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*Research article*

## Global existence of a strong solution to the planar radiative magnetohydrodynamic equations in the half-line

Hu Cheng<sup>1</sup>, Weizheng Li<sup>1</sup> and Rong Zhang<sup>1,2,\*</sup>

<sup>1</sup> School of Mathematics and Computer Sciences, Nanchang University, Nanchang 330031, P. R. China

<sup>2</sup> Institute of Mathematics and Interdisciplinary Sciences, Nanchang University, Nanchang 330031, P. R. China

\* **Correspondence:** Email: rzhang@ncu.edu.cn.

**Abstract:** The initial-boundary value problem of the planar radiative magnetohydrodynamic equations in the half-line was studied. We considered the Neumann boundary condition on the transverse magnetic field which was initially introduced by Kazhikhov in 1987. For the density-dependent viscosity and the degenerate heat-conductivity, we obtained the uniform upper and lower bounds of the density and the temperature, thus the global existence of a strong solution with large initial data to the magnetohydrodynamic system in the half-line was established.

**Keywords:** radiative MHD; density-dependent viscosity; degenerate heat-conductivity; global strong solution; half-line; Neumann boundary condition

**Mathematics Subject Classification:** 35Q35, 76N10, 76W05

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### 1. Introduction

Magnetohydrodynamics (MHD) concerns the study of the interaction between magnetic fields and electrically conducting fluids, which has been widely applied to astrophysics, geophysics, and plasma physics in practice, see [1–5] and the references therein. The governing equations of the motion of a planar magnetohydrodynamic compressible flow in the Lagrange variables read as follows:

$$v_t = u_x, \quad (1.1)$$

$$u_t + \left(P + \frac{1}{2}|\mathbf{b}|^2\right)_x = \left(\mu \frac{u_x}{v}\right)_x, \quad (1.2)$$

$$\mathbf{w}_t - \mathbf{b}_x = \left(\lambda \frac{\mathbf{w}_x}{v}\right)_x, \quad (1.3)$$

$$(\nu \mathbf{b})_t - \mathbf{w}_x = \left( \nu \frac{\mathbf{b}_x}{\nu} \right)_x, \quad (1.4)$$

$$\begin{aligned} & \left( e + \frac{u^2 + |\mathbf{w}|^2 + \nu |\mathbf{b}|^2}{2} \right)_t + \left( u \left( P + \frac{1}{2} |\mathbf{b}|^2 \right) - \mathbf{w} \cdot \mathbf{b} \right)_x \\ & = \left( \mu \frac{uu_x}{\nu} + \lambda \frac{\mathbf{w} \cdot \mathbf{w}_x}{\nu} + \nu \frac{\mathbf{b} \cdot \mathbf{b}_x}{\nu} + \kappa \frac{\theta_x}{\nu} \right)_x. \end{aligned} \quad (1.5)$$

Here,  $t \geq 0$  is time and  $x \in \Omega$  denotes the Lagrange mass coordinate. The unknown functions  $\nu > 0, u, \mathbf{w} \in \mathbb{R}^2, \mathbf{b} \in \mathbb{R}^2, e > 0, \theta > 0$ , and  $P$  are the specific volume of the gas, longitudinal velocity, transverse velocity, transverse magnetic field, internal energy, absolute temperature, and pressure, respectively.  $\mu$  and  $\lambda$  are the viscosities of the flow,  $\nu$  is the magnetic diffusivity of the magnetic field, and  $\kappa$  is the heat conductivity. According to the Stefan-Boltzmann radiative law (see [6]), the pressure  $P(\nu, \theta)$  and the internal energy  $e(\nu, \theta)$  satisfy

$$P(\nu, \theta) = \frac{R\theta}{\nu} + \frac{a\theta^4}{3}, \quad e = c_\nu \theta + a\nu\theta^4, \quad (1.6)$$

where  $R > 0$  is the perfect gas constant,  $a > 0$  is the Stefan-Boltzmann constant, and  $c_\nu > 0$  is the specific heat at constant volume. In this paper, we assume that  $\lambda$  and  $\nu$  are positive constants, and that  $\mu, \kappa$  satisfy

$$\mu = \tilde{\mu}_1 + \tilde{\mu}_2 \nu^{-\alpha}, \quad \kappa = \tilde{\kappa} \theta^\beta, \quad (1.7)$$

with constants  $\tilde{\mu}_1, \tilde{\kappa} > 0, \tilde{\mu}_2 \geq 0, \alpha \geq 0$ , and  $\beta \geq 0$ .

Let  $\Omega = (0, +\infty)$ . The system (1.1)–(1.7) is supplemented with the initial condition

$$(\nu, u, \theta, \mathbf{b}, \mathbf{w})(x, 0) = (\nu_0, u_0, \theta_0, \mathbf{b}_0, \mathbf{w}_0)(x), \quad x \in \Omega, \quad (1.8)$$

and the far-field and boundary conditions:

$$\begin{aligned} u(0, t) = 0, \quad \theta_x(0, t) = 0, \quad \mathbf{b}_x(0, t) = \mathbf{w}(0, t) = 0, \\ \lim_{x \rightarrow +\infty} (\nu, u, \theta, \mathbf{b}, \mathbf{w}) = (1, 0, 1, 0, 0), \quad t > 0, \end{aligned} \quad (1.9)$$

where the Neumann boundary condition assumed on the transverse magnetic field indicates that the magnetic field does not change on the boundary.

There is a large amount of literature on the studies of global existence and large time behavior of solutions to the compressible MHD system. In the special case where  $\mathbf{w} = \mathbf{b} = 0$ , system (1.1)–(1.5) reduces to the Navier-Stokes equations, which have been studied extensively in [7–14] (the ideal gas,  $a = 0$  in (1.6)), [15–18] (the radiative gas,  $a > 0$  in (1.6)), and the references therein.

For ideal compressible MHD flows (i.e.,  $a = 0$  in (1.6)), Kazhikhov [19] (see also Amosov and Zlotnik [20]) obtained the global existence of strong solutions with large initial data and constant coefficients. For non-degenerate heat-conductivity, there are many important results on the global solutions with large initial data, see [8, 10–12, 21] and the references therein. More precisely, Chen and Wang [9] established the existence and uniqueness for a global strong solution for  $q \geq 2$  and Fan et al. [10] obtained the global weak solution when  $q \geq 1$ . Considering the nonlinear viscosity and degenerate heat-conductivity (1.7), Hu and Ju [21] obtained global strong solutions with the initial data

$v_0 \in H^1$  and  $(u_0, \theta_0, \mathbf{w}_0, \mathbf{b}_0) \in H^2$  when  $\alpha = 0$  and  $\beta > 0$ , which was improved by Huang et al. [22] and Cao et al. [23] with more general initial data  $(v_0, u_0, \theta_0, \mathbf{w}_0, \mathbf{b}_0) \in H^1$  in bounded domains and unbounded ones, respectively, for  $\alpha \geq 0$  and  $\beta > 0$ .

Regarding radiative MHD flows (i.e.,  $a > 0$  in (1.6)), for the initial-boundary value problem in bounded domains, Zhang and Xie [24] proved the existence of global smooth solutions when the heat conductivity satisfies

$$0 < C^{-1}(1 + \theta^q) \leq \kappa(v, \theta) \leq C(1 + \theta^q) \quad (1.10)$$

with  $q > \frac{5}{2}$ . This result was subsequently improved by Qin et al. [25], who relaxed the exponent to  $q > 2$ , and further extended by Qin and Yao [26] to  $q \geq 0$ . In particular, [25] also addressed the large-time asymptotic behavior of strong solutions. For nonlinear viscosity and degenerate heat-conductivity (1.7), Wang et al. [27] and Xiao and Lu [28] obtained global smooth solutions in bounded domains for  $\alpha \geq 0$  and  $\beta > 1$ , and unbounded domains for  $\alpha \geq 0$  and  $\beta \geq 0$ . However, the methods used in [24–27] rely heavily on the boundedness of the domain and cannot be adapted directly to the case of unbounded domains, and it should be mentioned here that the boundary condition equipped on the transverse magnetic field  $\mathbf{b}$  is the Dirichlet one [28].

The aim of the present work is to show that initial-boundary value problems (IBVP) (1.1)–(1.9) with the Neumann boundary condition on  $\mathbf{b}$  in the half-line admit global strong solutions for large initial data. The key is to obtain the uniform upper and lower bounds of the density and temperature. Compared with the problem in the bounded domain [26, 27] and the ideal gas [23, 29], the current study faces additional challenges that arise from the unbounded domain  $\Omega$  and the fourth-order radiative term in (1.6). For the uniform bounds of the density, using the techniques introduced by Kazhikhov [19] with localization strategies due to Jiang [30, 31], we derive a crucial local representation for  $v$  (see (2.6)), see Lemma 2.3 for details. For the upper and lower bounds of the temperature, we adapt methodologies from Li et al. [13] and Liang and Li [7], as presented in Lemma 2.4. In order to overcome the difficulty caused by the fourth-order radiative term (see (2.50)), we test the temperature equation (2.4) with  $(\theta - 2)_+^7$  and then obtain the estimates of  $(\theta - 2)_+ \in L^\infty(0, T; L^8(\Omega))$  (see (2.57) in Lemma 2.6), which is crucial to obtain the upper bound of the temperature. Our result is stated as follows:

**Theorem 1.1.** *Let  $\alpha \geq 0$  and  $\beta \geq 0$ . Assume that the initial data  $(v_0, u_0, \theta_0, \mathbf{b}_0, \mathbf{w}_0)$  satisfies*

$$(v_0 - 1, u_0, \theta_0 - 1, \mathbf{b}_0, \mathbf{w}_0) \in H^1(\Omega), \quad \inf_{x \in \Omega} v_0(x) > 0, \quad \inf_{x \in \Omega} \theta_0(x) > 0, \quad (1.11)$$

*and are compatible with the boundary condition (1.9). Then the IBVP (1.1)–(1.9) admits a unique global strong solution  $(v, u, \theta, \mathbf{b}, \mathbf{w})$  satisfying*

$$\begin{cases} v - 1, u, \theta - 1, \mathbf{b}, \mathbf{w} \in L^\infty(0, T; H^1(\Omega)), \\ v_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ u_t, \theta_t, \mathbf{b}_t, \mathbf{w}_t, u_{xx}, \theta_{xx}, \mathbf{b}_{xx}, \mathbf{w}_{xx} \in L^2(\Omega \times (0, T)). \end{cases} \quad (1.12)$$

*Moreover, there exist some positive constant  $C$ , depending only on the data, and  $T$  such that for any  $(x, t) \in \Omega \times [0, T]$ ,*

$$C^{-1} \leq v(x, t) \leq C, \quad C^{-1} \leq \theta(x, t) \leq C. \quad (1.13)$$

**Remark 1.1.** *The Neumann boundary condition applied to the transverse magnetic field (see (1.9)) characterizes the continuity and smoothness of the magnetic field across the boundary, thereby preventing*

abrupt variations or extraneous surface currents. This formulation not only aligns more closely with physical principles in many practical contexts but also contributes to enhanced stability of the solution compared to the conventional Dirichlet boundary condition.

**Remark 1.2.** Theorem 1.1 can be regarded as a natural extension of both [26] ( $\alpha = \beta = 0$ ) and [27] ( $\alpha \geq 0, \beta > 1$ ) for the IBVP in bounded domains to the problem with general coefficients ( $\alpha \geq 0, \beta \geq 0$ ) in unbounded domains.

**Remark 1.3.** The large-time behavior of the solution obtained in Theorem 1.1 is still unknown and the global existence of the solution to the IBVP on bounded domains with the Neumann boundary condition on  $\mathbf{b}$  is also unsolved now. Both issues will be addressed in future work.

In the next section, we give details of the proof.

## 2. Proof of Theorem 1.1

We first state the following local existence lemma which can be obtained by using the Banach fixed point theorem on a small time interval (see [32–34]).

**Lemma 2.1.** Under the assumptions of Theorem 1.1, there exists a small time  $T_0 > 0$  depending on initial data such that the IBVP (1.1)–(1.9) has a unique strong solution  $(v, u, \theta, \mathbf{b}, \mathbf{w})$  with positive  $v(x, t)$  and  $\theta(x, t)$  satisfying (1.12).

Then, after obtaining some necessary a priori estimates (see (2.5), (2.26), (2.35), (2.68), and (2.69) below) where the constants depend only on the problem data, one can continue the local solution obtained by Lemma 2.1 to the whole interval  $[0, T]$  for any  $T > 0$ , which completes the proof of Theorem 1.1.

Next, without loss of generality, we assume that  $\tilde{\mu}_1 = \tilde{\kappa} = \lambda = \nu = R = c_\nu = a = 1$  and  $\tilde{\mu}_2 = \alpha$ . We first derive the following basic energy estimates.

**Lemma 2.2.** It holds that for any  $(x, t) \in \Omega \times [0, T]$ ,

$$\sup_{0 \leq t \leq T} E(t) + \int_0^T W(s) ds \leq E(0), \quad (2.1)$$

where

$$E(t) \triangleq \int_{\Omega} \left( \frac{u^2 + |\mathbf{w}|^2 + \nu |\mathbf{b}|^2}{2} + F(v) + F(\theta) + \frac{1}{3} \nu (\theta - 1)^2 (3\theta^2 + 2\theta + 1) \right) dx,$$

with

$$F(s) \triangleq s - \log s - 1, \quad \text{for any } s > 0, \quad (2.2)$$

and

$$W(t) \triangleq \int_{\Omega} \left( \frac{\theta^\beta \theta_x^2}{\nu \theta^2} + \frac{\mu u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2}{\nu \theta} \right) (x, t) dx. \quad (2.3)$$

*Proof.* We begin by rewriting (1.5) using (1.1)–(1.4) and (1.6) as

$$\theta_t + 4\nu\theta^3\theta_t + \left( \frac{\theta}{\nu} + \frac{4}{3}\theta^4 \right) u_x = \frac{\mu u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2}{\nu} + \left( \frac{\theta^\beta \theta_x}{\nu} \right)_x. \quad (2.4)$$

Multiplying (1.1) by  $1 - v^{-1}$ , (1.2) by  $u$ , (1.3) by  $\mathbf{w}$ , (1.4) by  $\mathbf{b}$ , and (2.4) by  $1 - \theta^{-1}$ , and summing the results, yields

$$\begin{aligned} & \left( \frac{u^2 + |\mathbf{w}|^2 + v|\mathbf{b}|^2}{2} + F(v) + F(\theta) + \frac{1}{3}v(\theta - 1)^2(3\theta^2 + 2\theta + 1) \right)_t \\ & + \frac{\mu u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2}{v\theta} + \frac{\theta^\beta \theta_x^2}{v\theta^2} \\ & = \left( \frac{\mu u u_x + \mathbf{w} \cdot \mathbf{w}_x + \mathbf{b} \cdot \mathbf{b}_x + (1 - \theta^{-1})\theta^\beta \theta_x}{v} - \left( \frac{\theta}{v} + \frac{\theta^4 - 4}{3} + \frac{|\mathbf{b}|^2}{2} \right) u + \mathbf{w} \cdot \mathbf{b} \right)_x. \end{aligned}$$

Integrating the above equality over  $\Omega \times [0, T]$  and applying (1.9), we derive (2.1).  $\square$

By making a slight modification to the method of Kazhikhov [19], we are able to prove the lower and upper bounds of  $v$ .

**Lemma 2.3.** *There exists a positive constant  $C$  such that*

$$C^{-1} \leq v(x, t) \leq C, \quad (2.5)$$

where (and in what follows)  $C$  denotes a generic positive constant depending only on  $T, \alpha, \beta, \|(v_0 - 1, u_0, \theta_0 - 1, \mathbf{b}_0, \mathbf{w}_0)\|_{H^1(\Omega)}, \inf_{x \in \Omega} v_0(x)$ , and  $\inf_{x \in \Omega} \theta_0(x)$ .

*Proof.* First, for any  $x \in \Omega$ , denoting  $N = [x]$ , we can obtain the following representation of  $v$ :

$$v(x, t) = D_N(x, t) Y_N(t) \exp\{v^{-\alpha}\} \left( 1 + \int_0^t \frac{\theta + \frac{1}{3}v\theta^4 + \frac{1}{2}v|\mathbf{b}|^2}{D_N(x, \tau) Y_N(\tau) \exp\{v^{-\alpha}\}} d\tau \right), \quad (2.6)$$

where

$$D_N(x, t) \triangleq v_0(x) \exp\{-v_0^{-\alpha}\} \exp\left\{ \int_N^x (u(y, t) - u_0(y)) dy \right\},$$

and

$$Y_N(t) \triangleq \exp\left\{ \int_0^t \sigma(N, s) ds \right\}, \quad \sigma \triangleq \frac{\mu u_x}{v} - P - \frac{1}{2}|\mathbf{b}|^2.$$

In fact,

$$\sigma = (\log v - v^{-\alpha})_t - \left( \frac{\theta}{v} + \frac{\theta^4}{3} \right) - \frac{1}{2}|\mathbf{b}|^2,$$

and integrating the equation  $u_t = \sigma_x$  over  $[N, x] \times [0, t]$ , we obtain

$$\int_N^x (u(y, t) - u_0) dy = \log v - v^{-\alpha} - \log v_0 + v_0^{-\alpha} - H(x, t) - \int_0^t \sigma(N, s) ds,$$

which leads to

$$v(x, t) = D_N(x, t) Y_N(t) \exp\{v^{-\alpha}\} \exp\{H(x, t)\}, \quad (2.7)$$

where

$$H(x, t) \triangleq \int_0^t \left( \frac{\theta}{v} + \frac{\theta^4}{3} + \frac{1}{2}|\mathbf{b}|^2 \right) ds.$$

From expression (2.7), we deduce

$$H_t = \frac{\theta + \frac{1}{3}v\theta^4 + \frac{1}{2}v|\mathbf{b}|^2}{v} = \frac{\theta + \frac{1}{3}v\theta^4 + \frac{1}{2}v|\mathbf{b}|^2}{D_N(x, t)Y_N(t) \exp\{v^{-\alpha}\} \exp\{H\}}, \quad (2.8)$$

which implies

$$\exp\{H\} = 1 + \int_0^t \frac{\theta + \frac{1}{3}v\theta^4 + \frac{1}{2}v|\mathbf{b}|^2}{D_N(x, \tau)Y_N(\tau) \exp\{v^{-\alpha}\}} d\tau. \quad (2.9)$$

Substituting this result back into (2.7) yields (2.6).

Observing that

$$\left| \int_N^x (u(y, t) - u_0(y)) dy \right| \leq \left( \int_N^{N+1} u^2 dy \right)^{1/2} + \left( \int_N^{N+1} u_0^2 dy \right)^{1/2} \leq C,$$

we have

$$C^{-1} \leq D_N(x, t) \leq C, \quad x \in [N, N + 1], \quad (2.10)$$

where in what follows,  $C$  is a constant independent of  $N$ .

Next, it follows from (2.1) that

$$\int_N^{N+1} (F(v) + F(\theta)) dx \leq E(0). \quad (2.11)$$

An application of Jensen's inequality to (2.11) yields

$$\alpha_1 \leq \int_N^{N+1} v(x, t) dx \leq \alpha_2, \quad \alpha_1 \leq \int_N^{N+1} \theta(x, t) dx \leq \alpha_2, \quad (2.12)$$

where  $0 < \alpha_1 < \alpha_2$  are two roots of  $F(s) = E(0)$ . Furthermore, inequality (2.11) implies that for any  $t > 0$ , there exists some  $b_N(t) \in [N, N + 1]$  such that

$$(v - \log v - 1 + \theta - \log \theta - 1)(b_N(t), t) \leq E(0),$$

from which we deduce

$$\alpha_1 \leq v(b_N(t), t) \leq \alpha_2, \quad \alpha_1 \leq \theta(b_N(t), t) \leq \alpha_2. \quad (2.13)$$

We proceed to estimate  $Y_N$ . On the one hand, integrating (2.6) over the interval  $[N, N + 1]$  and combining this with (2.12) and (2.10), we obtain

$$\frac{1}{Y_N(t)} \geq C. \quad (2.14)$$

On the other hand, it follows from (2.12) that

$$\int_N^{N+1} v(x, t) 1_{\{v > \frac{\alpha_1}{2}\}} dx \geq \frac{\alpha_1}{2}. \quad (2.15)$$

Then, multiplying (2.6) by  $\frac{1}{Y_N(t)} 1_{v > \frac{\alpha_1}{2}}$  and integrating over  $[N, N + 1]$ , we apply (2.10) and (2.15) to obtain

$$\begin{aligned} \frac{1}{Y_N(t)} &\leq \frac{C}{Y_N(t)} \int_N^{N+1} v(x, t) 1_{\{v > \frac{\alpha_1}{2}\}} dx \\ &\leq C \int_N^{N+1} D_N(x, t) \exp\{v^{-\alpha}\} 1_{\{v > \frac{\alpha_1}{2}\}} \left(1 + \int_0^t \frac{\theta + \frac{1}{3}v\theta^4 + \frac{1}{2}v|\mathbf{b}|^2}{D_N(x, \tau)Y_N(\tau)} d\tau\right) dx \\ &\leq C + C \int_0^t \frac{1}{Y_N(\tau)} \int_N^{N+1} \left(\theta + \frac{1}{3}v\theta^4 + \frac{1}{2}v|\mathbf{b}|^2\right) dx d\tau. \end{aligned} \quad (2.16)$$

Noticing that

$$(\theta - 1)^2 + (\theta - 1)^4 \leq C(\theta - 1)^2(3\theta^2 + 2\theta + 1), \quad (2.17)$$

one can obtain after using (2.1) that

$$\int_{\Omega} v(\theta - 1)^2 dx + \int_{\Omega} v(\theta - 1)^4 dx \leq C, \quad (2.18)$$

which together with (2.12) implies that

$$\begin{aligned} \int_N^{N+1} v\theta^4 dx &= \int_N^{N+1} v(\theta - 1 + 1)^4 dx \\ &\leq C \int_N^{N+1} v dx + C \int_N^{N+1} v(\theta - 1)^4 dx \\ &\leq C. \end{aligned} \quad (2.19)$$

Then using the above estimates and (2.1), (2.16) becomes

$$\frac{1}{Y_N(t)} \leq C + C \int_0^t \frac{1}{Y_N(\tau)} d\tau, \quad (2.20)$$

which together with the Grönwall's inequality and (2.14) implies

$$C^{-1} \leq Y_N(t) \leq C.$$

Combining this with (2.6) and (2.10), it follows that for  $(x, t) \in [N, N + 1] \times [0, T]$ ,

$$\begin{aligned} C^{-1} \leq v(x, t) &\leq C + C \int_0^t \left(\frac{\theta}{v} + \theta^4 + |\mathbf{b}|^2\right) v dt \\ &\leq C + C \int_0^t v \sup_{x \in [N, N+1]} (1 + \theta^4 + |\mathbf{b}|^2) dt. \end{aligned} \quad (2.21)$$

Then, direct calculations combining with (2.19) deduce that

$$\begin{aligned} \left| \theta^{\frac{\beta+4}{2}}(x, t) - \theta^{\frac{\beta+4}{2}}(b_N(t), t) \right| &= \left| \int_{b_N(t)}^x \left(\theta^{\frac{\beta+4}{2}}\right)_x dx \right| \\ &\leq C \left( \int_N^{N+1} \frac{\theta^\beta \theta_x^2}{\theta^2 v} dx \right)^{1/2} \left( \int_N^{N+1} v\theta^4 dx \right)^{1/2} \\ &\leq CW^{1/2}(t), \end{aligned}$$

which together with (2.13) yields that for any  $t > 0$ ,

$$\sup_{x \in [N, N+1]} (1 + \theta^4) \leq C + CW(t). \quad (2.22)$$

Furthermore, it follows from (2.1) and (2.18) that

$$\begin{aligned} \sup_{x \in \Omega} |\mathbf{b}|^2 &\leq C \int_{\Omega} |\mathbf{b} \cdot \mathbf{b}_x| dx \\ &\leq C \int_{\Omega} \frac{|\mathbf{b}_x|^2}{v\theta} dx + C \int_{\Omega} v|\mathbf{b}|^2 dx + C \int_{\Omega} (\theta - 1)v|\mathbf{b}|^2 dx \\ &\leq C \int_{\Omega} \frac{|\mathbf{b}_x|^2}{v\theta} dx + C + C \sup_{x \in \Omega} |\mathbf{b}| \left( \int_{\Omega} v(\theta - 1)^2 dx \right)^{1/2} \left( \int_{\Omega} v|\mathbf{b}|^2 dx \right)^{1/2} \\ &\leq C \int_{\Omega} \frac{|\mathbf{b}_x|^2}{v\theta} dx + C + \frac{1}{2} \sup_{x \in \Omega} |\mathbf{b}|^2, \end{aligned} \quad (2.23)$$

which implies

$$\sup_{x \in \Omega} |\mathbf{b}|^2 \leq C \int_{\Omega} \frac{|\mathbf{b}_x|^2}{v\theta} dx + C \leq CW(t) + C. \quad (2.24)$$

Substituting (2.22) and (2.24) into (2.21), we obtain

$$v(x, t) \leq C + C \int_0^t (1 + W(t)) v dt, \quad (2.25)$$

which together with Grönwall's inequality and (2.1) yields that

$$\sup_{(x,t) \in [N, N+1] \times [0, T]} v(x, t) \leq C.$$

This combined with (2.21) and the fact that  $C$  is independent of  $N$  gives (2.5). The proof of Lemma 2.3 is finished.  $\square$

To proceed, we are in a position to establish a lower bound on the temperature  $\theta$ .

**Lemma 2.4.** *There exists a positive constant  $C$  such that for any  $(x, t) \in \Omega \times [0, T]$ ,*

$$\theta(x, t) \geq C^{-1}. \quad (2.26)$$

*Proof.* First, denoting

$$\begin{aligned} (\theta > 2)(t) &= \{x \in \Omega \mid \theta(x, t) > 2\}, \\ (\theta < 1/2)(t) &= \{x \in \Omega \mid \theta(x, t) < 1/2\}, \end{aligned}$$

it follows from (2.1) that

$$\begin{aligned} E(0) &\geq \int_{(\theta < 1/2)(t)} (\theta - \log \theta - 1) dx + \int_{(\theta > 2)(t)} (\theta - \log \theta - 1) dx \\ &\geq (\log 2 - 1/2)|(\theta < 1/2)(t)| + (1 - \log 2)|(\theta > 2)(t)|, \end{aligned}$$

which shows that for any  $t \in [0, T]$ ,

$$|(\theta < 1/2)(t)| + |(\theta > 2)(t)| \leq C. \quad (2.27)$$

For any  $p \geq 0$ , we multiply (2.4) by  $\theta^{-5}w^p$ , where  $w \triangleq (\theta^{-1} - 2)_+$ . Integrating the result over  $\Omega$  and using (2.5), we find that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left( \frac{1}{p+4} w^{p+4} + \frac{6}{p+3} w^{p+3} + \frac{12}{p+2} w^{p+2} + \frac{8+4v}{p+1} w^{p+1} \right) dx \\ & + \int_{\Omega} \frac{(\mu u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) w^p}{v\theta^5} dx \\ & \leq C \int_{\Omega} |u_x| \theta^{-4} w^p dx + C \int_{\Omega} |u_x| \theta^{-1} w^p dx + C \int_{\Omega} |u_x| w^{p+1} dx \\ & \leq \frac{1}{2} \int_{\Omega} \frac{u_x^2 w^p}{v\theta^5} dx + C \int_{\Omega} (\theta^{-3} w^p + \theta^3 w^p + \theta^5 w^{p+2}) dx. \end{aligned}$$

It follows from (2.27) that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left( \frac{1}{p+4} w^{p+4} + \frac{6}{p+3} w^{p+3} + \frac{12}{p+2} w^{p+2} + \frac{8+4v}{p+1} w^{p+1} \right) dx \\ & + \int_{\Omega} \frac{(\mu u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) w^p}{v\theta^5} dx \\ & \leq \frac{1}{2} \int_{\Omega} \frac{u_x^2 w^p}{v\theta^5} dx + C \int_{(\theta < 1/2)} ([(\theta^{-1} - 2) + 2]^3 w^p + \theta^3 w^p + \theta^5 w^{p+2}) dx \\ & \leq \frac{1}{2} \int_{\Omega} \frac{u_x^2 w^p}{v\theta^5} dx + C \int_{(\theta < 1/2)} (w^p + w^{p+3}) dx \\ & \leq \frac{1}{2} \int_{\Omega} \frac{u_x^2 w^p}{v\theta^5} dx + C + \int_{\Omega} w^{p+4} dx. \end{aligned} \quad (2.28)$$

Integrating (2.28) over  $(0, T)$  and then combining with Grönwall's inequality, we have

$$\frac{1}{p+4} \int_{\Omega} w^{p+4} dx \leq C e^{C(p+4)}, \quad (2.29)$$

which implies

$$\left( \frac{1}{p+4} \int_{\Omega} w^{p+4} dx \right)^{\frac{1}{p+4}} \leq C, \quad (2.30)$$

where the positive constant  $C$  is independent of  $p$ . Finally, passing to the limit as  $p \rightarrow \infty$  in (2.30), we conclude that

$$\sup_{0 \leq t \leq T} \|(\theta^{-1} - 2)_+\|_{L^\infty(\Omega)} \leq C.$$

This establishes the desired estimate (2.26) and completes the proof of Lemma 2.4.  $\square$

**Lemma 2.5.** *There exists a positive constant  $C$  such that*

$$\int_0^T \int_{\Omega} (u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx dt \leq C. \quad (2.31)$$

*Proof.* First, adding (1.2) multiplied by  $u$ , (1.3) multiplied by  $\mathbf{w}$ , and (1.4) multiplied by  $\mathbf{b}$  together, and integrating the resultant equality over  $\Omega$ , one deduces from (2.24) and (2.5) that

$$\begin{aligned} & \frac{1}{2} \left( \int_{\Omega} (u^2 + v|\mathbf{b}|^2 + |\mathbf{w}|^2) dx \right)_t + \int_{\Omega} \frac{\mu u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2}{v} dx \\ &= \int_{\Omega} \frac{\theta}{v} u_x dx + \frac{1}{3} \int_{\Omega} \theta^4 u_x dx \\ &= \int_{\Omega} \frac{(\theta-1)}{v} u_x dx - \int_{\Omega} \frac{(v-1)u_x}{v} dx + \frac{1}{3} \int_{\Omega} ((\theta-1)+1)^4 u_x dx \\ &\leq C \int_{\Omega} (\theta-1)^2 dx + C \int_{\Omega} (v-1)^2 dx + C \int_{\Omega} (\theta-1)^8 dx + \frac{1}{4} \int_{\Omega} \frac{\mu u_x^2}{v} dx \\ &\leq C + C \sup_{x \in \Omega} (\theta^4 + 1) \int_{\Omega} (\theta-1)^4 dx + \frac{1}{4} \int_{\Omega} \frac{\mu u_x^2}{v} dx \\ &\leq C + C \sup_{x \in \Omega} \theta^4 + \frac{1}{4} \int_{\Omega} \frac{\mu u_x^2}{v} dx. \end{aligned} \quad (2.32)$$

Next, observing that

$$\sup_{x \in \Omega} \theta^{4+\beta} \leq C \sup_{x \in \Omega} |\theta-1|^{4+\beta} + C \leq C \int_{\Omega} |\theta-1|^{3+\beta} |\theta_x| dx + C,$$

which together with Hölder's inequality yields that

$$\begin{aligned} \sup_{x \in \Omega} \theta^{4+\beta} &\leq C \left( \int_{\Omega} |\theta-1|^{8+\beta} dx \right)^{1/2} \left( \int_{\Omega} \theta^{\beta-2} \theta_x^2 dx \right)^{1/2} + C \\ &\leq C \left( \sup_{x \in \Omega} \theta^{4+\beta} + 1 \right)^{1/2} \left( \int_{\Omega} (\theta-1)^4 dx \right)^{1/2} \left( \int_{\Omega} \theta^{\beta-2} \theta_x^2 dx \right)^{1/2} + C \\ &\leq \frac{1}{2} \sup_{x \in \Omega} \theta^{4+\beta} + C + C \int_{\Omega} (\theta-1)^4 dx \int_{\Omega} \theta^{\beta-2} \theta_x^2 dx, \end{aligned} \quad (2.33)$$

it follows from (2.18), (2.1), (2.5), and (2.26) that

$$\begin{aligned} & \int_0^T \sup_{x \in \Omega} \theta^{4+\beta} dt + \int_0^T \int_{\Omega} |\theta-1|^{8+\beta} dx dt \\ &\leq \int_0^T \sup_{x \in \Omega} \theta^{4+\beta} dt + C \int_0^T (\sup_{x \in \Omega} \theta^{4+\beta} + 1) \int_{\Omega} (\theta-1)^4 dx dt \\ &\leq C + C \int_0^T \int_{\Omega} \theta^{\beta-2} \theta_x^2 dx dt \\ &\leq C. \end{aligned} \quad (2.34)$$

Hence, the combination of (2.32) with (2.34), (2.1), and (2.5) gives directly (2.31), and thus completes the proof of Lemma 2.5.  $\square$

Next, we will prove the estimates on the  $L^\infty((0, T); L^2)$ -norm of  $(v_x, u_x, \mathbf{b}_x, \mathbf{w}_x)$ .

**Lemma 2.6.** *There exists a positive constant  $C$  such that*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\Omega} \left( (\theta - 2)_+^8 + (\theta - 2)_+^{11} + v_x^2 + u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2 \right) dx \\ & + \int_0^T \int_{\Omega} \left( \theta^\beta (\theta - 2)_+^6 \theta_x^2 + v_x^2 \right) dx dt \leq C. \end{aligned} \quad (2.35)$$

*Proof.* First, using (1.1), we can reformulate equation (1.2) as follows:

$$\left( \frac{\mu v_x}{v} - u \right)_t = \frac{\theta_x}{v} - \frac{\theta}{v^2} v_x + \frac{4}{3} \theta^3 \theta_x + \mathbf{b} \cdot \mathbf{b}_x. \quad (2.36)$$

Multiplying (2.36) by  $\frac{\mu v_x}{v} - u$  and integrating over  $\Omega$  yields:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \frac{\mu v_x}{v} - u \right)^2 dx + \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx \\ & = \int_{\Omega} \frac{\theta u v_x}{v^2} dx + \int_{\Omega} \left( \frac{\mu v_x}{v} - u \right) \frac{\theta_x}{v} dx \\ & \quad + \frac{4}{3} \int_{\Omega} \left( \frac{\mu v_x}{v} - u \right) \theta^3 \theta_x dx + \int_{\Omega} \left( \frac{\mu v_x}{v} - u \right) \mathbf{b} \cdot \mathbf{b}_x dx \\ & \triangleq I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (2.37)$$

A direct application of the a priori estimates (2.5), (2.1), (2.26), and (2.24) yields the subsequent series of estimates. A direct calculation gives the control:

$$I_1 \leq \varepsilon \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx + C \int_{\Omega} \frac{\theta u^2}{v \mu} dx \leq \varepsilon \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx + C \sup_{x \in \Omega} \theta. \quad (2.38)$$

For  $I_2$ , we obtain:

$$\begin{aligned} I_2 & = \int_{\Omega} \frac{\mu v_x \theta_x}{v^2} dx - \int_{\Omega} \frac{\theta_x u}{v} dx \\ & \leq \varepsilon \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx + C \int_{\Omega} \frac{\mu \theta^{-1} \theta_x^2}{v} dx + C \int_{\Omega} \frac{\theta^{-1} \theta_x^2}{v^2} dx + C \sup_{x \in \Omega} \theta \int_{\Omega} u^2 dx \\ & \leq \varepsilon \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx + C \int_{\Omega} \theta^{-1} \theta_x^2 dx + C \sup_{x \in \Omega} \theta. \end{aligned} \quad (2.39)$$

The estimate for  $I_3$  proceeds as follows:

$$\begin{aligned} I_3 & = \frac{4}{3} \int_{\Omega} \frac{\mu \theta^3 v_x \theta_x}{v} dx - \frac{4}{3} \int_{\Omega} u \theta^3 \theta_x dx \\ & \leq \varepsilon \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx + C \int_{\Omega} \mu v \theta^5 \theta_x^2 dx + C \int_{\Omega} \theta^6 \theta_x^2 dx + C \int_{\Omega} u^2 dx \\ & \leq \varepsilon \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx + C \int_{\Omega} \theta^6 \theta_x^2 dx + C. \end{aligned} \quad (2.40)$$

Finally,  $I_4$  is bounded by:

$$\begin{aligned}
 I_4 &= \int_{\Omega} \frac{\mu v_x}{v} \mathbf{b} \cdot \mathbf{b}_x dx - \int_{\Omega} u \mathbf{b} \cdot \mathbf{b}_x dx \\
 &\leq \varepsilon \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx + C \int_{\Omega} \mu v \theta^{-1} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx + C \int_{\Omega} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx + C \int_{\Omega} u^2 dx \\
 &\leq \varepsilon \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx + C \sup_{x \in \Omega} |\mathbf{b}|^2 \int_{\Omega} |\mathbf{b}_x|^2 dx + C \\
 &\leq \varepsilon \int_{\Omega} \frac{\mu \theta v_x^2}{v^3} dx + C \left( \int_{\Omega} |\mathbf{b}_x|^2 dx \right)^2 + C \int_{\Omega} |\mathbf{b}_x|^2 dx + C.
 \end{aligned} \tag{2.41}$$

Submitting (2.38)–(2.41) into (2.37), integrating over  $[0, T]$ , and then selecting a sufficiently small  $\varepsilon$  and using (2.31), we obtain

$$\sup_{0 \leq t \leq T} \int_{\Omega} v_x^2 dx + \int_0^T \int_{\Omega} v_x^2 dx dt \leq C + C \int_0^T \int_{\Omega} \theta^6 \theta_x^2 dx dt + C \int_0^T \left( \int_{\Omega} |\mathbf{b}_x|^2 dx \right)^2 dt, \tag{2.42}$$

where we have used

$$\int_{\Omega} \frac{\mu v_x}{v} u dx \leq \frac{1}{4} \int_{\Omega} \left( \frac{\mu v_x}{v} \right)^2 dx + C. \tag{2.43}$$

Next, integration by parts together with (1.1) gives

$$- \int_{\Omega} u_t \left( \frac{\mu u_x}{v} \right)_x dx = \frac{1}{2} \left( \int_{\Omega} \frac{\mu u_x^2}{v} dx \right)_t + \frac{1}{2} \int_{\Omega} \frac{v \mu'(v) - \mu}{v^2} u_x^3 dx. \tag{2.44}$$

Multiplying (1.2) by  $\left( \frac{\mu u_x}{v} \right)_x$  and integrating the resultant equality over  $\Omega$ , it follows from (2.44), (2.5), (2.26), (2.45), and (2.24) that

$$\begin{aligned}
 &\frac{1}{2} \left( \int_{\Omega} \frac{\mu u_x^2}{v} dx \right)_t + \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx \\
 &\leq C \int_{\Omega} |u_x|^3 dx + \frac{1}{4} \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx \\
 &\quad + C \int_{\Omega} \theta^6 \theta_x^2 dx + C \int_{\Omega} \theta_x^2 dx + C \int_{\Omega} \theta^2 v_x^2 dx + C \int_{\Omega} |\mathbf{b}|^2 |\mathbf{b}_x|^2 dx \\
 &\leq C \sup_{x \in \Omega} |u_x|^2 + C \left( \int_{\Omega} u_x^2 dx \right)^2 + \frac{1}{4} \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx + C \int_{\Omega} \theta^6 \theta_x^2 dx \\
 &\quad + C \sup_{x \in \Omega} \theta^2 \int_{\Omega} v_x^2 dx + C \sup_{x \in \Omega} |\mathbf{b}|^2 \int_{\Omega} |\mathbf{b}_x|^2 dx.
 \end{aligned}$$

Observing the fact that:

$$\begin{aligned}
 \sup_{x \in \Omega} u_x^2 &\leq C \sup_{x \in \Omega} \left( \frac{\mu u_x}{v} \right)^2 \leq C \left| \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx \right| \\
 &\leq C \left( \int_{\Omega} u_x^2 dx \right)^{1/2} \left( \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx \right)^{1/2},
 \end{aligned} \tag{2.45}$$

we finally have

$$\begin{aligned}
& \frac{1}{2} \left( \int_{\Omega} \frac{\mu u_x^2}{v} dx \right)_t + \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x dx \\
& \leq C \int_{\Omega} u_x^2 dx + C \left( \int_{\Omega} u_x^2 dx \right)^2 + \frac{1}{2} \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x dx + C \int_{\Omega} \theta^6 \theta_x^2 dx \\
& \quad + C \sup_{x \in \Omega} \theta^2 \int_{\Omega} v_x^2 dx + C \int_{\Omega} |\mathbf{b}_x|^2 dx + C \left( \int_{\Omega} |\mathbf{b}_x|^2 dx \right)^2.
\end{aligned} \tag{2.46}$$

Collecting the results of (2.46), (2.5), and (2.31) together, we derive

$$\begin{aligned}
& \int_{\Omega} u_x^2 dx + \int_0^T \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x dx dt \\
& \leq C + C \int_0^T \int_{\Omega} \theta^6 \theta_x^2 dx dt + C \int_0^T \left( \int_{\Omega} |\mathbf{b}_x|^2 dx \right)^2 dt \\
& \quad + C \int_0^T \left( \sup_{x \in \Omega} \theta^2 + \int_{\Omega} u_x^2 dx + 1 \right) \int_{\Omega} (v_x^2 + u_x^2) dx dt.
\end{aligned} \tag{2.47}$$

Furthermore, in view of (2.42), inequality (2.47) implies that

$$\begin{aligned}
& \int_{\Omega} (v_x^2 + u_x^2) dx + \int_0^T \int_{\Omega} \left( \left( \frac{\mu u_x}{v} \right)_x + v_x^2 \right) dx dt \\
& \leq C + C \int_0^T \int_{\Omega} \theta^6 \theta_x^2 dx dt + C \int_0^T \left( \int_{\Omega} |\mathbf{b}_x|^2 dx \right)^2 dt \\
& \quad + C \int_0^T \left( \sup_{x \in \Omega} \theta^2 + \int_{\Omega} u_x^2 dx + 1 \right) \int_{\Omega} (v_x^2 + u_x^2) dx dt.
\end{aligned} \tag{2.48}$$

It follows from (2.1), (2.5), and (2.26) that

$$\int_0^T \int_{\Omega} \theta^{-6} \theta_x^2 dx dt \leq C \int_0^T \int_{\Omega} \theta^{-2} \theta_x^2 dx dt \leq C \int_0^T \int_{\Omega} v^{-1} \theta^{\beta-2} \theta_x^2 dx dt \leq C. \tag{2.49}$$

Then for the second term on the right-hand side of (2.48), one has

$$\begin{aligned}
& \int_0^T \int_{\Omega} \theta^6 \theta_x^2 dx dt \leq C \int_0^T \int_{\Omega} ((\theta - 2)_+ + 2)^6 \theta_x^2 dx dt \\
& \leq C \int_0^T \int_{\Omega} (\theta - 2)_+^6 \theta_x^2 dx dt + C \int_0^T \int_{\Omega} \theta_x^2 dx dt \\
& \leq C \int_0^T \int_{\Omega} \theta^{\beta} (\theta - 2)_+^6 \theta_x^2 dx dt \\
& \quad + C \int_0^T \int_{\Omega} \theta^{-6} \theta_x^2 dx dt + \frac{1}{2} \int_0^T \int_{\Omega} \theta^6 \theta_x^2 dx dt \\
& \leq C + \frac{C_1}{2} \int_0^T \int_{\Omega} \theta^{\beta} (\theta - 2)_+^6 \theta_x^2 dx dt + \frac{1}{2} \int_0^T \int_{\Omega} \theta^6 \theta_x^2 dx dt.
\end{aligned} \tag{2.50}$$

Thus, combining (2.48) with (2.50) leads to

$$\begin{aligned}
& \int_{\Omega} (v_x^2 + u_x^2) dx + \int_0^T \int_{\Omega} \left( \left( \frac{\mu u_x}{v} \right)^2 + v_x^2 \right) dx dt \\
& \leq C + C_1 \int_0^T \int_{\Omega} \theta^\beta (\theta - 2)_+^6 \theta_x^2 dx dt + C \int_0^T \left( \int_{\Omega} |\mathbf{b}_x|^2 dx \right)^2 dt \\
& \quad + C \int_0^T \left( \sup_{x \in \Omega} \theta^2 + \int_{\Omega} u_x^2 dx + 1 \right) \int_{\Omega} (v_x^2 + u_x^2) dx dt.
\end{aligned} \tag{2.51}$$

To estimate the second term on the right-hand side of (2.51), we multiply (2.4) by  $(\theta - 2)_+^7$  and integrate over  $\Omega$ , which yields

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \left( \left( \frac{1}{8} + 4v \right) (\theta - 2)_+^8 + \frac{4v}{11} (\theta - 2)_+^{11} + \frac{12v}{5} (\theta - 2)_+^{10} + \frac{16v}{3} (\theta - 2)_+^9 \right) dx \\
& \quad + 7 \int_{\Omega} \frac{\theta^\beta (\theta - 2)_+^6 \theta_x^2}{v} dx \\
& \leq C \int_{\Omega} |u_x| \left( (\theta - 2)_+^2 + (\theta - 2)_+^{11} \right) dx + C \int_{\Omega} \frac{u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2}{v} (\theta - 2)_+^7 dx \\
& \leq C \int_{\Omega} |u_x|^2 dx + C \int_{\Omega} (\theta - 2)_+^4 dx + C \int_{\Omega} (\theta - 2)_+^{15} dx \\
& \quad + C \sup_{x \in \Omega} \left[ (u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) (\theta - 2)_+^3 \right] \int_{\Omega} (\theta - 2)_+^4 dx \\
& \leq C + C \int_{\Omega} |u_x|^2 dx + C \left( \sup_{x \in \Omega} \theta^4 + 1 \right) \int_{\Omega} (\theta - 2)_+^{11} dx \\
& \quad + C \sup_{x \in \Omega} \left[ (u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) (\theta - 2)_+^3 \right]
\end{aligned} \tag{2.52}$$

due to (2.5) and the following estimate:

$$\int_{\Omega} (\theta - 2)_+^4 dx = \int_{(\theta > 2)} (\theta - 2)^4 dx \leq C \int_{(\theta > 2)} (\theta - 1)^4 dx + C \leq C \tag{2.53}$$

owing to (2.18) and (2.27). Similarly to the proof of (2.45), we have

$$\begin{aligned}
\sup_{x \in \Omega} |\mathbf{b}_x|^2 + \sup_{x \in \Omega} |\mathbf{w}_x|^2 & \leq C \left( \int_{\Omega} \mathbf{b}_x^2 dx \right)^{1/2} \left( \int_{\Omega} \left( \frac{\mathbf{b}_x}{v} \right)^2 dx \right)^{1/2} \\
& \quad + C \left( \int_{\Omega} \mathbf{w}_x^2 dx \right)^{1/2} \left( \int_{\Omega} \left( \frac{\mathbf{w}_x}{v} \right)^2 dx \right)^{1/2}.
\end{aligned} \tag{2.54}$$

The straight calculations together with (2.5), (2.45), and (2.54) imply that

$$\begin{aligned}
& \sup_{x \in \Omega} \left[ (u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) (\theta - 2)_+^3 \right] \\
& \leq C \left( \int_{\Omega} u_x^2 dx \right)^{1/2} \left( \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx \right)^{1/2} \sup_{x \in \Omega} (\theta - 2)_+^3 \\
& \quad + C \left( \int_{\Omega} |\mathbf{b}_x|^2 dx \right)^{1/2} \left( \int_{\Omega} \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 dx \right)^{1/2} \sup_{x \in \Omega} (\theta - 2)_+^3 \\
& \quad + C \left( \int_{\Omega} |\mathbf{w}_x|^2 dx \right)^{1/2} \left( \int_{\Omega} \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 dx \right)^{1/2} \sup_{x \in \Omega} (\theta - 2)_+^3 \\
& \leq \varepsilon \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx + \varepsilon \int_{\Omega} \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 dx + \varepsilon \int_{\Omega} \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 dx \\
& \quad + C(\varepsilon) \int_{\Omega} (u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx \sup_{x \in \Omega} (\theta - 2)_+^6.
\end{aligned}$$

It follows from (2.53) that

$$\begin{aligned}
\sup_{x \in \Omega} (\theta - 2)_+^6 & \leq C \int_{\Omega} (\theta - 2)_+^5 |\theta_x| dx \\
& \leq C \left( \int_{\Omega} (\theta - 2)_+^4 dx \right)^{1/2} \left( \int_{\Omega} (\theta - 2)_+^6 \theta_x^2 dx \right)^{1/2} \\
& \leq C \left( \int_{\Omega} (\theta - 2)_+^6 \theta_x^2 dx \right)^{1/2},
\end{aligned} \tag{2.55}$$

and thus we have

$$\begin{aligned}
& \sup_{x \in \Omega} \left[ (u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) (\theta - 2)_+^3 \right] \\
& \leq \varepsilon \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx + \varepsilon \int_{\Omega} \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 dx + \varepsilon \int_{\Omega} \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 dx \\
& \quad + \eta \int_{\Omega} (\theta - 2)_+^6 \theta_x^2 dx + C \left( \int_{\Omega} (u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx \right)^2.
\end{aligned} \tag{2.56}$$

With the substitution of (2.56) into (2.52) and a sufficiently small choice of  $\eta$ , the estimates (2.5) and (2.26) lead to

$$\begin{aligned}
& \int_{\Omega} \left( (\theta - 2)_+^8 + (\theta - 2)_+^{11} \right) dx + \int_0^T \int_{\Omega} \theta^\beta (\theta - 2)_+^6 \theta_x^2 dx dt \\
& \leq C + \varepsilon \int_0^T \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx dt + C \int_0^T \left( \sup_{x \in \Omega} \theta^4 + 1 \right) \int_{\Omega} (\theta - 2)_+^{11} dx dt \\
& \quad + C \int_0^T \left( \int_{\Omega} (u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx \right)^2 dt.
\end{aligned} \tag{2.57}$$

In order to estimate the third term on the right-hand side of (2.51), using (1.4) and (1.1), we have

$$\mathbf{b}_t + \frac{\mathbf{b}u_x}{v} - \frac{\mathbf{w}_x}{v} = \left( \frac{\mathbf{b}_x}{v} \right)_x v^{-1}. \tag{2.58}$$

Adding (2.58) multiplied by  $\left(\frac{\mathbf{b}_x}{v}\right)_x$  and (1.3) multiplied by  $\left(\frac{\mathbf{w}_x}{v}\right)_x$  together, integrating the resultant equality over  $\Omega$  gives

$$\begin{aligned} & \frac{1}{2} \left( \int_{\Omega} \frac{|\mathbf{b}_x|^2}{v} dx + \int_{\Omega} \frac{|\mathbf{w}_x|^2}{v} dx \right)_t + \int_{\Omega} v^{-1} \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 dx + \int_{\Omega} \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 dx \\ & \leq C \int_{\Omega} |\mathbf{b}_x|^2 |u_x| dx + C \int_{\Omega} |\mathbf{b}|^2 u_x^2 dx + C \int_{\Omega} |\mathbf{w}_x|^2 dx \\ & \quad + C \int_{\Omega} |\mathbf{w}_x|^2 |u_x| dx + C \int_{\Omega} |\mathbf{b}_x|^2 dx + \frac{1}{4} \int_{\Omega} v^{-1} \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 dx + \frac{1}{4} \int_{\Omega} \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 dx \\ & \leq C \left( \sup_{\Omega} |u_x| + 1 \right) \int_{\Omega} (|\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx + C \sup_{\Omega} |\mathbf{b}|^2 \int_{\Omega} u_x^2 dx \\ & \quad + \frac{1}{4} \int_{\Omega} v^{-1} \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 dx + \frac{1}{4} \int_{\Omega} \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 dx. \end{aligned}$$

Then it follows from (2.45), (2.5), and (2.24) that

$$\begin{aligned} & \frac{1}{2} \left( \int_{\Omega} \frac{|\mathbf{b}_x|^2}{v} dx + \int_{\Omega} \frac{|\mathbf{w}_x|^2}{v} dx \right)_t + \int_{\Omega} v^{-1} \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 dx + \int_{\Omega} \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 dx \\ & \leq \varepsilon \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 + C + C \left( \int_{\Omega} (|\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx \right)^2 + C \left( \int_{\Omega} |\mathbf{b}_x|^2 dx + 1 \right) \int_{\Omega} u_x^2 dx \quad (2.59) \\ & \quad + \frac{1}{4} \int_{\Omega} v^{-1} \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 dx + \frac{1}{4} \int_{\Omega} \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 dx, \end{aligned}$$

and thus

$$\begin{aligned} & \int_{\Omega} (|\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx + \int_0^T \int_{\Omega} \left( \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 + \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 \right) dx dt \\ & \leq \varepsilon \int_0^T \int_{\Omega} \left( \frac{\mu u_x}{v} \right)_x^2 dx dt + C + C \int_0^T \left( \int_{\Omega} (|\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx \right)^2 dt \quad (2.60) \\ & \quad + C \int_0^T \left( \int_{\Omega} |\mathbf{b}_x|^2 dx + 1 \right) \int_{\Omega} u_x^2 dx dt. \end{aligned}$$

Now, adding (2.57) multiplied by  $C_2 = C_1 + 1$  with  $C_1$  defined in (2.51) and (2.60) to (2.51), one obtains, after choosing  $\varepsilon$  suitably small, that

$$\begin{aligned} & \int_{\Omega} \left( (\theta - 2)_+^8 + (\theta - 2)_+^{11} + v_x^2 + u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2 \right) dx \\ & \quad + \int_0^T \int_{\Omega} \left( \left( \frac{\mu u_x}{v} \right)_x^2 + v_x^2 + \theta^\beta (\theta - 2)_+^6 \theta_x^2 + \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 + \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 \right) dx dt \quad (2.61) \\ & \leq C + C \int_0^T \left( \sup_{x \in \Omega} \theta^4 + \int_{\Omega} (u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx + 1 \right) \\ & \quad \cdot \int_{\Omega} (v_x^2 + u_x^2 + (\theta - 2)_+^{11} + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dx dt. \end{aligned}$$

Combining this with Grönwall's inequality, (2.31), and (2.34), we have

$$\begin{aligned} & \int_{\Omega} \left( (\theta - 2)_+^8 + (\theta - 2)_+^{11} + v_x^2 + u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2 \right) dx \\ & + \int_0^T \int_{\Omega} \left( \left( \frac{\mu u_x}{v} \right)_x^2 + v_x^2 + \theta^\beta (\theta - 2)_+^6 \theta_x^2 + \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 + \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 \right) dx dt \leq C, \end{aligned}$$

which completes the proof.  $\square$

**Lemma 2.7.** *There exists a positive constant  $C$  such that*

$$\begin{aligned} & \int_0^T \int_{\Omega} \left( u_{xx}^2 + |\mathbf{b}_{xx}|^2 + |\mathbf{w}_{xx}|^2 + u_t^2 + |\mathbf{b}_t|^2 + |\mathbf{w}_t|^2 \right) dx dt \\ & + \int_0^T \sup_{x \in \Omega} \left( u_x^2 + u_x^4 + |\mathbf{b}_x|^2 + |\mathbf{b}_x|^4 + |\mathbf{w}_x|^2 + |\mathbf{w}_x|^4 \right) dt \leq C. \end{aligned} \quad (2.62)$$

*Proof.* First, it follows from (2.35), (2.45), and (2.54) that

$$\int_0^T \sup_{x \in \Omega} \left( u_x^2 + u_x^4 + |\mathbf{b}_x|^2 + |\mathbf{b}_x|^4 + |\mathbf{w}_x|^2 + |\mathbf{w}_x|^4 \right) dt \leq C. \quad (2.63)$$

Then by observing that

$$\begin{aligned} \frac{\mu u_{xx}}{v} + \frac{\mathbf{b}_{xx}}{v} + \frac{\mathbf{w}_{xx}}{v} &= \left( \frac{\mu u_x}{v} \right)_x + \left( \frac{\mathbf{b}_x}{v} \right)_x + \left( \frac{\mathbf{w}_x}{v} \right)_x \\ &+ \frac{\mu - v\mu'(v)}{v^2} u_x v_x + \frac{\mathbf{b}_x v_x}{v^2} + \frac{\mathbf{w}_x v_x}{v^2}, \end{aligned}$$

and using (2.5), (2.62), and (2.63), we obtain:

$$\begin{aligned} & \int_0^T \int_{\Omega} \left( u_{xx}^2 + |\mathbf{b}_{xx}|^2 + |\mathbf{w}_{xx}|^2 \right) dx dt \\ & \leq C \int_0^T \int_{\Omega} \left( \left( \frac{\mu u_x}{v} \right)_x^2 + \left| \left( \frac{\mathbf{b}_x}{v} \right)_x \right|^2 + \left| \left( \frac{\mathbf{w}_x}{v} \right)_x \right|^2 \right) dx dt \\ & + \int_0^T \sup_{x \in \Omega} \left( u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2 \right) \int_{\Omega} v_x^2 dx dt \\ & \leq C. \end{aligned} \quad (2.64)$$

Next, we rewrite the momentum equation (1.2) as

$$u_t = \left( \frac{\mu u_x}{v} \right)_x + \frac{\theta v_x}{v^2} - \frac{\theta_x}{v} - \frac{4}{3} \theta^3 \theta_x - \mathbf{b} \cdot \mathbf{b}_x. \quad (2.65)$$

The combination of (2.62), (2.50), (2.63), (2.34), and (2.24) gives

$$\begin{aligned}
 \int_0^T \int_0^1 u_t^2 dx dt &\leq C \int_0^T \int_{\Omega} \left( \left( \frac{\mu u_x}{v} \right)_x^2 + \theta^2 v_x^2 + (\theta^{-2} + \theta^6) \theta_x^2 + |\mathbf{b}|^2 |\mathbf{b}_x|^2 \right) dx dt \\
 &\leq C + C \int_0^T \sup_{x \in \Omega} \theta^2 \int_{\Omega} v_x^2 dx dt \\
 &\quad + C \int_0^T \int_{\Omega} \theta^{\beta} (\theta - 2)_+^6 \theta_x^2 dx dt + C \int_0^T \left( \int_0^1 |\mathbf{b}_x|^2 dx \right)^2 dt \\
 &\leq C.
 \end{aligned} \tag{2.66}$$

Moreover, from (2.58), (1.3), (2.5), and (2.62), it follows that

$$\begin{aligned}
 &\int_0^T \int_{\Omega} (|\mathbf{b}_t|^2 + |\mathbf{w}_t|^2) dx dt \\
 &\leq C \int_0^T \sup_{x \in \Omega} |\mathbf{b}|^2 \int_{\Omega} u_x^2 dx dt + \int_0^T \int_{\Omega} (|\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) dx dt + C \\
 &\leq C.
 \end{aligned} \tag{2.67}$$

Therefore, the collection of (2.63), (2.66), and (2.67) directly gives (2.62), and thus the proof of Lemma 2.7 is complete.  $\square$

Lastly, we are going to establish the higher-order estimates for  $\theta$  and thereby deduce the upper bound of  $\theta$ .

**Lemma 2.8.** *There exists a positive constant  $C$  such that*

$$\sup_{0 \leq t \leq T} \int_{\Omega} \theta_x^2 dx + \int_0^T \int_{\Omega} (\theta_t^2 + \theta_{xx}^2) dx dt \leq C, \tag{2.68}$$

and for all  $(x, t) \in \Omega \times [0, T]$ ,

$$\theta(x, t) \leq C. \tag{2.69}$$

*Proof.* First, multiplying (2.4) by  $\theta^{\beta} \theta_t$ , integrating the resultant equality over  $\Omega$ , and using (2.5) and

(2.26), we have

$$\begin{aligned}
& \frac{1}{2} \left( \int_{\Omega} \frac{(\theta^\beta \theta_x)^2}{v} dx \right)_t + \int_{\Omega} \theta^\beta (1 + 4v\theta^3) \theta_t^2 dx \\
&= -\frac{1}{2} \int_{\Omega} \frac{(\theta^\beta \theta_x)^2 u_x}{v^2} dx - \int_{\Omega} \left( \frac{\theta}{v} + \frac{4}{3} \theta^4 \right) \theta^\beta \theta_t u_x dx \\
&\quad + \int_{\Omega} \frac{\theta^\beta \theta_t (\mu u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2)}{v} dx \\
&\leq C \sup_{x \in \Omega} |u_x| \int_{\Omega} \frac{(\theta^\beta \theta_x)^2}{v} dx + \frac{1}{2} \int_{\Omega} \theta^\beta (1 + 4v\theta^3) \theta_t^2 dx \\
&\quad + C \int_{\Omega} (\theta^2 + \theta^5) \theta^\beta u_x^2 dx + C \int_{\Omega} \theta^\beta (u_x^4 + |\mathbf{b}_x|^4 + |\mathbf{w}_x|^4) dx \\
&\leq C \left( \sup_{x \in \Omega} |u_x|^2 + 1 \right) \int_{\Omega} \frac{(\theta^\beta \theta_x)^2}{v} dx + \frac{1}{2} \int_{\Omega} \theta^\beta (1 + 4v\theta^3) \theta_t^2 dx \\
&\quad + C \sup_{x \in \Omega} (1 + \theta^{2+2\beta} + \theta^{5+\beta} + u_x^4 + |\mathbf{b}_x|^4 + |\mathbf{w}_x|^4) \int_{\Omega} (u_x^2 + b_x^2 + w_x^2) dx,
\end{aligned} \tag{2.70}$$

where we have used the following fact:

$$\begin{aligned}
\int_{\Omega} \theta^\beta \theta_t \left( \frac{\theta^\beta \theta_x}{v} \right)_x dx &= - \int_{\Omega} \frac{\theta^\beta \theta_x}{v} (\theta^\beta \theta_t)_x dx \\
&= - \int_{\Omega} \frac{\theta^\beta \theta_x}{v} (\theta^\beta \theta_x)_t dx \\
&= -\frac{1}{2} \left( \int_{\Omega} \frac{(\theta^\beta \theta_x)^2}{v} dx \right)_t - \frac{1}{2} \int_{\Omega} \frac{(\theta^\beta \theta_x)^2 u_x}{v^2} dx,
\end{aligned}$$

due to integration by parts. It follows from (2.27), (2.26), and (2.53) that:

$$\begin{aligned}
& \sup_{x \in \Omega} (\theta^{2+2\beta} + \theta^{5+\beta}) \\
&\leq C \sup_{x \in \Omega} ((\theta - 2)_+^{2+2\beta} + (\theta - 2)_+^{5+\beta}) + C \\
&\leq C \int_{\Omega} (\theta - 2)_+^{1+2\beta} |\theta_x| dx + C \int_{\Omega} (\theta - 2)_+^{4+\beta} |\theta_x| dx + C \\
&\leq C \sup_{x \in \Omega} \theta^{1+\beta} \int_{(\theta>2)} \theta^\beta |\theta_x| dx + C \sup_{x \in \Omega} \theta^{\frac{5+\beta}{2}} \int_{(\theta>2)} (\theta - 2)_+^{3/2} \theta^\beta |\theta_x| dx + C \\
&\leq \frac{1}{2} \sup_{x \in \Omega} (\theta^{2+2\beta} + \theta^{5+\beta}) + C \left( 1 + \int_{(\theta>2)} (\theta - 2)_+^4 dx \right) \int_{\Omega} \frac{(\theta^\beta \theta_x)^2}{v} dx + C \\
&\leq \frac{1}{2} \sup_{x \in \Omega} (\theta^{2+2\beta} + \theta^{5+\beta}) + C \int_{\Omega} \frac{(\theta^\beta \theta_x)^2}{v} dx + C,
\end{aligned} \tag{2.71}$$

and together with (2.35) and (2.70) leads to

$$\begin{aligned} & \left( \int_{\Omega} \frac{(\theta^{\beta} \theta_x)^2}{v} dx \right)_t + \int_{\Omega} \theta^{\beta} (1 + 4v\theta^3) \theta_t^2 dx \\ & \leq C \left( \sup_{x \in \Omega} |u_x|^2 + 1 \right) \int_{\Omega} \frac{(\theta^{\beta} \theta_x)^2}{v} dx + C \sup_{x \in \Omega} (1 + u_x^4 + |\mathbf{b}_x|^4 + |\mathbf{w}_x|^4). \end{aligned} \quad (2.72)$$

Thus, by virtue of Grönwall's inequality, (2.72), and (2.62), we obtain

$$\sup_{0 \leq t \leq T} \int_{\Omega} (\theta^{\beta} \theta_x)^2 dx + \int_0^T \int_{\Omega} \theta^{\beta} \theta_t^2 dx dt \leq C, \quad (2.73)$$

which combined with (2.71) implies that (2.69). In addition, as a direct result of (2.26) and (2.73), we have

$$\sup_{0 \leq t \leq T} \int_{\Omega} \theta_x^2 dx + \int_0^T \int_{\Omega} \theta_t^2 dx dt \leq C. \quad (2.74)$$

Lastly, from (2.4), it follows that

$$\begin{aligned} \frac{\theta^{\beta} \theta_{xx}}{v} &= -\frac{\beta \theta^{\beta-1} \theta_x^2}{v} + \frac{\theta^{\beta} \theta_x v_x}{v^2} - \frac{\mu u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2}{v} \\ &\quad + \frac{(3\theta + 4v\theta^4)u_x}{3v} + (1 + 4v\theta^3)\theta_t, \end{aligned}$$

which together with (2.5), (2.26), (2.69), (2.74), and (2.35) gives

$$\begin{aligned} \int_0^T \int_{\Omega} \theta_{xx}^2 dx dt &\leq C \int_0^T \int_{\Omega} (\theta_x^4 + \theta_x^2 v_x^2 + u_x^4 + u_x^2 + |\mathbf{b}_x|^4 + |\mathbf{w}_x|^4 + \theta_t^2) dx dt \\ &\leq C + C \int_0^T \sup_{x \in \Omega} (\theta_x^2 + u_x^2 + |\mathbf{b}_x|^2 + |\mathbf{w}_x|^2) dt \\ &\leq C + C \int_0^T \int_{\Omega} \theta_x^2 dx dt + \frac{1}{2} \int_0^T \int_{\Omega} \theta_{xx}^2 dx dt. \end{aligned}$$

Combining this with (2.74) yields (2.68) and completes the proof of Lemma 2.8.  $\square$

### Author contributions

The authors have made the same contributions. All authors read and approved the final manuscript.

### Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflict of interest.

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