Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 18(2): 296–328.
DOI: 10.3934/cam.2026012
Received: 22 August 2025
Revised: 23 October 2025
Accepted: 15 December 2025
Published: 10 April 2026

Research article

Energy decay of damped Timoshenko–type beams on two-parameter elastic foundations

Marwa Jomaa[1], **Toufic El Arwadi**[1,*], **Samer Israwi**[2,3] **and Dilberto da S. Almeida Júnior**[4]

[1] Department of Mathematics and Computer Science, Faculty of Science, Beirut Arab University, P.O Box 11-5020, Beirut, Lebanon

[2] Arts, Sciences, and Technology University in Lebanon (AUL), CRAMS: Center for Research in Applied Mathematics and Statistics, Beirut, Lebanon

[3] Laboratoire de mathématiques-EDST, Faculté des sciences et EDST, Université Libanaise, Hadat, Lebanon

[4] PhD Program in Mathematics, Federal University of Pará, Belém, Pará, Brazil

* **Correspondence:** Email: t.elarwadi@bau.edu.lb.

Abstract: This work is devoted to the analysis of well-posedness and stabilization properties of both the classical and truncated Timoshenko beam models resting on a two-parameter elastic foundation. We rigorously investigate the effects of the elastic subgrade on the dynamic behavior of the systems and consider the influence of viscous damping mechanisms acting on the angular rotation. For each model, we establish the existence, uniqueness, and continuous dependence of solutions. Moreover, we analyze the long-time behavior of the solutions, proving exponential or polynomial energy decay depending on the parameters of the wave speeds. The results highlight the differences in stability between the classical and truncated models and contribute to the mathematical theory of damped elastic structures interacting with elastic media.

Keywords: Timoshenko beam; two-parameter elastic foundation; energy decay; exponential stability; polynomial stability; viscous damping; well-posedness; stabilization
Mathematics Subject Classification: 35L53, 35B40

1. Introduction

The analysis of beam-column stability and dynamics is a fundamental aspect of structural mechanics, relying on two primary theories: the Bernoulli–Euler beam theory for slender members and the Timoshenko beam theory for stockier sections. These frameworks address deformation and stress under loads [1] but face challenges in modeling soil-structure interactions. Soil-structure interaction plays a crucial role in various fields of structural and foundation engineering, including the analysis of buildings, highways, railroads, and submerged pipes.

Over the past century, numerous physical and mathematical models have been developed to approximate real soil behavior. These models are often characterized by one, two, or more parameters. One of the earliest and most influential models for soil-structure interaction was introduced by Emil Winkler in 1867. This model represents the soil as a series of independent, linear elastic springs that provide resistance proportional to beam deflection. The simplicity of this approach, often called the Winkler model or Winkler foundation, has made it widely popular among engineers, especially for analyzing settlement and designing foundations. Winkler's key assumption is that the soil reaction at any point depends solely on the local deflection and is independent of neighboring points, effectively modeling the soil as a series of uncoupled springs. Moreover, the Winkler model uses a single parameter to describe soil stiffness and is commonly used to model soil behavior despite its simplest form of elastic foundation. Timoshenko and Gere studied simply supported uniform beams coupled with the Winkler foundation. In Refs [2–5], they investigated free vibration and stability of beams coupled with the Winkler foundation.

Many researchers have sought to refine and extend the Winkler model, which is widely used for addressing soil-structure interaction challenges. These refined models go beyond the Winkler assumption of independent vertical springs by introducing an additional parameter that reflects the interaction between adjacent springs. Two-parameter extension models include the Pasternak model [6], the Filonienko–Borodich model [7], and the Vlasov–Leontiev model, which introduce coupling effects through shear or membrane interactions, offering a more realistic representation of soil behavior. Their main characteristics and distinctions are summarized in the following Table 1.

Recent studies have addressed the analytical modeling of beams on elastic foundations, providing insights into soil-structure interaction. Notably, Yue et al. [8] proposed two novel Vlasov models for bending analysis of finite-length beams embedded in elastic foundations, deriving explicit analytical solutions that capture the coupled beam–foundation behavior.

Numerous researchers have investigated the behavior of beams supported by two-parameter elastic foundations, as documented in studies [9, 10]. Most of these works utilize finite element methods to carry out their analyses. For instance, Naidu and Rao focused on the stability and vibrational characteristics of Bernoulli–Euler beams resting on a modified Pasternak foundation model. Their research explored how different boundary conditions influence buckling loads under the foundation's effect. Similarly, Yokoyama [11] applied finite element analysis to both Bernoulli–Euler and Timoshenko beam theories, incorporating factors such as axial loading, foundation stiffness coefficients, transverse shear deformation, and rotary inertia into the model.

The Timoshenko beam model, introduced in 1921, improves classical beam theories by incorporating rotational inertia and transverse shear through a hyperbolic system, accurately capturing shear and rotary inertia effects. This has led to increased interest in its dynamic analysis with various damping mechanisms. Soufyane [12] investigated a Timoshenko beam system with frictional damping applied solely to the angular deformation equation. The study demonstrated that the system achieves exponential stability under the equal wave speeds condition. If this condition is not met, the system is only polynomially stable, as established by Racke and Rivera [13].

A notable feature of the model is the existence of two natural frequencies, causing a physical paradox known as the second spectrum, which was identified in later analytical studies. To address this, Elishakoff [14] proposed a truncated version of the classical Timoshenko system. Recently, this truncated system has attracted significant research attention, with different damping strategies employed to stabilize it. Almeida Júnior and Ramos [15] introduced linear frictional damping in the transverse displacement equation and demonstrated that this damping exponentially stabilizes the system independently of wave speed conditions. Similar stabilization results using various dissipation types have been reported in other studies [16–20].

Table 1. Key characteristics of Winkler, Pasternak, Filonienko-Borodich, and Vlasov-Leontiev models.

Model	Parameters	Physical Interpretation	Key Features	Soil Interaction Type
Winkler	Single parameter k_1	Independent vertical springs	Soil reaction proportional to local deflection; no shear interaction between points	No coupling, springs act independently
Pasternak [6]	Two parameters k_1, k_2	Vertical springs + shear layer	Second parameter models shear modulus of continuous shear layer connecting springs	Includes lateral shear transfer between springs
Filonienko [7] Borodich	Two parameters k_1, k_2	Vertical springs + elastic membrane tension	Second parameter models tension in elastic membrane linking spring tops	Coupling via elastic membrane tension
Vlasov Leontiev	Two parameters k_1, k_2	Elastic layer on rigid base	Incorporates bending and shear effects in foundation response	More general interaction including bending

Despite these contributions, detailed analytical studies focusing on the stability and vibration of beams supported by elastic foundations remain relatively limited in the existing literature. Almeida Júnior et al. in their paper [21], extended beyond many previous works by rigorously analyzing the stability and energy decay of Timoshenko beams resting on a Winkler foundation, incorporating damping effects applied to angular rotation. The authors analyzed the decay behavior of the classical system's energy. They demonstrated that while exponential decay may be achieved under equal wave speed conditions, it can fail when wave speeds differ, alongside polynomial decay rates. For the truncated system, the manuscript showed exponential decay of energy solutions. In [22], the authors analyzed the Timoshenko beam behavior on the Winkler elastic foundation coupled with a Kelvin-Voight damping. Their research established that the fully viscoelastic system, where both bending moment and shear stress exhibit damping, demonstrates analytic behavior, leading to exponential stability. In contrast, partial viscoelastic systems are found to be polynomially stable. The study also explored cases without a second spectrum, proving exponential stability.

In this work, we analyze the vibrational behavior of the Timoshenko beam model with initial length L resting on a two-parameter elastic foundation, while assuming viscous damping acts only on the angle of rotation. The Timoshenko beam is made of an isotropic homogeneous linear elastic material characterized by the Young modulus E, shear modulus G, and Poisson ratio ν. The transverse cross-section is doubly symmetric, with cross-sectional area A, and the second moment of the cross-sectional area I. The beam has uniform mass density ρ per unit length. The shear coefficient k accounts for the shape of the cross-section. The total transverse shear force is denoted by $S(x,t)$, where $t \in (0,\infty)$ is the time and $x \in (0,L)$ is the distance along the center line of the beam structure, depending on the vertical deflection φ, and the rotation of the cross section due to bending ψ, as illustrated in Figure 1, expressed as

$$S(x,t) = k(\varphi_x + \psi)(x,t), \tag{1.1}$$

where $k = JGA$. J represents the transverse shear factor. Additionally, the bending moment in this model, denoted by $M(x,t)$ (see Figure 1), is given by

$$M(x,t) = b\psi_x(x,t), \tag{1.2}$$

where $b = EI$.

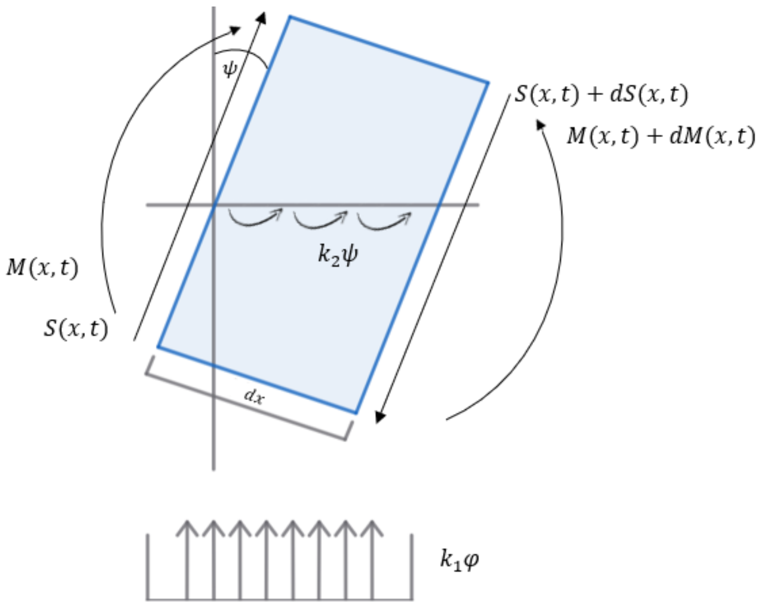

Figure 1. Internal forces and foundation reactions in a Timoshenko beam on a two-parameter elastic foundation, where k_1 represents vertical (Winkler-type) stiffness and k_2 denotes soil shear interaction linked to beam rotation.

The Timoshenko system is written as

$$\begin{cases} \rho_1\varphi_{tt}(x,t) = S_x(x,t), \\ \rho_2\psi_{tt}(x,t) = M_x(x,t) - S(x,t). \end{cases} \tag{1.3}$$

Substituting $S(x,t)$ and $M(x,t)$, we arrive

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k(\varphi_x + \psi)(x,t) = 0. \end{cases} \tag{1.4}$$

The elastic foundation is modeled using a two-parameter approach, characterized by the vertical foundation modulus k_1 (also known as the Winkler parameter), which characterizes the vertical stiffness of the foundation, and the horizontal foundation modulus k_2, which introduces an additional parameter to account for the shear interaction within the soil that the Winkler model does not capture. When $k_2 = 0$, this model simplifies to the traditional Winkler model. The foundation's response is captured through two reactions: the vertical ground reaction, which is proportional to the vertical displacement of the beam, and given by $k_1\varphi$, and the horizontal reaction, which is proportional to the horizontal displacement of the beam's extreme fibers, and given by $k_2\psi$.

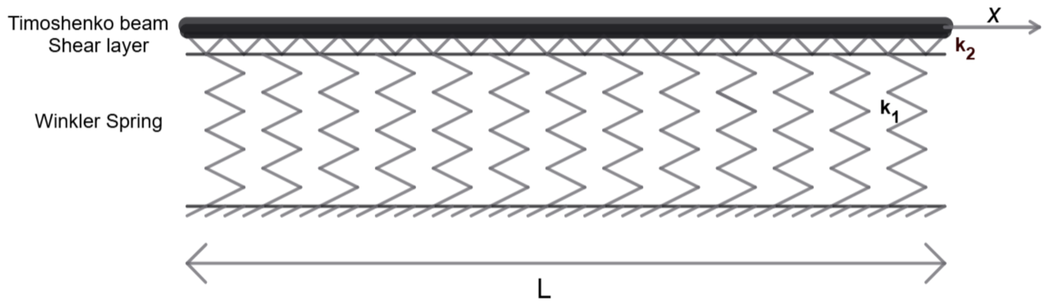

Figure 2. Timoshenko beam resting on a two-parameter elastic foundation, where k_1 represents vertical stiffness (Winkler), and k_2 represents the shear layer stiffness along the beam.

The transverse and rotational equilibrium equations are as follows :

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k(\varphi_x + \psi)_x(x,t) + k_1\varphi(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k(\varphi_x + \psi)(x,t) + k_2\psi(x,t) = 0. \end{cases} \tag{1.5}$$

Introducing damping on the rotational angle in the classical Timoshenko beam model with a two-parameter elastic foundation (1.5) effectively prevents the blow-up of the second phase velocity at low wave numbers, making the model more physically realistic. Alternatively, a truncated version of the model simplifies analysis by reducing to a single-phase velocity, removing the requirement for equal wave speed assumptions while ensuring exponential energy decay solutions.

Our proposal relies on the study of the well-posedness and stabilization properties of the classical and truncated Timoshenko systems resting on a two-parameter elastic foundation, subject to the viscous damping effect acting on the angular rotation. The layout of this paper is as follows.

- In Section 2, we study the well-posedness of the classical Timoshenko system resting on a two-parameter elastic foundation coupled with viscous damping acting on angle rotation (2.1). In order to do this, we use the semigroup theory.
- In Section (3), we demonstrate the conditions that lead to a lack of exponential decay of the classical Timoshenko system. We use the well-known Gearhart–Herbst–Prüss–Huang theorem for dissipative systems.
- In Section (4), we establish exponential decay under non-equal wave speeds.
- In Section (5), we examine the polynomial decay behavior and ascertain the optimality of the decay rate.
- In Section (6), we focus on the truncated system (6.1), also coupled with viscous damping on angle rotation.

- In Section (7), we prove well-posedness using the Faedo–Galerkin method.
- In Section (8), we demonstrate exponential decay for energy solutions via the energy method, removing the requirement for equal wave speed assumptions.
- In Section (9), we present the main conclusions of the study.

2. Semigroup framework

In this section, we are concerned with the existence and uniqueness of a solution for the dissipative Timoshenko-type system given by

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + k_1\varphi = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + k_2\psi + \mu\psi_t = 0, \end{cases} \tag{2.1}$$

where $\mu > 0$ characterizes the coefficient of viscous damping due to internal friction forces.
Consider the initial conditions demonstrated by

$$\varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), \quad x \in (0,L), \tag{2.2}$$

and consider the mixed boundary conditions where we use the Dirichlet condition on φ and the Neumann condition on ψ given by

$$\varphi(0,t) = \varphi(L,t) = 0, \ \psi_x(0,t) = \psi_x(L,t) = 0, \forall t \geq 0. \tag{2.3}$$

The system (2.1)–(2.3) has a dissipative nature. In that direction, the respective energy is defined by

$$E(t) := \frac{1}{2}\int_0^L \left[\rho_1|\varphi_t|^2 + k|\varphi_x + \psi|^2 + k_1|\varphi|^2 + \rho_2|\psi_t|^2 + b|\psi_x|^2 + k_2|\psi|^2\right] dx, \ \forall t > 0, \tag{2.4}$$

from where we have a first result.

Lemma 2.1. *The energy* $E(t)$ (2.6) *of the system* (2.1) − (2.3) *satisfies the energy dissipation law given by*

$$\frac{d}{dt}E(t) = -\mu\int_0^L |\psi_t|^2 dx, \ t > 0. \tag{2.5}$$

Proof. Indeed, multiply $(2.1)_1$ by φ_t and $(2.1)_2$ by ψ_t, then integrate over $(0,L)$. We derive

$$\begin{aligned} \rho_1\int_0^L \varphi_{tt}\varphi_t dx - k\int_0^L (\varphi_x + \psi)_x\varphi_t dx + k_1\int_0^L \varphi\varphi_t dx = 0, \\ \rho_2\int_0^L \psi_{tt}\psi_t dx - b\int_0^L \psi_{xx}\psi_t dx + k\int_0^L (\varphi_x + \psi)\psi_t dx + k_2\int_0^L \psi\psi_t dx + \mu\int_0^L |\psi_t|^2 dx = 0. \end{aligned} \tag{2.6}$$

Now, using integration by parts over $(0,L)$, taking into account the boundary conditions (2.3), then adding up Eqs $(2.6)_1$ and $(2.6)_2$, after applying basic identities, we get

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L |\varphi_t|^2 dx + \frac{k}{2}\frac{d}{dt}\int_0^L |\varphi_x + \psi|^2 dx + \frac{k_2}{2}\frac{d}{dt}\int_0^L |\varphi|^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^L |\psi_t|^2 dx + \frac{b}{2}\frac{d}{dt}\int_0^L |\psi_x|^2 dx = -\mu\int_0^L |\psi_t|^2 dx, \tag{2.7}$$

and this leads to the dissipation law given by

$$\frac{d}{dt}E(t) = -\mu \int_0^L |\psi_t|^2 dx, \tag{2.8}$$

where the energy is represented as follows:

$$E(t) = \frac{\rho_1}{2}\int_0^L |\varphi_t|^2 dx + \frac{k}{2}\int_0^L |\varphi_x + \psi|^2 dx + \frac{k_2}{2}\int_0^L |\varphi|^2 dx + \frac{\rho_2}{2}\int_0^L |\psi_t|^2 dx + \frac{b}{2}\int_0^L |\psi_x|^2 dx. \tag{2.9}$$

□

We now turn our attention to investigating the system's well-posedness. By defining the auxiliary variables $z = \varphi_t$ and $y = \psi_t$, the system can be reformulated as an abstract evolution equation of the form

$$\begin{cases} \dfrac{d}{dt}U(t) = \mathcal{A}U(t), \\ U(t=0) = U_0, \end{cases} \tag{2.10}$$

where $U = (\varphi, z, \psi, y)^T$ and $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1)^T \in \mathcal{H}$. The operator $\mathcal{A}$ is expressed by

$$\mathcal{A}U(t) = \begin{pmatrix} \varphi_t \\ \frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{k_1}{\rho_1}\varphi \\ \psi_t \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) - \frac{k_2}{\rho_2}\psi - \frac{\mu}{\rho_2}\psi_t \end{pmatrix}. \tag{2.11}$$

Now we consider the Hilbert space $\mathcal{H}$ defined by

$$\mathcal{H} := H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L_*^2(0,L),$$

induced with the inner product as defined below:

$$\begin{aligned} \left\langle Z_1, Z_2 \right\rangle :=& \frac{1}{2}\Big(\rho_1 \int_0^L z_1\bar{z}_2 dx + k\int_0^L (\varphi_{1,x} + \psi_1)\overline{(\varphi_{2,x} + \psi_2)} dx + k_1 \int_0^L \varphi_1\bar{\varphi}_2 dx + \rho_2 \int_0^L y_1\bar{y}_2 dx \\ &+ b\int_0^L \psi_{1,x}\bar{\psi}_{2,x} dx + k_2 \int_0^L \psi_1\bar{\psi}_2 dx\Big), \end{aligned} \tag{2.12}$$

where $Z_1 = (\varphi, z_1, \psi_1, y_1)^T$ and $Z_2 = (\varphi_2, z_2, \psi_2, y_2)^T$, and the norm is given by

$$\|Z_1\|_{\mathcal{H}}^2 := \langle Z_1, Z_1 \rangle.$$

We consider the spaces $L_*^2(0,L)$, $H_*^1(0,L)$ and $H_*^2(0,L)$ as given below:

$$\begin{aligned} L_*^2(0,L) &:= \left\{ g \in L^2(0,L); \int_0^L g(x)dx = 0 \right\}, \\ H_*^1 &:= H^1(0,L) \cap L_*^2(0,L), \\ H_*^2(0,L) &:= \left\{ g \in H^2(0,L) | g_x(0) = g_x(L) = 0 \right\}. \end{aligned}$$

The maximal domain of the operator $\mathcal{A}$ is given by

$$D(\mathcal{A}) := \{Z \in \mathcal{H}; \varphi \in H_0^1(0,L) \cap H^2(0,L); z \in H_0^1(0,L); \psi \in H_*^1(0,L) \cap H_*^2(0,L); y \in H_*^1(0,L)\}.$$

Note that $D(\mathcal{A})$ is dense in $\mathcal{H}$. The upcoming lemmas will help us prove that our system is well posed.

Lemma 2.2. *The operator $\mathcal{A}$ satisfies the dissipating property, i.e., for all $Z \in \mathcal{H}$*

$$Re\,\langle \mathcal{A}Z, Z\rangle_{\mathcal{H}} \leq 0.$$

Proof. From (2.12), for all $Z = (\varphi, z, \psi, y) \in \mathcal{H}$, we have

$$\begin{aligned}\langle AZ, Z\rangle_{\mathcal{H}} &= \frac{\rho_1}{2}\int_0^L \left(\frac{k}{\rho_1}(\varphi_x+\psi)_x - \frac{k_1}{\rho_1}\varphi\right) z dx + \frac{k}{2}\int_0^L (z_x + y)(\varphi_x+\psi)dx + \frac{k_1}{2}\int_0^L z\varphi dx \\ &+ \frac{\rho_2}{2}\int_0^L \left(\frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x+\psi) - \frac{k_2}{\rho_2}\psi - \frac{\mu}{\rho_2}\psi_t\right) y dx + \frac{b}{2}\int_0^L y_x\psi_x dx + \frac{k_2}{2}\int_0^L y\psi dx. \quad (2.13)\\ &= -\mu\int_0^L \psi_t^2 dx \leq 0,\end{aligned}$$

ensuring the dissipative nature of $\mathcal{A}$ in space $\mathcal{H}$. □

The upcoming lemma helps us to show the maximality of the operator $I - \mathcal{A}$.

Lemma 2.3. *The operator $I - \mathcal{A}$ is surjective.*

Proof. Consider $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$. We have to find $U(t) = (\varphi, z, \psi, y) \in D(\mathcal{A})$ satisfying the following equation:

$$-\mathcal{A}U(t) = F. \quad (2.14)$$

As a direct result, we have

$$\begin{aligned} z &= -f_1,\\ k(\varphi_x+\psi)_x - k_1\varphi &= -\rho_1 f_2,\\ y &= -f_3,\\ b\psi_{xx} - k(\varphi_x+\psi) - k_2\psi - \mu y &= -\rho_2 f_4.\end{aligned} \quad (2.15)$$

Substituting $(2.15)_3$ by $(2.15)_4$, we get

$$\begin{aligned} k(\varphi_x+\psi)_x - k_1\varphi &= -\rho_1 f_2,\\ b\psi_{xx} - k(\varphi_x+\psi) - k_2\psi &= -\rho_2 f_4 - \mu f_3.\end{aligned} \quad (2.16)$$

Multiplying $(2.16)_1$ by $\tilde{\varphi} \in H_0^1(0,L)$ and (2.16) by $\tilde{\psi} \in H_*^1(0,L)$, then integrating over $(0,L)$, we get

$$\begin{aligned} k\int_0^L (\varphi_x+\psi)_x\tilde{\varphi}dx - k_1\int_0^L \varphi\tilde{\varphi}dx &= -\rho_1\int_0^L f_2\tilde{\varphi}dx,\\ b\int_0^L \psi_{xx}\tilde{\psi}dx - k\int_0^L (\varphi_x+\psi)\tilde{\psi}dx - k_2\int_0^L \psi\tilde{\psi}dx &= -\rho_2\int_0^L f_4\tilde{\psi}dx - \mu\int_0^L f_3\tilde{\psi}dx.\end{aligned} \quad (2.17)$$

Adding up the two equations, we get

$$\mathcal{E}((\varphi,\psi),(\tilde{\varphi},\tilde{\psi})) = \mathcal{K}(\tilde{\varphi},\tilde{\psi}), \quad (2.18)$$

where $\mathcal{E}$ is a bilinear form defined over $V \times V$, where $V = (H_0^1(0,L), H_*^1(0,L))$, given by

$$\mathcal{E}((\varphi,\psi),(\tilde{\varphi},\tilde{\psi})) = k\int_0^L (\varphi_x+\psi)(\tilde{\varphi}_x+\tilde{\psi})dx + k_1\int_0^L \varphi\tilde{\varphi}dx + b\int_0^L \psi_x\tilde{\psi}_x dx + k_2\int_0^L \psi\tilde{\psi}dx,$$

and $\mathcal{K}$ is a continuous functional defined over V and given by

$$\mathcal{K}(\tilde{\varphi},\tilde{\psi}) = \rho_1 \int_0^L f_2\tilde{\varphi}dx + \rho_2 \int_0^L f_4\tilde{\psi}dx + \mu \int_0^L f_3\tilde{\psi}dx.$$

Note that $\mathcal{E}$ is continuous over its domain. Now, let us show coercivity. It is noted that the space V is endowed with the following norm:

$$\|(\varphi,\psi)\|_V^2 := \|\varphi\|^2 + \|\varphi_x\|^2 + \|\psi\|^2 + \|\psi_x\|^2.$$

Now, we have

$$\begin{aligned}\mathcal{E}((\varphi,\psi),(\varphi,\psi)) &= k\int_0^L (\varphi_x+\psi)^2dx + k_1\int_0^L \varphi^2dx + b\int_0^L \psi_x^2dx + k_2\int_0^L \psi^2dx \\ &\geq \frac{k}{2}\int_0^L \varphi_x^2dx + \frac{k}{2}\int_0^L \psi^2dx + k_1\int_0^L \varphi^2dx + b\int_0^L \psi_x^2dx + k_2\int_0^L \psi^2dx,\end{aligned}$$

so

$$\mathcal{E}((\varphi,\psi),(\varphi,\psi)) \geq \alpha\|(\varphi,\psi)\|_V^2,$$

where $\alpha = min\{k, k_1, k_2, b\}$. Now, by the Lax–Milgram theorem, there exists a unique solution $(\varphi,\psi) \in V$ satisfying the following equation:

$$\mathcal{E}((\varphi,\psi),(\tilde{\varphi},\tilde{\psi})) = \mathcal{K}(\tilde{\varphi},\tilde{\psi}), \forall(\tilde{\varphi},\tilde{\psi}) \in V. \tag{2.19}$$

We consider the test function as following $(\varphi_1, 0)$ with $\varphi_1 \in \mathcal{D}(0,L)$ and $(0,\psi_1)$ with $\psi_1 \in \mathcal{D}(0,L)$, from where (φ,ψ) is a weak solution of (2.15). Moreover, we have

$$\begin{aligned}\varphi_{xx} &= -\psi_x + \frac{k_1}{k}\varphi - \frac{\rho_1}{k}f_2 \quad \in L^2(0,L),\\ \psi_{xx} &= \frac{k}{b}(\varphi_x+\psi) + \frac{k_2}{b}\psi + \frac{\mu}{k}y - \frac{\rho_2}{k}f_4 \quad \in L^2(0,L).\end{aligned} \tag{2.20}$$

Also, we have $z = -f_1 \in H_0^1(0,L)$ and $y = -f_3 \in H_*^1(0,L)$, then $U(t) \in D(\mathcal{A})$.
Since $D(\mathcal{A})$ is dense in $\mathcal{H}$, then applying the Lumer–Phillips theorem, $\mathcal{A}$ generates a C_0-semigroups of contractions over the space $\mathcal{H}$. Now, we have the well-posedness result of our system treated in the next theorem.

Theorem 2.4. *The problem* (2.1) *has only one weak solution. Whenever the initial data* $U_0 = (\varphi_0,\varphi_1,\psi_0,\psi_1)$ *belongs to the space* $\mathcal{H}$*, then*

$$U \in C([0,\infty[,\mathcal{H}). \tag{2.21}$$

System (2.1) *also has a strong solution. Whenever the initial data* $U_0 = (\varphi_0,\varphi_1,\psi_0,\psi_1)$ *belongs to the domain of* $\mathcal{A}$*, then*

$$U \in C\left([0,\infty[, D(\mathcal{A})\right) \cap C^1\left([0,\infty[,\mathcal{H}\right). \tag{2.22}$$

□

In the following section, we investigate the limitations of the decay properties of the system, showing, in particular, that exponential decay does not occur under general parameter settings.

3. Lack of exponential decay

In this section, we will start by showing that the semigroup $S(t)$ associated with the system (2.1) is not exponentially stable as long as condition (3.3) is satisfied, with the help of the Gearhart–Herbst–Prüss–Huang theorem for dissipative systems; see [23]. Physically, the second condition in (3.3) identifies a critical case related to the interaction between the two deformation fields of the beam resting on the elastic foundation, in which the coupling between them becomes unbalanced. When this mismatch occurs, certain frequencies fail to dissipate energy efficiently. Instead of exponential decay, energy may persist, reflect, or oscillate within the system, slowing the return to equilibrium.

Theorem 3.1. *The C_0-semigroup of contractions $S(t)$ defined on a Hilbert space $\mathcal{H}$ is exponentially stable if and only if we have*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{3.1}$$

and we have

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{A})} < \infty. \tag{3.2}$$

The main result of this section is given by the upcoming theorem.

Theorem 3.2. *The semigroup $S(t)$ associated with the system* (2.1) *is not exponentially stable, whenever we consider*

$$\begin{aligned} \frac{k}{\rho_1} &\neq \frac{b}{\rho_2}, \\ k_1 &\neq \frac{k^2\rho_1}{b\rho_2 - k\rho_2}. \end{aligned} \tag{3.3}$$

Proof. Our proof will be based on showing that (3.2) does not hold. In other words, we will show that there exist a sequence of numbers $(\lambda_n)_{n\in\mathbb{N}} \subset i\mathbb{R}$ and a sequence of data $(F_n)_{n\in\mathbb{N}} \subset \mathcal{H}$, with $\|F_n\|_{\mathcal{H}} < \infty$ such that the norm of the resolvent operator $\|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}$ tends to infinity as $n \to \infty$, with the resolvent equation

$$\lambda_n U_n - \mathcal{A}U_n = F_n. \tag{3.4}$$

Now, rewriting the resolvent equation related to our system (2.1) in terms of the components of $U_n = (\varphi^n, z^n, \psi^n, y^n)^T$ and $F_n = (f_1, f_2, f_3, f_4)^T$, we have

$$\begin{aligned} \lambda_n\varphi^n - z^n &= f_1, \\ \lambda_n\rho_1 z^n - k(\varphi^n_x + \psi^n)_x + k_1\varphi^n &= \rho_1 f_2, \\ \lambda_n\psi^n - y^n &= f_3, \\ \lambda_n\rho_2 y^n - b\psi^n_{xx} + k(\varphi^n_x + \psi^n) + k_2\psi^n + \mu y^n &= \rho_2 f_4. \end{aligned} \tag{3.5}$$

If we choose F_n in way such that $f_1 = f_3 = 0$, so we have $\lambda_n\varphi^n = z^n$ and $\lambda_n\psi^n = y^n$, and we can rewrite $(3.5)_2$ and $(3.5)_4$ as the following:

$$\begin{aligned} \lambda_n^2\rho_1\varphi^n - k(\varphi^n_x + \psi^n)_x + k_1\varphi^n &= \rho_1 f_2, \\ \lambda_n^2\rho_2\psi^n - b\psi^n_{xx} + k(\varphi^n_x + \psi^n) + k_2\psi^n + \mu\lambda_n\psi &= \rho_2 f_4. \end{aligned} \tag{3.6}$$

Due to (2.3), we choose φ^n and ψ^n as the following:

$$\varphi^n = A\sin(\beta_n x) \quad \textbf{and} \quad \psi^n = B\cos(\beta_n x),$$

as well as

$$f_2 = \rho_1^{-1}\sin(\beta_n x) \quad \textbf{and} \quad f_4 = \rho_2^{-1}\cos(\beta_n x).$$

The choice of U_n and F_n solves the system (6.9) if A and B satisfy

$$\begin{aligned} (\lambda_n^2\rho_1 + k\beta_n^2 + k_1)A + k\beta_n B = 1, \\ k\beta_n A + (\lambda_n^2\rho_2 + b\beta_n^2 + k + k_1 + \mu\lambda_n)B = 1. \end{aligned} \tag{3.7}$$

Now, we take

$$\lambda_n := i\sqrt{(k/\rho_1)}\beta_n, \quad \forall n \in \mathbb{N},$$

and we consider $\beta_n = n\pi/L$, where i is the unit imaginary number. Substituting λ_n in system (3.7), we get

$$\begin{aligned} (\lambda_n^2\rho_1 + k\beta_n^2 + k_1)A + \beta_n B = 1, \\ \beta_n A + (\lambda_n\rho_2 + b\beta_n^2 + k + k_2 + \mu\lambda_n)B = 1. \end{aligned} \tag{3.8}$$

After solving system (3.8), we obtain A as given below:

$$A = \frac{\frac{\beta_n^2}{\rho_1}(b\rho_1 - k\rho_2) + \frac{\beta_n}{\rho_1}\left(i\mu\sqrt{k\rho_1} - k\rho_1\right) + k + k_2}{\frac{\beta_n^2}{\rho_1}\left[k_1(b\rho_1 - k\rho_2) - k^2\rho_1\right] + \frac{k_1}{\rho_1}\left(i\mu\sqrt{k\rho_1}\beta_n + k\rho_1 + k_2\rho_1\right)}.$$

Taking $k_1 \neq \frac{k^2\rho_1}{(b\rho_1 - k\rho_2)}$, and under all the considered conditions, we have

$$A \approx \frac{b\rho_1 - k\rho_2}{k_1(b\rho_1 - k\rho_2) - k^2\rho_1}.$$

Moreover, from $(3.5)_1$, we have

$$\begin{aligned} z^n(x) &= \lambda_n\varphi^n(x) \\ &= \lambda_n A\sin(\beta_n x). \end{aligned}$$

Integrating over $(0, L)$ and using the fact that

$$\begin{aligned} \int_0^L \sin(\beta_n x)^2 dx &= \frac{1}{2}\int_0^L (1 - \cos(2\beta_n x))dx \\ &= \frac{L}{2} - \frac{\sin(2\beta_n x)}{4\beta_n}\Big|_0^L \\ &= \frac{L}{2} \end{aligned} \tag{3.9}$$

we get

$$\begin{aligned} \int_0^L |z^n(x)|^2 dx &= \left|\frac{\frac{\beta_n^2}{\rho_1}(b\rho_1 - k\rho_2) + \frac{\beta_n}{\rho_1}\left(i\mu\sqrt{k\rho_1} - k\rho_1\right) + k + k_2}{\frac{\beta_n^2}{\rho_1}\left[k_1(b\rho_1 - k\rho_2) - k^2\rho_1\right] + \frac{k_1}{\rho_1}\left(i\mu\sqrt{k\rho_1}\beta_n + k\rho_1 + k_2\rho_1\right)}\lambda_n\right|^2 \frac{L}{2} \\ &= \frac{Lk}{2\rho_1}\beta_n^2\left|\frac{\frac{\beta_n^2}{\rho_1}(b\rho_1 - k\rho_2) + \frac{\beta_n}{\rho_1}\left(i\mu\sqrt{k\rho_1} - k\rho_1\right) + k + k_2}{\frac{\beta_n^2}{\rho_1}\left[k_1(b\rho_1 - k\rho_2) - k^2\rho_1\right] + \frac{k_1}{\rho_1}\left(i\mu\sqrt{k\rho_1}\beta_n + k\rho_1 + k_2\rho_1\right)}\right|^2. \end{aligned}$$

Taking into consideration the equal speed limits assumption where we have $\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$, we get

$$\int_0^L |z^n(x)|^2 dx \approx \frac{Lk}{2\rho_1} \beta_n^2 \left| \frac{b\rho_1 - k\rho_2}{k_1(b\rho_1 - k\rho_2) - k^2\rho_1} \right|^2 \to +\infty. \tag{3.10}$$

Therefore, we get

$$\lim_{n\to\infty} \|U_n\|_{\mathcal{H}}^2 \geq \lim_{n\to\infty} \|z^n\|_{L^2(0,L)}^2 \to \infty.$$

Finally, we conclude that the second assertion (3.2) of Theorem does not hold, and thus our system (2.1) is not exponentially stable under condition (3.3). □

4. Exponential decay

In this section, we will show the exponential stability of our system under a particular relationship between the parameters of the system. The assumption of the coefficients given by

$$\frac{k}{\rho_1} = \frac{b}{\rho_2} \tag{4.1}$$

plays a crucial role in achieving our results. To proceed, we present the theorem that articulates our main finding.

Theorem 4.1. *The semigroup associated with system* (2.1) *is exponentially stable under the above condition*

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}. \tag{4.2}$$

To establish this result, we consider the resolvent equation given by

$$\begin{aligned} \lambda\varphi - z &= f_1, \\ \lambda\rho_1 z - k(\varphi_x + \psi)_x + k_1\varphi &= \rho_1 f_2, \\ \lambda\psi - y &= f_3, \\ \lambda\rho_2 y - b\psi_{xx} + k(\varphi_x + \psi) + k_2\psi + \mu y &= \rho_2 f_4. \end{aligned} \tag{4.3}$$

Multiplying (4.3) by $U = (\varphi, z, \psi, y)^T \in D(\mathcal{A})$, taking the real part, and using (2.13), we get

$$\int_0^L \psi_t^2 dx \leq \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \tag{4.4}$$

We proceed by formulating the following technical lemma, which is essential for the proof of the main theorem.

Lemma 4.2. *The imaginary axis $i\mathbb{R}$ is included in the resolvent set of the operator $\mathcal{A}$, i.e.,*

$$\rho(\mathcal{A}) \supset i\mathbb{R}. \tag{4.5}$$

Proof. Due to the regularity property of $\mathcal{A}$, as $\mathcal{A}$ has a compact resolvent, the compactness of the resolvent guarantees a discrete spectrum. So in order to prove (4.5), it suffices to demonstrate the injectivity of $\lambda I - \mathcal{A}$. Therefore, from (4.4) and the fact that $F = 0$, we have $\int_0^L \psi_t^2 dx = 0$, so we guarantee that $\psi_t = y = 0$. Now, substitute this value of y in Eq $(4.3)_3$, and we get $\psi = 0$. Using $(4.3)_4$, we get $\varphi_x = 0$, and applying Poincaré inequality, we get $\varphi = 0$. Substituting the deduced result in $(4.3)_1$, we guarantee that $z = 0$. So, the desired result of injectivity is obtained where we have $U = 0$. □

The upcoming lemmas are critical in demonstrating the exponential stability of the system.

Lemma 4.3. *Based on the stated conditions, the required inequality holds consistently:*

$$\int_0^L \varphi^2 dx \leq \frac{1}{\lambda^2}\|U\|_{\mathcal{H}}^2 + \frac{C}{|\lambda|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{4.6}$$

for $|\lambda|$ large enough, and C is a positive constant.

Proof. Multiply $(4.3)_1$ by $\tilde{\varphi}$, then after integrating over $(0, L)$, we get

$$|\lambda| \int_0^L \varphi^2 dx \leq |z||\varphi| dx + \int_0^L |f_1||\varphi| dx.$$

Using Young's inequality for $\varepsilon = |\lambda|$, our equation becomes

$$\frac{|\lambda|}{2}\int_0^L |\varphi|^2 dx \leq \frac{1}{2|\lambda|}\int_0^L z^2 dx + \int_0^L |f_1||\varphi| dx.$$

Using Schwarz's inequality, we get

$$\int_0^L \varphi^2 dx \leq \frac{1}{\lambda^2}\|U\|_{\mathcal{H}}^2 + \frac{C}{|\lambda|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

So, the desired result is obtained. □

We now establish the following lemma.

Lemma 4.4. *Based on the stated conditions, the required inequality holds consistently:*

$$\rho_1 \int_0^L |z|^2 dx \leq \frac{3k}{2}\rho_1 \int_0^L (\varphi_x + \psi)^2 dx + \frac{kc_p}{2}\rho_1 \int_0^L |\psi_x|^2 dx + k_1 \int_0^L \varphi^2 dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{4.7}$$

for $|\lambda|$ large enough, and C is a positive constant.

Proof. Multiply $(4.3)_2$ by $\tilde{\varphi}$, then after integrating over $(0, L)$, we get

$$\lambda\rho_1 \int_0^L z\tilde{\varphi} dx + k\int_0^L (\varphi_x + \psi)\tilde{\varphi}_x dx + k_1 \int_0^L \varphi^2 dx = \rho_1 \int_0^L f_2\tilde{\varphi} dx.$$

Let $I_1 := \lambda\rho_1 \int_0^L z\tilde{\varphi} dx$. From $(4.3)_1$ we have $\varphi = (1/\lambda)z + (1/\lambda)f_1$. Substituting it in I_1, we get

$$\rho_1 \int_0^L z^2 dx = -\rho_1 \int_0^L z\tilde{f}_1 dx + k\int_0^L (\varphi_x + \psi)^2 dx - k\int_0^L (\varphi_x + \psi)\tilde{\psi} dx + k_1 \int_0^L \varphi^2 dx - \rho_1 \int_0^L f_2\tilde{\varphi} dx.$$

Now, using Poincaré's inequality, we get

$$\rho_1 \int_0^L z^2 dx \le \frac{3k}{2}\rho_1 \int_0^L (\varphi_x + \psi)^2 dx + \frac{kc_p}{2}\rho_1 \int_0^L |\psi_x|^2 dx + k_1 \int_0^L \varphi^2 dx - \rho_1 \int_0^L z\tilde{f}_1 dx - \rho_1 \int_0^L f_2 \tilde{\varphi} dx.$$

Using Cauchy–Schwarz's inequality, we get

$$\rho_1 \int_0^L z^2 dx \le \frac{3k}{2}\rho_1 \int_0^L (\varphi_x + \psi)^2 dx + \frac{kc_p}{2}\rho_1 \int_0^L |\psi_x|^2 dx + k_1 \int_0^L \varphi^2 dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

So, the desired result is obtained. □

We now establish the following lemma.

Lemma 4.5. *Based on the stated conditions, the required inequality holds consistently:*

$$\frac{k}{2}\int_0^L (\varphi_x+\psi)^2 dx \le \lambda b\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^L \psi_x \bar{z} dx + C_\varepsilon \int_0^L \varphi^2 dx + \left(\varepsilon + \frac{c_p k_2^2}{k}\right)\int_0^L \psi_x^2 dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \quad (4.8)$$

Proof. Multiply Eq $(4.3)_4$ by $\overline{(\varphi_x + \psi)}$, then after integrating over $(0, L)$, we get

$$\begin{aligned}
&\lambda\rho_2 \int_0^L y\bar{\varphi}_x dx + \lambda\rho_2 \int_0^L y\bar{\psi} dx + b\int_0^L \psi_x \overline{(\varphi_x + \psi)}_x dx + k\int_0^L (\varphi_x + \psi)^2 dx + k_2 \int_0^L \psi\overline{(\varphi_x + \psi)} dx \\
&+ \mu \int_0^L \overline{(\varphi_x + \psi)} dx = \rho_2 \int_0^L f_4 \overline{(\varphi_x + \psi)} dx.
\end{aligned} \quad (4.9)$$

Now, let us consider I_2, I_3, and I_4 as given below:

$$I_2 := \lambda\rho_2 \int_0^L y\bar{\psi} dx,$$

$$I_3 := \lambda\rho_2 \int_0^L y\bar{\varphi}_x dx,$$

$$I_4 := b\int_0^L \psi_x \overline{(\varphi_x + \psi)}_x dx.$$

Then we substitute $\bar{\psi}$ given $(4.3)_3$ into I_2, and we get

$$I_2 = -\rho_2 \int_0^L y^2 dx - \rho_2 \int_0^L y\bar{f}_3 dx.$$

Moreover, we replace y given by $(4.3)_3$ and $\bar{\varphi}_x$ given by $(4.3)_1$ into I_3, and we arrive at

$$\begin{aligned}
I_3 &= -\rho_2 \int_0^L y\bar{z}_x dx - \rho_2 \int_0^L y\bar{f}_{1,x} dx \\
&= \lambda\rho_2 \int_0^L \psi_x \bar{z} dx - \rho_2 \int_0^L f_{3,x}\bar{z} dx - \rho_2 \int_0^L y\bar{f}_{1,x} dx.
\end{aligned}$$

On the other hand, replace in I_4 the term $\overline{(\varphi_x+\psi)_x}$ given by $(4.3)_2$, then we arrive at

$$I_4 = -\frac{\lambda b\rho_1}{k}\int_0^L \psi_x\bar{z}dx + \frac{bk_1}{k}\int_0^L \psi_x\bar{\varphi}dx - \frac{b\rho_1}{k}\int_0^L \psi_x\bar{f}_2dx.$$

Finally, by substituting I_2, I_3, and I_4 into (4.9), the following result is obtained:

$$\begin{aligned}
k\int_0^L (\varphi_x+\psi)^2dx &= -I_2 - I_3 - I_4 - k_2\int_0^L \psi\overline{(\varphi_x+\psi)}dx - \mu\int_0^L y\overline{(\varphi_x+\psi)}dx + \rho_2\int_0^L f_4\overline{(\varphi_x+\psi)}dx \\
&= \rho_2\int_0^L y^2dx + \rho_2\int_0^L y\bar{f}_3dx - \rho_2\lambda\int_0^L \psi_x\bar{z}dx + \rho_2\int_0^L f_{3,x}\bar{z}dx + \rho_2\int_0^L y\bar{f}_{1,x}dx \\
&+ \frac{\lambda b\rho_1}{k}\int_0^L \psi_x\bar{z}dx - \frac{bk_1}{k}\int_0^L \psi_x\bar{\varphi}dx + \frac{b\rho_1}{k}\int_0^L \psi_x\bar{f}_2dx - k_2\int_0^L \psi\overline{(\varphi_x+\psi)}dx \\
&- \mu\int_0^L y\overline{(\varphi_x+\psi)}dx + \rho_2\int_0^L f_4\overline{(\varphi_x+\psi)}dx.
\end{aligned}$$

Performing the necessary simplifications, we arrive at

$$\begin{aligned}
k\int_0^L (\varphi_x+\psi)^2dx &= \lambda b\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^L \psi_x\bar{z}dx + \rho_2\int_0^L y^2dx - \frac{bk_1}{k}\int_0^L \psi_x\bar{\varphi}dx \\
&- \mu\int_0^L y\overline{(\varphi_x+\psi)}dx + \rho_2\int_0^L y\bar{f}_{1,x}dx + \frac{\rho_1 b}{k}\int_0^L \psi_x\bar{f}_2dx + \rho_2\int_0^L f_{3,x}\bar{z}dx \\
&+ \rho_2\int_0^L y\bar{f}_3dx + \rho_2\int_0^L f_4\overline{(\varphi_x+\psi)}dx - k_2\int_0^L \psi\overline{(\varphi_x+\psi)}dx.
\end{aligned}$$

Then, using Young's inequality, we get

$$\begin{aligned}
\frac{k}{2}\int_0^L (\varphi_x+\psi)^2dx &\le \lambda b\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^L \psi_x\bar{z}dx + \left(\rho_2+\frac{\mu^2}{k}\right)\int_0^L y^2dx + C_\varepsilon\int_0^L \varphi^2dx + \left(\varepsilon+\frac{c_pk_2^2}{k}\right)\int_0^L \psi_x^2dx \\
&+ \rho_2\int_0^L y\bar{f}_{1,x}dx + \frac{\rho_1 b}{k}\int_0^L \psi_x\bar{f}_2dx + \rho_2\int_0^L f_{3,x}\bar{z}dx + \rho_2\int_0^L y\bar{f}_3dx + \rho_2\int_0^L f_4\overline{(\varphi_x+\psi)}dx.
\end{aligned}$$

Finally, using Cauchy–Schwarz's inequality and estimating (4.4), the desired result is obtained. □

We now establish the following lemma.

Lemma 4.6. *Based on the stated conditions, the required inequality holds consistently*

$$\frac{b}{2}\int_0^L |\psi_x|dx \le \frac{k^2c_p}{b}\int_0^L (\varphi_x+\psi)^2dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{4.10}$$

where $c_p > 0$ is Poincaré's constant.

Proof. Multiply Eq $(4.3)_4$ by $\bar{\psi}$, then after integrating by parts over $(0, L)$, we obtain

$$\lambda\rho_2\int_0^L y\bar{\psi}dx + b\int_0^L \psi_x^2dx + k\int_0^L (\varphi_x+\psi)\bar{\psi}dx + k_2\int_0^L \psi^2dx + \mu\int_0^L y\bar{\psi}dx = \rho_2\int_0^L f_4\bar{\psi}dx. \tag{4.11}$$

Replace $\bar{\psi}$ as given by $(4.3)_3$ in Eq (4.11), and we get

$$b\int_0^L \psi_x^2 dx = \rho_2\int_0^L y^2 dx + \rho_2\int_0^L y\bar{f}_3 dx - k\int_0^L (\varphi_x+\psi)\bar{\psi}dx - k_2\int_0^L \psi^2 dx - \mu\int_0^L y\bar{\psi}dx + \rho_2\int_0^L f_4\bar{\psi}dx.$$

Using the inequalities of Young and Poincaré, we have

$$b\int_0^L \psi_x^2 dx \le (\rho_2+\varepsilon\mu^2)\int_0^L y^2 dx + k^2\varepsilon\int_0^L (\varphi_x+\psi)^2 dx + \frac{c_p}{2\varepsilon}\int_0^L \psi_x^2 dx + \rho_2\int_0^L y\bar{f}_3 dx - k_2\int_0^L \psi^2 dx.$$

Then, setting $\varepsilon = c_p/b$ and utilizing Eq (4.4), the conclusion naturally follows. □

We now return to the main theorem of this section and proceed to prove it by leveraging the lemmas we have just established.

Proof of Theorem (4.1). We can conclude after summing the inequalities from Lemmas (4.3)–(4.6) that for sufficiently small $\varepsilon > 0$, we have

$$\|U\|_{\mathcal{H}}^2 \le C|\lambda|\left|\frac{\rho_1}{k} - \frac{\rho_2}{b}\right|\left|\int_0^L \bar{z}\psi_x dx\right| + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C_\varepsilon}{|\lambda|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C_\varepsilon}{|\lambda|^2}\|U\|^2. \tag{4.12}$$

By selecting $|\lambda|$ sufficiently large and utilizing the fact that (4.2) is satisfied, we can deduce the existence of a positive constant M such that

$$\|U\|_{\mathcal{H}} \le M\|F\|_{\mathcal{H}}.$$

This controls the solution by bounding it. Under the assumptions of Prüss's theorem, this typically translates to exponential decay and thus the desired result.

5. Polynomial energy decay with optimality analysis

In this section, we show that the solution to the system described by Eqs (2.1) gradually decreases to zero over time at a polynomial rate, provided the wave propagation speed condition in Eq (4.2) is not met.

Theorem 5.1. *The C_0-semigroup generated by the system* (2.1) *exhibits the following decay behavior:*

$$\|S(t)U_0\|_{\mathcal{H}} \le \frac{1}{\sqrt{t}}\|U_0\|_{\mathcal{D}(\mathcal{A})}. \tag{5.1}$$

Proof. Assuming $k/\rho_1 \neq b/\rho_2$, we can substitute $(4.3)_1$ and $(4.3)_3$ in (4.12). After some calculations, we obtain the following result:

$$\|U\|_{\mathcal{H}}^2 \le C_\varepsilon|\lambda|^2\int_0^L |y|^2 dx + \varepsilon\|U\|_{\mathcal{H}}^2 + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C_\varepsilon}{|\lambda|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C_\varepsilon}{|\lambda|^2}\|U\|_{\mathcal{H}}^2.$$

Use (4.4), then perform algebraic manipulation to isolate $\|U\|_{\mathcal{H}}^2$ on one side, eventually obtaining by λ large enough

$$\|U\|_{\mathcal{H}}^2 \le C|\lambda|^4\|F\|_{\mathcal{H}}^2. \tag{5.2}$$

This can be expressed as

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le C|\lambda|^2.$$

As a result, using Borichev and Tomilov's theorem (see Theorem 2.4 in [24]), we show that the solution gradually fades over time, following a slow polynomial pattern of $t^{-1/2}$ as time goes to infinity. □

Now, we aim to establish that the polynomial decay rate derived in the previous section is indeed optimal.

Theorem 5.2. *The polynomial decay rate established in the previous theorem is optimal.*

Proof. We proceed by contradiction, assuming that the decay rate can be improved to $t^{-\frac{1}{2-\delta}}$ for some $\delta > 0$. Under this assumption, we have

$$\frac{1}{|\lambda|^{2-\delta}}\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \tag{5.3}$$

which corresponds to

$$\frac{1}{|\lambda|^{2-\delta}}\|U\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}},$$

for $F \in \mathcal{H}$ and $|\lambda|$ large. Now, we can construct a sequence, $(\lambda_n)_{n\in\mathbb{N}} \subset i\mathbb{R}, (U_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{A})$, and $(F_n)_{n\in\mathbb{N}} \subset \mathcal{H}$, such that

$$\|U_n\|_{\mathcal{H}}^2 \geq |\lambda_n|^2\|F_n\|_{\mathcal{H}}^2.$$

Consequently, it follows that

$$\frac{1}{|\lambda|^{2-\delta}}\|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{H}} \geq C|\lambda_n|^{\delta-1} \to \infty, \quad \text{as n } \to \infty.$$

Nevertheless, this is impossible due to the constraint imposed by inequality (5.3), thus concluding the proof.

To clarify the optimality argument for the decay rates, we note that prior numerical studies provide supporting evidence for the expected decay behavior. In particular, Chebbi and Hamouda [25] developed a discretizations scheme combining finite element and finite difference methods for linear and nonlinear damped Timoshenko systems. Their simulations reproduced key properties of discrete energy, including positivity, conservation, and polynomial decay rates. The observed polynomial decay trends are consistent with the theoretical predictions presented here, providing numerical confirmation of the general decay behavior. □

6. Truncated version

Here, we extend the work of Almeida Júnior et al. [21] by considering a modified Timoshenko–Ehrenfest model in the second spectrum-free case, which includes additional physical effects. Specifically, we introduce a stiffness term $k_2\psi$ in the equation for transverse displacement. $k_2\psi$ represents the foundation's resistance to horizontal shear deformation, complementing the vertical support provided by k_1. By including this term, the model accounts for the way the foundation opposes rotational or bending-induced shear, leading to a more realistic depiction of the beam's deformation, energy dissipation, and stability behavior.

The system is described by

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + k_1\varphi = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k(\varphi_x + \psi) + k_2\psi + \mu\psi_t = 0, \end{cases} \tag{6.1}$$

with the appropriate initial conditions

$$\varphi(x,0) = \varphi_0(x),\ \ \varphi_t(x,0) = \varphi_1(x),\ \ \psi(x,0) = \psi_0,\ \ \forall\ x \in (0,L), \tag{6.2}$$

accompanied by Dirichlet–Neumann boundary conditions,

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0,\ \ \forall t \geq 0. \tag{6.3}$$

We define the functional energy of the solution of our system as

$$\begin{aligned} \mathcal{E}(t) = &\left(\frac{\rho_1}{2} + \frac{k_1\rho_2}{2k}\right)\int_0^L |\varphi_t|^2 dx + \frac{k}{2}\int_0^L |\varphi_x + \psi|^2 dx + \frac{k_1}{2}\int_0^L |\varphi|^2 dx \\ &+ \frac{\rho_1\rho_2}{2k}\int_0^L |\varphi_{tt}|^2 dx + \frac{\rho_2}{2}\int_0^L |\varphi_{tx}|^2 dx + \frac{b}{2}\int_0^L |\psi_x|^2 dx + \frac{k_2}{2}\int_0^L |\psi|^2 dx. \end{aligned} \tag{6.4}$$

From this, we obtain the dissipative nature of our energy, stated below.

Lemma 6.1. *It is established that the energy $\mathcal{E}(t)$ of the system* (6.1) *adheres to a dissipation law, as shown by the following equation:*

$$\frac{d}{dt}\mathcal{E}(t) = -\mu\int_0^L |\psi_t|^2 dx,\ \ \forall t \geq 0. \tag{6.5}$$

Proof. Multiplying Eq $(6.1)_1$ by φ_t using the boundary conditions (6.3), we get

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L |\varphi_t|^2 dx + k\int_0^L (\varphi_x + \psi)\varphi_{tx} dx + \frac{k_1}{2}\int_0^L |\varphi|^2 dx = 0. \tag{6.6}$$

Similarly, for $(6.1)_2$, we have

$$\rho_2\int_0^L \varphi_{tt}\psi_{tx} dx + \frac{b}{2}\frac{d}{dt}\int_0^L \psi_x^2 dx + k\int_0^L (\varphi_x + \psi)\psi_t dx + \frac{k_2}{2}\frac{d}{dt}\int_0^L \psi^2 dx + \mu\int_0^L \psi_t^2 dx = 0. \tag{6.7}$$

Using Eq $(6.1)_1$, we can replace ψ_{tx} in Eq (6.7) with $\psi_{tx} = \frac{\rho_1}{k}\varphi_{ttt} - \varphi_{txx} + \frac{k_1}{k}\varphi_t$, which allows us to rewrite Eq (6.7) as the following:

$$\begin{aligned} &\frac{\rho_1\rho_2}{2k}\frac{d}{dt}\int_0^L |\varphi_{tt}|^2 dx + \frac{\rho_2 k_1}{2k}\frac{d}{dt}\int_0^L |\varphi_t|^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^L |\varphi_{tx}|^2 dx + \frac{b}{2}\frac{d}{dt}\int_0^L |\psi_x|^2 dx \\ &+ \frac{k_2}{2}\frac{d}{dt}\int_0^L |\psi|^2 dx + k\int_0^L (\varphi_x + \psi)\psi_t dx = -\mu\int_0^L |\psi_t|^2 dx. \end{aligned} \tag{6.8}$$

By adding Eqs (6.6) and (6.8), we arrive at the desired result. □

Remark 6.2. Multiply $(6.1)_2$ by k, then differentiate by x, and we establish

$$-\rho_2 k\varphi_{ttxx} - kb\psi_{xxx} + k^2(\varphi_x + \psi)_x + kk_2\psi_x + \mu k\psi_{tx} = 0. \tag{6.9}$$

Now, multiply $(6.1)_1$ by k, then sum it with (6.9), we derive

$$\rho_1 k\varphi_{tt} - \rho_2 k\varphi_{ttxx} - bk\psi_{xxx} + kk_2\psi_x + \mu k\psi_{tx} + kk_1\varphi = 0.$$

Eq $(6.1)_1$ provides $k\psi_{xxx} = \rho_1\varphi_{ttxx} - k\varphi_{xxxx} + k_1\varphi_{xx}$. Substituting this expression into the previous equation yields

$$\rho_1 k\varphi_{tt} - (b\rho_1 + k\rho_2)\varphi_{ttxx} + bk\varphi_{xxxx} - bk_1\varphi_{xx} + kk_2\psi_x + \mu k\psi_{tx} + kk_1\varphi = 0.$$

Now, we can reformulate the system to be written as

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + k_1\varphi = 0, \\ B\varphi_{tt} + bk\varphi_{xxxx} - bk_1\varphi_{xx} + kk_2\psi_x + \mu k\psi_{tx} + kk_1\varphi = 0, \end{cases}$$

and the compatibility condition

$$B\left(k(\varphi_{0x} + \psi_0)_x - k_1\varphi_0\right) = -\rho_1 bk\varphi_{0xxxx} + \rho_1 bk_1\varphi_{0xx} - \rho_1 kk_2\psi_{0x} - \rho_1\mu k\psi_{1x} - \rho_1 kk_1\varphi_0.$$

Now, we can define φ_{tt} at $t = 0$ as follows:

$$\begin{aligned} \varphi_{tt}(x, 0) &:= \frac{1}{\rho_1}\left(k(\varphi_x + \psi)_x - k_1\varphi\right) \\ &= -B^{-1}\left(bk\varphi_{0xxxx} - bk_1\varphi_{0xx} + kk_2\psi_{0x} + \mu k\psi_{1x} + kk_1\varphi_0\right). \end{aligned} \tag{6.10}$$

Consequently, $E(0)$ makes sense.

7. Well-posedness

This section establishes the existence and uniqueness of a weak solution to the system (6.1). The proof employs the standard Faedo–Galerkin method combined with a priori estimates, followed by passage to the limit via compactness techniques. A similar approach was taken by Ramos et al. [26]. The analysis will be conducted within an appropriate Hilbert space framework. Consider

$$\mathcal{H} = H_0^1(0, L) \times H_0^1(0, L) \times H_*^1(0.L) \times L_*^2(0, L), \tag{7.1}$$

and we define the Hilbert space $\mathcal{H}_1$ as

$$\mathcal{H}_1 = (H^2(0, L) \cap H_0^1(0, L)) \times (H^2(0, L) \cap H_0^L(0, L)) \times H_*^2(0, L) \times H_*^1(0, L). \tag{7.2}$$

Furthermore, H_*^1 and H_*^2 are defined as

$$H_*^1(0, L) = H^1(0, L) \cap L_*^2(0, L), \quad H_*^2(0, L) = H^2(0, L) \cap H_*^1(0, L), \tag{7.3}$$

where

$$L_*^2(0, L) = \left\{ u \in L^2(0, L); \int_0^L u(x)dx = 0 \right\}. \tag{7.4}$$

We start by stating a clear and rigorous characterization of what constitutes a weak solution for the system described by Eqs (6.1).

Definition 7.1. Given the initial data $U_0 = (\varphi_0, \varphi_1, \psi_0) \in \mathcal{H}$, a function $U = (\varphi, \varphi_t, \psi) \in C([0,T]; \mathcal{H})$ is said to be a weak solution of (6.1) if, for almost everywhere $t \in [0,T]$,

$$\begin{aligned} &\rho_1 \frac{d}{dt}(\varphi_{tt}, z) + k(\varphi_x + \psi, z_x) + k_1(\varphi, z) = 0, \\ &\rho_2 \frac{d}{dt}(\varphi_t, v_x) + b(\psi_x, v_x) + k(\varphi_x + \psi, v) + k_2(\psi, v) + \mu(\psi_t, v) = 0, \\ &\rho_2 \frac{d}{dt}(\varphi_{tt}, y_x) + b(\psi_{tx}, y_x) + k(\varphi_{tx} + \psi_t, y) + k_2(\psi_t, y) + \mu(\psi_{tt}, y) = 0. \end{aligned} \tag{7.5}$$

$\forall z, y \in H_0^1(0,L),\;\; v \in H_*^1(0,L)$, and

$$(\varphi(0), \varphi_t(0), \psi(0), \psi_t(0)) = (\varphi_0, \varphi_1, \psi_0, \psi_1). \tag{7.6}$$

Theorem 7.2. *The system* (6.1) *has a unique weak solution if the initial data of this system* $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1)$ *belongs to the space* $\mathcal{H}$*, and this solution has the following regularity:*

$$\begin{aligned} \varphi &\in L^\infty(0,T; H_0^1(0,L)), \\ \varphi_t &\in L^\infty(0,T; H_0^1(0,L)), \\ \psi &\in L^2(0,T; H_*^1(0,L)), \\ \psi_t &\in L^2(0,T; L_*^2(0,L)). \end{aligned} \tag{7.7}$$

Furthermore, the solution $U = (\varphi, \varphi_t, \psi, \psi_t)$ *exhibits a continuous dependence on the initial data within the space* $\mathcal{H}$*.*

Theorem 7.3. *The system* (6.1) *has a unique stronger weak solution if the initial data of this system* $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1)$ *belongs to the space* $\mathcal{H}_1$*, and this solution has the following regularity:*

$$\begin{aligned} \varphi &\in L^\infty(0,T; H^2(0,L) \cap H_0^1(0,L)), \\ \varphi_t &\in L^\infty(0,T; H^2(0,L) \cap H_0^1(0,L)), \\ \psi &\in L^2(0,T; H_*^2(0,L)), \\ \psi_t &\in L^2(0,T; H_*^1(0,L)). \end{aligned} \tag{7.8}$$

The solution $U = (\varphi, \varphi_t, \psi, \psi_t)$ *exhibits continuous dependence on the initial data within the space* $\mathcal{H}_1$*.*

Proof. The proof is established using the Faedo–Galerkin method. We provide only a brief overview of the six main steps.

First Step. Estimated solution. Let $U_0 = (\varphi_0, \varphi_1, \psi_0) \in \mathcal{H}$. Let $\{w_j\}_{j=1}^\infty$ and $\{\mu_j\}_{j=1}^\infty$ be orthonormal bases for $H^2(0,L) \cap H_0^1(0,L)$ and $H_*^1(0,L)$, respectively, as both basis are orthonormal in $L^2(0,L)$. Let $H_n = span\{w_1, w_2, ..., w_n\}$ and $V_n = \{\mu_1, \mu_1, ...\mu_n\}$ for any integer $n \in \mathbb{N}$. Now, let's introduce

$$y^n(x,t) = \sum_{j=1}^n a_{j,n} w_j(x), \text{ and } \psi^n(x,t) = \sum_{j=1}^n b_{j,n} \mu_j(x),$$

two approximate solutions that satisfy the approximated problem

$$\begin{aligned} \rho_1(\varphi^n_{tt}, z) + k(\varphi^n_x + \psi^n, z_x) + k_1(\varphi^n, z) &= 0,\\ \rho_2(\varphi^n_{tt}, v_x) + b(\psi^n_x, v_x) + k(\varphi^n_x + \psi^n, v) + k_2(\psi^n, v) + \mu(\psi^n_t, v) &= 0,\\ \rho_2(\varphi^n_{ttt}, y_x) + b(\psi^n_{tx}, y_x) + k(\varphi^n_{tx} + \psi^n, y) + k_2(\psi^n_t, y) + \mu(\psi^n_{tt}, y) &= 0. \end{aligned} \tag{7.9}$$

for all $z \in H_n$, and $v, y \in V_n$ with initial conditions

$$(\varphi^n(0), \varphi^n_t(0), \psi^n(0), \psi^n_t(0)) = (\varphi^n_0, \varphi^n_1, \psi^n_0, \psi^n_1), \tag{7.10}$$

satisfying

$$(\varphi^n_0, \varphi^n_1, \psi^n_0, \psi^n_1) \to (\varphi_0, \varphi_1, \psi_0, \psi_1) \text{ strongly in } \mathcal{H}. \tag{7.11}$$

Using the standard theory of ordinary differential equations, system (6.1) possesses a local solution on the maximal interval $[0, t_n)$, with $0 < t_n \leq T$ for every $n \in N$.

Second Step. A priori estimate.

Let $z = \varphi^n_t$, $y = \varphi^n_{tt}$, and $v = \psi^n_t$. Now substitute them in system (7.9), and we obtain

$$\begin{aligned} \rho_1(\varphi^n_{tt}, \varphi^n_t) + k(\varphi^n_x + \psi^n, \varphi^n_{tx}) + k_1(\varphi^n, \varphi^n_t) &= 0,\\ \rho_2(\varphi^n_{tt}, \psi^n_{tx}) + b(\psi^n_x, \psi^n_{tx}) + k(\varphi^n_x + \psi^n, \psi^n_t) + k_2(\psi^n, \psi^n_t) + \mu(\psi^n_t, \psi^n_t) &= 0. \end{aligned} \tag{7.12}$$

Taking into consideration from Eq $(6.1)_1$ that

$$\psi_{tx} = \frac{\rho_1}{k}\varphi_{ttt} - \varphi_{txx} + \frac{k_1}{k}\varphi_t, \tag{7.13}$$

system (7.12) becomes

$$\begin{aligned} \rho_1(\varphi^n_{tt}, \varphi^n_t) + k(\varphi^n_x + \psi^n, \varphi^n_{tx}) + k_1(\varphi^n, \varphi^n_t) &= 0,\\ \frac{\rho_1\rho_2}{k}(\varphi^n_{ttt}, \varphi^n_{tt}) + \rho_2(\varphi^n_{ttx}, \varphi^n_{tx}) + \frac{k_1\rho_2}{k}(\varphi^n_{tt}, \varphi^n_t) + b(\psi^n_x, \psi^n_{tx}) + k(\varphi^n_x + \psi^n, \psi^n_t) + k_2(\psi^n, \psi^n_t) + \mu(\psi^n_t, \psi^n_t) &= 0, \end{aligned} \tag{7.14}$$

which is equivalent to

$$\frac{d}{dt}E^n(t) + \mu \int_0^L |\psi_t|^2 dx = 0, \tag{7.15}$$

where the estimated energy $E^n(t)$ is defined by

$$E^n(t) := \left(\frac{\rho_1}{2} + \frac{k_1\rho_2}{2k}\right)\|\varphi^n_t\|^2 + \frac{k}{2}\|\varphi^n_x + \psi^n\|^2 + \frac{k_1}{2}\|\varphi^n\|^2 + \frac{\rho_1\rho_2}{2k}\|\varphi^n_{tt}\|^2 + \frac{\rho_2}{2}\|\varphi^n_{tx}\|^2 + \frac{b}{2}\|\psi^n_x\|^2 + \frac{k_2}{2}\|\psi^n\|^2. \tag{7.16}$$

Now, integrating from 0 to $t < t_n$, we obtain from the choice of the initial data that

$$E^n(t) + \mu \int_0^t \int_0^L |\psi_t|^2 dxds \leq K \tag{7.17}$$

for all $t \in [0, T]$ and for every $n \in \mathbb{N}$, where $K > 0$. As a result, approximate solutions are established over the entire interval $[0, T]$.

Third step. Passing to the limit. From (7.17) and the definition of $E^n(t)$, we obtain

$$\begin{aligned}\{\varphi^n\} &\text{ is bounded in } L^\infty(0,T;H_0^1(0,L)),\\ \{\varphi_t^n\} &\text{ is bounded in } L^\infty(0,T;H_0^1(0,L)),\\ \{\psi^n\} &\text{ is bounded in } L^\infty(0,T;H_*^1(0,L)),\\ \{\psi_t^n\} &\text{ is bounded in } L^2(0,T;L_*^2(0,L)).\end{aligned}$$

We can extract a subsequence from $\{\varphi^n\}$ and $\{\psi^n\}$, still denoted by $\{\varphi^n\}$ and $\{\psi^n\}$, and ensuring consistency in notation, such that

$$\begin{aligned}\varphi^n \to \varphi &\text{ weakly star in } L^\infty(0,T;H_0^1(0,L)),\\ \varphi_t^n \to \varphi_t &\text{ weakly star in } L^\infty(0,T;H_0^1(0,L)),\\ \psi^n \to \psi &\text{ weakly star in } L^\infty(0,T;H_*^1(0,L)),\\ \psi_t^n \to \psi &\text{ weakly in } L^2(0,T;L_*^2(0,L)).\end{aligned}$$

Consequently, the aforementioned limits enable us to take the limit in the approximate problem (6.1), resulting in a weak solution that satisfies

$$\begin{aligned}\varphi &\in L^\infty(0,T;H_0^1(0,L)),\\ \varphi_t &\in L^\infty(0,T;H_0^1(0,L)),\\ \psi &\in L^\infty(0,T;H_*^1(0,L)),\\ \psi_t &\in L^2(0,T;L_*^2(0,L)).\end{aligned}$$

Fourth Step. Initial data. Given that

$$H_0^1(0,L) \subset L^2(0,L) \subset H^{-1}(0,L),$$

where H^{-1} is the dual of $H_0^1(0,L)$, by using the Aubin–Lions lemma, we obtain that $L^\infty(0,T;H_0^1(0,L))$ is compactly embedded in $C(0,T;L^2(0,L))$, and this implies that

$$\begin{aligned}\varphi^n \to \varphi &\text{ strongly in } C([0,T];L^2(0,L)),\\ \varphi_t^n \to \varphi_t &\text{ strongly in } C([0,T];L^2(0,L)).\end{aligned} \tag{7.18}$$

Consequently,

$$(\varphi(0),\varphi_t(0)) = (\varphi_0,\varphi_1).$$

In a similar way, we can get that

$$\psi(0) = \psi_0.$$

Now, we multiply $(7.9)_3$ by a test function

$$\eta \in H^1(0,L),\ \ \eta(0) = 1,\ \ \eta(T) = 0$$

and integrate the result over $[0,T]$ to obtain

$$\rho_2\int_0^T(\psi_{ttt}^n,y_x)\eta dx+b\int_0^T(\psi_{tx}^n,y_x)\eta dx+k\int_0^T(\psi_{tx}^n+\psi^n,y)\eta dx+k_2\int_0^T(\psi_t^n,y)\eta dx-\mu(\psi_1^n,y)-\mu\int_0^T(\psi_t^n,y)\eta dx = 0. \tag{7.19}$$

Taking the limit as n tends to infinity, we obtain

$$\rho_2 \int_0^T (\psi_{ttt}, y_x)\eta dx + b \int_0^T (\psi_{tx}, y_x)\eta dx + k \int_0^T (\psi_{tx} + \psi, y)\eta dx + k_2 \int_0^T (\psi_t, y)\eta dx - \mu(\psi_1, y) - \mu \int_0^T (\psi_t, y)\eta dx = 0. \tag{7.20}$$

On the other hand, multiplying $(7.5)_3$ by η and integrating the result over $[0, T]$, we obtain

$$\rho_2 \int_0^T (\psi_{ttt}, y_x)\eta dx + b \int_0^T (\psi_{tx}, y_x)\eta dx + k \int_0^T (\psi_{tx} + \psi, y)\eta dx + k_2 \int_0^T (\psi_t, y)\eta dx - \mu(\psi_t(0), y) - \mu \int_0^T (\psi_t, y)\eta dx = 0. \tag{7.21}$$

Combining (7.19) and (7.21), we conclude the following:

$$\psi_t(0) = \psi_1.$$

Fifth step. Stronger solutions. Assume the initial conditions in the approximate scheme (6.1) satisfy

$$(\varphi_0, \varphi_1, \psi_0) \in \mathcal{H}_1 \text{ and } (\varphi_0^n, \varphi_1^n, \psi_0^n) \to (\varphi_0, \varphi_1, \psi_0) \text{ strongly in } \mathcal{H}_1. \tag{7.22}$$

Now, let $z = -\varphi_{txx}^n$, $y = -\varphi_{ttxx}^n$ and $v = -\psi_{txx}^n$. After substituting them in system (7.12), we obtain

$$\frac{d}{dt} F^n(t) + \mu \int_0^L |\psi_{tx}^n|^2 dx = 0, \tag{7.23}$$

where

$$F^n(t) := \left(\frac{\rho_1}{2} + \frac{k_1 \rho_2}{2k}\right)\|\varphi_{tx}^n\|^2 + \frac{k}{2}\|(\varphi_x^n + \psi^n)_x\|^2 + \frac{k_1}{2}\|\varphi_x^n\|^2 + \frac{\rho_1\rho_2}{2k}\|\varphi_{ttx}^n\|^2 + \frac{\rho_2}{2}\|\varphi_{txx}^n\|^2 + \frac{b}{2}\|\psi_{xx}^n\|^2 + \frac{k_2}{2}\|\psi_x^n\|^2. \tag{7.24}$$

Integrating (7.23) from 0 to $t < t_n$, we obtain from the choice of the initial data

$$F^n(t) + \mu \int_0^t \int_0^L |\psi_t|^2 dx dx \leq K_1 \tag{7.25}$$

for all $t \in [0, T]$, and for every $n \in \mathbb{N}$, where $K_1 > 0$. From (7.25), we deduce that

$$\begin{aligned}
&\{\varphi^n\} \text{ is bounded in } L^\infty(0, T; H^2(0, L) \cap H_0^1(0, L)),\\
&\{\varphi_t^n\} \text{ is bounded in } L^\infty(0, T; H^2(0, L) \cap H_0^1(0, L)),\\
&\{\psi^n\} \text{ is bounded in } L^\infty(0, T; H_*^2(0, L)),\\
&\{\psi_t^n\} \text{ is bounded in } L^2(0, T; H_*^1(0, L)).
\end{aligned}$$

We can extract a subsequence from $\{\varphi^n\}$ and $\{\psi^n\}$, still denoted by $\{\varphi^n\}$ and $\{\psi^n\}$, and ensuring consistency in notation, such that

$$\begin{aligned}
&\varphi^n \to \varphi \text{ weakly star } L^\infty(0, T; H^2(0, L) \cap H_0^1(0, L)),\\
&\varphi_t^n \to \varphi_t \text{ weakly star } L^\infty(0, T; H^2(0, L) \cap H_0^1(0, L)),\\
&\psi^n \to \psi \text{ weakly star } L^\infty(0, T; H_*^2(0, L)),\\
&\psi_t^n \to \psi_t \text{ weakly in } L^2(0, T; H_*^1(0, L)).
\end{aligned}$$

Consequently, the aforementioned limits enable us to take the limit in the approximate problem (6.1), resulting in a weak solution that satisfies

$$\begin{aligned} \varphi &\in L^\infty(0,T;H^2(0,L)\cap H_0^1(0,L)),\\ \varphi_t &\in L^\infty(0,T;H^2(0,L)\cap H_0^1(0,L)),\\ \psi &\in L^\infty(0,T;H_*^2(0,L)),\\ \psi_t &\in L^2(0,T;H_*^1(0,L)). \end{aligned}$$

Sixth step. Continuous dependence on initial data. First, consider the case of stronger solutions. Let

$$U(t) = (\varphi, \varphi_t, \psi),$$

$$V(t) = (\tilde{\varphi}, \tilde{\varphi}_t, \tilde{\psi})$$

represent two strong solutions to the system (6.1) with initial data

$$U(0) = (\varphi_0, \varphi_1, \psi_0) \in \mathcal{H}_1,$$

$$V(0) = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\psi}_0) \in \mathcal{H}_1.$$

Now, define the difference variables

$$(\Phi, \Phi_t, \Psi) = U(t) - V(t).$$

These differences satisfy the following governing equations:

$$\begin{aligned} \rho_1\Phi_{tt} - k(\Phi_x + \Psi)_x + k_1\Phi &= 0,\\ -\rho_2\Psi_{ttx} - b\Psi_{xx} + k(\Phi_x + \Psi) + k_2\Psi + \mu\Psi_t &= 0, \end{aligned} \tag{7.26}$$

with initial data $(\Phi(0), \Phi_t(0), \Psi(0)) = U(0) - V(0)$. Now, multiply $(7.26)_1$ by Φ_t and $(7.26)_2$ by Ψ_t, then in the result over $(0, L)$, we arrive at

$$\frac{d}{dt}\hat{E}(t) = -\mu \int_0^L |\Psi_t|^2 dx, \tag{7.27}$$

where $\hat{E}(t)$ is the energy corresponding to $U(t) - V(t)$ defined by

$$\hat{E}(t) := \left(\frac{\rho_1}{2} + \frac{k_1\rho_2}{2k}\right)\|\Phi_t\|^2 + \frac{k}{2}\|\Phi_x + \Psi\|^2 + \frac{k_1}{2}\|\Phi\|^2 + \frac{\rho_1\rho_2}{2k}\|\Phi_{tt}\|^2 + \frac{\rho_2}{2}\|\Phi_{tx}\|^2 + \frac{b}{2}\|\Psi_x\|^2 + \frac{k_2}{2}\|\Psi\|^2. \tag{7.28}$$

Integrating over $(0, L)$, we obtain that there exist a positive constant K_τ such that, for any $t \in [0, T]$,

$$\hat{E}(t) \le K_\tau \hat{E}(0). \tag{7.29}$$

This implies that the stronger weak solution depends continuously on the initial data, which in turn means that the stronger weak solution of problem (6.1) is unique. Furthermore, weak solutions are essentially limits of stronger weak solutions, so by employing density arguments, we establish the continuous dependence and uniqueness of weak solutions. □

8. Exponential decay

In this section, we establish the exponential stability of the truncated Timoshenko model. No restrictions are imposed on the physical parameters of the system. The proof relies on the energy method combined with suitable multiplier techniques. The main result is presented in the following theorem.

Theorem 8.1. *The energy $\mathcal{E}(t)$ of the system* (6.1)*, given by* (6.4)*, decays exponentially as time tends to infinity under the following conditions*

$$\begin{aligned} k_1 &< k^2\rho_2/(2c_p b\rho_1), \\ k_2 &< 2\Big(\frac{k_1 bc_p}{k} + \frac{k_1^2 b}{k^2}\Big). \end{aligned} \tag{8.1}$$

Then there exists two positive constants M and η such that

$$\mathcal{E}(t) \leq M\mathcal{E}(0)e^{-\eta t}, \ \ \forall t \geq 0. \tag{8.2}$$

Initially, we establish the following functional:

$$\mathcal{F}(t) := -\rho_1 \int_0^L \varphi_t \varphi dx. \tag{8.3}$$

Lemma 8.2. *Let $U(t) = (\varphi, \psi)$ be a solution of system* (6.1)*. Then we have*

$$\frac{d}{dt}\mathcal{F}(t) \leq -\rho_1 \int_0^L |\varphi_t|^2 dx + 2\varepsilon c_p k \int_0^L |\psi_x|^2 dx + c(\varepsilon) \int_0^L |\varphi_x + \psi|^2 dx + \kappa_1 \int_0^L |\varphi|^2 dx, \tag{8.4}$$

where c_p is Poincaré's constant, and $c(\varepsilon) = \frac{k}{4\varepsilon} + 2k\varepsilon$.

Proof. Multiplying $(6.1)_1$ by φ, while considering the boundary conditions (6.3), and using integration by parts over $(0, L)$, we get a new form of the equation

$$\rho_1 \int_0^L \varphi_{tt}\varphi dx + k \int_0^L (\varphi_x + \psi)\varphi_x dx + k_1 \int_0^L |\varphi|^2 dx = 0. \tag{8.5}$$

Utilizing the identity $\varphi_{tt} = \frac{d}{dt}(\varphi_t\varphi) - |\varphi_t|$, then applying Young's inequality, we derive

$$-\frac{d}{dt}\Big(\rho_1 \int_0^L \varphi_t\varphi dx\Big) \leq -\rho_1 \int_0^L |\varphi_t|^2 dx + \frac{k}{4\varepsilon} \int_0^L |\varphi_x + \psi|^2 dx + \varepsilon k \int_0^L |\varphi_x|^2 dx + k_1 \int_0^L |\varphi|^2 dx, \tag{8.6}$$

for $\varepsilon > 0$. Additionally, we have

$$\int_0^L |\varphi_x|^2 dx \leq 2 \int_0^L |\varphi_x + \psi|^2 dx + 2c_p \int_0^L |\psi_x|^2 dx, \tag{8.7}$$

where c_p is a positive Poincaré's constant. Now, substitute the inequality (8.7) in (8.6), and the last one can be expressed as

$$-\frac{d}{dt}\Big(\rho_1 \int_0^L \varphi_t\varphi dx\Big) \leq -\rho_1 \int_0^L |\varphi_t|^2 dx + 2\varepsilon c_p k \int_0^L |\psi_x|^2 dx + \Big(\frac{k}{4\varepsilon} + 2k\varepsilon\Big) \int_0^L |\varphi_x + \psi|^2 dx + \kappa_1 \int_0^L |\varphi|^2 dx. \tag{8.8}$$

Consequently, this completes the proof of the lemma. □

Now, we proceed to introduce the second functional

$$\mathcal{G}(t) := \rho_2 \int_0^L \varphi_{tx}\varphi_x dx + \frac{\mu}{2}\int_0^L |\psi|^2 dx + \frac{\rho_2 k_1}{k}\int_0^L \varphi_t \varphi dx. \tag{8.9}$$

Lemma 8.3. *Let $U(t) = (\varphi, \psi)$ be a solution of system* (6.1)*, then we have*

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t) \leq &-\frac{\rho_1\rho_2}{k}\int_0^L |\varphi_{tt}|^2 dx - \frac{b}{2}\int_0^L |\psi_x|^2 dx + \left(1 + \frac{k_1 c_p}{k}\right)\rho_2 \int_0^L |\varphi_{tx}|^2 dx \\ &+ \frac{k^2 c_p}{2b}\int_0^L |\varphi_x + \psi|^2 dx - k_2 \int_0^l |\psi|^2 dx.\end{aligned} \tag{8.10}$$

Proof. Multiplying $(6.1)_2$ by ψ, while considering the boundary conditions (6.3) and using integration by parts over $(0, L)$, we get

$$\rho_2 \int_0^L \varphi_{tt}\psi_x dx + b\int_0^L |\psi_x|^2 dx + k\int_0^L (\varphi_x + \psi)\psi dx + k_2 \int_0^L |\psi|^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^L |\psi|^2 dx = 0. \tag{8.11}$$

Using Eq $(6.1)_1$, we get $\psi_x = \frac{\rho_1}{k}\varphi_{tt} - \varphi_{xx} + \frac{k_1}{k}\varphi$. Now, replacing ψ_x in Eq (8.11), we get the following:

$$\begin{aligned}\frac{d}{dt}\left(\rho_2 \int_0^L \varphi_{tx}\varphi_x dx + \frac{\mu}{2}\int_0^L |\psi|^2 dx + \frac{\rho_2 k_1}{k}\int_0^L \varphi_t\varphi dx\right) - \rho_2 \int_0^L |\varphi_{tx}|^2 dx + \frac{\rho_1\rho_2}{k}\int_0^L |\varphi_{tt}|^2 dx \\ + b\int_0^L |\psi_x|^2 dx + k\int_0^L (\varphi_x + \psi)\psi dx - \frac{\rho_2 k_1}{k}\int_0^L |\varphi_t|^2 dx + k_2 \int_0^l |\psi|^2 dx = 0.\end{aligned} \tag{8.12}$$

Using Poincaré and Young's inequalities, we get

$$\begin{aligned}\frac{d}{dt}\left(\rho_2 \int_0^L \varphi_{tx}\varphi_x dx + \frac{\mu}{2}\int_0^L |\psi|^2 dx + \frac{\rho_2 k_1}{k}\int_0^L \varphi_t\varphi dx\right) \leq -\frac{\rho_1\rho_2}{k}\int_0^L |\varphi_{tt}|^2 dx - \left(b - \frac{c_p}{2\varepsilon_2}\right)\int_0^L |\psi_x|^2 dx \\ + \left(1 + \frac{k_1 c_p}{k}\right)\rho_2 \int_0^L |\varphi_{tx}|^2 dx + \frac{k^2 \varepsilon_2}{2}\int_0^L |\varphi_x + \psi|^2 dx - k_2 \int_0^l |\psi|^2 dx.\end{aligned} \tag{8.13}$$

Now, take $\varepsilon_2 = \frac{c_p}{b}$, and we derive

$$\begin{aligned}\frac{d}{dt}\left(\rho_2 \int_0^L \varphi_{tx}\varphi_x dx + \frac{\mu}{2}\int_0^L |\psi|^2 dx + \frac{\rho_2 k_1}{k}\int_0^L \varphi_t\varphi dx\right) \leq -\frac{\rho_1\rho_2}{k}\int_0^L |\varphi_{tt}|^2 dx - \frac{b}{2}\int_0^L |\psi_x|^2 dx \\ + \left(1 + \frac{k_1 c_p}{k}\right)\rho_2 \int_0^L |\varphi_{tx}|^2 dx + \frac{k^2 c_p}{2b}\int_0^L |\varphi_x + \psi|^2 dx - k_2 \int_0^l |\psi|^2 dx.\end{aligned} \tag{8.14}$$

Consequently, this completes the proof of the lemma. □

Now, we proceed to introduce the third functional

$$\mathcal{H}(t) := -\rho_2 \int_0^L \varphi_{tx}(\varphi_x + \psi) dx - \frac{b\rho_1}{k}\int_0^L \varphi_{tx}\psi dx - \frac{k_1 b\rho_1}{k^2}\int_0^L \varphi_t\varphi dx. \tag{8.15}$$

Lemma 8.4. *Let $U(t) = (\varphi, \psi)$ be a solution of system* (6.1)*, then we have*

$$\frac{d}{dt}\mathcal{H}(t) \leq -C_{k_1}\frac{\rho_2}{2}\int_0^L |\varphi_{tx}|^2 dx - \left(\frac{k_2}{2} + \frac{k}{2}\right)\int_0^L |\varphi_x + \psi|^2 dx - \left(\frac{k_1 b c_p}{k} + \frac{k_1^2 b}{k^2} - \frac{k_2}{2}\right)\int_0^L |\varphi|^2 dx + C\int_0^L |\psi_t|^2 dx, \tag{8.16}$$

with $k_1 < k^2\rho_2/(2b\rho_1 c_p)$, $k_2 < 2\left(\frac{k_1 b c_p}{k} + \frac{k_1^2 b}{k^2}\right)$, $C_{k_1} = 1 - \frac{2k_1 b\rho_1 c_p}{k^2\rho_2}$, *and* $C = b^2\left(\frac{\rho_1}{k} + \frac{\rho_2}{b}\right)^2/(2\rho_2) + \frac{\mu^2}{2k}$.

Proof. Multiplying $(6.1)_2$ by $(\varphi_x + \psi)$, while considering the boundary conditions (6.3), and using integration by parts over $(0, L)$, we get

$$-\rho_2\int_0^L \varphi_{ttx}(\varphi_x+\psi)dx + b\int_0^L \psi_x(\varphi_x+\psi)_x dx + k\int_0^L |\varphi_x+\psi|^2 dx + k_2\int_0^L \psi(\varphi_x+\psi)dx + \mu\int_0^L \psi_t(\varphi_x+\psi)dx = 0. \tag{8.17}$$

Using Young's inequality, we get

$$-\rho_2\int_0^L \varphi_{ttx}(\varphi_x+\psi)dx + b\int_0^L \psi_x(\varphi_x+\psi)_x dx + k_2\int_0^L \psi(\varphi_x+\psi)dx \leq -\frac{k}{2}\int_0^L |\varphi_x+\psi|^2 dx + \frac{\mu^2}{2k}\int_0^L |\varphi_t|^2 dx. \tag{8.18}$$

From $(6.1)_1$, we get $(\varphi_x + \psi)_x = \frac{\rho_1}{k}\varphi_{tt} + \frac{k_1}{k}\varphi$, which allows us to reformulate the preceding inequality as

$$-\rho_2\int_0^L \varphi_{ttx}(\varphi_x + \psi)dx + \frac{b\rho_1}{k}\int_0^L \varphi_{tt}\psi_x dx + \frac{k_1 b}{k}\int_0^L \varphi\psi_x dx + k_2\int_0^L \psi(\varphi_x + \psi)dx \leq -\frac{k}{2}\int_0^L |\varphi_x + \psi|^2 dx + \frac{\mu^2}{2k}\int_0^L |\varphi_t|^2 dx. \tag{8.19}$$

In addition, $\varphi_{ttx}(\varphi_x + \psi)$ can be expressed as $\varphi_{ttx}(\varphi_x + \psi) = \frac{d}{dt}[\varphi_{tx}(\varphi_x + \psi)] - \varphi_{tx}(\varphi_x + \psi)_t$. From this we derive

$$-\rho_2\frac{d}{dt}\int_0^L \varphi_{tx}(\varphi_x + \psi)dx \leq -\rho_2\int_0^L |\varphi_{tx}|^2 dx - \rho_2\int_0^L \varphi_{tx}\psi_t dx + \frac{b\rho_1}{k}\int_0^L \varphi_{ttx}\psi dx - \frac{k}{2}\int_0^L |\varphi_x + \psi|^2 dx - \frac{k_1 b}{k}\int_0^L \varphi\psi_x dx + \frac{\mu^2}{2k}\int_0^L |\varphi_t|^2 dx - k_2\int_0^L \psi(\varphi_x + \psi)dx. \tag{8.20}$$

Now, utilizing the identity $\varphi_{ttx}\psi = \frac{d}{dt}(\varphi_{tx}\psi) - \varphi_{tx}\psi_t$, we derive

$$-\frac{d}{dt}\left(\rho_2\int_0^L \varphi_{tx}(\varphi_x + \psi)dx + \frac{b\rho_1}{k}\int_0^L \varphi_{tx}\psi dx\right) \leq -\rho_2\int_0^L |\varphi_{tx}|^2 dx - b\left(\frac{\rho_1}{k} + \frac{\rho_2}{b}\right)\int_0^L \varphi_{tx}\psi_t dx - \frac{k}{2}\int_0^L |\varphi_x + \psi|^2 dx - \frac{k_1 b}{k}\int_0^L \varphi\psi_x dx + \frac{\mu^2}{2k}\int_0^L |\varphi_t|^2 dx - k_2\int_0^L \psi(\varphi_x + \psi)dx. \tag{8.21}$$

Using $(6.1)_1$, we get $\psi_x = \frac{\rho_1}{k}\varphi_{tt} - \varphi_{xx} + \frac{k_1}{k}\varphi$. Now, replace ψ_x in Eq (8.21), and using Young and

Poincaré's inequalities, we derive

$$-\frac{d}{dt}\left(\rho_2\int_0^L \varphi_{tx}(\varphi_x+\psi)dx+\frac{b\rho_1}{k}\int_0^L \varphi_{tx}\psi dx-\frac{k_1 b\rho_1}{k^2}\int_0^L \varphi_t\varphi dx\right)\le -\frac{\rho_2}{2}\int_0^L |\varphi_{tx}|^2dx-\frac{k}{2}\int_0^L|\varphi_x+\psi|^2dx$$
$$+\frac{k_1 b\rho_1}{k^2}\int_0^L|\varphi_t|^2dx-\left(\frac{k_1 bc_p}{k}+\frac{k_1^2 b}{k^2}\right)\int_0^L|\varphi|^2dx+\left(b^2\left(\frac{\rho_1}{k}+\frac{\rho_2}{b}\right)^2/(2\rho_2)+\frac{\mu^2}{2k}\right)\int_0^L|\psi_t|^2dx-k_2\int_0^L\psi(\varphi_x+\psi)dx. \quad (8.22)$$

Now, use the identity

$$-k_2\int_0^L(\varphi_x+\psi)\psi dx=-k_2\int_0^L|\varphi_x+\psi|^2dx+k_2\int_0^L(\varphi_x+\psi)\varphi dx. \quad (8.23)$$

Through the use of Young's inequality, we get

$$-\frac{d}{dt}\left(\rho_2\int_0^L \varphi_{tx}(\varphi_x+\psi)dx+\frac{b\rho_1}{k}\int_0^L \varphi_{tx}\psi dx+\frac{k_1 b\rho_1}{k^2}\int_0^L \varphi_t\varphi dx\right)\le -\frac{\rho_2}{2}\int_0^L |\varphi_{tx}|^2dx-\frac{k}{2}\int_0^L|\varphi_x+\psi|^2dx$$
$$+\frac{k_1 b\rho_1}{k^2}\int_0^L|\varphi_t|^2dx-\left(\frac{k_1 bc_p}{k}+\frac{k_1^2 b}{k^2}\right)\int_0^L|\varphi|^2dx+\left(b^2\left(\frac{\rho_1}{k}+\frac{\rho_2}{b}\right)^2/(2\rho_2)+\frac{\mu^2}{2k}\right)\int_0^L|\psi_t|^2dx$$
$$-\frac{k_2}{2}\int_0^L|\varphi_x+\psi|^2dx+\frac{k_2}{2}\int_0^L|\varphi|^2dx. \quad (8.24)$$

Through the use of Poincaré's inequality, we derive

$$\frac{d}{dt}\mathcal{H}(t)\le-\left(1-\frac{2k_1 b\rho_1 c_p}{k^2\rho_2}\right)\frac{\rho_2}{2}\int_0^L|\varphi_{tx}|^2dx-\left(\frac{k_2}{2}+\frac{k}{2}\right)\int_0^L|\varphi_x+\psi|^2dx-\left(\frac{k_1 bc_p}{k}+\frac{k_1^2 b}{k^2}-\frac{k_2}{2}\right)\int_0^L|\varphi|^2dx$$
$$+\left(b^2\left(\frac{\rho_1}{k}+\frac{\rho_2}{b}\right)^2/(2\rho_2)+\frac{\mu^2}{2k}\right)\int_0^L|\psi_t|^2dx, \quad (8.25)$$

with $k_1<k^2\rho_2/(2b\rho_1L^2)$ and $k_2<2\left(\frac{k_1bc_p}{k}+\frac{k_1^2b}{k^2}\right)$. Finally, we get

$$\frac{d}{dt}\mathcal{H}(t)\le-C_{k_1}\frac{\rho_2}{2}\int_0^L|\varphi_{tx}|^2dx-\left(\frac{k_2}{2}+\frac{k}{2}\right)\int_0^L|\varphi_x+\psi|^2dx-\left(\frac{k_1 bc_p}{k}+\frac{k_1^2 b}{k^2}-\frac{k_2}{2}\right)\int_0^L|\varphi|^2dx$$
$$+C\int_0^L|\psi_t|^2dx, \quad (8.26)$$

where $C_{k_1}=1-\frac{2k_1b\rho_1c_p}{k^2\rho_2}$ and $C=b^2\left(\frac{\rho_1}{k}+\frac{\rho_2}{b}\right)^2/(2\rho_2)+\frac{\mu^2}{2k}$. Therefore, we finish the proof. □

Now, we want to prove our main theorem. For this purpose, we introduce the Lyapunov functional $\mathcal{L}$ defined by

$$\mathcal{L}(t):=N_1\mathcal{E}(t)+\mathcal{F}(t)+N_2\mathcal{G}(t)+N_3\mathcal{H}(t), \quad (8.27)$$

where N_1, N_2, and N_3 are positive constants to be determined, and the functionals $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ referenced here are defined in prior lemmas.

Lemma 8.5. *There exist $\alpha_1>0$ and $\alpha_2>0$ such that*

$$\alpha_1\mathcal{E}(t)\le\mathcal{L}(t)\le\alpha_2\mathcal{E}(t)\ \ \forall t\ge 0. \quad (8.28)$$

Proof. From (8.27), we have

$$\begin{aligned}\mathcal{L}(t) - N_1\mathcal{E}(t) &= \mathcal{F}(t) + N_2\mathcal{G}(t) + N_3\mathcal{H}(t)\\ &= -\rho_1\int_0^L \varphi_t\varphi dx + N_2\rho_2\int_0^L \varphi_{tx}\varphi_x dx + N_2\frac{\mu}{2}\int_0^L |\psi|^2dx + N_2\frac{\rho_2 k_1}{k}\int_0^L \varphi_t\varphi dx\\ &\quad - N_3\rho_2\int_0^L \varphi_{tx}(\varphi_x+\psi)dx - N_3\frac{b\rho_1}{k}\int_0^L \varphi_{tx}\psi dx - N_3\frac{k_1 b\rho_1}{k^2}\int_0^L \varphi_t\varphi dx.\end{aligned}$$

Now utilizing Poincaré and Young's inequalities, we derive

$$\begin{aligned}|\mathcal{L}(t) - N_1\mathcal{E}(t)| \le &\left(\frac{\rho_1}{2} + N_2\frac{\rho_2 k_1}{2k} + N_3\frac{k_1 b\rho_1}{2k^2}\right)\int_0^L |\varphi_t|^2dx\\ &+\left(\frac{\rho_1 c_p}{2} + N_2\frac{\rho_2}{2} + N_2\frac{\rho_2 k_1 c_p}{2k} + N_3\frac{k_1 b\rho_1 c_p}{2k^2}\right)\int_0^L |\varphi_x|^2dx\\ &+\left(N_2\frac{\rho_2}{2} + N_3\frac{\rho_2}{2} + N_3\frac{b\rho_1}{2k}\right)\int_0^L |\varphi_{tx}|^2dx\\ &+\left(N_2\frac{\mu}{2} + N_3\frac{b\rho_1}{2k}\right)\int_0^L |\psi|^2dx\\ &+N_3\frac{\rho_2}{2}\int_0^L |\varphi_x+\psi|^2dx.\end{aligned}$$

We proceed by using equality (8.7), then we derive

$$\begin{aligned}|\mathcal{L}(t) - N_1\mathcal{E}(t)| \le &\left(\frac{\rho_1}{2} + N_2\frac{\rho_2 k_1}{2k} + N_3\frac{k_1 b\rho_1}{2k^2}\right)\int_0^L |\varphi_t|^2dx\\ &+\left(N_2\frac{\rho_2}{2} + N_3\frac{\rho_2}{2} + N_3\frac{b\rho_1}{2k}\right)\int_0^L |\varphi_{tx}|^2dx\\ &+\left(N_2\frac{\mu}{2} + N_3\frac{b\rho_1}{2k} + \rho_1 c_p + N_2\rho_2 + N_2\frac{\rho_2 k_1 c_p}{k} + N_3\frac{k_1 b\rho_1 c_p}{k^2}\right)\int_0^L |\psi|^2dx\\ &+\left(N_3\frac{\rho_2}{2} + \rho_1 c_p + N_2\rho_2 + N_2\frac{\rho_2 k_1 c_p}{k} + N_3\frac{k_1 b\rho_1 c_p}{k^2}\right)\int_0^L |\varphi_x+\psi|^2dx.\end{aligned}$$

Therefore, there exist $N_0 > 0$ such that

$$|\mathcal{L}(t) - N_1\mathcal{E}(t)| < N_0\mathcal{E}(t) \quad \forall t \ge 0.$$

By choosing $\alpha_1 = N_1 - N_0$, $\alpha_2 = N_1 + N_0$, and $N_1 > N_0$, the desired result is accomplished. □

It follows from Lemmas (8.2), (8.3), and (8.4) that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \le &-\rho_1\int_0^L |\varphi_t|^2dx - N_2\frac{\rho_1\rho_2}{k}\int_0^L |\varphi_{tt}|^2dx - \left[C_{k_1}N_3 - 2\left(1+\frac{k_1 c_p}{k}\right)N_2\right]\frac{\rho_2}{2}\int_0^L |\varphi_{tx}|^2dx\\ &-\left(N_2 - \frac{4\varepsilon c_p k}{b}\right)\frac{b}{2}\int_0^L |\psi_x|^2dx - \left(\left(\frac{k_1^2 b}{k^2} + \frac{k_1 b c_p}{k} - \frac{k_2}{2}\right)N_3 - k_1\right)\int_0^L |\varphi|^2dx\\ &-\left(\left(1+\frac{k_2}{k}\right)N_3 - \frac{k c_p}{b}N_2 - \frac{2c(\varepsilon)}{k}\right)\frac{k}{2}\int_0^L |\varphi_x+\psi|^2dx - (\mu N_1 - CN_3)\int_0^L |\psi_t|^2dx.\end{aligned} \tag{8.29}$$

Now, we proceed to fix the constants carefully. Selecting

$$\varepsilon = \frac{b}{4kc_p}, \tag{8.30}$$

we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq & -\rho_1 \int_0^L |\varphi_t|^2 dx - N_2 \frac{\rho_1\rho_2}{k} \int_0^L |\varphi_{tt}|^2 dx - k_2 N_3 \int_0^L |\psi|^2 dx - \Big[C_{k_1} N_3 - 2\Big(1 + \frac{k_1 c_p}{k}\Big) N_2\Big] \frac{\rho_2}{2} \int_0^L |\varphi_{tx}|^2 dx \\ & - (N_2 - 1)\frac{b}{2} \int_0^L |\psi_x|^2 dx - \left(\left(\frac{k_1^2 b}{k^2} + \frac{k_1 b c_p}{k} - \frac{k_2}{2}\right) N_3 - k_1\right) \int_0^L |\varphi|^2 dx \\ & - \left(\left(1 + \frac{k_2}{k}\right) N_3 - \frac{kc_p}{b} N_2 - \frac{2c(\varepsilon)}{k}\right) \frac{k}{2} \int_0^L |\varphi_x + \psi|^2 dx - (\mu N_1 - C N_3) \int_0^L |\psi_t|^2 dx.\end{aligned} \tag{8.31}$$

Now, we select N_2 such that

$$\begin{aligned}& N_2 > 1, \\ & N_3 > max\Big\{2\left(1 + \frac{k_1 c_p}{k}\right) C_{k_1}^{-1} N_2, \left(\frac{c(\varepsilon)}{k} + \frac{kc_p}{b} N_2\right)(1 + k_2/k)^{-1}, k_1 \left(\frac{2k_1 b}{k^2} + \frac{2bc_p}{k} - \frac{k}{k_1}\right)^{-1}\Big\}, \\ & N_1 > (C/\mu) N_3.\end{aligned}$$

It is now evident that there exists a positive constant β such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\beta \mathcal{E}(t), \quad \forall t \geq 0, \tag{8.32}$$

where $\mathcal{E}(t)$ is the energy defined for our system. As a consequence from equivalence relation between $\mathcal{E}(t)$ and $\mathcal{L}(t)$ given by lemma (8.5), we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\eta \mathcal{L}(t),$$

where $\eta = \frac{\beta}{\alpha_2}$. Now, integrating the preceding inequality across the interval $(0, t)$, we derive

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\eta t}.$$

Furthermore, the equivalence in Lemma (8.5) gives the desired result

$$\mathcal{E}(t) \leq M \mathcal{E}(0) e^{-\eta t},$$

with $M = \alpha_2/\alpha_1$. Consequently, this completes the proof of our theorem.

9. Conclusion

In this work, we extended previous studies on the Timoshenko beam model by analyzing the system resting on a two-parameter elastic foundation and incorporating a viscous damping mechanism acting solely on the angle of rotation. We treated both the classical and the truncated models and conducted a thorough investigation of their stability behavior.

For the classical model, we established the well-posedness and proved that the system is not exponentially stable under the parameter conditions (3.3). However, under the specific condition of equal wave propagation speeds, we demonstrated exponential stability using semigroup theory and resolvent estimates. When this condition is not met, the system exhibits optimal polynomial decay.

We then turned our attention to the truncated system, and it was studied under the second spectrum–free case. We established the well-posedness, then we proved exponential decay without the need for wave speed matching.

This work builds upon and extends the contributions of Almeida Júnior by not only modifying the structural formulation but also analyzing the model under more general and realistic spectral conditions.

Author contributions

Marwa Jomaa: conceptualization, writing-review & editing, writing-original draft; Toufic El Arwadi: conceptualization, writing-review & editing; Samer Israwi: conceptualization, writing-review; Dilberto D. S Almeida Junior: conceptualization, writing-review.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable remarks and comments, which helped improve the article.

Conflict of interest

This work does not have any conflicts of interest.

References

1. S. P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Phil Mag Ser*, **41** (1921), 744–746. https://doi.org/10.1080/14786442108636264
2. D. Thambiratnam, Y. Zhuge, Free vibration analysis of beams on elastic foundation, *Comput. Struct.*, **60** (1996), 971–980. https://doi.org/10.1016/0045-7949(96)00053-3
3. C. N. Chen, DQEM vibration analyses of non-prismatic shear deformable beams resting on elastic foundations, *J. Sound and Vib.*, **255** (2002), 989–999. https://doi.org/10.1006/jsvi.2001.4176
4. M. A. De Rosa, Stability and dynamics of beams on Winkler elastic foundation, *Earthquake Eng. Struct. Dyn.*, **18** (1989), 377–388. https://doi.org/10.1002/eqe.4290180306

5. S. H. Fargitaly, K. M. Zeid, An exact frequency equation for an axially loaded beam–mass–spring system resting on a Winkler elastic foundation, *J. Sound Vib.*, **185** (1995), 357–363. https://doi.org/10.1006/jsvi.1995.0384
6. P. L. Pasternak, *On a new method of analysis of an elastic foundation by means of two foundation constants,* Gos. Izd. Lit. po Strait i Arkh, 1954.
7. M. M. Filonienko-Borodich, *Some approximate theories of elastic foundation,* Uchenyie Zapiski Moskovskogo Gosudarstvennogo Universiteta, Mechanika, **46** (1940), 3–18.
8. F. Yue, Z. Wu, Z. Fan, H. Li, Two novel Vlasov models for bending analysis of finite-length beams embedded in elastic foundations, *Buildings*, **12** (2022), 1122. https://doi.org/10.3390/buildings12081122
9. N. R. Naidu, G. V. Rao, Stability behaviour of uniform columns on a class of two parameter elastic foundation, *Comput. struct.*, **57** (1995), 551–553. https://doi.org/10.1016/0045-7949(94)00636-H
10. Y. Ying, C. F. Lü, W. Q. Chen, Two-dimensional elasticity solutions for functionally graded beams resting on elastic foundations, *Compos. Struct.*, **84** (2008), 209–219. https://doi.org/10.1016/j.compstruct.2007.07.004
11. T. Yokoyama, Vibration analysis of Timoshenko beam–columns on two–parameter elastic foundations, *Comput. Struct.*, **61** (1995), 995–1007. https://doi.org/10.1016/0045-7949(96)00107-1
12. A. Soufyane, Stabilisation de la poutre de Timoshenko, *C. R. Acad. Sci. Paris, Sér. I Math.*, **328** (1999), 731–734. https://doi.org/10.1016/S0764-4442(99)80244-4
13. J. E. Muñoz Rivera, R. Racke, Global stability for damped Timoshenko systems, *Discrete Contin. Dyn. Syst.*, **9** (2003), 1625–1639. https://doi.org/10.3934/dcds.2003.9.1625
14. I. Elishakoff, An equation both more consistent and simpler than the Bresse-Timoshenko equation, In: *Advances in Mathematical Modeling and Experimental Methods for Materials and Structures, Solid Mech. Appl.,* Springer, Dordrecht, **168** (2010), 249–254. https://doi.org/10.1007/978-90-481-3467-0_19
15. D. S. Almeida Júnior, A. J. A. Ramos, On the nature of dissipative Timoshenko systems at light of the second spectrum of frequency, *Z. Angew. Math. Phys.* **68** (2017), 145. https://doi.org/10.1007/s00033-017-0881-x
16. D. S. Almeida Júnior, B. Feng, M. Afilal, A. Soufyane, The optimal decay rates for viscoelastic Timoshenko type system in the light of the second spectrum of frequency, *Z. Angew. Math. Phys.*, **72** (2021), 147. https://doi.org/10.1007/s00033-021-01574-y
17. D. S. Almeida Júnior, A. J. A. Ramos, A. Soufyane, M. L. Cardoso, M. L. Santos, Issues related to the second spectrum, Ostrogradsky's energy and the stabilization of Timoshenko–Ehrenfest-type systems, *Acta. Mech.*, **231** (2020), 3565–3581. https://doi.org/10.1007/s00707-020-02730-7
18. H. Zougheib, T. E. Arwadi, A. Soufyane, Asymptotic behavior of the porous elastic system with dual phase lag model: classical versus second spectrum perspectives, *Stud. Appl. Math.*, **151** (2023), 1136–1165. https://doi.org/10.1111/sapm.12622
19. H. Zougheib, T. E. Arwadi, M. El-Hindi, A. Soufyane, Energy decay analysis for Porous elastic system with microtemperature: Classical vs second spectrum approach, *Partial Differ. Equ. Appl.*, **5** (2024), 6. https://doi.org/10.1007/s42985-024-00273-3

20. H. Zougheib, T. El. Arwadi, R. L. R. Madureira, M. A. Rincon, Asymptotic analysis of a porous elastic system with Kelvin–Voigt damping from the second spectrum perspective, *Comp. Appl. Math.*, **43** (2024), 215. https://doi.org/10.1007/s40314-024-02621-w
21. D. S. Almeida Júnior, M. M. Freitas, A. J. A. Ramos, A. Soufyane, M. L. Cardoso, A. D, S. Campelo, Stabilization of Timoshenko–Ehrenfest type systems, *Comp. Appl. Math.*, **41** (2022), 28. https://doi.org/10.1007/s40314-021-01723-z
22. M. Jomaa, T. E. Arwadi, S. Israwi, S. Boulaaras, On the Stability of Timoshenko System With Foundation From the Classical and Second Spectrum Perspectives, *Math. Methods Appl. Sci.*, **49** (2026), 366–391. https://doi.org/10.1002/mma.70157
23. F. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Diff. Eqs.*, **1** (1985), 43–56.
24. A. Borichev, Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, *Math. Ann.*, **347** (2009), 455–478. https://doi.org/10.1007/s00208-009-0439-0
25. C. Sabrine, H. Makram, Discrete energy behavior of a damped Timoshenko system, *Comp. Appl. Math.*, **39** (2020), 4. https://doi.org/10.1007/s40314-019-0982-6
26. A. J. A. Ramos, M. Aouadi, D. S. Almeida Júnior, M. M. Freitas, M. L. Araújo, A new stabilization scenario for Timoshenko systems with thermo-diffusion effects in second spectrum perspective, *Arch. Math.*, **116** (2021), 203–219. https://doi.org/10.1007/s00013-020-01526-4

AIMS Press