma(\eta)}\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\\ &\quad\times\Big[\mathtt{A}\mathfrak{y}_{\varsigma} + F\Big(\varsigma,\mathfrak{y}_{\varsigma},\int_0^{\varsigma} e(\mathtt{z},s,\mathfrak{y}_s)ds\Big) + \mathtt{B}\nu(\varsigma)\Big]d\varsigma.\end{aligned}$$

This completes the proof. □

Lemma 3.2. *The integral equation* (3.1) *is satisfied*

$$\begin{aligned}\mathfrak{y}(\mathtt{z}) =&\mathcal{S}_{\eta,\zeta}(\mathtt{z})[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)] + \int_0^{\mathtt{z}} \mathtt{K}_{\eta}(\mathtt{z}-\varsigma)\underline{\nu}(\varsigma)d\varsigma \\ &+ \int_0^{\mathtt{z}} \mathtt{K}_{\eta}(\mathtt{z}-\varsigma)F\Big(\varsigma,\mathfrak{y}(\varsigma),\int_0^{\varsigma} e(\mathtt{z},s,\mathfrak{y}_s)ds\Big)d\varsigma, \qquad \mathtt{z}\in\mathcal{I},\end{aligned}\tag{3.2}$$

where $\mathcal{S}_{\eta,\zeta}(\mathtt{z}) = I_0^{\zeta(1-\eta)}\mathtt{K}_{\eta}(\mathtt{z}),\ \mathtt{K}_{\eta}(\mathtt{z}) = \mathtt{z}^{\eta-1}\mathtt{Q}_{\eta}(\mathtt{z}).$

Proof. Let $\lambda > 0$, we apply the Laplace transform on (3.1) and take the following

$$\begin{aligned}&\mathtt{G}_1(\lambda) = \int_0^{\infty} e^{-\lambda\varsigma}\mathfrak{y}(\varsigma)d\varsigma,\ \mathtt{G}_2(\lambda) = \int_0^{\infty} e^{-\lambda\varsigma}F\Big(\varsigma,\mathfrak{y}(\varsigma),\int_0^{\varsigma} e(\mathtt{z},s,\mathfrak{y}_s)ds\Big)d\varsigma\\ &\mathtt{G}_3(\lambda) = \int_0^{\infty} e^{-\lambda\varsigma}\nu(\varsigma)d\varsigma.\end{aligned}$$

We can write

$$\begin{aligned}&\mathtt{G}_1(\lambda) = \lambda^{(1-\eta)(1-\zeta)-1}[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)] + \frac{1}{\lambda^{\eta}}\mathtt{A}\mathtt{G}_1(\lambda) + \frac{1}{\lambda^{\eta}}\mathtt{G}_2(\lambda) + \frac{1}{\lambda^{\eta}}\mathtt{G}_3(\lambda)\\ &\mathtt{G}_1(\lambda) = \lambda^{\zeta(\eta-1)}(\lambda^{\eta}I-\mathtt{A})^{-1}[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)] + (\lambda^{\eta}I-\mathtt{A})^{-1}\mathtt{G}_2(\lambda) + (\lambda^{\eta}I-\mathtt{A})^{-1}\mathtt{B}\mathtt{G}_3(\lambda)\end{aligned}$$

$$\begin{aligned}\mathtt{G}_1(\lambda) = \lambda^{\zeta(1-\eta)}&\int_0^{\infty} e^{-\lambda^{\eta}\varsigma}\mathtt{T}(\varsigma)d\varsigma[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)] + \int_0^{\infty} e^{-\lambda^{\eta}\varsigma}\mathtt{T}(\varsigma)d\varsigma\mathtt{G}_2(\lambda)\\ &+ \int_0^{\infty} e^{-\lambda^{\eta}\varsigma}\mathtt{T}(\varsigma)d\varsigma\mathtt{B}\mathtt{G}_3(\lambda).\end{aligned}\tag{3.3}$$

Let $\rho_{\eta}(\theta) = \frac{\eta}{\theta^{\eta+1}}W_{\eta}(\theta^{-\eta})$, where $W_{\eta}(\theta)$ is the Wright function and the Laplace transform of the function ρ_{η} is written as:

$$\int_0^{\infty} e^{-\lambda\theta}\rho_{\eta}(\theta)d\theta = e^{\lambda^{\eta}},\ \text{where } 0<\eta<1.\tag{3.4}$$

From Eq. (3.4), we obtain

$$\int_0^\infty e^{-\lambda^\eta \varsigma}\mathtt{T}(\varsigma)[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)]d\varsigma = \int_0^\infty \eta \mathtt{t}^{\eta-1}e^{-(\lambda \mathtt{t})^\eta}\mathtt{T}(\mathtt{t}^\eta)[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)]d\mathtt{t}$$
$$= \int_0^\infty\int_0^\infty \eta\rho_\eta(\theta)e^{-(\lambda \mathtt{t}\theta)}\mathtt{T}(\mathtt{t}^\eta)\mathtt{t}^{\eta-1}[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)]d\theta d\mathtt{t}$$
$$= \int_0^\infty\int_0^\infty \eta\rho_\eta(\theta)e^{-\lambda \mathtt{t}}\mathtt{T}\Big(\frac{\mathtt{t}^\eta}{\theta^\eta}\Big)\frac{\mathtt{t}^{\eta-1}}{\theta^\eta}[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)]d\theta d\mathtt{t}$$
$$= \int_0^\infty e^{-\lambda \mathtt{t}}\Big(\eta\int_0^\infty \rho_\eta(\theta)\mathtt{T}\Big(\frac{\mathtt{t}^\eta}{\theta^\eta}\Big)\frac{\mathtt{t}^{\eta-1}}{\theta^\eta}[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)]d\theta\Big)d\mathtt{t}$$
$$= \int_0^\infty e^{-\lambda \mathtt{t}}\mathtt{t}^{\eta-1}\mathtt{Q}_\eta(\mathtt{t})[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)]d\mathtt{t}. \tag{3.5}$$

$$\int_0^\infty e^{-\lambda^\eta\varsigma}\mathtt{T}(\varsigma)d\varsigma\mathtt{G}_2(\lambda) = \int_0^\infty\int_0^\infty \eta\mathtt{t}^{\eta-1}e^{-(\lambda \mathtt{t})^\eta}\mathtt{T}(\mathtt{t}^\eta)e^{-\lambda\varsigma}F\Big(\varsigma,\mathfrak{y}(\varsigma),\int_0^\varsigma e(\mathtt{z},s,\mathfrak{y}_s)ds\Big)d\varsigma d\mathtt{t}$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty \eta\rho_\eta(\theta)e^{-(\lambda\mathtt{t}\theta)}\mathtt{T}(\mathtt{t}^\eta)e^{-\lambda\varsigma}\mathtt{t}^{\eta-1}F\Big(\varsigma,\mathfrak{y}(\varsigma),\int_0^\varsigma e(\mathtt{z},s,\mathfrak{y}_s)ds\Big)d\theta d\varsigma d\mathtt{t}$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty \eta\rho_\eta(\theta)e^{-\lambda(\mathtt{t}+\varsigma)}\mathtt{T}\Big(\frac{\mathtt{t}^\eta}{\theta^\eta}\Big)\frac{\mathtt{t}^{\eta-1}}{\theta^\eta}F\Big(\varsigma,\mathfrak{y}(\varsigma),\int_0^\varsigma e(\mathtt{z},s,\mathfrak{y}_s)ds\Big)d\theta d\varsigma d\mathtt{t}$$
$$= \int_0^\infty e^{-\lambda\mathtt{t}}\Big[\int_0^{\mathtt{t}}\int_0^\infty \eta\rho_\eta(\theta)\mathtt{T}\Big(\frac{(\mathtt{t}-\varsigma)^\eta}{\theta^\eta}\Big)\frac{(\mathtt{t}-\varsigma)^{\eta-1}}{\theta^\eta}F\Big(\varsigma,\mathfrak{y}(\varsigma),\int_0^\varsigma e(\mathtt{z},s,\mathfrak{y}_s)ds\Big)d\theta d\varsigma\Big]d\mathtt{t}$$
$$= \int_0^\infty e^{-\lambda\mathtt{t}}\Big[\int_0^{\mathtt{t}}(\mathtt{t}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{t}-\varsigma)F\Big(\varsigma,\mathfrak{y}(\varsigma),\int_0^\varsigma e(\mathtt{z},s,\mathfrak{y}_s)ds\Big)d\varsigma\Big]d\mathtt{t}. \tag{3.6}$$

Similarly,

$$\int_0^\infty e^{-\lambda^\eta\varsigma}\mathtt{T}(\varsigma)d\varsigma\mathtt{B}\mathtt{G}_3(\lambda) = \int_0^\infty e^{-\lambda\mathtt{t}}\Big[\int_0^{\mathtt{t}}(\mathtt{t}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{t}-\varsigma)\mathtt{B}v(\varsigma)\Big]d\mathtt{t}. \tag{3.7}$$

Thus, from (3.5), (3.6), and (3.7), we obtain Eq. (3.3);

$$L(\mathfrak{y}(\mathtt{z})) = L(L^{-1}(\lambda^{\zeta(1-\eta)})*\mathtt{K}_\eta)(\varsigma)[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)]$$
$$+ L\Big(\mathtt{K}_\eta * F\Big(\varsigma,\mathfrak{y}(\varsigma),\int_0^\varsigma e(\mathtt{z},s,\mathfrak{y}_s)ds\Big)\Big)(\varsigma) + \mathtt{B}L(\mathtt{K}_\eta * v)(\varsigma).$$

Using inverse of Laplace transform, we get the mild solution of the system (1.1) – (1.2)

$$\mathfrak{y}(\mathtt{z}) = \mathcal{S}_{\eta,\zeta}(\mathtt{z})[\mathfrak{y}(0)+\xi(\mathtt{z}_1,\mathtt{z}_2,\dots,\mathtt{z}_n)] + \int_0^{\mathtt{z}}\mathtt{K}_\eta(\mathtt{z}-\varsigma)\underline{v}(\varsigma)d\varsigma$$
$$+ \int_0^{\mathtt{z}}\mathtt{K}_\eta(\mathtt{z}-\varsigma)F\Big(\varsigma,\mathfrak{y}(\varsigma),\int_0^\varsigma e(\mathtt{z},s,\mathfrak{y}_s)ds\Big)d\varsigma,$$

where $\mathcal{S}_{\eta,\zeta}(\mathtt{z}) = I_{0^+}^{\zeta(1-\eta)}\mathtt{K}_\eta(\mathtt{z}),\ \mathtt{K}_\eta(\mathtt{z}) = \mathtt{z}^{\eta-1}\mathtt{Q}_\eta(\mathtt{z})$. □

Definition 3.3. *The mild solution of the Cauchy problem* (1.1) – (1.2)*, is a function* $\mathfrak{y}(\mathtt{z}) \in C(\mathcal{I}', Y)$*, that satisfies*

$$\mathfrak{y}(\mathtt{z}) = \mathcal{S}_{\eta,\zeta}(\mathtt{z})[\mathfrak{y}(0) + \xi(\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_n)] + \int_0^{\mathtt{z}} \mathtt{K}_\eta(\mathtt{z} - \varsigma)\underline{v}(\varsigma)d\varsigma \qquad (3.8)$$
$$+ \int_0^{\mathtt{z}} \mathtt{K}_\eta(\mathtt{z} - \varsigma)F\Big(\varsigma, \mathfrak{y}(\varsigma), \int_0^{\varsigma} e(\mathtt{z}, s, \mathfrak{y}_s)ds\Big)d\varsigma, \qquad \mathtt{z} \in \mathcal{I}'.$$

Lemma 3.4. *[39] For any fixed* $\mathtt{z} > 0$, $\mathtt{Q}_\eta(\mathtt{z})$, $\mathtt{K}_\eta(\mathtt{z})$, *and* $\mathcal{S}_{\eta,\zeta}(\mathtt{z})$ *are linear operators, and for any* $\mathfrak{y} \in Y$

$$\Big\|\mathtt{Q}_\eta(\mathtt{z})\mathfrak{y}\Big\| \le L'\mathtt{z}^{-\eta+\eta\vartheta}\|\mathfrak{y}\|, \ \Big\|\mathtt{K}_\eta(\mathtt{z})\mathfrak{y}\Big\| \le L'\mathtt{z}^{-1+\eta\vartheta}\|\mathfrak{y}\|, \ \Big\|\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathfrak{y}\Big\| \le L''\mathtt{z}^{-1+\zeta-\eta\zeta+\eta\vartheta}\|\mathfrak{y}\|,$$

where

$$L' = \kappa_0\frac{\Gamma(\vartheta)}{\Gamma(\eta\vartheta)}, \ L'' = \kappa_0\frac{\Gamma(\vartheta)}{\Gamma(\zeta(1-\eta) + \eta\vartheta)}.$$

Lemma 3.5. *[39] Let* $\{\mathtt{T}(\mathtt{z})\}_{\mathtt{z}>0}$ *be equicontinuous, then* $\{\mathtt{Q}_\eta(\mathtt{z})\}_{\mathtt{z}>0}$, $\{\mathtt{K}_\eta(\mathtt{z})\}_{\mathtt{z}>0}$*, and* $\{\mathcal{S}_{\eta,\zeta}(\mathtt{z})\}_{\mathtt{z}>0}$ *are the strongly continuous, i.e., for any* $\mathfrak{y} \in Y$ *and* $\mathtt{z}_2 > \mathtt{z}_1 > 0$,

$$\Big\|\mathtt{Q}_\eta(\mathtt{z}_2)\mathfrak{y} - \mathtt{Q}_\eta(\mathtt{z}_1)\mathfrak{y}\Big\| \to 0, \ \Big\|\mathtt{K}_\eta(\mathtt{z}_2)\mathfrak{y} - \mathtt{K}_\eta(\mathtt{z}_1)\mathfrak{y}\Big\| \to 0$$
$$\Big\|\mathcal{S}_{\eta,\zeta}(\mathtt{z}_2)\mathfrak{y} - \mathcal{S}_{\eta,\zeta}(\mathtt{z}_1)\mathfrak{y}\Big\| \to 0, \ as\ \mathtt{z}_2 \to \mathtt{z}_1.$$

Lemma 3.6. *[40] Let* $\mathcal{I}$ *be a compact real interval, and* $\mathcal{P}_{bd,cv,cl}(Y)$ *is the set of all nonempty, bounded, convex, and closed subset of* Y*. Let* F *be the* L^1*-Caratheodory multivalued map, measurable to* $\mathtt{z}$ *for each* $\mathfrak{y} \in Y$*, u.s.c. to* $\mathfrak{y}$ *for each* $\mathtt{z} \in C(\mathcal{I}, Y)$*, the set*

$$S_{F,\mathfrak{y}} = \Big\{f \in L^1(\mathcal{I}, Y) : f(\mathtt{z}) \in F\Big(\mathtt{z}, \mathfrak{y}_{\mathtt{z}}, \int_0^{\mathtt{z}} e(\mathtt{z}, s, \mathfrak{y}_s)ds\Big), \quad \mathtt{z} \in \mathcal{I}\Big\}, \qquad (3.9)$$

is nonempty. Let Ξ *be the linear continuous function from* $L^1(\mathcal{I}, Y)$ *to* $\complement$*, then*

$$\Xi \circ S_F : \complement \to \complement, \quad \mathfrak{y} \to (\Xi \circ S_F(\mathfrak{y}) = \Xi(S_{F,\mathfrak{y}}), \qquad (3.10)$$

is closed graph operator in $\complement \times \complement$.

Lemma 3.7. *[41][Bohnenblust – Karlin fixed – point theorem] Let* D *be a nonempty subset of* Y *which is bounded, closed, and convex. Suppose that* $F : D \to 2^Y\backslash\{\emptyset\}$ *is upper semicontinuous with closed, convex values such that* $F(D) \subset D$ *and* $F(D)$ *is compact, then* F *has a fixed point.*

4. Approximate controllability

We will use some hypotheses as

(H_1) $\{\mathtt{T}(\mathtt{z}),\ \mathtt{z} \ge 0\}$ is a semigroup generated by the almost sectorial operators $\mathtt{A}$ such that $\|\mathtt{T}(\mathtt{z})\| \le M_0\mathtt{z}^{\vartheta-1}$ for some $M_0 > 0$ and $\|\alpha R(\alpha, \mathfrak{T}_0^b)\| \le 1$, $\forall\, \alpha > 0$.

(H_2) Let $F : \mathcal{I} \times \mathfrak{D}_h \times Y \to \mathcal{P}_{bd,cv,cl}(Y)$ be measurable to $\mathtt{z}$ for every $\mathfrak{y} \in Y$, u.s.c. to $\mathfrak{y}$ for each $\mathtt{z} \in \mathcal{I}$ and for every $\mathfrak{y} \in \complement$;

$$S_{F,\mathfrak{y}} = \left\{ f \in L^1(\mathcal{I}, Y) : f(\mathtt{z}) \in F\left(\mathtt{z}, \mathfrak{y}_{\mathtt{z}}, \int_0^{\mathtt{z}} e(\mathtt{z}, s, \mathfrak{y}_s)ds\right),\ \mathtt{z} \in \mathcal{I} \right\},$$

is nonempty.

(H_3) For $\mathtt{z} \in \mathcal{I}$, $F(\mathtt{z}, \cdot, \cdot) : \mathfrak{D}_h \times Y \to \mathcal{P}_{bd,cv,cl}(Y)$, $e(\mathtt{z}, s, \cdot) : \mathfrak{D}_h \to Y$ are continuous and for every $\mathfrak{y} \in \mathcal{X}$, $F(\cdot, \mathfrak{y}_{\mathtt{z}}, \int e(\mathtt{z}, s, \mathfrak{y}_{\mathtt{z}})ds) : \mathcal{I} \to Y$ and $e(\cdot, \cdot, \mathfrak{y}_{\mathtt{z}}) : \mathcal{I} \times \mathcal{I} \to Y$ are strongly measurable.

(H_4) There exists a non – decreasing function $\psi^* : \mathbb{R}^+ \to (0, \infty)$ and $L_{F,P}(\cdot) \in L^1(\mathcal{I}', \mathbb{R})$ such that $\left\|F(\mathtt{z}, \gamma_1, \gamma_2)\right\| \le L_{F,P}(\mathtt{z})\psi^*(\|\gamma_1\|_{\mathfrak{D}_h} + \|\gamma_2\|)$ for all $(\mathtt{z}, \gamma_1, \gamma_2) \in \mathcal{I} \times \mathfrak{D}_h \times Y$.

(H_5) There exists a constant $E_0 > 0$ st: $\|e(\mathtt{z}, s, \gamma)\| \le E_0(1 + \|\gamma\|_{\mathfrak{D}_h})$ for all $(\mathtt{z}, s, \gamma) \in \mathcal{I} \times \mathcal{I} \times \mathfrak{D}_h$.

(H_6) The continuous function $\xi : \mathfrak{D}_h^n \to \mathfrak{D}_h$ and $\Xi_n(\xi) > 0$ are so that

$$\left\|\xi(a_1, a_2, a_3, \ldots, a_n) - \xi(b_1, b_2, b_3, \ldots, b_n)\right\| \le \sum_{k=0}^{n} \Xi_k(\xi)\|a - b\|_{\mathfrak{D}_h},$$

for every $a_n, b_n \in \mathfrak{D}_h$ and take $\mathcal{P}_m = \sup\{\|\xi(a_1, a_2, a_3, \ldots, a_n)\| : a_j \in \mathfrak{D}_h\}$.

Before; prove the non–linear control system (1.1) – (1.2), we first look at the linear control system;

$$\begin{cases} D_{0+}^{\eta,\zeta}\mathfrak{y}(\mathtt{z}) \in \mathtt{A}\mathfrak{y}(\mathtt{z}) + (\underline{\mathtt{v}})(\mathtt{z}),\ \mathtt{z} \in \mathcal{I}' = (0, b] \\ I_{0+}^{(1-\eta)(1-\zeta)}\mathfrak{y}(0) = \mathfrak{y}_0, \end{cases} \tag{4.1}$$

where $\mathtt{B} : U \to Y$ is the linear bounded operator, $v \in L^2(\mathcal{I}, U)$.

Consider, the terms:

$$\mathfrak{T}_0^b = \int_0^b (b - \varsigma)^{\eta-1}\mathtt{Q}_\eta(b - \varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}_\eta^*(b - \varsigma)d\varsigma,$$
$$R(\alpha, \mathfrak{T}_0^b) = (\alpha I + \mathfrak{T}_0^b)^{-1},\ \alpha > 0,$$

$\mathtt{B}^*$ and $\mathtt{Q}_\eta^*$ are the adjoint of $\mathtt{B}$ and $\mathtt{Q}_\eta$. $\mathfrak{T}_0^b$ is linear and bounded.

Moreover, for every $\alpha > 0$, and $\mathfrak{y}_1 \in Y$, set

$$v(\mathtt{z}) = \mathtt{B}^*\mathtt{Q}_\eta^*(b - \mathtt{z})R(\alpha, \mathfrak{T}_0^b)P(\mathfrak{y}(\cdot)),$$

where

$$P(\mathfrak{y}(\cdot)) = \mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}\mathtt{z}_1, \mathfrak{y}\mathtt{z}_2, \ldots, \mathfrak{y}\mathtt{z}_n)] - \int_0^b (b - \varsigma)^{\eta-1}\mathtt{Q}_\eta(b - \varsigma)F\left(\varsigma, \mathfrak{y}_\varsigma, \int_0^\varsigma e(\varsigma, s, \mathfrak{y}_s)ds\right)d\varsigma.$$

Theorem 4.1. *Assume* (H_1) – (H_6) *holds, then the Hilfer fractional differential system* (1.1) – (1.2) *has a solution on* $\mathcal{I}$ *, provided;*

$$\frac{b^{1-\zeta+\eta\zeta-\eta\vartheta}\left[M^{**} + \frac{b^{\eta(2\vartheta-1)}(L'M_B)^2}{\alpha(\eta(2\vartheta-1))}[\mathfrak{y}_1 - M^{**}]\right]}{P} < 1,$$

where

$$M^{**} = L'' b^{-1+\zeta-\eta\zeta+\eta\vartheta}(\mathfrak{y}_0 + \mathcal{P}) + L' L_{F,P}(b)\frac{b^{\eta\vartheta}}{\eta\vartheta}.$$

and $\mathfrak{y}(0) \in D(\mathtt{A}^{\theta})$ *with* $\theta > 1 - \vartheta$.

Proof. Consider the operator $\Psi : \mathfrak{D}'_w \to 2^{\mathfrak{D}'_w}$, defined

$$\Psi(\mathfrak{y}(\mathtt{z})) = \begin{cases} \Psi_1(\mathtt{z}) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n})(\mathtt{z}),\ (-\infty, 0], \\ \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\mathcal{S}_{\eta,\zeta}(\mathtt{z})[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n})] + \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma) \\ \quad \times F\Big(\varsigma, \mathfrak{y}_\varsigma, \int_0^\varsigma e(\varsigma, s, \mathfrak{y}_s)ds\Big)d\varsigma + \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\underline{\mathtt{v}}(\varsigma)d\varsigma\Big], \quad \mathtt{z} \in (0, b]. \end{cases} \tag{4.2}$$

For $\Psi_1 \in \mathfrak{D}_w$, we define $\widehat{\Psi}$

$$\widehat{\Psi}(\mathtt{z}) = \begin{cases} \Psi_1(\mathtt{z}) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n}), & \mathtt{z} \in (-\infty, 0], \\ \mathcal{S}_{\eta,\mathtt{z}eta}(\mathtt{z})[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n})(0)], & \mathtt{z} \in \mathcal{I}, \end{cases}$$

then $\widehat{\Psi} \in \mathfrak{D}'_w$. Let $\mathfrak{y}_{\mathtt{z}} = [y_{\mathtt{z}} + \widehat{\Psi}_{\mathtt{z}}],\ \infty < \mathtt{z} \le b$. It can be easily shown that $\mathfrak{y}$ satisfies (3.8) if and only if y satisfies y_0 and

$$\begin{aligned} y(\mathtt{z}) = &\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F\Big(\varsigma, (y_\varsigma + \widehat{\Psi}_\varsigma), \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s)ds\Big)d\varsigma \\ &+ \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}^*_\eta(d-\varsigma)R(\alpha, \mathfrak{T}_0^b)\Big[\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n})] \\ &- \int_0^b (b-r)^{\eta-1}\mathtt{Q}_\eta(b-r)F\Big(r, y_r + \widehat{\Psi}_r, \int_0^r e(r, s, y_s + \widehat{\Psi}_s)ds\Big)dr\Big]. \end{aligned}$$

Assume; $\mathfrak{D}''_w = \{y \in \mathfrak{D}'_w : y_0 \in \mathfrak{D}_w\}$. $\forall\ y \in \mathfrak{D}'_w$;

$$\begin{aligned} \|y\|_b &= \|y_0\|_{\mathfrak{D}_w} + \sup\{\|y(\varsigma)\| : 0 \le \varsigma \le b\} \\ &= \sup\{\|y(\varsigma)\| : 0 \le \varsigma \le b\}. \end{aligned}$$

Therefore, $(\mathfrak{D}''_w, \|\cdot\|)$ is a Banach space.

For $P > 0$, choose $\mathfrak{D}_P = \{y \in \mathfrak{D}''_w : \|y\|_b \le P\}$, then $\mathfrak{D}_P \subset \mathfrak{D}''_w$ is uniformly bounded, $y \in \mathfrak{D}_P$, from Lemma 2.3

$$\begin{aligned} \big\|y_{\mathtt{z}} + \widehat{\Psi}_{\mathtt{z}}\big\|_{\mathfrak{D}_w} &\le \big\|y_{\mathtt{z}}\big\|_{\mathfrak{D}_w} + \big\|\widehat{\Psi}_{\mathtt{z}}\big\|_{\mathfrak{D}_w} \\ &\le l\Big(P + L''b^{-1+\zeta-\eta\zeta+\eta\vartheta}[\mathfrak{y}_0 + \mathcal{P}]\Big) + \|\Psi_1\|_{\mathfrak{D}_w} + \|\xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n})\|_{\mathfrak{D}_w} \\ &= P'. \end{aligned}$$

Consider the operator $\Phi : \mathfrak{D}''_w \to 2^{\mathfrak{D}''_w}$, defined as

$$\Phi(y(\mathtt{z})) = \begin{cases} 0,\ \mathtt{z} \in (-\infty, 0], \\ \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s)ds\Big)d\varsigma \\ + \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\underline{\mathtt{v}}(\mathtt{z})d\varsigma, \quad \mathtt{z} \in \mathcal{I}. \end{cases}$$

Now, prove that Φ has a fixed point.

Step:1 For every $y \in \mathfrak{D}''_w$, $\Phi(y)$ is convex.

Consider $\psi_1, \psi_2 \in \{\Psi y(\mathtt{z})\}$ and $f_1, f_2 \in S_{F,y}$ such that $\mathtt{z} \in \mathcal{I}$. We know that

$$\begin{aligned}\psi_i =& \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F_i\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s+\widehat{\Psi}_s)ds\Big)d\varsigma \\ &+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}^*_\eta(b-\varsigma)R(\alpha,\mathfrak{T}_0^b)\Big[\mathfrak{y}_1-\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}_0+\xi(\mathfrak{y}_{\mathtt{z}_1},\mathfrak{y}_{\mathtt{z}_2},\ldots,\mathfrak{y}_{\mathtt{z}_n})] \\ &-\int_0^b(b-r)^{\eta-1}\mathtt{Q}_\eta(b-r)F_i\Big(r, y_r+\widehat{\Psi}_r, \int_0^r e(r,s,y_s+\widehat{\Psi}_s)ds\Big)dr\Big]\Big], \qquad i=1,2.\end{aligned}$$

Let $0 \le \chi \le 1$, then for each $\mathtt{z} \in \mathcal{I}$, we get

$$\begin{aligned}&\chi\psi_1+(1-\chi)\psi_2(\mathtt{z}) \\ &= \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}^*_\eta(b-\varsigma)R(\alpha,\mathfrak{T}_0^b)\Big[\mathfrak{y}_1-\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}_0+\xi(\mathfrak{y}_{\mathtt{z}_1},\mathfrak{y}_{\mathtt{z}_2},\ldots,\mathfrak{y}_{\mathtt{z}_n})] \\ &\quad-\int_0^b(b-r)^{\eta-1}\mathtt{Q}_\eta(b-r)[\chi f_1+(1-\chi)f_2]dr\Big]+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)[\chi f_1+(1-\chi)f_2]d\varsigma\Big].\end{aligned}$$

We know that $S_{F,\mathfrak{y}}$ is convex. $\chi f_1+(1-\chi)f_2 \in S_{F,y}$.
Therefore;

$$\chi\psi_1+(1-\chi)\psi_2 \in \Phi\mathfrak{y}(\mathtt{z}).$$

Hence Φ is convex.

Step 2: To prove $P > 0$ such that $\Phi(\mathfrak{D}_P) \subseteq \mathfrak{D}_P$. It is enough to show that there exists $y_p \in \mathfrak{D}_P$, but $\Phi(y_p) \notin \mathfrak{D}_P$ ie.,$\|\Phi(y_p)(\mathtt{z})\| = \sup\{\|\psi(\mathtt{z})\|_b : \psi_p \in \Phi(y_p)\} \ge p$. Assume; for every $y \in \mathfrak{D}_P(\mathcal{I})$, we can get

$$\begin{aligned}P &\le \|\Phi(y_p)(\mathtt{z})\| \\ &\le \sup\ \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big\|\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)ds\Big)d\varsigma \\ &\quad+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}^*_\eta(b-\varsigma)R(\alpha,\mathfrak{T}_0^b)\Big[\mathfrak{y}_1-\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0)+\xi(\mathfrak{y}_{\mathtt{z}_1},\mathfrak{y}_{\mathtt{z}_2},\ldots,\mathfrak{y}_{\mathtt{z}_n}) \\ &\quad-\int_0^b(b-r)^{\eta-1}\mathtt{Q}_\eta(b-r)F\Big(r,y_r+\widehat{\Psi}_r,\int_0^r e(r,s,y_s+\widehat{\Psi}_s)ds\Big)dr\Big]d\varsigma\Big\| \\ &\le b^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\sup\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\Big\|\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\Big\|\Big\|F\Big(\varsigma,(y_\varsigma+\widehat{\Psi}_\varsigma),\int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)\Big)\Big\|d\varsigma \\ &\quad+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{2\eta\vartheta-\eta-1}L'^2M_B^2\frac{1}{\alpha}\Big[\mathfrak{y}_1-\sup\Big\|\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0)-\xi(\mathfrak{y}_{\mathtt{z}_1},\mathfrak{y}_{\mathtt{z}_2},\ldots,\mathfrak{y}_{\mathtt{z}_n})]\Big\| \\ &\quad-\int_0^b(b-r)^{\eta-1}\Big\|\mathtt{Q}_\eta(b-r)\Big\|\Big\|F\Big(r,y_r+\widehat{\Psi}_r,\int_0^r e(r,s,y_s+\widehat{\Psi}_r)ds\Big)\Big\|dr\Big]\Big]d\varsigma \\ &\le b^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[M^{**}+\frac{b^{\eta(2\vartheta-1)}(L'M_B)^2}{\alpha(\eta(2\vartheta-1))}\Big(\mathfrak{y}_1-L''b^{-1+\zeta-\eta\zeta+\eta\vartheta}[\mathfrak{y}_0+\mathcal{P}_m]-M^{**}\Big)\Big]\end{aligned}$$

where

$$M^{**} = L' L_{F,P}(b) \frac{b^{\eta\vartheta}}{\eta\vartheta} \psi^*(P' + bE_0(1 + P')).$$

Dividing, P both sides, we get a contradiction to our assumption. Therefore, $\Phi(\mathfrak{D}_P) \subseteq \mathfrak{D}_P$.
Step 3: Show; the equicontinuous of $\Phi(y)$ in $\mathfrak{D}''_w(\mathcal{I})$.
Consider $0 < \mathtt{z}_1 < \mathtt{z}_2 \leq b$ and there exists $F \in S_{F,y}$, we get

$$\begin{aligned}
&\Big\|\psi(\mathtt{z}_2) - \psi(\mathtt{z}_1)\Big\| \\
&\leq \Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathtt{z}_2} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma \\
&\quad + \int_0^{\mathtt{z}_2} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) \underline{\mathtt{v}}(\varsigma) d\varsigma\Big] \\
&\quad - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_1 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s)\Big) d\varsigma \\
&\quad + \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_1 - \varsigma) \underline{\mathtt{v}}(\varsigma) d\varsigma\Big]\Big\| \\
&\leq \Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma \\
&\quad + \mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathtt{z}_1}^{\mathtt{z}_2} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma \\
&\quad - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_1 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma\Big\| \\
&\quad + \Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) \mathtt{B} v(\varsigma) d\varsigma \\
&\quad + \mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathtt{z}_1}^{\mathtt{z}_2} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) \underline{\mathtt{v}}(\varsigma) d\varsigma \\
&\quad - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_1 - \varsigma) \underline{\mathtt{v}}(\varsigma) d\varsigma\Big\| \\
&\leq \Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathtt{z}_1}^{\mathtt{z}_2} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma\Big\| \\
&\quad + \Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma \\
&\quad - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma\Big\| \\
&\quad + \Big\|\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma \\
&\quad - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_1 - \varsigma) F\Big(\varsigma, y_m + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma\Big\| \\
&\quad + \Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathtt{z}_1}^{\mathtt{z}_2} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) \underline{\mathtt{v}}(\varsigma) d\varsigma\Big\|
\end{aligned}$$

$$
\begin{aligned}
&+ \left\| \mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) \underline{\mathtt{v}}(\varsigma) d\varsigma \right. \\
&\left. - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) \mathtt{B} v(\varsigma) d\varsigma \right\| \\
&+ \left\| \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) \underline{\mathtt{v}}(\varsigma) d\varsigma \right. \\
&\left. - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_1 - \varsigma) \underline{\mathtt{v}}(\varsigma) d\varsigma \right\| \\
&= \sum_{i=1}^{6} I_i.
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \left\| \mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathtt{z}_1}^{\mathtt{z}_2} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_\varsigma + \widehat{\Psi}_\varsigma) ds\Big) d\varsigma \right\| \\
&\leq L' \left| \mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathtt{z}_1}^{\mathtt{z}_2} (\mathtt{z}_2 - \varsigma)^{\eta\vartheta-1} L_{F,P}(\varsigma) \psi^*(P' + bE_0(1 + P')) d\varsigma \right|
\end{aligned}
$$

Integrating $\mathtt{z}_2 \to \mathtt{z}_1 \implies I_1 = 0$;

$$
\begin{aligned}
I_2 &= \left\| \mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_2 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma \right. \\
&\quad \left. - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}_1} (\mathtt{z}_1 - \varsigma)^{\eta-1} \mathtt{Q}_\eta(\mathtt{z}_2 - \varsigma) F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma \right\| \\
&\leq L' \int_0^{\mathtt{z}_1} (\mathtt{z}_2 - \varsigma)^{-\eta+\eta\vartheta} \left| \mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_2 - \varsigma)^{\eta-1} - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_1 - \varsigma)^{\eta-1} \right| \\
&\quad \times L_{F,P}(\varsigma) \psi^*(P' + bE_0(1 + P')) d\varsigma,
\end{aligned}
$$

Consider

$$
\begin{aligned}
&(\mathtt{z}_2 - \varsigma)^{-\eta+\eta\vartheta} \left| \mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_2 - \varsigma)^{\eta-1} - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_1 - \varsigma)^{\eta-1} \right| L_{F,P}(\varsigma) \psi(P' + bE_0(1 + P')) \\
&\leq \Big[\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_2 - \varsigma)^{\eta\vartheta-1} + \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_1 - \varsigma)^{\eta-1} (\mathtt{z}_2 - \varsigma)^{-\eta+\eta\vartheta} \Big] L_{F,P}(\varsigma) \psi(P' + bE_0(1 + P')) \\
&\leq \Big[\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_2 - \varsigma)^{\eta\vartheta-1} + \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_1 - \varsigma)^{\eta\vartheta-1} \Big] L_{F,P}(\varsigma) \psi(P' + bE_0(1 + P')) \\
&\leq 2\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_1 - \varsigma)^{\eta\vartheta-1} L_{F,P}(\varsigma) \psi^*(P' + bE_0(1 + P')).
\end{aligned}
$$

$\int_0^{\mathtt{z}_1} 2\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_1 - \varsigma)^{\eta\vartheta-1} L_{F,P}(\varsigma) \psi^*(P' + bE_0(1 + P')) d\varsigma$ exists ($\varsigma \in (0, \mathtt{z}_1]$), then by using the Lebesgue's dominated convergence theorem, we get

$$
\int_0^{\mathtt{z}_1} (\mathtt{z}_2 - \varsigma)^{-\eta+\eta\vartheta} \left| \mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_2 - \varsigma)^{\eta-1} - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathtt{z}_1 - \varsigma)^{\eta-1} \right| L_{F,P}(\varsigma) \psi(P' + bE_0(1 + P')) d\varsigma \to 0
$$

as $z_2 \to z_1$, so we conclude $\lim_{z_2 \to z_1} I_2 = 0$.
For any positive value ϵ, we write

$$
\begin{aligned}
I_3 &= \left\| \int_0^{z_1} z_1^{1-\zeta+\eta\zeta-\eta\vartheta} [Q_\eta(z_2-\varsigma) - Q_\eta(z_1-\varsigma)](z_1-\varsigma)^{\eta-1} F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s) ds\Big) d\varsigma \right\| \\
&\leq \int_0^{z_1-\epsilon} z_1^{1-\zeta+\eta\zeta-\eta\vartheta} \left\| Q_\eta(z_2-\varsigma) - Q_\eta(z_1-\varsigma) \right\| (z_1-\varsigma)^{\eta-1} L_{F,P}(\varsigma)\psi(P' + bE_0(1+P')) d\varsigma \\
&\quad + \int_{z_1-\epsilon}^{z_1} z_1^{1-\zeta+\eta\zeta-\eta\vartheta} \left\| Q_\eta(z_2-\varsigma) - Q_\eta(z_1-\varsigma) \right\| (z_1-\varsigma)^{\eta-1} L_{F,P}(\varsigma)\psi(P' + bE_0(1+P')) d\varsigma \\
&\leq z_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{z_1-\epsilon} (z_1-\varsigma)^{\eta-1} L_{F,P}(\varsigma)\psi(P' + bE_0(1+P')) d\varsigma \sup_{w\in[0,z_1-\epsilon]} \left\| Q_\eta(z_2-\varsigma) - Q_\eta(z_1-\varsigma) \right\| \\
&\quad + L' \int_{z_1-\epsilon}^{z_1} z_1^{1-\zeta+\eta\zeta-\eta\vartheta} [(z_2-\varsigma)^{-\eta+\eta\vartheta} + (z_1-\varsigma)^{-\eta+\eta\vartheta}](z_1-\varsigma)^{\eta-1} L_{F,P}(\varsigma)\psi^*(P' + bE_0(1+P')) d\varsigma \\
&\leq z_1^{1-\zeta+\eta\zeta-\eta\vartheta+\eta(1+\vartheta)} \int_0^{z_1} (z_1-\varsigma)^{\eta\vartheta-1} L_{F,P}(\varsigma)\psi^*(P' + bE_0(1+P')) d\varsigma \\
&\quad \times \sup_{\varsigma\in[0,z_1-\epsilon]} \left\| Q_\eta(z_2-\varsigma) - Q_\eta(z_1-\varsigma) \right\| \\
&\quad + 2L' \int_{z_1-\epsilon}^{z_1} z_1^{1-\zeta+\eta\zeta-\eta\vartheta} (z_1-\varsigma)^{\eta\vartheta-1} L_{F,P}(\varsigma)\psi^*(P' + bE_0(1+P')) d\varsigma.
\end{aligned}
$$

From Theorem 3.5 and $\lim_{z_2 \to z_1} I_1 = 0$, we obtain $I_3 \to 0$ independently of $y \in \mathfrak{D}_w''(\mathcal{I})$ as $z_2 \to z_1$, $\epsilon \to 0$.

$$
\begin{aligned}
I_4 &= \left\| z_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{z_1}^{z_2} (z_2-\varsigma)^{\eta-1} Q_\eta(z_2-\varsigma) \underline{v}(\varsigma) d\varsigma \right\| \\
&\leq z_2^{1-\zeta+\eta\zeta-\eta\vartheta} L' M_B \left\| \int_{z_1}^{z_2} (z_2-\varsigma)^{\eta\vartheta-1} v(\varsigma) d\varsigma \right\|
\end{aligned}
$$

I_4 tends to zero as $z_2 \to z_1$.

$$
\begin{aligned}
I_5 &= \left\| \int_0^{z_1} z_2^{1-\zeta+\eta\zeta-\eta\vartheta} [(z_2-\varsigma)^{\eta-1} - z_1^{1-\zeta+\eta\zeta-\eta\vartheta}(z_1-\varsigma)^{\eta-1}] Q_\eta(z_2-\varsigma) \underline{v}(\varsigma) \right\| \\
&\leq L' M_B \int_0^{z_1} z_2^{1-\zeta+\eta\zeta-\eta\vartheta} [(z_2-\varsigma)^{\eta-1} - z_1^{1-\zeta+\eta\zeta-\eta\vartheta}(z_1-\varsigma)^{\eta-1}](z_2-\varsigma)^{-\eta+\eta\vartheta} v(\varsigma) d\varsigma \\
I_6 &= \left\| z_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{z_1} (z_1-\varsigma)^{\eta-1} [Q_\eta(z_2-\varsigma) - Q_\eta(z_1-1)] \underline{v}(\varsigma) d\varsigma \right\| \\
&\leq M_B\, z_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{z_1} (z_1-\varsigma)^{\eta-1} v(\varsigma) d\varsigma \, \sup \left\| Q_\eta(z_2-\varsigma) - Q_\eta(z_1-\varsigma) \right\|.
\end{aligned}
$$

Similar, proof of I_2 and I_3, obtain I_5 and I_6 tends zero.
Therefore, $\|\psi(z_2) - \psi(z_1)\| \to 0$ independently of $\psi \in \Psi(y)$ as $z_2 \to z_1$. This implies that $\{\Psi y(z) : y \in \mathfrak{D}_w''(\mathcal{I})\}$ is equicontinuous on $\mathcal{I}$.
Step 4: The relatively compact of $V(z) = \{\psi(z) : z \in \Phi(y(z)),\ z \in \mathfrak{D}_w(\mathcal{I})\}$ in Y.

For $\alpha \in (0, \mathsf{z})$ and $q > 0$, consider the operator $\psi(\mathsf{z})$ on $\mathfrak{D}_w(\mathcal{I})$

$$\begin{aligned}
\psi_{\alpha,q}(\mathsf{z}) &= \mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathsf{z}-\alpha}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s+\widehat{\Psi}_s)ds\Big)d\varsigma \\
&\quad + \int_0^{\mathsf{z}-\alpha}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)\mathsf{B}\nu(\varsigma)d\varsigma\Big] \\
&= \mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathsf{z}-\alpha}\int_q^\infty \eta\theta\rho_\eta(\theta)(\mathsf{z}-\varsigma)^{\eta-1}T((\mathsf{z}-\varsigma)^\eta\theta) \\
&\quad \times F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s+\widehat{\Psi}_s)ds\Big)d\theta d\varsigma \\
&\quad + \int_0^{\mathsf{z}-\alpha}\int_q^\infty \eta\theta\rho_\eta(\theta)(\mathsf{z}-\varsigma)^{\eta-1}T((\mathsf{z}-\varsigma)^\eta\theta)\mathsf{B}\nu(\varsigma)d\theta d\varsigma\Big] \\
&= \eta\mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}T(\alpha^\eta q)\int_0^{\mathsf{z}-q}\int_q^\infty \theta\rho_\eta(\theta)(\mathsf{z}-\varsigma)^{\eta-1} \\
&\quad \times T((\mathsf{z}-\varsigma)^\eta\theta-\alpha^\eta q)\Big[F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s+\widehat{\Psi}_s)ds\Big)+\mathsf{B}\nu(\varsigma)\Big]d\theta d\varsigma.
\end{aligned}$$

Hence $V_{\alpha,\vartheta}(\mathsf{z}) = \{(\psi(\mathsf{z}))_{\alpha,q} : \mathsf{z} \in \mathfrak{D}_w(\mathcal{I})\}$ is precompact in Y, for every $\alpha \in (0, \mathsf{z})$ and $q > 0$ due to the compactness of $T(\alpha^\eta q)$. For every $\mathsf{z} \in \mathfrak{D}_w(\mathcal{I})$, we get

$$\begin{aligned}
&\Big\|\psi(\mathsf{z})-\psi_{\alpha,q}(\mathsf{z})\Big\| \\
&\le \Big\|\eta\mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathsf{z}}\int_0^q \theta\rho_\eta(\theta)(\mathsf{z}-\varsigma)^{\eta-1}T((\mathsf{z}-\varsigma)^\eta\theta) \\
&\qquad \Big[F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s+\widehat{\Psi}_s)ds\Big)+\mathsf{B}\nu(\varsigma)\Big]d\theta d\varsigma\Big\| \\
&\quad + \Big\|\eta\mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\mathsf{z}-\alpha}^{\mathsf{z}}\int_q^\infty (\mathsf{z}-\varsigma)^{\eta-1}\theta\rho_\eta(\theta)T((\mathsf{z}-\varsigma)^\eta\theta) \\
&\qquad \Big[F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s+\widehat{\Psi}_s)\Big)+\mathsf{B}\nu(\varsigma)\Big]d\theta d\varsigma\Big\| \\
&\le \eta M_0\mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(\int_0^{\mathsf{z}}\int_0^q \theta\rho_\eta(\theta)(\mathsf{z}-\varsigma)^{\eta-1}(\mathsf{z}-\varsigma)^{\eta\vartheta-\eta}\theta^{\vartheta-1} \\
&\quad \times [L_{F,P}(\varsigma)\psi^*(P'+bE_0(1+P'))+M_B\|\nu\|]d\theta d\varsigma \\
&\quad + \int_{\mathsf{z}-\alpha}^{\mathsf{z}}\int_q^\infty (\mathsf{z}-\varsigma)^{\eta-1}\theta\rho_\eta(\theta)(\mathsf{z}-\varsigma)^{\eta\vartheta-\eta}\theta^{\vartheta-1}[L_{F,P}(\varsigma)\psi^*(P'+bE_0(1+P'))+M_B\|\nu\|]d\varsigma\Big) \\
&\le \eta M_0\mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(\int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta\vartheta-1}[L_{F,P}(\varsigma)\psi^*(P'+bE_0(1+P'))+M_B\|\nu\|]d\varsigma\int_0^q \theta^\vartheta\rho_\eta(\theta)d\theta \\
&\quad + \int_{\mathsf{z}-\alpha}^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta\vartheta-1}[L_{F,P}(\varsigma)\psi(P'+bE_0(1+P'))+M_B\|\nu\|]d\varsigma\int_0^\infty \theta^\vartheta\rho_\eta(\theta)d\theta\Big) \\
&\to 0 \text{ as } \alpha \to 0, q \to 0.
\end{aligned}$$

Thus, $V_{\alpha,q}(\mathsf{z}) = \{\psi_{\alpha,q}(\mathsf{z}) : \mathsf{z} \in \mathfrak{D}''_w(\mathcal{I})\}$ is arbitrary and is closed to $V(\mathsf{z}) = \{\psi(\mathsf{z}) : \mathsf{z} \in \mathfrak{D}''_w(\mathcal{I})\}$. Hence, $\{\psi(\mathsf{z}) : \mathsf{z} \in \mathfrak{D}_P(\mathcal{I})\}$ is relatively compact. Therefore, the continuity of $\psi(\mathsf{z})$ and relative compactnes of

$\{\psi(\mathsf{z}) : \mathsf{z} \in \mathfrak{D}''_w(\mathcal{I})\}$ imply that $\psi(\mathsf{z})$ is completely continuous.

Step 5: Closed graph of Φ.

Let $y_k \to y_*$ as $k \to \infty$, $\psi_k(\mathsf{z}) \in \Phi(y_k)$ and $\psi_k \to \psi_*$ as $k \to \infty$, to prove that $\psi_* \in \Phi(y_*)$. Since $\psi_k \in \Phi(y_k)$, then there exists a function $F_k \in S_{F,y_k}$ such that

$$\psi_k(\mathsf{z}) = \mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)F_k(\varsigma)d\varsigma + \int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)\mathsf{B}\mathsf{B}^*\mathsf{Q}^*_\eta(b-\varsigma)$$
$$\times R(\alpha, \mathfrak{T}_0^b)\Big(\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathsf{z}_1}, \mathfrak{y}_{\mathsf{z}_2}, \ldots, \mathfrak{y}_{\mathsf{z}_n})] - \int_0^b (b-r)^{\eta-1}\mathsf{Q}_\eta(b-r)F_k(r)dr\Big)d\varsigma\Big].$$

We prove there exists an $F_* \in S_{F,y_*}$;

$$\psi_*(\mathsf{z}) = \mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)F_*(\varsigma)d\varsigma + \int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)\mathsf{B}\mathsf{B}^*\mathsf{Q}^*_\eta(d-\varsigma)$$
$$\times R(\alpha, \mathfrak{T}_0^b)\Big(\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathsf{z}_1}, \mathfrak{y}_{\mathsf{z}_2}, \ldots, \mathfrak{y}_{\mathsf{z}_n})] - \int_0^b (b-r)^{\eta-1}\mathsf{Q}_\eta(b-r)F_*(r)dr\Big)d\varsigma\Big].$$

Clearly,

$$\Big\|\psi_k(\mathsf{z}) - \mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)\mathsf{B}\mathsf{B}^*\mathsf{Q}^*_\eta(d-\varsigma)R(\alpha, \mathfrak{T}_0^b)\Big(\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)$$
$$[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathsf{z}_1}, \mathfrak{y}_{\mathsf{z}_2}, \ldots, \mathfrak{y}_{\mathsf{z}_n})]\Big)\Big] - \Big[z_*(\mathsf{z}) - \mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)\mathsf{B}\mathsf{B}^*\mathsf{Q}^*_\eta(d-\varsigma)$$
$$\times R(\alpha, \mathfrak{T}_0^b)\Big(\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathsf{z}_1}, \mathfrak{y}_{\mathsf{z}_2}, \ldots, \mathfrak{y}_{\mathsf{z}_n})]\Big)\Big]\Big]\Big\| \to 0 \text{ as } n \to \infty.$$

Now, we consider $\Xi : L^1(\mathcal{I}, Y) \to \mathcal{X}$ as

$$\Xi(F)(\mathsf{z}) = \int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)F(\varsigma)d\varsigma + \int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)\mathsf{B}\mathsf{B}^*R(0, \mathfrak{T}_0^b)$$
$$\times \int_0^b (b-r)^{\eta-1}\mathsf{Q}_\eta(b-r)F_k(r)dr.$$

From Lemma 3.6, $(\Xi \circ S_{F,y})$ is a closed graph operator. Thus, from Ξ, we obtain

$$\Big[\psi_k(\mathsf{z}) - \mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)\mathsf{B}\mathsf{B}^*\mathsf{Q}^*_\eta(b-\varsigma)$$
$$\times R(\alpha, \mathfrak{T}_0^b)\Big(\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathsf{z}_1}, \mathfrak{y}_{\mathsf{z}_2}, \ldots, \mathfrak{y}_{\mathsf{z}_n})]\Big)\Big] \in \Xi(S_{F,y_k}),$$

since $F_k \to F_*$, from Lemma 3.6 that

$$\Big[\psi_*(\mathsf{z}) - \mathsf{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathsf{z}}(\mathsf{z}-\varsigma)^{\eta-1}\mathsf{Q}_\eta(\mathsf{z}-\varsigma)\mathsf{B}\mathsf{B}^*\mathsf{Q}^*_\eta(b-\varsigma)$$
$$\times R(\alpha, \mathfrak{T}_0^b)\Big(\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathsf{z}_1}, \mathfrak{y}_{\mathsf{z}_2}, \ldots, \mathfrak{y}_{\mathsf{z}_n})]\Big)\Big] \in \Xi(S_{F,y_*}).$$

Hence Φ is closed graph.

From **Step 1 – 5** along with the Arzela – Ascoli theorem, we find out that Φ is compact and upper semi-continuous with closed convex values. Therefore, by Lemma 3.7, Φ has a fixed point, which is the mild solution of (1.1) – (1.2). □

Definition 4.2. *[36] The system* (1.1) – (1.2) *is called approximately controllable on* $\mathcal{I}$ $\forall$ $\mathfrak{y}_0 \in Y$, *there is some control* $v \in L^2(\mathcal{I}, U)$, $\overline{R(b, \mathfrak{y}_0)} = Y$, *where* $R(b, \mathfrak{y}_0) = \{\mathfrak{y}(b, v); v \in L^2(\mathcal{I}, U), \mathfrak{y}(0, v) = \mathfrak{y}_0\}$*; which is the reachable set of system* (1.1) – (1.2) *with the initial value* $\mathfrak{y}_0$ *at the terminal time b.*

Theorem 4.3. *Assume* $(H_1) - (H_6)$ *hold and the function F is uniformly bounded. Assume that the corresponding linear equation* (4.1) *is approximately controllable on* $\mathcal{I}$. *Then* (1.1) – (1.2) *is approximately controllable on* $\mathcal{I}$.

Proof. Let $\mathfrak{y}^\alpha(\cdot)$ be a fixed point of Ψ in $B_P(\mathcal{I})$. By using (4.1), any fixed point of Ψ is the mild solution of (1.1) – (1.2). Further, by Dunford - Pettis Theorem, we can use there is a subsequence $\{F^\alpha(\varsigma)\}$ that converges weakly to $F(\varsigma)$ in $L^1(\mathcal{I}, Y)$. $\forall$ $\alpha > 0$, then there exists $F^\alpha \in S_{F,\mathfrak{y}}$;

$$\begin{aligned}\mathfrak{y}^\alpha(\mathrm{z}) =& S_{\eta,\zeta}(\mathrm{z})[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathrm{z}_1}, \mathfrak{y}_{\mathrm{z}_2}, \dots, \mathfrak{y}_{\mathrm{z}_n})] + \int_0^{\mathrm{z}} (\mathrm{z} - \varsigma)^{\eta-1} \mathrm{Q}_\eta(\mathrm{z} - \varsigma) F^\alpha\Big(\varsigma, \mathfrak{y}^\alpha_\varsigma, \int_0^\varsigma e(\varsigma, s, \mathfrak{y}^\alpha_\varsigma) ds\Big) d\varsigma \\ &+ \int_0^{\mathrm{z}} (\mathrm{z} - \varsigma)^{\eta-1} \mathrm{Q}_\eta(\mathrm{z} - \varsigma) \mathrm{B}\mathrm{B}^* \mathrm{Q}^*_\eta(b - \varsigma) R(\alpha, \mathfrak{T}^b_0) P(\mathfrak{y}^\alpha(\mathrm{z})) d\varsigma,\end{aligned}$$

where

$$V^\alpha(\mathrm{z}) = \mathrm{B}^* \mathrm{Q}^*_\eta(b - \mathrm{z}) R(\alpha, \mathfrak{T}^b_0) P(\mathfrak{y}^\alpha)(\mathrm{z}),$$

and

$$P(\mathfrak{y}^\alpha) =\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathrm{z}_1}, \mathfrak{y}_{\mathrm{z}_2}, \dots, \mathfrak{y}_{\mathrm{z}_n)}] - \int_0^b (b - r)^{\eta-1} \mathrm{Q}_\eta(b - r) F^\alpha\Big(r, \mathfrak{y}^\alpha_r, \int_0^r e(r, s, \mathfrak{y}^\alpha_s) ds\Big) dr.$$

$(I - \mathfrak{T}^d_0 R(\alpha, \mathfrak{T}^b_0)) = \alpha R(\alpha, \mathfrak{T}^b_0)$, we obtain

$$\begin{aligned}\mathfrak{y}^\alpha(b) &= S_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathrm{z}_1}, \mathfrak{y}_{\mathrm{z}_2}, \dots, \mathfrak{y}_{\mathrm{z}_n})] + \int_0^b (b - \varsigma)^{\eta-1} \mathrm{Q}_\eta(d - \varsigma) F^\alpha\Big(\varsigma, \mathfrak{y}^\alpha_\varsigma, \int_0^\varsigma e(\varsigma, s, \mathfrak{y}^\alpha_s) ds\Big) d\varsigma \\ &\quad + \int_0^b (b - \varsigma)^{\eta-1} \mathrm{Q}_\eta(b - \varsigma) \mathrm{B}\mathrm{B}^* \mathrm{Q}^*_\eta(b - \varsigma) R(\alpha, \mathfrak{T}^b_0) P(\mathfrak{y}^\alpha(b)) d\varsigma \\ &= S_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathrm{z}_1}, \mathfrak{y}_{\mathrm{z}_2}, \dots, \mathfrak{y}_{\mathrm{z}_n})] + \int_0^b (b - \varsigma)^{\eta-1} \mathrm{Q}_\eta(b - \varsigma) F^\alpha\Big(\varsigma, \mathfrak{y}^\alpha_\varsigma, \int_0^\varsigma e(\varsigma, s, \mathfrak{y}^\alpha_s) ds\Big) d\varsigma \\ &\quad + \mathfrak{T}^b_0 R(\alpha, \mathfrak{T}^b_0) P(\mathfrak{y}^\alpha) \\ &= S_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathrm{z}_1}, \mathfrak{y}_{\mathrm{z}_2}, \dots, \mathfrak{y}_{\mathrm{z}_n})] + \int_0^b (b - \varsigma)^{\eta-1} \mathrm{Q}_\eta(b - \varsigma) F^\alpha\Big(\varsigma, \mathfrak{y}^\alpha_\varsigma, \int_0^\varsigma e(\varsigma, s, u^\alpha_s) ds\Big) d\varsigma \\ &\quad + P(\mathfrak{y}^\alpha) - \alpha R(\alpha, \mathfrak{T}^b_0) P(\mathfrak{y}^\alpha) \\ &= \mathfrak{y}_1 - \alpha R(\alpha, \mathfrak{T}^b_0) P(\mathfrak{y}^\alpha), \text{ for all } F \in S_{F,\mathfrak{y}}.\end{aligned}$$

Moreover, by using Hypothesis, $\|F^\alpha(\varsigma)\| \le L_{F,P}$, where $L_{F,P} < \infty$ a constant. Consequently, the sequence $\{F^\alpha(\varsigma)\}$ has subsequence defined as $\{F^\alpha(\varsigma)\}$; weakly converges to, say $F(\varsigma)$.

$$W = \mathfrak{y}_1 - S_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathrm{z}_1}, \mathfrak{y}_{\mathrm{z}_2}, \dots, \mathfrak{y}_{\mathrm{z}_n}] - \int_0^b (b - \varsigma)^{\eta-1} \mathrm{Q}_\eta(b - \varsigma) F\Big(\varsigma, \mathfrak{y}_\varsigma, \int_0^\varsigma e(\varsigma, s, \mathfrak{y}_\varsigma) ds\Big) d\varsigma$$

and

$$\|P(\mathfrak{y}^{\alpha}) - W\| = \left\| \int_0^b (b-\varsigma)^{\eta-1} \mathtt{Q}_{\eta}(b-\varsigma)[F^{\alpha}(\varsigma) - F(\varsigma)]d\varsigma \right\|.$$

The compactness of the operator $\mathtt{Q}_{\eta}(\mathtt{z}), \mathtt{z} > 0$ is deduced and the uniform boundness of $F^{\alpha}(\varsigma)$ such that $\exists$ some $F(\varsigma) \in L(\mathcal{I}, Y)$ as $\alpha \to 0^+$,

$$\mathtt{Q}_{\eta}(b-\varsigma)F^{\alpha}(\varsigma) \to \mathtt{Q}_{\eta}(b-\varsigma)F(\varsigma).$$

Hence for every $\mathtt{z} \in [0, b]$, we get $\|P(\mathfrak{y}^{\alpha}) - \varsigma\| \to 0$. Besides, by approximate controllability of system (4.1), we obtain $\alpha R(\alpha, \mathfrak{T}_0^b) \to 0$ as $\alpha \to 0^+$ in the strong topology. Thus, we obtain that $\alpha \to 0^+$,

$$\begin{aligned} \|\mathfrak{y}^{\alpha}(b) - \mathfrak{y}_1\| &= \left\| \alpha R(\alpha, \mathfrak{T}_0^b) P(\mathfrak{y}^{\alpha}) \right\| \\ &\leq \left\| \alpha R(\alpha, \mathfrak{T}_0^b)\varsigma \right\| + \left\| \alpha R(\alpha, \mathfrak{T}_0^b) \right\| \left\| (P(\mathfrak{y}^{\alpha}) - \varsigma) \right\| \\ &\leq \left\| \alpha R(\alpha, \mathfrak{T}_0^b)\varsigma \right\| + \left\| (P(\mathfrak{y}^{\alpha}) - \varsigma) \right\| \to 0. \end{aligned}$$

As a result, system (1.1) – (1.2) is approximately controllable on I. □

5. Neutral systems

Neutral systems, which involve delay terms in their equations, have indeed gained significant attention across various domains of applied mathematics due to their relevance in modeling real-world phenomena. These systems are particularly crucial in scenarios where the past history of the system influences its present behavior, such as in heat flow, wave propagation, and visco-elasticity.

In the context of differential equations, neutral systems can be challenging to analyze but offer valuable insights into dynamic processes. Understanding and effectively dealing with neutral systems require a thorough grasp of both ordinary and delay differential equations, as well as techniques specific to neutral systems.

Consider the Hilfer fractional neutral differential inclusions of the forms;

$$D_{0^+}^{\eta,\zeta}[\mathfrak{y}(\mathtt{z}) - \mathcal{K}(\mathtt{z}, \mathfrak{y}_{\mathtt{z}})] \in \mathtt{A}\mathfrak{y}(\mathtt{z}) + \underline{\mathtt{v}}(\mathtt{z}) + F\Big(\mathtt{z}, \mathfrak{y}_{\mathtt{z}}, \int_0^{\mathtt{z}} e(\mathtt{z}, s, \mathfrak{y}_{\mathtt{z}})ds\Big), \qquad \mathtt{z} \in \mathcal{I}' = (0, b], \tag{5.1}$$

$$I_{0^+}^{(1-\eta)(1-\zeta)}\mathfrak{y}(0) = \xi_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n}) \in L^2(D, \mathfrak{D}_h), \quad \mathtt{z} \in (-\infty, 0], \tag{5.2}$$

where $\mathcal{K} : \mathcal{I} \times \mathfrak{D}_w \to Y$ is an appropriate function.

Proposition 5.1. *[5] Suppose that* $0 < \eta < 1,\ 0 < q \leq 1$, *and for all* $\mathfrak{y} \in D(\mathtt{A})$, *then there exists a* $\kappa_q > 0$ *such that*

$$\left\| \mathtt{A}^q \mathtt{Q}_{\eta}(\mathtt{z})\mathfrak{y} \right\| \leq \frac{\eta \kappa_q \Gamma(2-q)}{\mathtt{z}^{\eta q}\Gamma(1 + \eta(1-q))} \|\mathfrak{y}\|,\ 0 < \mathtt{z} < b.$$

Consider the following hypotheses;

(H_7) The function $\mathcal{K} : \mathcal{I} \times \mathfrak{D}_w \to Y$ is continuous and $\exists\ q > 0,\ 0 < q < 1$ such that $\mathcal{K} \in D(\mathtt{A}^q)$ for any $\mathfrak{y} \in Y,\ \mathtt{z} \in \mathcal{I},\ \mathtt{A}^q\mathcal{K}(\cdot, \mathfrak{y})$ is strongly measurable, then $\exists\ M_w > 0,\ M'_w > 0$ such that $\gamma_1, \gamma_2 \in Y$ and $\mathtt{A}^q\mathcal{K}(\mathtt{z}, \cdot)$ satisfy the following:

$$\left\|\mathtt{A}^q\mathcal{K}(\mathtt{z}, \gamma_1(\mathtt{z})) - \mathtt{A}^q\mathcal{K}(\mathtt{z}, \gamma_2(\mathtt{z}))\right\| \leq M_w\|\gamma_1(\mathtt{z}) - \gamma_2(\mathtt{z})\|_{\mathfrak{D}_w},$$
$$\left\|\mathtt{A}^q\mathcal{K}(\mathtt{z}, \mathfrak{y}(\mathtt{z}))\right\| \leq M'_w(1 + \|\mathfrak{y}\|_{\mathfrak{D}_w}).$$

Take $\|\mathtt{A}^{-q}\| = M_0$ and $\|\mathtt{A}^{1-q}\| \leq M'_0$.

Definition 5.2. *A continuous function* $\mathfrak{y} : (-\infty, b] \to Y$ *is said to be a mild solution of* (5.1) – (5.2)*; that*

$$\mathfrak{y}(\mathtt{z}) = \mathcal{S}_{\eta,\zeta}(\mathtt{z})[\xi_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathtt{z}_n)(0) - \mathcal{K}(0, \mathfrak{y}_0)] + \mathcal{K}(\mathtt{z}, \mathfrak{y}_\mathtt{z}) + \int_0^\mathtt{z} (\mathtt{z} - \varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z} - \varsigma)\mathcal{K}(\varsigma, \mathfrak{y}_\varsigma)d\varsigma$$
$$+ \int_0^\mathtt{z} (\mathtt{z} - \varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z} - \varsigma)\underline{v}(\varsigma)d\varsigma + \int_0^\mathtt{z} (\mathtt{z} - \varsigma)^{(\eta-1)}\mathtt{Q}_\eta(\mathtt{z} - \varsigma)F\Big(\varsigma, \mathfrak{y}_\varsigma, \int_0^\varsigma e(\varsigma, s, u_s)ds\Big)d\varsigma.$$

Theorem 5.3. *Suppose* $(H_1) - (H_7)$ *holds, then the Hilfer fractional differential system* (5.1) – (5.2) *has a solution on* $\mathcal{I}$ *provided;*

$$\frac{1}{P}\Big[b^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[MM'_w(1 + P')M^{***} + M^{**} + \frac{b^{\eta(2\vartheta-1)}(L'M_B)^2}{\alpha(\eta(2\vartheta - 1))}$$
$$\Big(\mathfrak{y}_1 - L''b^{-1+\zeta-\eta\zeta+\eta\vartheta}[\mathfrak{y}_0 + \mathcal{P}_m] - MM'_w(1 + P')M^{***} - M^{**}\Big)\Big] < 1,$$

and $\theta > 1 - \vartheta$.

Proof. Consider the operator $\Psi : \mathfrak{D}'_w \to 2^{\mathfrak{D}'_w}$, defined as

$$\Psi(\mathfrak{y}(\mathtt{z})) = \begin{cases} \Psi_1(\mathtt{z}),\ (-\infty, 0], \\ \mathcal{S}_{\eta,\zeta}(\mathtt{z})[\xi_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathtt{z}_n)(0) - \mathcal{K}(0, \mathfrak{y}_0)] + \mathcal{K}(\mathtt{z}, \mathfrak{y}_\mathtt{z}) \\ \quad + \int_0^\mathtt{z}(\mathtt{z} - \varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z} - \varsigma)\mathcal{K}(\varsigma, \mathfrak{y}_\varsigma)d\varsigma + \int_0^\mathtt{z}(\mathtt{z} - \varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z} - \varsigma)\underline{v}(\varsigma)d\varsigma \\ \quad + \int_0^\mathtt{z}(\mathtt{z} - \varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z} - \varsigma)F\Big(\varsigma, \mathfrak{y}_\varsigma, \int_0^\varsigma e(\varsigma, s, u_s)ds\Big)d\varsigma, \quad \mathtt{z} \in \mathcal{I}. \end{cases}$$

For $\Psi_1 \in \mathfrak{D}_w$, we define $\widehat{\Psi}$

$$\widehat{\Psi}(\mathtt{z}) = \begin{cases} \Psi_1(\mathtt{z}), & \mathtt{z} \in (-\infty, 0], \\ \mathcal{S}_{\eta,\zeta}(\mathtt{z})\xi_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n}), & \mathtt{z} \in \mathcal{I}, \end{cases}$$

then $\widehat{\Psi} \in \mathfrak{D}'_w$. Let $\mathfrak{y}_\mathtt{z} = [y_\mathtt{z} + \widehat{\Psi}_\mathtt{z}],\ \infty < \mathtt{z} \leq b$, $\mathfrak{y}$ fulfills from a simple standpoint from (5.2) if and only if y satisfies y_0 and

$$y(\mathtt{z}) = -\,\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0, \mathfrak{y}_0) + \mathcal{K}(\mathtt{z}, v_\mathtt{z} + \widehat{\Psi}_\mathtt{z}) + \int_0^\mathtt{z} (\mathtt{z} - \varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z} - \varsigma)\mathcal{K}(\varsigma, v_\varsigma + \widehat{\Psi}_\varsigma)d\varsigma$$
$$+ \int_0^\mathtt{z} (\mathtt{z} - \varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z} - \varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}^*_\eta(b - \varsigma)R(\alpha, \mathfrak{T}^b_0)\Big[\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n})$$

$$+ \mathcal{K}(0, \mathfrak{y}_0)] - K(b, y_b + \widehat{\Psi}_b) - \int_0^b (b-r)\mathtt{AQ}_\eta(b-r)\mathcal{K}(r, y_r + \widehat{\Psi}_r)dr$$
$$- \int_0^b (b-r)^{\eta-1} F\Big(r, y_r + \widehat{\Psi}_r, \int_0^r e(r, s, y_s + \widehat{\Psi}_r)ds\Big)dr\Big](\varsigma)d\varsigma$$
$$+ \int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s)ds\Big)d\varsigma.$$

We define the operator $\Phi : \mathfrak{D}''_w \to 2^{\mathfrak{D}''_w}$:

$$\Phi y(\mathtt{z}) = \begin{cases} 0, \ \mathtt{z} \in (-\infty, 0], \\ -\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0, \mathfrak{y}_0) + \mathcal{K}(\mathtt{z}, v_{\mathtt{z}} + \widehat{\Psi}_{\mathtt{z}}) + \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{AQ}_\eta(\mathtt{z}-\varsigma)\mathcal{K}(\varsigma, v_\varsigma + \widehat{\Psi}_\varsigma)d\varsigma \\ \quad + \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{BB}^*\mathtt{Q}^*_\eta(b-\varsigma)R(\alpha, \mathfrak{T}_0^b)\Big[\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \dots, \mathfrak{y}_{\mathtt{z}_n}) \\ \quad + \mathcal{K}(0, \mathfrak{y}_0)] - K(b, y_b + \widehat{\Psi}_b) - \int_0^b (b-r)\mathtt{AQ}_\eta(b-r)\mathcal{K}(r, y_r + \widehat{\Psi}_r)dr \\ \quad - \int_0^b (b-r)^{\eta-1}F\Big(r, y_r + \widehat{\Psi}_r, \int_0^r e(r, s, y_s + \widehat{\Psi}_r)ds\Big)dr\Big](\varsigma)d\varsigma \\ \quad + \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s)ds\Big)d\varsigma, \qquad \mathtt{z} \in \mathcal{I}. \end{cases}$$

Now, we prove Φ has a fixed point.
Step 1: For every $y \in \mathfrak{D}''_w$, $\Phi(y)$ is convex.

Consider $\varphi_1, \varphi_2 \in \{\Psi y(\mathtt{z})\}$ and $f_1, f_2 \in S_{F,y}$ such that $\mathtt{z} \in \mathcal{I}$. We know that

$$\varphi_i = -\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0, \mathfrak{y}_0) + \mathcal{K}(\mathtt{z}, v_{\mathtt{z}} + \widehat{\Psi}_{\mathtt{z}}) + \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{AQ}_\eta(\mathtt{z}-\varsigma)\mathcal{K}(\varsigma, v_\varsigma + \widehat{\Psi}_\varsigma)d\varsigma$$
$$+ \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{BB}^*\mathtt{Q}^*_\eta(b-\varsigma)R(\alpha, \mathfrak{T}_0^b)\Big(\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \dots, \mathfrak{y}_{\mathtt{z}_n})$$
$$+ \mathcal{K}(0, \mathfrak{y}_0)] - K(b, y_b + \widehat{\Psi}_b) - \int_0^b (b-r)^{\eta-1}\mathtt{AQ}_\eta(b-r)\mathcal{K}(r, y_r + \widehat{\Psi}_r)dr$$
$$- \int_0^b (b-r)^{\eta-1} f_i(r)dr\Big)(\varsigma)d\varsigma + \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{(\eta-1)}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)f_i(\varsigma)d\varsigma.$$

Take $\chi \in [0, 1]$ then each of $\mathtt{z} \in \mathcal{I}$, we can get

$$\chi\varphi_1 + (1-\chi)\varphi_2(\mathtt{z})$$
$$= \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[- \mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0, \mathfrak{y}_0) + \mathcal{K}(\mathtt{z}, v_{\mathtt{z}} + \widehat{\Psi}_{\mathtt{z}})$$
$$+ \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{BB}^*\mathtt{Q}^*_\eta(b-\varsigma)R(\alpha, \mathfrak{T}_0^b)\Big(\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \dots, \mathfrak{y}_{\mathtt{z}_n})]$$
$$- \mathcal{S}_{\eta,\zeta}(b)\mathcal{K}(0, \mathfrak{y}_0) + \mathcal{K}(b, v_b + \widehat{\Psi}_b) - \int_0^b (b-r)^{\eta-1}\mathtt{Q}_\eta(b-r)[\chi f_1 + (1-\chi)f_2]dr\Big)$$
$$+ \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)[\chi f_1 + (1-\chi)f_2]d\varsigma\Big].$$

We know that $S_{F,\mathfrak{y}}$ is convex. So, $\chi f_1 + (1-\chi) f_2 \in S_{F,y}$.
Therefore,

$$\chi\varphi_1 + (1-\chi)\varphi_2 \in \Phi\mathfrak{y}(\mathtt{z});$$

Hence Φ is convex.
Step 2: To prove $P > 0$ such that $\Phi(\mathfrak{D}_P) \subseteq \mathfrak{D}_P$; it is enough to show that there exists $y_p \in \mathfrak{D}_P$, but $\Phi(y_p) \notin \mathfrak{D}_P$ ie., $\|\Phi(y_p)(\mathtt{z})\| = \sup\{\|\psi(\mathtt{z})\|_b : \psi_p \in \Phi(y_p)\} \geq p$. Assume, for every $y \in \mathfrak{D}_P(\mathcal{I})$;

$$\begin{aligned}
P &< \left\|\Phi(y(\mathtt{z}))\right\| \\
&\leq \sup\ \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big\| - \mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0,\mathfrak{y}_0) + \mathcal{K}(\mathtt{z}, y_{\mathtt{z}} + \widehat{\Psi}_{\mathtt{z}}) \\
&\quad + \int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathcal{K}(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma)d\varsigma \\
&\quad + \int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}_\eta^*(b-\varsigma)R(\alpha, \mathfrak{T}_0^b)\Big[\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n}) \\
&\quad + \mathcal{K}(0,\mathfrak{y}_0)] - K(b, y_b + \widehat{\Psi}_b) - \int_0^b (b-r)\mathtt{A}\mathtt{Q}_\eta(b-r)\mathcal{K}(r, y_r + \widehat{\Psi}_r)dr \\
&\quad - \int_0^b (b-r)^{\eta-1}F\Big(r, y_r + \widehat{\Psi}_r, \int_0^r e(r,s,y_s + \widehat{\Psi}_s)ds\Big)dr\Big](\varsigma)d\varsigma \\
&\quad + \int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{(\eta-1)}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F\Big(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s)ds\Big)d\varsigma\Big\| \\
&\leq b^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\left\|\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0,\mathfrak{y}_0)\right\| + \left\|\mathcal{K}(\mathtt{z}, y_{\mathtt{z}} + \widehat{\Psi}_{\mathtt{z}})\right\| + \int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}\left\|\mathtt{A}^{1-q}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\right\| \\
&\quad \times \left\|\mathtt{A}^q\mathcal{K}(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma)\right\|d\varsigma + \sup\int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}\left\|\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\right\| \\
&\quad \times \left\|F\Big(\varsigma, (y_\varsigma + \widehat{\Psi}_\varsigma), \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s)\Big)\right\|d\varsigma \\
&\quad + \int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}L'^2M_B^2\frac{1}{\alpha}\Big[\mathfrak{y}_1 - \sup\left\|\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) - \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n})]\right\| \\
&\quad + \left\|\mathcal{S}_{\eta,\zeta}(b)\mathcal{K}(0,\mathfrak{y}_0)\right\| + \left\|\mathcal{K}(b, y_b + \widehat{\Psi}_b)\right\| + \int_0^b (b-r)^{\eta-1}\left\|\mathtt{A}^{1-q}\mathtt{Q}_\eta(b-r)\right\|\left\|\mathtt{A}^q\mathcal{K}(b, y_b + \widehat{\Psi}_b)\right\|d\varsigma \\
&\quad - \int_0^b (b-r)^{\eta-1}\left\|\mathtt{Q}_\eta(b-r)\right\|\left\|F(s, y_s + \widehat{\Psi}_s, \int_0^s e(s,r,y_r + \widehat{\Psi}_r)dr)\right\|dr\Big]d\varsigma\Big] \\
&\leq b^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[MM_w'(1+P')\Big(L''b^{-1+\zeta-\eta\zeta+\eta\vartheta} + 1 + \kappa_{1-q}b^{\eta q}\frac{\Gamma(1+q)}{q\Gamma(1+\eta q)}\Big) + M^{**} \\
&\quad + \frac{b^{\eta(2\vartheta-1)}(L'M_B)^2}{\alpha(\eta(2\vartheta-1))}\Big(\mathfrak{y}_1 - L''b^{-1+\zeta-\eta\zeta+\eta\vartheta}[\mathfrak{y}_0 + \mathcal{P}_m] - MM_w'(1+P') \\
&\quad \times\Big(L''b^{-1+\zeta-\eta\zeta+\eta\vartheta} + 1 + \kappa_{1-q}b^{\eta q}\frac{\Gamma(1+q)}{q\Gamma(1+\eta q)}\Big) - M^{**}\Big)\Big] \\
&\leq b^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[MM_w'(1+P')M^{***} + M^{**} + \frac{b^{\eta(2\vartheta-1)}(L'M_B)^2}{\alpha(\eta(2\vartheta-1))}
\end{aligned}$$

$$\times\Big(\mathfrak{y}_1 - L''b^{-1+\zeta-\eta\zeta+\eta\vartheta}[\mathfrak{y}_0+\mathcal{P}_m] - MM'_w(1+P')M^{***} - M^{**}\Big),$$

where $M^{***} = \Big(L''b^{-1+\zeta-\eta\zeta+\eta\vartheta}+1+\kappa_{1-q}b^{\eta q}\frac{\Gamma(1+q)}{q\Gamma(1+\eta q)}\Big)$. Dividing both sides by P, we obtained a contradiction. Thus Φ is bounded.

Step 3: Next, we need to show that equicontinuous of Φ.

For $\mathfrak{y} \in \mathfrak{D}_P(\mathcal{I})$, and $0 \le \mathtt{z}_1 < \mathtt{z}_2 \le b$, we have

$$\begin{aligned}
&\big\|\Phi\mathfrak{y}(\mathtt{z}_2) - \Phi\mathfrak{y}(\mathtt{z}_1)\big\| \\
&= \Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[-\mathcal{S}_{\eta,\zeta}(\mathtt{z}_2)\mathcal{K}(0,\mathfrak{y}_0)+\mathcal{K}(\mathtt{z}_2, y_{\mathtt{z}_2}+\widehat{\Psi}_{\mathtt{z}_2}) + \int_0^{\mathtt{z}_2}(\mathtt{z}_2-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma) \\
&\quad\times\mathcal{K}(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma + \int_0^{\mathtt{z}_2}(\mathtt{z}_2-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)\underline{\mathtt{v}}(\varsigma)d\varsigma \\
&\quad+\int_0^{\mathtt{z}_2}(\mathtt{z}_2-\varsigma)^{(\eta-1)}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)ds\Big)d\varsigma\Big] \\
&\quad-\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[-\mathcal{S}_{\eta,\zeta}(\mathtt{z}_1)\mathcal{K}(0,\mathfrak{y}_0)+\mathcal{K}(\mathtt{z}_1, y_{\mathtt{z}_1}+\widehat{\Psi}_{\mathtt{z}_1}) + \int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}_1-\varsigma) \\
&\quad\times\mathcal{K}(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma + \int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}_1-\varsigma)\underline{\mathtt{v}}(\varsigma)d\varsigma \\
&\quad+\int_0^{\mathtt{z}}(\mathtt{z}_1-\varsigma)^{(\eta-1)}\mathtt{Q}_\eta(\mathtt{z}_1-\varsigma)F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)ds\Big)d\varsigma\Big]\Big\| \\
&\le \Big\|[\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\mathtt{z}_2) - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\mathtt{z}_1)]\mathcal{K}(0,\mathfrak{y}_0)\Big\| \\
&\quad+\Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{K}(\mathtt{z}_2,([y_{\mathtt{z}_2}+\widehat{\Psi}_{\mathtt{z}_2}])) - \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{K}(\mathtt{z}_1,([y_{\mathtt{z}_1}+\widehat{\Psi}_{\mathtt{z}_1}]))\Big\| \\
&\quad+\Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_2-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)\mathcal{K}(\varsigma,[y_\varsigma+\widehat{\Psi}_\varsigma])d\varsigma \\
&\quad-\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)\mathcal{K}(\varsigma,[y_\varsigma+\widehat{\Psi}_\varsigma])d\varsigma\Big\| \\
&\quad+\Big\|\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)\mathcal{K}(\varsigma,[y_\varsigma+\widehat{\Psi}_\varsigma])d\varsigma \\
&\quad-\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}_1-\varsigma)\mathcal{K}(\varsigma,[y_\varsigma+\widehat{\Psi}_\varsigma])d\varsigma\Big\| \\
&\quad+\Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\mathtt{z}_1}^{\mathtt{z}_2}(\mathtt{z}_2-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)\mathcal{K}(\varsigma,[y_\varsigma+\widehat{\Psi}_\varsigma])d\varsigma\Big\| \\
&\quad+\Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_2-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)ds\Big)d\varsigma \\
&\quad-\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)ds\Big)d\varsigma\Big\| \\
&\quad+\Big\|\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)F(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)ds)d\varsigma
\end{aligned}$$

$$
\begin{aligned}
&- z_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{z_1}(z_1-\varsigma)^{\eta-1}Q_\eta(z_1-\varsigma F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma,s,\widehat{\Psi}_s)ds\Big)d\varsigma\Big\|\\
&+\Big\|z_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{z_1}^{z_2}(z_2-\varsigma)^{\eta-1}Q_\eta(z_2-\varsigma)F\Big(\varsigma, y_\varsigma+\widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)ds\Big)d\varsigma\Big\|\\
&+\Big\|z_2^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(\int_0^{z_1}(z_2-\varsigma)^{\eta-1}Q_\eta(z_2-\varsigma)\underline{v}(\varsigma)d\varsigma-\int_0^{z_1}(z_2-\varsigma)^{\eta-1}Q_\eta(z_1-\varsigma)\underline{v}(\varsigma)d\varsigma\Big)\Big\|\\
&+\Big\|\Big(z_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{z_1}(z_2-\varsigma)^{\eta-1}-z_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{z_1}(z_1-\varsigma)^{\eta-1}\Big)Q_\eta(z_1-\varsigma)\underline{v}(\varsigma)d\varsigma\Big\|\\
&+\Big\|z_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{z_1}^{z_2}(z_2-\varsigma)^{\eta-1}Q_\eta(z_2-\varsigma)\underline{v}(\varsigma)\Big\|\\
\le&\sum_{i=1}^{11} I_i.
\end{aligned}
$$

$$
I_1=\Big\|[z_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(z_2)-z_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(z_1)]\mathcal{K}(0,\mathfrak{y}_0)\Big\|,
$$

by the strong continuous of $\mathcal{S}_{\eta,\zeta}(z)$ and (3.5), we obtain I_1 tends to zero as $z_2\to z_1$.

$$
I_2=\Big\|z_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{K}(z_2,(z_2^{1-\zeta+\eta\zeta-\eta\vartheta}[y_{z_2}+\widehat{\Psi}_{z_2}]))-z_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{K}(z_1,(z_1^{1-\zeta+\eta\zeta-\eta\vartheta}[y_{z_1}+\widehat{\Psi}_{z_1}]))\Big\|.
$$

By (H_7), we obtain I_2 tends to zero as $z_2\to z_1$.

$$
\begin{aligned}
I_3=&\Big\|z_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{z_1}(z_2-\varsigma)^{\eta-1}AQ_\eta(z_2-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma\\
&- z_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{z_1}(z_1-\varsigma)^{\eta-1}AQ_\eta(z_2-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma\Big\|\\
\le&\Big\|\Big(\int_0^{z_1}z_2^{1-\zeta+\eta\zeta-\eta\vartheta}(z_2-\varsigma)^{\eta-1}-z_1^{1-\zeta+\eta\zeta-\eta\vartheta}(z_1-\varsigma)^{\eta-1}\Big)AQ_\eta(z_2-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma\Big\|\\
\le&\Big\|\Big(\int_0^{z_1}z_2^{1-\zeta+\eta\zeta-\eta\vartheta}(z_2-\varsigma)^{\eta-1}-z_1^{1-\zeta+\eta\zeta-\eta\vartheta}(z_1-\varsigma)^{\eta-1}\Big)\Big\|\\
&\qquad\times\Big\|A^{1-q}Q_\eta(z_2-\varsigma)\Big\|\Big\|A^q\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)\Big\|d\varsigma\\
\le&\frac{M'_w\kappa_{1-q}\eta\Gamma(1+q)}{q\Gamma(1+\eta q)}(1+P')\\
&\qquad\times\Big\|\Big(\int_0^{z_1}(z_2-\varsigma)^{\eta(q-1)}[z_2^{1-\zeta+\eta\zeta-\eta\vartheta}(z_2-\varsigma)^{\eta-1}-z_1^{1-\zeta+\eta\zeta-\eta\vartheta}(z_1-\varsigma)^{\eta-1}]\Big)d\varsigma\Big\|.
\end{aligned}
$$

$I_3\to 0$ *as* $z_2\to z_1$.

$$
I_4=\Big\|z_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{z_1}(z_1-\varsigma)^{\eta-1}AQ_\eta(z_2-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma
$$

$$
\begin{aligned}
&- \mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}_1-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma\Big\|\\
&\le \Big\|\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}[\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)-\mathtt{Q}_\eta(\mathtt{z}_1-\varsigma)]\mathtt{A}\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma\Big\|\\
&\le M_0M'_w(1+P')\Big\|\mathtt{z}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}_1}(\mathtt{z}_1-\varsigma)^{\eta-1}[\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)-\mathtt{Q}_\eta(\mathtt{z}_1-\varsigma)]d\varsigma\Big\|.
\end{aligned}
$$

Since $\mathtt{Q}_\eta(\mathtt{z})$ is uniformly continuous in the operator norm topology, we obtain $I_4 \to 0\ as\ \mathtt{z}_2 \to \mathtt{z}_1$.

$$
\begin{aligned}
I_5 &= \Big\|\mathtt{z}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\mathtt{z}_1}^{\mathtt{z}_2}(\mathtt{z}_2-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma\Big\|\\
&\le b^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\mathtt{z}_1}^{\mathtt{z}_2}(\mathtt{z}_2-\varsigma)^{\eta-1}\Big\|\mathtt{A}^{1-q}\mathtt{Q}_\eta(\mathtt{z}_2-\varsigma)\Big\|\Big\|\mathtt{A}^q\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)\Big\|d\varsigma\\
&\le M'_w\kappa_{1-q}\eta\frac{\Gamma(1+q)}{q\Gamma(1+\eta q)}(1+P')b^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\mathtt{z}_1}^{\mathtt{z}_2}(\mathtt{z}_2-\varsigma)^{\eta q-1}d\varsigma.
\end{aligned}
$$

Integrating and $\mathtt{z}_2 \to \mathtt{z}_1 \Longrightarrow I_5 = 0$.

From similar proof of equicontinuous in step 3, (4.1), we obtain $I_6 - I_{11}$ is zero. Therefore, Φ is equicontinuous on $\mathcal{I}$.

Step 4: Prove that $V(\mathtt{z}) = \{\psi(\mathtt{z}) : z \in \Phi(y(\mathtt{z})),\ \mathtt{z} \in \mathfrak{D}_w(\mathcal{I})\}$ is relatively compact in Y

For $\alpha \in (0, \mathtt{z})$ and $q > 0$, assume $y(\mathtt{z})$ is the operator on $\mathfrak{D}_w(\mathcal{I})$

$$
\begin{aligned}
\psi_{\alpha,q}(\mathtt{z}) &= \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(-\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0,\mathfrak{y}_0)+\mathcal{K}(\mathtt{z},y_\mathtt{z}+\widehat{\Psi}_\mathtt{z})\\
&\quad+\int_0^{\mathtt{z}-\alpha}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma+\int_0^{\mathtt{z}-\alpha}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\underline{\mathtt{v}}(\varsigma)d\varsigma\\
&\quad+\int_0^{\mathtt{z}-\alpha}(\mathtt{z}-\varsigma)^{(\eta-1)}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F\Big(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma,\int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)ds\Big)d\varsigma\Big)\\
&= \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[-\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0,\mathfrak{y}_0)+\mathcal{K}(\mathtt{z},y_\mathtt{z}+\widehat{\Psi}_\mathtt{z})\\
&\quad+\mathtt{A}\int_0^{\mathtt{z}-\alpha}\int_q^\infty\eta\theta\rho_\eta(\theta)(\mathtt{z}-\varsigma)^{\eta-1}T((\mathtt{z}-\varsigma)^\eta\theta)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma\\
&\quad+\int_0^{\mathtt{z}-\alpha}\int_q^\infty\eta\theta\rho_\eta(\theta)(\mathtt{z}-\varsigma)^{\eta-1}T((\mathtt{z}-\varsigma)^\eta\theta)F\Big(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma,\int_0^\varsigma e(\varsigma,s,y_s+\widehat{\Psi}_s)\Big)d\theta d\varsigma\\
&\quad+\int_0^{\mathtt{z}-\alpha}\int_q^\infty\eta\theta\rho_\eta(\theta)(\mathtt{z}-\varsigma)^{\eta-1}T((\mathtt{z}-\varsigma)^\eta\theta)\mathtt{B}v(\varsigma)d\theta d\varsigma\Big]\\
&= \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[-\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0,\mathfrak{y}_0)+\mathcal{K}(\mathtt{z},y_\mathtt{z}+\widehat{\Psi}_\mathtt{z})+\eta T(\alpha^\eta q)\int_0^{\mathtt{z}-q}\int_q^\infty\theta\rho_\eta(\theta)(\mathtt{z}-\varsigma)^{\eta-1}\\
&\quad\times T((\mathtt{z}-\varsigma)^\eta\theta-\alpha^\eta q)[\mathtt{A}\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)+F\Big(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma,\int_0^\varsigma e(\varsigma,s,y_\varsigma+\widehat{\Psi}_\varsigma)\Big)+\mathtt{B}v(\varsigma)]d\theta d\varsigma\Big].
\end{aligned}
$$

Hence $V_{\alpha,\vartheta}(\mathtt{z}) = \{(\psi(\mathtt{z}))_{\alpha,q}\ : \mathtt{z} \in \mathfrak{D}_w(\mathcal{I})\}$ is precompact in Y for all $\alpha \in (0, \mathtt{z})$ and $q > 0$ due to the compactness of $T(\alpha^\eta q)$. For every $y \in \mathfrak{D}_w(\mathcal{I})$, we get

$$
\begin{aligned}
&\left\|\psi(\mathtt{z}) - \psi_{\alpha,q}(\mathtt{z})\right\| \\
&\leq \left\|\eta \mathtt{z}^{1-\mathtt{z}eta+\eta\zeta-\eta\vartheta} \int_0^{\mathtt{z}} \int_0^q \theta\rho_\eta(\theta)(\mathtt{z}-\varsigma)^{\eta-1} T((\mathtt{z}-\varsigma)^\eta\theta)\right. \\
&\qquad\left.\left[\mathtt{A}\mathcal{K}(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma) + F\left(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s)ds\right) + \mathtt{B}v(\varsigma)\right] d\theta d\varsigma\right\| \\
&+ \left\|\eta \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathtt{z}-\alpha}^{\mathtt{z}} \int_q^\infty (\mathtt{z}-\varsigma)^{\eta-1}\theta\rho_\eta(\theta) T((\mathtt{z}-\varsigma)^\eta\theta)\right. \\
&\qquad\left.\left[\mathtt{A}\mathcal{K}(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma) + F\left(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma, \int_0^\varsigma e(\varsigma, s, y_s + \widehat{\Psi}_s)\right) + \mathtt{B}v(\varsigma)\right] d\theta d\varsigma\right\| \\
&\leq \eta M_0 \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\left(\int_0^{\mathtt{z}} \int_0^q \theta\rho_\eta(\theta)(\mathtt{z}-\varsigma)^{\eta-1}(\mathtt{z}-\varsigma)^{\eta\vartheta-\eta}\theta^{\vartheta-1}\right. \\
&\quad \times [M_0 M_0'(1+P') + L_{F,P}(\varsigma)\psi^*(P' + bE_0(1+P')) + M_B\|v\|] d\theta d\varsigma \\
&\quad + \int_{\mathtt{z}-\alpha}^{\mathtt{z}} \int_q^\infty (\mathtt{z}-\varsigma)^{\eta-1}\theta\rho_\eta(\theta)(\mathtt{z}-\varsigma)^{\eta\vartheta-\eta}\theta^{\vartheta-1} \\
&\quad \left.\times [M_0 M_0'(1+P') L_{F,P}(\varsigma)\psi(P' + bE_0(1+P')) + M_B\|v\|] d\varsigma\right) \\
&\leq \eta M_0 \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\left(\int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta\vartheta-1}[M_0 M_0'(1+P) + L_{F,P}(\varsigma)\psi^*(P' + bE_0(1+P'))\right. \\
&\quad + M_B\|v\|] d\varsigma \int_0^q \theta^\vartheta \rho_\eta(\theta) d\theta + \int_{\mathtt{z}-\alpha}^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta\vartheta-1}[M_0 M_0'(1+P') \\
&\quad \left.+ L_{F,P}(\varsigma)\psi^*(P' + bE_0(1+P')) + M_B\|v\|] d\varsigma \int_0^\infty \theta^\vartheta \rho_\eta(\theta) d\theta\right) \\
&\leq \eta M_0 \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\left(\int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta\vartheta-1}[M_0 M)'(1+P') + L_{F,P}(\varsigma)\psi^*(P' + bE_0(1+P'))\right. \\
&\quad + M_B\|v\|] d\varsigma \int_0^q \theta^\vartheta \rho_\eta(\theta) d\theta \\
&\quad \left.+ \frac{\Gamma(1-\vartheta)}{\Gamma(1-\eta\vartheta)} \int_{\mathtt{z}-\alpha}^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta\vartheta-1}[M_0 M_0'(1+P') + L_{F,P}(\varsigma)\psi^*(P' + bE_0(1+P')) + M_B\|v\|] d\varsigma\right) \\
&\to 0 \text{ as } \alpha \to 0, q \to 0.
\end{aligned}
$$

Therefore, we conclude $\psi(\mathtt{z})$ is completely continuous.

Step 5: Φ has closed graph.

Consider $\psi_k \to y_*$ as $k \to \infty$, $\psi_k(\mathtt{z}) \in \Phi(y_k)$, and $\psi_k \to \psi_*$ as $k \to \infty$, we need to prove that $\psi_* \in \Phi(y_*)$. Since $\psi_k \in \Phi(y_k)$ then there exists a function $F_k \in S_{F,y_k}$ such that

$$
\begin{aligned}
\psi_k(\mathtt{z}) = \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[&- \mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0, \mathfrak{y}_0) + \mathcal{K}(\mathtt{z}, y_{\mathtt{z}} + \widehat{\Psi}_{\mathtt{z}}) + \int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathcal{K}(\varsigma, y_\varsigma + \widehat{\Psi}_\varsigma) d\varsigma \\
&+ \int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F_k(\varsigma)d\varsigma + \int_0^{\mathtt{z}} (\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}_\eta^*(d-\varsigma) \\
&\times R(\alpha, \mathfrak{T}_0^b)\Big(\mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0) + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n})] - \int_0^b (b-r)^{\eta-1}\mathtt{Q}_\eta(b-r)F_k(r)dr\Big) d\varsigma\Big].
\end{aligned}
$$

We can prove, there exists $F_* \in S_{F,y_*}$,

$$\psi_*(\mathtt{z}) = \mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[-\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0,\mathfrak{y}_0)+\mathcal{K}(\mathtt{z},y_{\mathtt{z}}+\widehat{\Psi}_{\mathtt{z}})+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma$$
$$+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F_*(\varsigma)d\varsigma+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}_\eta^*(d-\varsigma)$$
$$\times R(\alpha,\mathfrak{T}_0^b)\Big(\mathfrak{y}_1-\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0)+\xi(\mathfrak{y}_{\mathtt{z}_1},\mathfrak{y}_{\mathtt{z}_2},\ldots,\mathfrak{y}_{\mathtt{z}_n})]-\int_0^b(b-r)^{\eta-1}\mathtt{Q}_\eta(b-r)F_*(r)dr\Big)d\varsigma\Big].$$

Clearly,

$$\Big\|\psi_k(\mathtt{z})-\mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[-\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0,\mathfrak{y}_0)+\mathcal{K}(\mathtt{z},y_{\mathtt{z}}+\widehat{\Psi}_{\mathtt{z}})+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma$$
$$+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}_\eta^*(d-\varsigma)R(\alpha,\mathfrak{T}_0^b)\Big(\mathfrak{y}_1-\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0)+\xi(\mathfrak{y}_{\mathtt{z}_1},\mathfrak{y}_{\mathtt{z}_2},\ldots,\mathfrak{y}_{\mathtt{z}_n})]\Big)$$
$$-\Big[z_*(\mathtt{z})-\mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[-\mathcal{S}_{\eta,\zeta}(\mathtt{z})\mathcal{K}(0,\mathfrak{y}_0)+\mathcal{K}(\mathtt{z},y_{\mathtt{z}}+\widehat{\Psi}_{\mathtt{z}})$$
$$+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathcal{K}(\varsigma,y_\varsigma+\widehat{\Psi}_\varsigma)d\varsigma+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}_\eta^*(d-\varsigma)$$
$$\times R(\alpha,\mathfrak{T}_0^b)\Big(\mathfrak{y}_1-\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0)+\xi(\mathfrak{y}_{\mathtt{z}_1},\mathfrak{y}_{\mathtt{z}_2},\ldots,\mathfrak{y}_{\mathtt{z}_n})]\Big)\Big]\Big]\Big]\Big\|\to 0 \text{ as } n\to\infty.$$

Now, we consider an operator $\Xi : L^1(\mathcal{I},Y)\to\mathcal{X}$

$$\Xi(F)(\mathtt{z}) = \int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F(\varsigma))d\varsigma+\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*R(0,\mathfrak{T}_0^b)$$
$$\times\int_0^b(b-r)^{\eta-1}\mathtt{Q}_\eta(b-r)F_k(r)dr.$$

Lemma 3.6, $(\Xi\circ S_{F,y})$ is closed graph operator. So by referring Ξ, we have

$$\Big[\psi_k(\mathtt{z})-\mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}_\eta^*(b-\varsigma)$$
$$\times R(\alpha,\mathfrak{T}_0^b)\Big(\mathfrak{y}_1-\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0)+\xi(\mathfrak{y}_{\mathtt{z}_1},\mathfrak{y}_{\mathtt{z}_2},\ldots,\mathfrak{y}_{\mathtt{z}_n)})]\Big)\Big]\in\Xi(S_{F,y_k});$$

Since $F_k\to F_*$, from Lemma 3.6 that

$$\Big[z_*(\mathtt{z})-\mathtt{z}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big(\int_0^{\mathtt{z}}(\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}_\eta^*(d-\varsigma)$$
$$\times R(\alpha,\mathfrak{T}_0^b)\Big(\mathfrak{y}_1-\mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}(0)+\xi(\mathfrak{y}_{\mathtt{z}_1},\mathfrak{y}_{\mathtt{z}_2},\ldots,\mathfrak{y}_{\mathtt{z}_n)})]\Big)\in\Xi(S_{F,y_*}).$$

Hence Φ is closed graph.

From **Step 1 – 5** in conjunction with the Arzela–Ascoli theorem, we conclude that Φ is a compact multivalued map, upper semicontinuous with closed convex values. Therefore, by Lemma 3.7 the operator Φ has a fixed point, which is the mild solution of the problem (5.1) – (5.2). □

Theorem 5.4. *If the hypotheses* $(H_1) - (H_7)$ *are satisfied and a multivalued function* F *is uniformly bounded. Moreover, suppose that linear Eq.* (4.1) *is approximately controllable on* $\mathcal{I}$*, then* (5.1) – (5.2) *is approximately controllable on* $\mathcal{I}$.

Proof. Let $\mathfrak{y}^{\alpha}(\cdot)$ be a fixed point of Ψ in $B_P(\mathcal{I})$. Theorem 5.3, any fixed point of Ψ is the mild solution of (5.1) – (5.2). Further results on Dunford–Pettis Theorem, we conclude there is a subsequence $\{F^{\alpha}(\varsigma)\}$ that converges weakly to $F(\varsigma)$ in $L^1(\mathcal{I}, Y)$. for all $\alpha > 0$ then, there exists $F^{\alpha} \in S_{F,\mathfrak{y}}$;

$$\begin{aligned}\mathfrak{y}^{\alpha}(\mathtt{z}) =& S_{\eta,\zeta}(\mathtt{z})[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n)} + \mathcal{K}(0, \mathfrak{y}_0) + \mathcal{K}(\mathtt{z}, \mathfrak{y}^{\alpha}_{\mathtt{z}})] + \int_0^{\mathtt{z}} (\mathtt{z} - \varsigma)^{\eta-1} \mathtt{AQ}_{\eta}(\mathtt{z} - \varsigma)\mathcal{K}(\varsigma, \mathfrak{y}^{\alpha}_{\varsigma})d\varsigma \\ &+ \int_0^{\mathtt{z}} (\mathtt{z} - \varsigma)^{\eta-1} \mathtt{Q}_{\eta}(\mathtt{z} - \varsigma)F^{\alpha}\Big(\varsigma, \mathfrak{y}^{\alpha}_{\varsigma}, \int_0^{\varsigma} e(\varsigma, s, \mathfrak{y}^{\alpha}_{\varsigma})ds\Big)d\varsigma \\ &+ \int_0^{\mathtt{z}} (\mathtt{z} - \varsigma)^{\eta-1} \mathtt{Q}_{\eta}(\mathtt{z} - \varsigma)\mathtt{BB}^*\mathtt{Q}^*_{\eta}(d - \varsigma)R(\alpha, \mathfrak{T}^b_0)P(\mathfrak{y}^{\alpha}(\mathtt{z}))d\varsigma,\end{aligned}$$

where

$$V^{\alpha}(\mathtt{z}) = B^*\mathtt{Q}^*_{\eta}(b - \mathtt{z})R(\alpha, \mathfrak{T}^b_0)P(\mathfrak{y}^{\alpha})(\mathtt{z}),$$

and

$$\begin{aligned}P(\mathfrak{y}^{\alpha}) =& \mathfrak{y}_1 - \mathcal{S}_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n)} + \mathcal{K}(0, \mathfrak{y}_0)] - \mathcal{K}(b, \mathfrak{y}^{\alpha}_1) - \int_0^b (b - \varsigma)^{\eta-1} \mathtt{AQ}_{\eta}(b - \varsigma) \\ &\times \mathcal{K}(\varsigma, \mathfrak{y}^{\alpha}_{\varsigma})d\varsigma - \int_0^b (b - r)^{\eta-1} \mathtt{Q}_{\eta}(b - r)F^{\alpha}\Big(r, \mathfrak{y}^{\alpha}_r, \int_0^r e(r, s, \mathfrak{y}^{\alpha}_s)ds\Big)dr.\end{aligned}$$

$(I - \mathfrak{T}^b_0 R(\alpha, \mathfrak{T}^b_0)) = \alpha R(\alpha, \mathfrak{T}^b_0)$, we obtain

$$\begin{aligned}\mathfrak{y}^{\alpha}(b) =& S_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n)}] + \mathcal{K}(0, \mathfrak{y}_0) + \mathcal{K}(b, \mathfrak{y}^{\alpha}_1) + \int_0^b (b - \varsigma)^{\eta-1} \mathtt{AQ}_{\eta}(b - \varsigma) \\ &\times \mathcal{K}(\varsigma, \mathfrak{y}^{\alpha}_{\varsigma})d\varsigma + \int_0^b (b - \varsigma)^{\eta-1} \mathtt{Q}_{\eta}(b - \varsigma)F^{\alpha}\Big(\varsigma, \mathfrak{y}^{\alpha}_{\varsigma}, \int_0^{\varsigma} e(\varsigma, s, \mathfrak{y}^{\alpha}_s)\Big)d\varsigma \\ &+ \int_0^b (b - \varsigma)^{\eta-1} \mathtt{Q}_{\eta}(b - \varsigma)\mathtt{BB}^*\mathtt{Q}^*_{\eta}(b - \varsigma)R(\alpha, \mathfrak{T}^b_0)P(\mathfrak{y}^{\alpha}(b))d\varsigma \\ =& S_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n)} + \mathcal{K}(0, \mathfrak{y}_0)] + \mathcal{K}(b, \mathfrak{y}^{\alpha}_1) + \int_0^b (b - \varsigma)^{\eta-1} \mathtt{AQ}_{\eta}(b - \varsigma) \\ &\times \mathcal{K}(\varsigma, \mathfrak{y}^{\alpha}_{\varsigma})d\varsigma + \int_0^b (b - \varsigma)^{\eta-1} \mathtt{Q}_{\eta}(b - \varsigma)F^{\alpha}\Big(\varsigma, \mathfrak{y}^{\alpha}_{\varsigma}, \int_0^{\varsigma} e(\varsigma, s, \mathfrak{y}^{\alpha}_s)\Big)d\varsigma + \mathfrak{T}^b_0 R(\alpha, \mathfrak{T}^b_0)P(\mathfrak{y}^{\alpha}) \\ =& S_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathfrak{y}_{\mathtt{z}_n)} + \mathcal{K}(0, u_0)] + \mathcal{K}(b, \mathfrak{y}^{\alpha}_1) + \int_0^b (b - \varsigma)^{\eta-1} \mathtt{AQ}_{\eta}(b - \varsigma)\mathcal{K}(\varsigma, \mathfrak{y}^{\mathtt{A}}_{\varsigma}lpha)d\varsigma \\ &+ \int_0^b (b - \varsigma)^{\eta-1} \mathtt{Q}_{\eta}(b - \varsigma)F^{\alpha}\Big(\varsigma, \mathfrak{y}^{\alpha}_{\varsigma}, \int_0^{\varsigma} e(\varsigma, s, u^{\alpha}_s)ds\Big)d\varsigma + P(\mathfrak{y}^{\alpha}) - \alpha R(\alpha, \mathfrak{T}^b_0)P(\mathfrak{y}^{\alpha}) \\ =& \mathfrak{y}_1 - \alpha R(\alpha, \mathfrak{T}^b_0)P(\mathfrak{y}^{\alpha}), \text{ for all } F \in S_{F,\mathfrak{y}}.\end{aligned}$$

Moreover, there exists a constant $L_{F,P} < \infty$ st: $\|F^{\alpha}(\varsigma)\| \leq L_{F,P}$. Consequently, the sequence $\{F^{\alpha}(\varsigma)\}$ has subsequence still denoted by $\{F^{\alpha}(\varsigma)\}$; weakly converges to, say $F(\varsigma)$.

$$W = \mathfrak{y}_1 - S_{\eta,\zeta}(b)[\mathfrak{y}_0 + \xi(\mathfrak{y}_{z_1}, \mathfrak{y}_{z_2}, \ldots, \mathfrak{y}_{z_n}) + \mathcal{K}(0, \mathfrak{y}_0)] - \mathcal{K}(b, \mathfrak{y}_1) - \int_0^b (b-\varsigma)^{\eta-1} \mathtt{A}\mathtt{Q}_\eta(b-\varsigma) \times \mathcal{K}(\varsigma, \mathfrak{y}_\varsigma) d\varsigma - \int_0^b (b-\varsigma)^{\eta-1} \mathtt{Q}_\eta(b-\varsigma) F\Big(\varsigma, \mathfrak{y}_\varsigma, \int_0^\varsigma e(\varsigma, s, \mathfrak{y}_\varsigma) ds\Big) d\varsigma$$

and

$$\Big\|P(\mathfrak{y}^\alpha) - W\Big\| = \Big\|\mathcal{K}(b, \mathfrak{y}_1^\alpha) - \mathcal{K}(b, \mathfrak{y}_1)\Big\| + \Big\|\int_0^b (b-\varsigma)^{\eta-1} \mathtt{A}\mathtt{Q}_\eta(b-\varsigma)[\mathcal{K}(b, \mathfrak{y}_1^\alpha) - \mathcal{K}(b, \mathfrak{y}_1)]\Big\| + \Big\|\int_0^b (b-\varsigma)^{\eta-1} \mathtt{Q}_\eta(t-\varsigma)[F^\alpha(\varsigma) - F(\varsigma)] d\varsigma\Big\|.$$

From (H_7) first two terms of the above equation become zero. The compactness of the operator $\mathtt{Q}_\eta(\mathtt{z}), \mathtt{z} > 0$ is deduced and the uniform boundness of the $F^\alpha(\varsigma)$, there exists some $F(\varsigma) \in L^(\mathcal{I}, Y)$ such as $\alpha \to 0^+$,

$$\mathtt{Q}_\eta(b-\varsigma) F^\alpha(\varsigma) \to \mathtt{Q}_\eta(b-\varsigma) F(\varsigma).$$

Hence, for every $\mathtt{z} \in [0, b]$, we get $\|P(\mathfrak{y}^\alpha) - \varsigma\| \to 0$. Besides, by approximate controllability of (4.1), we can get $\alpha R(\alpha, \mathfrak{T}_0^b) \to 0$ as α converges to 0^+ in the strong topology. As a result, we obtain $\alpha \to 0^+$;

$$\begin{aligned} \Big\|\mathfrak{y}^\alpha(b) - \mathfrak{y}_1\Big\| &= \Big\|\alpha R(\alpha, \mathfrak{T}_0^b) P(\mathfrak{y}^\alpha)\Big\| \\ &\le \Big\|\alpha R(\alpha, \mathfrak{T}_0^b)\varsigma\Big\| + \Big\|\alpha R(\alpha, \mathfrak{T}_0^b)\Big\|\Big\|(P(\mathfrak{y}^\alpha) - \varsigma)\Big\| \\ &\le \Big\|\alpha R(\alpha, \mathfrak{T}_0^b)\varsigma\Big\| + \Big\|(P(\mathfrak{y}^\alpha) - \varsigma)\Big\| \to 0. \end{aligned}$$

As a result, System (5.1) – (5.2) is approximately controllable on I. □

6. Example

6.1. Example

Consider the following Hilfer fractional neutral integro-differential inclusion of the form

$$\begin{cases} D_{0^+}^{\frac{4}{7},\zeta}\Big[\rho(\mathtt{z},\tau) - \int_{-\infty}^0 a(\theta,\tau)\rho(\mathtt{z},\theta)d\theta\Big] \in \frac{\partial^2}{\partial\tau^2}\rho(\mathtt{z},\tau) + \mathtt{W}\beta(\mathtt{z},\tau) \\ \quad +\Upsilon\Big(\mathtt{z}, \int_{-\infty}^0 \varphi(s-\mathtt{z})\varrho(s,\tau)ds, \int_0^{\mathtt{z}}\int_{-\infty}^0 e_1(s,\tau,r-s)\Theta(\rho(r,\tau))drds\Big)\ \mathtt{z} \in (0,b],\ \tau \in [0,\pi], \\ I_{0^+}^{(1-\frac{4}{7})(1-\zeta)}\rho(0,\tau) = \phi(\mathtt{z},\tau) + \sum_{k=0}^n M_k\,\rho(\mathtt{z}_k+\tau),\ \tau \in [0,\pi], \\ \rho(\mathtt{z},0) = \rho(\mathtt{z},\pi) = 0, \qquad \mathtt{z} \in [0,b], \end{cases} \tag{6.1}$$

where $D_{0^+}^{\frac{4}{7},\zeta}$ is the Hilfer fractional derivative of order $\frac{4}{7}$ and type ζ, $I_{0^+}^{(1-\frac{4}{7})(1-\zeta)}$ is the R-L integral of order $\frac{3}{7}(1-\zeta), \beta(\mathtt{z}, a),\ \Upsilon,$ and e_1 are the required functions. To change the framework into abstract structure, let $U = Y = L^2[0,\pi]$ and $\mathtt{A}$ be an almost sectorial operator defined by $\mathtt{A}\tau = \frac{\partial^2}{\partial\tau^2}$ with the domain

$$D(\mathtt{A}) = \Big\{\rho \in Y : \rho_{\mathtt{z}}, \rho_{\mathtt{zz}} \text{ are absolutely continuous in Y, such that } \rho(\mathtt{z},0) = \rho(\mathtt{z},\pi) = 0\Big\}.$$

We need to verify the condition $\|\alpha \mathtt{R}(\alpha I + \mathfrak{T}_0^b)^{-1}\| \leq 1;$ for $\alpha > 0$. Let $\mathtt{B} : U \to L^2([0,\pi])$ be defined as

$$(\mathtt{B}v(\mathtt{z}))(\tau) = \mathtt{W}\beta(\mathtt{z},\tau),$$
$$\mathtt{B}^*\mathtt{v} = \sum_{k=1}^{\infty}\langle b, \mathtt{e}_k\rangle\langle v, \mathtt{e}_k\rangle,$$

where $0 \leq \tau \leq \pi$.

$$\mathtt{Q}_{\frac{4}{7}}(\mathtt{z}) = \frac{4}{7}\int_0^{\infty} \tau M_{\frac{4}{7}}(\tau)T(\mathtt{z}^{\frac{4}{7}}\tau d\tau),$$
$$\mathtt{Q}_{\frac{4}{7}}(\mathtt{z})\mathfrak{y} = \frac{4}{7}\sum_{k=1}^{\infty}\int_0^{\infty} \tau M_{\frac{4}{7}}(\tau)exp(-k^2\mathtt{z}^{\frac{4}{7}}\tau)d\tau\langle \mathfrak{y}, \mathtt{e}_k\rangle\mathtt{e}_k.$$

We have

$$\mathfrak{T}_0^b = \int_0^b (b-\varsigma)^{\eta-1}\mathtt{Q}_\eta(b-\varsigma)\mathtt{B}\mathtt{B}^*\mathtt{Q}_\eta^*(b-\varsigma)d\varsigma,$$
$$R(\alpha, \mathfrak{T}_0^b) = (\alpha I + \mathfrak{T}_0^b)^{-1}, \ \alpha > 0,$$

$\mathtt{B}^*$ and $\mathtt{Q}_\eta^*$ are the adjoint of $\mathtt{B}$ and $\mathtt{Q}_\eta$ respectively, also $\mathfrak{T}_0^b$ be the linear bounded operator.
We have to prove that;

$$(b-s)^{\eta-1}\mathtt{B}^*\mathtt{Q}_\eta(b-s)\mathfrak{y} = 0 \Longrightarrow \mathfrak{y} = 0.$$

Indeed, observe that

$$\begin{aligned}(b-s)^{\eta-1}\mathtt{B}^*\mathtt{Q}_\eta(b-s)\mathfrak{y} &= (b-s)^{\frac{4}{7}-1}\sum_{k=1}^{\infty}\langle b, \mathtt{e}_k\rangle\frac{4}{7}(\tau)M_{\frac{4}{7}}(\tau)exp(-k^2\mathtt{z}^{\frac{4}{7}}\tau)d\tau\langle \mathfrak{y}, \mathtt{e}_k\rangle \\ &= (b-s)^{\frac{4}{7}-1}\frac{4}{7}\sum_{k=1}^{\infty}(\tau)M_{\frac{4}{7}}(\tau)exp(-k^2\mathtt{z}^{\frac{4}{7}}\tau)d\tau\langle b, \mathtt{e}_k\rangle\langle \mathfrak{y}, \mathtt{e}_k\rangle = 0 \\ \Longrightarrow \langle \mathfrak{y}, \mathtt{e}_k\rangle &= 0 \\ \Longrightarrow \mathfrak{y} &= 0,\end{aligned}$$

provided that $\langle \mathfrak{y}, \mathtt{e}_k\rangle = \int_0^\pi \beta(\mathtt{z},\tau)\mathtt{e}_k(\tau)d\tau \neq 0$ for $k = 1, 2, 3, \cdots$. Therefore, the corresponding linear system is approximately controllable provided that $\int_0^\pi \beta(\mathtt{z},\tau)\mathtt{e}_k(\tau)d\tau \neq 0$. Because of the compactness of the semigroup $T(\mathtt{z})$ and function ϵ satisfied the conditions. So, for every $\mathfrak{y} \in Y$, $\alpha(\alpha I + \mathfrak{T}_0^b)(\mathfrak{y}) \to 0$ as $\alpha \to 0^+$ is strong topology. Thus, we conclude that the system (6.1) satisfy condition in (H_1).

Now for any $\mathfrak{y} \in Y = L^2[0,\pi]$, $\tau \in [0,\pi]$, we define the function F;

$$F\Big(\mathtt{z}, \mathfrak{y}_\mathtt{z}, \int_0^\mathtt{z} e(\mathtt{z}, s, \mathfrak{y}_s)ds\Big) = \Upsilon\Big(\mathtt{z}, \int_{-\infty}^0 \varphi(s-\mathtt{z})\varrho(s,\tau)ds, \int_0^\mathtt{z}\int_{-\infty}^0 e_1(s,\tau,r-s)\Theta(\rho(r,\tau))drds\Big),$$

where

$$e(\mathtt{z}, s, \mathfrak{y}_s)(\tau) = \int_{-\infty}^0 e_1(s,\tau,r-s)\Theta(\rho(r,\tau))dr.$$

Consider the following:

(i) The function $\mathfrak{y}psilon(\cdot,\cdot,\cdot)$ is continuous and measurable in $[0,b]\times\mathfrak{D}_w\times L^2[0,\pi]$ and uniformly bounded.
(ii) $\varphi(\cdot,\cdot)$ is continuous in $[0,b]\times[0,\pi]\times L^2([0,\pi])$.
(iii) $e_1(\cdot,\cdot,\cdot)$ is continuous and measurable in $[0,b]\times[0,\pi]\times\mathfrak{D}_w$ and $e_1(s,\tau,r)\geq 0,\ \int_{-\infty}^0 e_1(s,\tau,r)dr = \kappa_1(s,\tau)<\infty$.
(iv) The function $\Theta(\cdot)$ is continuous and $0\leq\Theta(\rho(r)(\tau))\leq \mathtt{Q}(\int_{-\infty}^0 e^{2s}\|\rho(s,\cdot)\|_{L^2}ds)$, $\mathtt{Q}:[0,\infty)\to(0,\infty)$ is an increasing and continuous function.
(v) $\mathcal{K}$ is continuous and measurable mapping in $[0,b]\times\mathfrak{D}_w$, defined by $\mathcal{K}(\mathsf{z},\mathfrak{y}_{\mathsf{z}})=\int_{-\infty}^0 a(\theta,\tau)\rho(\mathsf{z},\theta)d\theta$.
(vi) Fix the nolocal conditions, $\xi(\mathfrak{y}_{\mathsf{z}_1},\mathfrak{y}_{\mathsf{z}_2},\dots,u_{\mathsf{z}_n})=\sum_{k=0}^n M_k\,\rho(\mathsf{z}_k+\tau),\ \tau\in[0,\pi]$, is the continuous function.

Now we take $\delta(\nu)(\tau)=\delta(\nu,\tau);\ (\nu,\tau)\in(-\infty,0]\times[0,\pi]$, then we verify the following:

$$
\begin{aligned}
e(\mathsf{z},\delta)(\tau) &= \int_{-\infty}^0 e_1(\mathsf{z},\tau,\nu)\Theta(\delta(\tau))d\nu\\
\Longrightarrow \left|e(\mathsf{z},\delta)(\tau)\right|_{L^2} &= \Big[\int_0^\pi\Big(\int_{-\infty}^0 e_1(\mathsf{z},\tau,\nu)\Theta(\delta(\tau))d\nu\Big)^2 d\tau\Big]^{\frac{1}{2}}\\
&\leq \Big[\int_0^\pi\Big(\int_{-\infty}^0 e_1(\mathsf{z},\tau,\nu)\mathtt{Q}\Big(\int_{-\infty}^0 e^{2s}\|\delta(s)(\cdot)\|_{L^2}ds\Big)d\nu\Big)^2 d\tau\Big]^{\frac{1}{2}}\\
&\leq \Big[\int_0^\pi\Big(\int_{-\infty}^0 e_1(\mathsf{z},\tau,\nu)\Big)^2 d\tau\Big]^{\frac{1}{2}}\mathtt{Q}(1+\|\delta\|_{\mathfrak{D}_w})\\
&= \Big[\int_0^\pi(\kappa_1(\mathsf{z},\tau))^2 d\tau\Big]^{\frac{1}{2}}\mathtt{Q}(1+\|\delta\|_{\mathfrak{D}_w})\\
&= \kappa^*(\mathsf{z})\mathtt{Q}(\|\delta\|_{\mathfrak{D}_w}).
\end{aligned}
$$

Therefore we observe,

$$
\begin{aligned}
\left|F(\mathsf{z},\cdot,\cdot)\right|_{L^2} &= \Big|\Upsilon\Big(\mathsf{z},\int_{-\infty}^0\varphi(s)\varrho(s,\tau)ds,\int_0^{\mathsf{z}}\int_{-\infty}^0 e_1(s,\tau,\nu)\Theta(\delta(\nu)(\tau))d\nu d\tau\Big)\Big|\\
&\leq \Big[\int_0^\pi\Big(\Upsilon\Big(\mathsf{z},\int_{-\infty}^0\varphi(s)\varrho(s,\tau)ds,\int_0^{\mathsf{z}}\int_{-\infty}^0 e_1(s,\tau,\nu)\Theta(\delta(s)(\cdot))ds\Big)d\nu\Big)^2 d\tau\Big]^{\frac{1}{2}}\\
&\leq \mathtt{M}^*\varsigma\Big(\left\|\mathtt{L}\right\|+\kappa^*(\mathsf{z})\mathtt{Q}(\|\delta\|_{\mathfrak{D}_w})\Big)
\end{aligned}
$$

where $\mathtt{M}^*=L_{F,P}(\mathsf{z})$ and $\varsigma,\ \mathtt{Q}$ are an increasing functions in Hypotheses (H_4) and (H_5). Thus we conclude that the system (6.1) satisfied the hypotheses $(H_2)-(H_7)$.

Therefore, fractional system (6.1) written as the nonlinear Cauchy problem (5.1) – (5.2). Clearly, $F(\cdot,\cdot,\cdot)$ is uniformly bounded. Then the hypotheses $(H_1)-(H_7)$ are satisfied. On the other hand, linear system corresponding to (6.1) is approximately controllable and Theorem 4.1 hold. Therefore, all conditions of the Theorem 4.3 are satisfied. So, the Hilfer fractional differential inclusion (5.1) – (5.2) is approximately controllable on $[0,b]$.

6.2. Numerical study

In this section we studied numerical results of the Hilfer fractional system (1.1); consider the following problem.

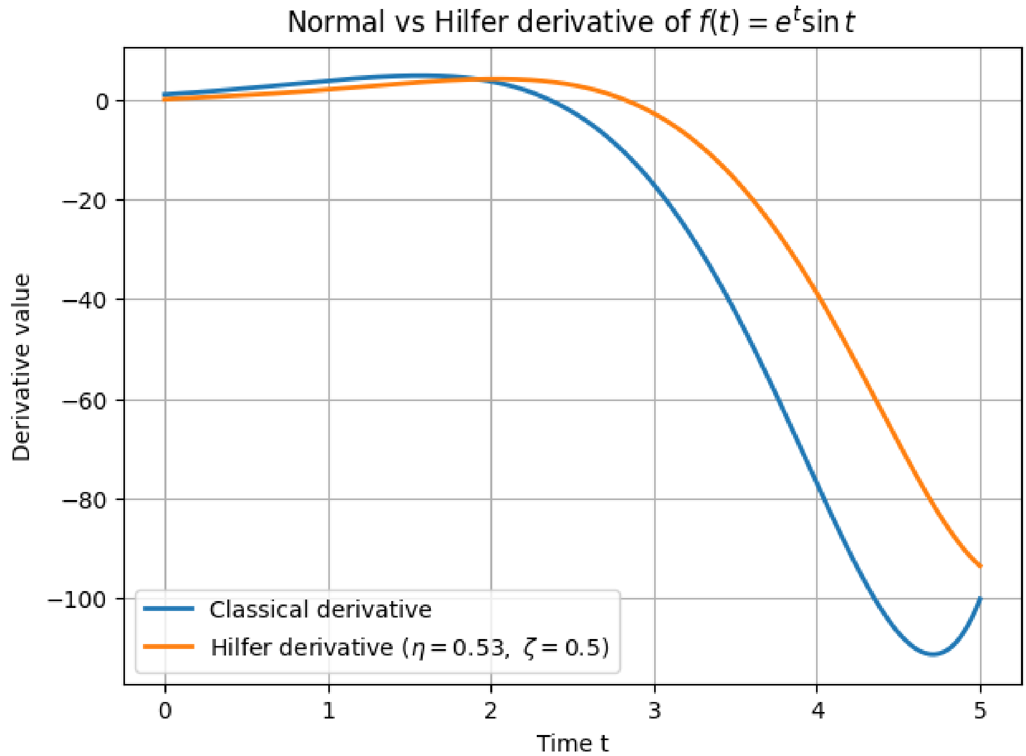

Figure 1. Fractional derivative vs normal derivative.

$$\begin{cases} D_{0^+}^{\frac{7}{13},\frac{1}{2}}\Big[\rho(\mathrm{z},\tau) - C_0 \sin(\rho(\mathrm{z},\tau))\Big] \in \frac{\partial^2}{\partial\tau^2}\rho(\mathrm{z},\tau) + \mathtt{W}\beta(\mathrm{z},\tau) \\ \qquad +C_1\frac{e^{-\mathrm{z}}}{1+e^{-\mathrm{z}}}\big(\sin(\rho(\mathrm{z},\tau)) + C_2\int_0^{\mathrm{z}} \sin(\rho(\mathrm{z},\tau))d\tau\big),\ \mathrm{z} \in (0,b],\ \tau \in [0,\pi], \\ I_{0^+}^{\frac{3}{7},\frac{1}{2}}\rho(0,\tau) = \phi(\mathrm{z},\tau) + \sum_{k=0}^{n} M_k\, \rho(\mathrm{z}_k + \tau),\ \tau \in [0,\pi], \\ \rho(\mathrm{z},0) = \rho(\mathrm{z},\pi) = 0, \qquad \mathrm{z} \in [0,b], \end{cases} \tag{6.2}$$

where $D_{0^+}^{\frac{7}{13},\frac{1}{2}}$ is the Hilfer fractional differential equation of order $\frac{7}{13}$ and type $\frac{1}{2}$. Let the almost sectorial operator defined as $\mathtt{A}(\eta(z)) = \frac{\partial^2\rho(\mathrm{z},\tau)}{\partial\tau^2}$, where $\rho((z),\tau)$ is an absolutely continuous function. The almost sectorial operators generate the semigroup $\mathtt{T}(\mathrm{z})$. We define the following functions:

$$K(\mathrm{z},\eta_z) = C_0 \sin(\rho(\mathrm{z},\tau))$$

$$F\Big(\mathrm{z},\mathfrak{y}_{\mathrm{z}},\int_0^{\mathrm{z}} e(\mathrm{z},s,\mathfrak{y}_{\mathrm{z}})ds\Big) = C_1\frac{e^{-\mathrm{z}}}{1+e^{-\mathrm{z}}}\big(\sin(\rho(\mathrm{z},\tau)) + C_2\int_0^{\mathrm{z}} \sin(\rho(\mathrm{z},\tau))d\tau\big)$$

$$\mathtt{V}(\mathrm{z}) = \beta(\mathrm{z},\tau)$$

$$\xi(\eta_{\mathrm{z}_1},\eta_{\eta_2},\ldots,\eta_{\mathrm{z}_n}) = \sum_{k=0}^{n} M_k\, \rho(\mathrm{z}_k + \tau),$$

where C_1, C_2, and M_k are the constants. We obtain the following:

$$L' = \kappa_0\frac{\Gamma(1/2)}{\Gamma(7/26)}, \quad L'' = \kappa_1\frac{\Gamma(1/2)}{\Gamma(19/26)}.$$

Take $b = 1, \vartheta = 1/2$, and appropriate constants values, we obtained

$$M^{**} = L'' b^{-1+\zeta-\eta\zeta+\eta\vartheta}(\mathfrak{y}_0 + \mathcal{P}) + L' L_{F,P}(b)\frac{b^{\eta\vartheta}}{\eta\vartheta} = K\kappa_1 \frac{\Gamma(1/2)}{\Gamma(19/26)} + K'\kappa_0 \frac{\Gamma(1/2)}{\Gamma(7/26)}$$

then

$$\frac{\left[M^{**} + \frac{b^{\eta(2\vartheta-1)}(L'M_B)^2}{\alpha(\eta(2\vartheta-1))}[\mathfrak{y}_1 - M^{**}]\right]}{P} < 1,$$

and

$$\frac{1}{P}\Bigg[\Bigg[MM'_w(1+P')M^{***} + M^{**} + \frac{(L'M_B)^2}{\alpha(\eta(2\vartheta-1))}$$
$$\Big(\mathfrak{y}_1 - L''[\mathfrak{y}_0 + \mathcal{P}_m] - MM'_w(1+P')M^{***} - M^{**}\Big)\Bigg] < 1;$$

$$\begin{aligned}\left\|\mathtt{A}^q\mathcal{K}(\mathtt{z},\gamma_1(\mathtt{z})) - \mathtt{A}^q\mathcal{K}(\mathtt{z},\gamma_2(\mathtt{z}))\right\| &\le C_0\left\|\left(\frac{\partial^2}{\partial\tau_1^2}\right)^q \sin(\rho(\mathtt{z},\tau_1)) - \left(\frac{\partial^2}{\partial\tau_2^2}\right)^q \sin(\rho(\mathtt{z},\tau_2))\right\| \\ &\le M_w\|\rho(\tau_1) - \rho(\tau_2)\|, \\ \left\|\mathtt{A}^q\mathcal{K}(\mathtt{z},\mathfrak{y}(\mathtt{z}))\right\| &\le M'_w\left\|\sin(\rho(\mathtt{z},\tau))\right\| \le M'_w.\end{aligned}$$

From the example we have verified the hypothesis, therefore the system (6.2) is approximate controllable.

Remark: The exponential term leads the classical derivative to rise quickly, whereas the Hilfer derivative exhibits memory effects, which result in smoother behaviour, a slower growth rate, and a significant departure from the classical derivative as t increases. Figure 1 makes this very evident.

6.3. Filter system

This section looks at the Hilfer fractional differential equation-based initial value problem and shows the potential benefits of fractional derivatives with regard to another function. Examine the system's mild solution (1.1);

$$\begin{aligned}\mathfrak{y}(\mathtt{z}) = \mathcal{S}_{\eta,\zeta}(\mathtt{z})[\xi_0 + \xi(\mathfrak{y}_{\mathtt{z}_1}, \mathfrak{y}_{\mathtt{z}_2}, \ldots, \mathtt{z}_n)(0) - \mathcal{K}(0,\mathfrak{y}_0)] + \mathcal{K}(\mathtt{z},\mathfrak{y}_\mathtt{z}) + \int_0^\mathtt{z} (\mathtt{z}-\varsigma)^{\eta-1}\mathtt{A}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\mathcal{K}(\varsigma,\mathfrak{y}_\varsigma)d\varsigma \\ + \int_0^\mathtt{z} (\mathtt{z}-\varsigma)^{\eta-1}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)\underline{\mathtt{v}}(\varsigma)d\varsigma + \int_0^\mathtt{z} (\mathtt{z}-\varsigma)^{(\eta-1)}\mathtt{Q}_\eta(\mathtt{z}-\varsigma)F\Big(\varsigma,\mathfrak{y}_\varsigma,\int_0^\varsigma e(\varsigma,s,u_s)ds\Big)d\varsigma.\end{aligned}$$

We propose the digital filter system corresponding to the mild solution in Figure 2, the foundation of each signal processing application is a digital filter. These days, a lot of biomedical signals pertaining to the human body are obtained for a variety of informative feature extractions. In general, the majority of the signals indicated have low frequencies by nature. These signals provide information on a variety of diseases and ailments for which accuracy is crucial. Any digital signal processing filtering system's effectiveness depends on its capacity to reject noise.

Figure 2 describes the following:

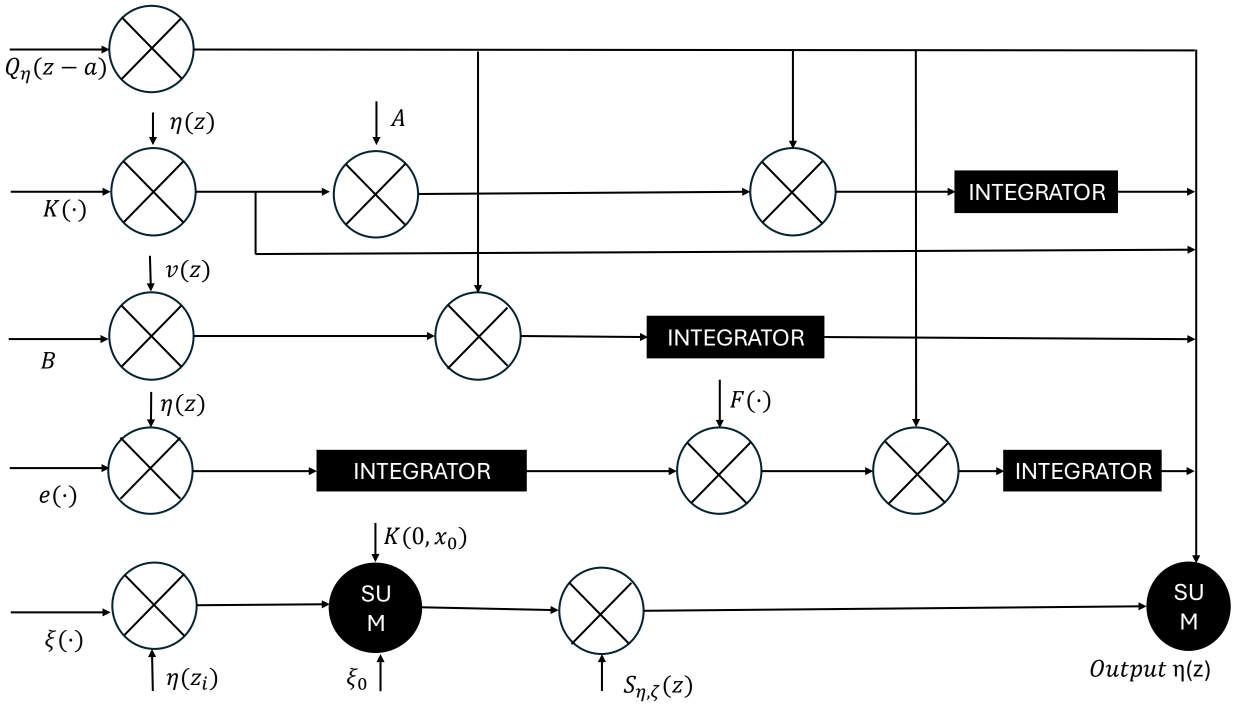

Figure 2. Filter system model.

1. The product modulator 1 accepts the input $\eta(z)$, and $\mathtt{K}(\cdot)$ produces the output $\mathtt{K}(zo, \eta_z)$.
2. The product modulator 2 accepts the input $\mathtt{K}(\cdot, \eta_z)$, and $\mathtt{A}$ and gives out put $\mathtt{AK}(\cdot)$.
3. The product modulator 3 accepts the input $Q^\eta_\Psi(z-\varsigma)$, and $\mathtt{AK}(\cdot, \cdot)$ produces the output $Q_\eta(z-\varsigma)\mathtt{AK}(\cdot)$.
4. The product modulator 4 accepts the input $v(z)$ and $\mathtt{B}$, produces the output $\mathtt{B}v(z)$.
5. The product modulator 5 accepts the input $Q_\eta(z-\varsigma)$, and $\mathtt{B}v(z)$, give the output $\mathtt{B}Q^\eta_\Psi(\rho, \vartheta)v(\rho)$
6. The product modulator 6 accepts the input $\eta(z)$ and $\mathtt{e}(\cdot)$, give the output $e(\cdot, \eta_z)$ over the period $(0, z)$.
7. The integrator exicute the input $\mathtt{F}(\cdots)$, and $\int e(\cdot, \eta_z)$, the product modulator 7 produces the output $\mathtt{F}\Big(z, \eta_z, \int_0^z e(z, s, \eta_s)ds\Big)$ over the period of time $(0, z)$, $\forall\ z \in [0, b]$.
8. The product modulator 8 accepts the input $Q^\eta_\Psi(\rho, \vartheta)$, and $\mathtt{F}\Big(z, \eta_z, \int_0^z e(z, s, \eta_s)ds\Big)$ gives the output $Q_\eta(\rho, \varsigma)\mathtt{F}\Big(z, \eta_z, \int_0^z e(z, s, \eta_s)ds\Big)$.
9. The product modulator 9 accepts $[\xi_0 + \xi(\mathfrak{y}_{z_1}, \mathfrak{y}_{z_2}, \ldots, z_n)(0) - \mathtt{K}(0, \eta(0))]$, and $S_{\eta,\xi}(\rho, 0)$ at time $\rho = 0$, produces $\mathcal{S}_{\eta,\xi}(\rho, 0)[\xi_0 + \xi(\mathfrak{y}_{z_1}, \mathfrak{y}_{z_2}, \ldots, z_n)(0) - \mathtt{K}(0, \eta(0))]$.
10. The integrators exicute the value:

$$\mathcal{S}_{\eta,\xi}(z)[\xi_0 + \xi(\mathfrak{y}_{z_1}, \mathfrak{y}_{z_2}, \ldots, z_n)(0) - \mathcal{K}(0, \mathfrak{y}_0)] + \mathcal{K}(z, \mathfrak{y}_z) + \int_0^z (z-\varsigma)^{\eta-1}\mathtt{A}Q_\eta(z-\varsigma)\mathcal{K}(\varsigma, \mathfrak{y}_\varsigma)d\varsigma$$
$$+ \int_0^z (z-\varsigma)^{\eta-1}Q_\eta(z-\varsigma)\underline{v}(\varsigma)d\varsigma + \int_0^z (z-\varsigma)^{(\eta-1)}Q_\eta(z-\varsigma)F\Big(\varsigma, \mathfrak{y}_\varsigma, \int_0^\varsigma e(\varsigma, s, u_s)ds\Big)d\varsigma.$$

produces the integral value over the period ρ. Finally, we turn all output from the integrators to summer network and output of $\mathfrak{y}(\rho)$ is obtained; it is bounded and approximately controllable.

7. Conclusions

We established the sufficient conditions for approximate controllability of Hilfer fractional neutral differential equations with infinite delay via almost sectorial operator in this study. Firstly, we evaluate the mild solution of the system by Laplace transform method. Next, we proved approximate controllability of Hilfer fractional diffeential systems, then extended to neutral term of system. The major conclusions are established by applying the results and ideas belonging to almost sectorial operators, fractional calculus, measure of noncompactness and fixed - point method. Finally, to explain the principle, we offered three applications have to explained our results via theoretical, numerical and filter system. In the future, we will study the exact controllability of Hilfer fractional differential system with infinite delay via almost sectorial operators using the fixed - point method.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

This work does not have any conflicts of interest.

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