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Research article

Stochastic pseudo-parabolic equation with logarithmic nonlinearity

Yang Cao[1], **Chengyuan Qu**[2,*] **and Benhui Wang**[1]

[1] School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China

[2] School of Mathematical Sciences, Dalian Minzu University, Dalian 116600, P. R. China

* **Correspondence:** Email: qcyuan@dlnu.edu.cn.

Abstract: This manuscript was dedicated to exploring the existence and uniqueness of global solutions pertaining to a specific class of stochastic pseudo-parabolic equations. These equations have logarithmic nonlinearity and are driven by Brownian motion. By utilizing the Galerkin method, Prokhorov's theorem, and Skorohod's embedding theorem, we proved the existence of a global solution in the weak probabilistic sense. Subsequently, by leveraging the uniqueness of solutions and applying the Yamada-Watanabe theorem, the global existence and uniqueness of a probability strong solution were established. A notable finding, in contrast to the stochastic heat equation, was that as the pseudo-parabolic coefficient μ increased, the growth condition imposed on the noise coefficient within the stochastic pseudo-parabolic equation could be significantly relaxed.

Keywords: stochastic pseudo-parabolic equation; logarithmic nonlinearity; global solution; Galerkin method; non-local
Mathematics Subject Classification: 35K70, 35R60

1. Introduction

In recent times, there has been a surge of interest and concern regarding both deterministic and stochastic differential equation models. This heightened attention is a direct response to the advancements observed in fields such as physics, control engineering, economics, and the social sciences. In particular, the study of pseudo-parabolic equations has found widespread applications across a diverse array of physical phenomena. These include fluid flow through rock cracks, soil moisture migration, heat conduction in various media, and shear in second-order fluids, among others. In contemporary mathematical terms, the pseudo-parabolic equation denotes partial differential equations that incorporate a combination of spatial derivatives and the highest-order time derivatives [1]. Here, we consider the following stochastic pseudo-parabolic equation

$$\begin{cases} du(x,t) - \mu d\Delta u(x,t) = \Delta u(x,t)dt + u(x,t)\log|u(x,t)|\,dt + f(u(x,t))dB_t, \ x \in D, t > 0, \\ u(x,t) = 0, \ x \in \partial D, \ t > 0, \\ u(x,0) = u_0(x), \ x \in D, \end{cases} \tag{1.1}$$

where the domain $D \subset \mathbb{R}^n$ is bounded and has smooth boundary, $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous and deterministic, and B_t is a 1-dimensional standard Brownian motion, adapted to the filtration $\mathcal{F}_t$ on the complete probability space $(\Omega, \mathcal{F}, P)$, and the initial data $u_0(x) \in H_0^1(D)$.

The term "logarithmic nonlinearity", represented as $u \log|u|$, emerges in certain partial differential equations arising in physics; see [2] and the references there in. It also appears in the theory of continuous-state branching processes, where a parabolic equation featuring logarithmic nonlinearity is related to the Neveu branching mechanism [3]. As explained above, the logarithmic nonlinearity $u \log|u|$ is interesting in its own right. From a mathematical perspective, it is also intriguing to explore the effects of the logarithmic nonlinearity $u \log|u|$ on the behavior of solutions. In particular, comparing the effects of the logarithmic nonlinearity $u \log|u|$ and the power-type nonlinearity u^p reveals significant differences in the qualitative properties of solutions. Chen and Tian [4] were the first to investigate a deterministic pseudo-parabolic equation incorporating logarithmic nonlinearity, given by

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} - \Delta u = u \log|u|. \tag{1.2}$$

Their research revealed phenomena such as infinite-time blow-up and decay of solutions. When comparing these findings with those of Xu and Su [5], who studied the same equation (1.2) but with a power-like source term u^p, it becomes evident that the critical condition for finite-time blow-up is $p = 1 + \varepsilon$, where $\varepsilon > 0$. Furthermore, in the case of (1.2) with a linear source term $m(x,t)u$, the asymptotic behavior of solutions–whether they decay to zero or blow up at infinite time–depends on the characteristics of $m(x,t)$. This suggests that, in terms of blow-up profiles, the logarithmic nonlinearity behaves more similarly to a linear source than to a power-like one. In addition to infinite-time blow-up and decay to zero, Ji, Yin, and Cao [6] demonstrated that (1.2) may also admit globally periodic solutions, and further pointed out that for the existence of periodic solutions, logarithmic nonlinearity behaves similar to power-like source. For those who are interested in delving deeper into the study of deterministic pseudo-parabolic equations with nonlinear source, we recommend consulting the works of [4, 6–11].

In recent years, stochastic pseudo-parabolic equations have been studied. In [12], Liaskos, Stratis, and Yannacopoulos presented existence and uniqueness of mild and strong solutions for a general class of linear pseudo-parabolic equations with additive noise. In addition, they also considered a related perturbed Cauchy problem and investigated the continuity of the solution with respect to a small parameter. Thach and Tuan [13] considered the fractional stochastic pseudo-parabolic equation driven by fractional Brownian motion, and established the existence, uniqueness, and regularity of mild solutions. Subsequently, Tuan, Caraballo, and Thach [14] studied the fractional stochastic pseudo-parabolic equation driven by fractional Brownian motion with bounded and unbounded delays. Under the condition that the nonlinear source term satisfies local Lipschitz conditions, they established the global existence, uniqueness, and regularity results for mild solutions. Moreover, they proved that, as the coefficient of the pseudo-parabolic term tends to zero, the mild solution of the stochastic fractional pseudo-parabolic equation converges to that of the stochastic fractional parabolic equation in a certain sense.

As far as we know, there has been no relevant research on stochastic pseudo-parabolic equations with logarithmic nonlinearity. Even for the logarithmic stochastic heat equation, the literature remains remarkably scarce. To our knowledge, the only work in this direction is by Shang and Zhang [15]; they have established the global existence and uniqueness of strong solutions in the probabilistic sense on a bounded domain. Motivated by [15], the primary objective of this paper is to investigate the global existence and uniqueness of solutions to the stochastic pseudo-parabolic equation (1.1). We aim to explore the impact of the interplay between the mixed partial derivative term $\mu\Delta u_t$, the logarithmic nonlinearity, and the noise term on the existence of solutions. Our findings reveal that, for stochastic pseudo-parabolic equations with a logarithmic term, the spatial regularity of solutions is consistent with that of the initial data. This aligns with the behavior of their deterministic counterparts. However, in contrast to stochastic heat equations in [15], the temporal regularity of solutions to stochastic pseudo-parabolic equations exhibits an improvement, elevating from L^2 to L^∞; see Remark 1. Additionally, when compared with stochastic heat equations in [15], as the coefficient μ of the pseudo-parabolic term increases, stochastic pseudo-parabolic equations allow for more relaxed growth conditions on the noise term. This suggests that the third-order term, to some extent, plays a facilitating role in ensuring the global existence of solutions; see Remark 2.

Let $H := L^2(D)$ equipped with the norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$, and let $V := H_0^1(D)$. Define $V^* := H^{-1}(D)$, which is evidently the dual space of V. We use $\langle f, v\rangle$ to represent the dual pairing between $f \in V^*$ and $v \in V$. Next, we define the norm in V as follows

$$\|u\|_V = \left\{\sum_{i=1}^{\infty} \lambda_i (u(x), e_i)^2\right\}^{\frac{1}{2}},$$

where $\{e_i\}_1^\infty$ forms an orthogonal basis in V and satisfies

$$\Delta e_i = -\lambda_i e_i, \quad e_i|_{\partial D} = 0, \quad i \in \mathbb{N},$$

with $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_i < \ldots$, and $\lambda_i \to \infty$ as $i \to \infty$. Set

$$u(t)(x) := u(x,t), \quad (u(t)\log|u(t)|)(x) := u(x,t)\log|u(x,t)|, \quad f(u(t))(x) := f(u(x,t)).$$

We can rewrite (1.1) as the equivalent form

$$\begin{cases} u(t) - \mu\Delta u - (u_0 - \mu\Delta u_0) = \int_0^t (\Delta u(s) + u(s)\log|u(s)|)ds + \int_0^t f(u(s))dB_s, \\ u(0) = u_0 \in H_0^1(D). \end{cases} \tag{1.3}$$

With the above preparations, we are ready to state the definition of a solution.

Definition 1.1. *An $H-$valued continuous and $\mathcal{F}_t-$adapted stochastic process u is called a weak solution in the probabilistic sense to* (1.3)*, if there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and a Brownian motion B_t on this basis such that the process $u : [0,T] \times \Omega \to H$ is progressively measurable and satisfies the following conditions*

(i) $\forall T > 0$, $u \in L^\infty([0,T]; V)$, P-a.s.;

(ii) $\forall t \geq 0$, *u fulfills* (1.3) *within the dual space* V^*, *P-a.s.*

We now present our assumptions concerning the diffusion coefficient f. (H1) is for the uniqueness, and (H2) and (H3) are for the different existence results.

(H1) For any $u, v \in \mathbb{R}$,

$$|f(u) - f(v)| \leq L_1 |u - v| + L_2 |u - v| (\log_+(|u| \vee |v|))^{\frac{1}{2}}, \tag{1.4}$$

where $L_1 > 0$, $L_2 > 0$, $\log_+ u := \log(1 \vee u)$.

(H2) For any $u \in \mathbb{R}$,

$$|f(u)| \leq C_1 + C_2(1 + \sqrt{\mu}) |u|^\theta, \tag{1.5}$$

where $C_1 > 0$, $C_2 > 0$, $\theta \in (0, 1)$.

(H3) For any $u \in \mathbb{R}$,

$$|f(u)| \leq C_3 + C_4(1 + \sqrt{\mu}) |u| (\log_+ |u|)^{\frac{1}{2}}, \tag{1.6}$$

where $C_3 > 0$, $C_4 > 0$.

To control $\int_0^t u(s) \log |u(s)|\, ds$ in (1.3), one needs to use the inequality of logarithmic Sobolev form

$$\int_D |\xi(x)|^2 \log |\xi(x)|\, dx \leq \epsilon \|\xi\|_V^2 + \left(\frac{d}{4} \log \frac{1}{\epsilon}\right) \|\xi\|^2 + \|\xi\|^2 \log \|\xi\|, \tag{1.7}$$

which holds for all $\epsilon > 0$, $\xi \in V$.

The remaining part of this paper is structured as follows. In Section 2, we demonstrate the uniqueness of solutions to (1.1). Section 3 focuses on establishing the global existence under the assumption that the coefficient $f(u)$ satisfies a sublinear condition. To this end, we employ the Galerkin method, Prokhorov's theorem, and Skorohod's embedding theorem to derive a moment estimate for the solution. Finally, in Section 4, we explore the global existence of solutions when $f(u)$ is superlinear.

2. Uniqueness

Under (H1), we establish the pathwise uniqueness property for the solution of (1.3).

Theorem 2.1. *Suppose (H1) holds. Then, the solution of (1.3) is pathwise unique within* $L^2(D)$.

Proof. We suppose that u and v are different global solutions of (1.3), and introduce the stopping times

$$\begin{aligned} \tau_M &: = \inf\left\{t > 0 : \|u(t)\|_V^2 \vee \|v(t)\|_V^2 > M\right\}, \\ \tau^\delta &: = \inf\left\{t > 0 : \|u(t) - v(t)\| > \delta\right\}, \\ \tau_M^\delta &: = \tau_M \wedge \tau^\delta \end{aligned}$$

with any given $M > 0$ and $\delta \in (0, 1]$. It is easy to see that as $M \to \infty$, $\tau_M \to \infty$, $P - a.s.$ Set $w(t) := u(t) - v(t)$. Applying Itô's formula and using (1.4), we have

$$\left\|w(t \wedge \tau_M^\delta)\right\|^2 + \mu \left\|w(t \wedge \tau_M^\delta)\right\|_V^2 + 2 \int_0^{t \wedge \tau_M^\delta} \|w(s)\|_V^2\, ds$$

$$
\begin{aligned}
=&2\int_0^{t\wedge\tau_M^\delta}(u\log|u|-v\log|v|\,,u-v)ds+2\int_0^{t\wedge\tau_M^\delta}(f(u)-f(v),u-v)dB_s\\
&+\int_0^{t\wedge\tau_M^\delta}(f(u)-f(v),(I-\mu\Delta)^{-1}(f(u)-f(v)))ds\\
\le&2\int_0^{t\wedge\tau_M^\delta}\left[(u\log|u|-v\log|v|\,,u-v)+L_1^2\|w(s)\|^2+L_2^2\int_D|u-v|^2\log_+(|u|\vee|v|)dx\right]ds\\
&+2\int_0^{t\wedge\tau_M^\delta}(f(u)-f(v),u-v)dB_s. \qquad (2.1)
\end{aligned}
$$

Based on Lemma 3.1 (with $\epsilon=\frac{1}{4}$) and Lemma 3.2 (with $\epsilon=\frac{1}{4L_2^2}$) from [15], we conclude that

$$
\begin{aligned}
&\left\|w(t\wedge\tau_M^\delta)\right\|^2+\mu\left\|w(t\wedge\tau_M^\delta)\right\|_V^2+\int_0^{t\wedge\tau_M^\delta}\|w(s)\|_V^2\,ds\\
\le&C\int_0^{t\wedge\tau_M^\delta}\|w(s)\|^2\,ds+(2+2L_2^2)\int_0^{t\wedge\tau_M^\delta}\|w(s)\|^2\log\|w(s)\|\,ds\\
&+\frac{(1+L_2^2)e^{-1}}{1-\alpha}\int_0^{t\wedge\tau_M^\delta}\|u(s)\|^{2(1-\alpha)}\|w(s)\|^{2\alpha}\,ds+\frac{(1+L_2^2)e^{-1}}{1-\alpha}\int_0^{t\wedge\tau_M^\delta}\|v(s)\|^{2(1-\alpha)}\|w(s)\|^{2\alpha}\,ds\\
&+\frac{L_2^2e^{-1}}{1-\alpha}\int_0^{t\wedge\tau_M^\delta}4^{1-\alpha}\mathrm{mes}^{1-\alpha}(D)\,\|w(s)\|^{2\alpha}\,ds+2\int_0^{t\wedge\tau_M^\delta}(f(u)-f(v),u-v)dB_s, \qquad (2.2)
\end{aligned}
$$

which means

$$
\int_0^{t\wedge\tau_M^\delta}\|w(s)\|^2\log\|w(s)\|\,ds\le 0,\qquad \sup_{t\in[0,t\wedge\tau_M^\delta]}\left\{\|u(t)\|^2\vee\|v(t)\|^2\right\}\le M. \qquad (2.3)
$$

Substituting (2.3) into (2.2) and computing the expectation for (2.2), we get

$$
\begin{aligned}
&E\left\|w(t\wedge\tau_M^\delta)\right\|^2+\mu E\left\|w(t\wedge\tau_M^\delta)\right\|_V^2+E\int_0^{t\wedge\tau_M^\delta}\|w(s)\|_V^2\,ds\\
\le&CE\int_0^{t\wedge\tau_M^\delta}\|w(s)\|^2\,ds+\frac{2(1+L_2^2)e^{-1}}{1-\alpha}M^{1-\alpha}E\int_0^{t\wedge\tau_M^\delta}\|w(s)\|^{2\alpha}\,ds\\
&+\frac{L_2^2e^{-1}4^{1-\alpha}\mathrm{mes}^{1-\alpha}(D)}{1-\alpha}E\int_0^{t\wedge\tau_M^\delta}\|w(s)\|^{2\alpha}\,ds,
\end{aligned}
$$

where $E\int_0^{t\wedge\tau_M^\delta}(f(u)-f(v),u-v)dB_s=0$. Letting $Y(t):=E\left\|w(t\wedge\tau_M^\delta)\right\|^2$, then

$$
Y(t)\le C\int_0^t Y(s)ds+\frac{2(1+L_2^2)M^{1-\alpha}e^{-1}+L_2^24^{1-\alpha}\mathrm{mes}^{1-\alpha}(D)e^{-1}}{1-\alpha}\int_0^t Y(s)^\alpha ds.
$$

By applying the nonlinear Grönwall's inequality (see XII.9 Theorem 1 in [16]), we derive the following estimate

$$
Y(t)\le\left\{\int_0^t(2(1+L_2^2)M^{1-\alpha}e^{-1}+L_2^24^{1-\alpha}\mathrm{mes}^{1-\alpha}(D)e^{-1})\times e^{(1-\alpha)\times C(t-s)}ds\right\}^{\frac{1}{1-\alpha}}
$$

$$\leq \frac{1}{2}\left\{\left[4(1+L_2^2)t^\alpha e^{-1}\right]^{\frac{1}{1-\alpha}} \times M + \left[2L_2^2 t^\alpha e^{-1}\right]^{\frac{1}{1-\alpha}} \times 4\mathrm{mes}(D)\right\} \times \left(\int_0^t e^{Cs} ds\right).$$

Now, define $\overline{T} := \left(\frac{e}{4(1+L_2^2)}\right)^2$. By taking the limit as $\alpha \to 1$, we obtain

$$Y(t) = 0,$$

for any $t \in [0, \overline{T}]$. Noting that $\overline{T}$ does not depend on u_0, therefore, employing similar reasoning from $\overline{T}$ to $2\overline{T}$, one can deduce that $Y(t)$ equals 0 for all t. Carrying out this procedure iteratively, we can conclude that for any $t \geq 0$,

$$E\left\|w(t \wedge \tau_M \wedge \tau^\delta)\right\|^2 = 0. \tag{2.4}$$

Taking the limit as $M \to \infty$ in (2.4), for any $t \geq 0$,

$$E\left\|w(t \wedge \tau^\delta)\right\|^2 = 0.$$

This indicates that $P(\tau^\delta > t) = 1$ for all $t \geq 0$ and $\delta > 0$. Thus, for all $t \geq 0$, $u(t)$ equals to $v(t)$, $P - a.s.$. Combining the path continuity of u, v in H, we conclude that the pathwise uniqueness holds. □

3. Existence of global solutions: Part I

In this section, we assume that condition (1.5) is fulfilled. Our strategy is to apply the Galerkin method to verify the existence of solutions. Initially, we demonstrate that the Galerkin approximation equation admits a global solution. Subsequently, to confirm the existence of a solution to equation (1.3), it is essential to demonstrate the compactness property of the Galerkin approximate solutions. By employing the method of taking limits, we are capable of establishing the global existence of the weak solutions. Finally, when we combine the uniqueness with the Yamada - Watanabe theorem, we can prove that the strong solution exists uniquely. We will divide the proof into 3 steps.

Remark on local existence. Although the local-in-time existence of weak solutions is inherently embedded in our global existence proof via the Galerkin approximation procedure, we would like to emphasize that local existence plays a fundamental role in the theory of nonlinear evolution equations. For this reason, and to provide a more comprehensive theoretical framework, we will also briefly present the local existence of mild solutions to the stochastic pseudo-parabolic equation (1.1). We adopt the semigroup approach of [17, 18], specifically Theorem 3.3 in [18], which establishes the local existence and uniqueness of solutions for a class of nonlocal stochastic parabolic equations.

Let us define the operator $\mathcal{L} = I - \mu\Delta$ with domain $D(\mathcal{L}) = H_0^1(D) \cap H^2(D)$. The pseudo-parabolic structure allows us to rewrite (1.1) in the following abstract form

$$du = \mathcal{L}^{-1}\Delta u dt + K(u)dt + F(u)dB_t,$$

where $K(u) = \mathcal{L}^{-1} u \log|u|$, $F(u) = \mathcal{L}^{-1} f(u)$. From [17, 19], any weak solution in Definition 1.1 is also a mild solution, which satisfies the following equation in H,

$$u(t) = G(t)u_0 + \int_0^t G(t-s)K(u(s))ds + \int_0^t G(t-s)F(u(s))dB_s,$$

where $G(t) = e^{t\mathcal{L}^{-1}\Delta}$ is the C_0-semigroup generated by $\mathcal{L}^{-1}\Delta$; see [1]. From the fact that $\mathcal{L}^{-1}$ is a bounded operator on H and the assumption (H1)-(H3), we can also have the locally Lipschitz condition stated in [18], namely, for any $u_0 \in V$, there exist $\delta > 0$, $C_L > 0$, $C_f > 0$, and $C_F > 0$ such that for any $u, v \in B_{u_0,\delta} = \{w \in V : \|w - u_0\|_\infty < \delta\}$, there holds

$$\|u \log |u| - v \log |v|\|_H \leq C_L \|u - v\|_H, \tag{3.1}$$

$$\|f(u) - f(v)\|_H \leq C_f \|u - v\|_H, \tag{3.2}$$

$$\|F(u) - F(v)\|_H \leq C_F \|u - v\|_H, \tag{3.3}$$

Denote

$$S_T = \{u; u \in L^2(\Omega \times [0, T]; L^\infty(D) \cap V)\}$$

with the norm

$$\|u\|_{\gamma,\delta} = E\left[\int_0^T e^{-\gamma t}(\|u(t)\|_H^2 + \delta\|\nabla u(t)\|_H^2)dt\right],$$

where $\gamma > 0$ and $\delta > 0$ will be determined later. Then, the mapping

$$\Phi(u)(t) = G(t)u_0 + \int_0^t G(t-s)K(u(s))ds + \int_0^t G(t-s)F(u(s))dB_s,$$

is from S_T to S_T. Further, Φ is a contraction operator, namely, we can find positive γ, δ, and $0 < \kappa < 1$ such that

$$\|\Phi(u)(t) - \Phi(v)(t)\|_{\gamma,\delta} \leq \kappa\|u - v\|_{\gamma,\delta}.$$

Actually, if we set

$$\upsilon(t) = \Phi(u)(t) - \Phi(v)(t) = \int_0^t G(t-s)\left(K(u(s)) - K(v(s))\right)ds + \int_0^t G(t-s)\left(F(u(s)) - F(v(s))\right)dB_s,$$

then it satisfies

$$d\upsilon = \mathcal{L}^{-1}\Delta\upsilon dt + \left(K(u(t)) - K(v(t))\right)dt + \left(F(u(t)) - F(v(t))\right)dB_t.$$

Let $w(t) = \upsilon(t)e^{-\frac{\gamma}{2}t}$, which satisfies

$$dw - \mu d\Delta w - (1 + \frac{\gamma}{2}\mu)\Delta w dt = -\frac{\gamma}{2}w + \left(u \log |u| - v \log |v|\right)e^{-\frac{\gamma}{2}t}dt + \left(f(u) - f(v)\right)dB_t.$$

Implementing Ito's formula, we can get

$$\begin{aligned} &e^{-\gamma t}\|\upsilon(T)\|_H^2 + \mu e^{-\gamma t}\|\nabla\upsilon(T)\|_H^2 + \int_0^T \gamma e^{-\gamma t}\|\upsilon\|_H^2 dt + \int_0^T (2 + \gamma\mu)e^{-\gamma t}\|\nabla\upsilon\|_H^2 dt \\ &\leq (\varepsilon_1 C_L + \varepsilon_2 C_f + C_F)\int_0^T e^{-\gamma t}\|u(t) - v(t)\|_H^2 dt + (C_{\varepsilon_1} + C_{\varepsilon_1})\int_0^T e^{-\gamma t}\|\upsilon(t)\|_H^2 dt, \end{aligned} \tag{3.4}$$

where we use (3.1), (3.2), (3.3), and the following Young's equalities

$$2\int_0^T e^{-\gamma t}(\upsilon(t), u \log |u| - v \log |v|)dt \leq \varepsilon_1 C_L \int_0^T e^{-\gamma t}\|u(t) - v(t)\|_H^2 dt + C_{\varepsilon_1}\int_0^T e^{-\gamma t}\|\upsilon(t)\|_H^2 dt$$

and

$$2\int_0^T e^{-\gamma t}(\upsilon(t), f(u)-f(v))dt \leq \varepsilon_2 C_f \int_0^T e^{-\gamma t}\|u(t)-v(t)\|_H^2 dt + C_{\varepsilon_2}\int_0^T e^{-\gamma t}\|\upsilon(t)\|_H^2 dt.$$

Taking the expectation on both sides of (3.4) and choosing $\gamma > C_{\varepsilon_1} + C_{\varepsilon_2}$, we can derive

$$\begin{aligned} &E\left[e^{-\gamma t}\|\upsilon(T)\|_H^2\right] + E\left[\mu e^{-\gamma t}\|\nabla\upsilon(T)\|_H^2\right] + (\gamma - C_{\varepsilon_1} - C_{\varepsilon_2})E\left[\int_0^T e^{-\gamma t}\|\upsilon\|_H^2 dt\right] \\ &\qquad + E\left[\int_0^T (2+\gamma\mu)e^{-\gamma t}\|\nabla\upsilon\|_H^2 dt\right] \\ \leq &(\varepsilon_1 C_L + \varepsilon_2 C_f + C_F)E\left[\int_0^T e^{-\gamma t}\|u(t)-v(t)\|_H^2 dt\right], \end{aligned}$$

and, further,

$$\begin{aligned} &E\left[\int_0^T e^{-\gamma t}\|\upsilon\|_H^2 dt\right] + \frac{2+\gamma\mu}{\gamma - C_{\varepsilon_1} - C_{\varepsilon_2}}E\left[\int_0^T e^{-\gamma t}\|\nabla\upsilon\|_H^2 dt\right] \\ \leq &\frac{\varepsilon_1 C_L + \varepsilon_2 C_f + C_F}{\gamma - C_{\varepsilon_1} - C_{\varepsilon_2}}E\left[\int_0^T e^{-\gamma t}\|u(t)-v(t)\|_H^2 dt\right]. \end{aligned}$$

If we choose sufficiently large γ and suitable ε_i $(i = 1, 2)$ such that $0 < \frac{\varepsilon_1 C_L+\varepsilon_2 C_f + C_F}{\gamma - C_{\varepsilon_1} - C_{\varepsilon_2}} = \kappa < 1$, we have

$$E\left[\int_0^T e^{-\gamma t}\left(\|\upsilon\|_H^2 dt + \delta\|\nabla\upsilon\|_H^2\right)dt\right] \leq \kappa E\left[\int_0^T e^{-\gamma t}\left(\|u(t)-v(t)\|_H^2 + \delta\|\nabla u(t) - \nabla v(t)\|_H^2\right)dt\right],$$

for $\delta = \frac{2+\gamma\mu}{\gamma - C_{\varepsilon_1} - C_{\varepsilon_2}}$. The above inequality means

$$\|\Phi(u) - \Phi(v)\|_{\gamma,\delta} \leq \kappa\|u - v\|_{\gamma,\delta}, \quad 0 < \kappa < 1.$$

Then, by Banach's fixed point theorem, there exists a unique local-in-time solution $u \in S_T$. This completes the remark.

Step 1 Finite-dimensional approximation.

Let H_k represent the k-dimensional subspace of H that is generated by the set $\{e_1, e_2, \cdots, e_k\}$, and let the projection $P_k : V^* \to H_k$ be defined by

$$P_k g := \sum_{i=1}^{k} \langle g, e_i\rangle e_i.$$

For each finite $k \in \mathbb{N}$, we now examine the following stochastic differential equation within the subspace H_k

$$\begin{cases} du_k(t) - \mu d\Delta u_k(t) = \Delta u_k(t)dt + P_k[u_k(t)\log|u_k(t)|]dt + P_k f(u_k(t))dB_t, \quad t \geq 0, \\ u_k(0) = P_k u_0, \end{cases} \tag{3.5}$$

where $u_k(t) = \sum_{i=1}^{k} g_{ik}(t)e_i$. It is straightforward to observe that u_k is a solution to (3.5) precisely when the sequence $\left\{g_{jk}\right\}_{j=1}^{k}$ satisfies the corresponding system

$$\begin{aligned} dg_{jk}(t) = &-\frac{\lambda_j}{1+\mu\lambda_j} g_{jk}(t)dt + \frac{1}{1+\mu\lambda_j}\left(\sum_{i=1}^{k} g_{ik}(t)e_i \log\left|\sum_{i=1}^{k} g_{ik}(t)e_i\right|, e_j\right)dt \\ &+ \frac{1}{1+\mu\lambda_j}\left(f\left(\sum_{i=1}^{k} g_{ik}(t)e_i\right), e_j\right)dB_t. \end{aligned} \tag{3.6}$$

Set $x = (x_1, \ldots, x_k) = (g_{1k}, \ldots, g_{kk})$. Then, we define

$$F_j := \frac{1}{1+\mu\lambda_j}\left(\sum_{i=1}^{k} x_i e_i \log\left|\sum_{i=1}^{k} x_i e_i\right|, e_j\right), \qquad G_j := \frac{1}{1+\mu\lambda_j}\left(f\left(\sum_{i=1}^{k} x_i(t)e_i\right), e_j\right).$$

Let $z_1(x) = \sum_{i=1}^{k} x_i e_i(x)$, $z_2(x) = \sum_{i=1}^{k} y_i e_i(x)$. Then,

$$\begin{aligned} &\left|F_j(x_1, \ldots, x_k) - F_j(y_1, \ldots, y_k)\right| \\ =&\frac{1}{1+\mu\lambda_j}\left|\int_{\{|z_1|>|z_2|\}} e_j(z_1 - z_2)\log|z_1|\,dx + \int_{\{|z_1|>|z_2|\}} e_j z_2(\log|z_1| - \log|z_2|)dx \right.\\ &\left. + \int_{\{|z_1|<|z_2|\}} e_j(z_1 - z_2)\log|z_2|\,dx + \int_{\{|z_1|<|z_2|\}} e_j z_1(\log|z_1| - \log|z_2|)dx\right| \\ \leq&\left|\int_{\{|z_1|>|z_2|\}} e_j(z_1 - z_2)\log|z_1|\,dx + \int_{\{|z_1|>|z_2|\}} e_j z_2(\log|z_1| - \log|z_2|)dx \right.\\ &\left. + \int_{\{|z_1|<|z_2|\}} e_j(z_1 - z_2)\log|z_2|\,dx + \int_{\{|z_1|<|z_2|\}} e_j z_1(\log|z_1| - \log|z_2|)dx\right|. \end{aligned}$$

Thus, by adopting a proof strategy analogous to the one employed in [15], we can arrive at the following conclusions.

Lemma 3.1. *(i) For any $x, y \in \mathbb{R}^k$, there exist constants $C, \delta > 0$, such that whenever $|x - y| < \delta$, the following inequality holds*

$$\begin{aligned} &\left|F_j(x_1, \ldots, x_k) - F_j(y_1, \ldots, y_k)\right| \\ \leq& C\left(|x - y| + |x - y|\log_+(|x| \vee |y|) + |x - y|\log\frac{1}{|x - y|}\right). \end{aligned} \tag{3.7}$$

(ii) For any $x \in \mathbb{R}^k$, there exists a constant $C > 0$, such that

$$\left|F_j(x_1, \ldots, x_k)\right| \leq C(1 + |x|\log_+|x|).$$

Lemma 3.2. *(i) Assuming (1.4) holds. Then, there exists a constant C such that for any $x, y \in \mathbb{R}^k$,*

$$\left|G_j(x_1, \ldots, x_k) - G_j(y_1, \ldots, y_k)\right| \leq C\left(|x - y| + |x - y|\,(\log_+(|x| \vee |y|))^{\frac{1}{2}}\right).$$

(ii) Assuming (1.6) holds. Then, there exists a constant C such that for any $x \in \mathbb{R}^k$,

$$\left|G_j(x_1, \ldots, x_k)\right| \leq C(1 + |x|\,(\log_+|x|)^{\frac{1}{2}}).$$

Relying on Lemma 3.1 and Lemma 3.2, we are in a position to directly utilize Theorem A, Theorem B, and Theorem D presented in [20] to obtain the following conclusion.

Theorem 3.1. *If Assumption (1.4) and (1.6) are satisfied, then the stochastic differential equation (3.5) possesses a unique global probabilistic strong solution.*

Step 2 Priori estimates.

Lemma 3.3. *Define* $T_p := \log \frac{p}{p-1+\theta}$ *with* θ *given in (1.5). It can be shown that* T_p *decreases monotonically as* p *varies over* $[2,\infty)$. *Furthermore, under the hypothesis (1.5), for arbitrary* $p \geq 2$,

$$\sup_k E\left[\sup_{t\in[0,T_p]}\left(\|u_k(s)\|^2 + \mu\,\|u_k(s)\|_V^2\right)^{\frac{p}{2}} + \int_0^{T_p}\left(\|u_k(s)\|^2 + \mu\,\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1}\|u_k(s)\|_V^2\,ds\right]$$
$$\leq C_{p,\theta}\left[1 + (\|u_0\|^2 + \mu\,\|u_0\|_V^2)^{\frac{p^2}{2(p-1+\theta)}}\right] < \infty.$$

Proof. Define

$$\tau_M^k := \inf\{t \geq 0 : \|u_k(t)\| > M\} \wedge T_p, \quad \forall k \in \mathbb{N}, \forall M > 0.$$

Given that u_k does not explode, $\tau_M^k \to T_p$, almost surely, when M tends to ∞. Using Itô's formula on (3.6), we have, for $t \leq \tau_M^k$, $j = 1,\ldots,k$,

$$dg_{jk}^2(t) = -\frac{2\lambda_j}{1+\mu\lambda_j}g_{jk}^2(t)dt + \frac{2}{1+\mu\lambda_j}g_{jk}(t)\left(\sum_{i=1}^k g_{ik}(t)e_i\log\left|\sum_{i=1}^k g_{ik}(t)e_i\right|, e_j\right)dt$$
$$+ \frac{2}{1+\mu\lambda_j}g_{jk}\left(f\left(\sum_{i=1}^k g_{ik}(t)e_i\right), e_j\right)dB_t + \frac{1}{(1+\mu\lambda_j)^2}\left(f\left(\sum_{i=1}^k g_{ik}(t)e_i\right), e_j\right)^2 dt.$$

Multiply both sides by $1+\mu\lambda_j$, sum from $j=1$ to k, and it is derived that

$$\sum_{j=1}^k dg_{jk}^2(t) + \mu\sum_{j=1}^k \lambda_j dg_{jk}^2(t) = -2\sum_{j=1}^k \lambda_j g_{jk}^2(t)dt + 2\sum_{j=1}^k g_{jk}(t)\left(u_k(x,t)\log|u_k(x,t)|, e_j\right)dt$$
$$+ 2\sum_{j=1}^k g_{jk}\left(f(u_k(x,t)), e_j\right)dB_t + \sum_{j=1}^k \frac{1}{1+\mu\lambda_j}\left(f(u_k(x,t)), e_j\right)^2 dt,$$

i.e.,

$$d\left(\|u_k(t)\|^2 + \mu\,\|u_k(t)\|_V^2\right) = -2\,\|u_k(t)\|_V^2\,dt + 2\,(u_k(t)\log|u_k(t)|, u_k(t))\,dt$$
$$+ 2\,(f(u_k(t)), u_k(t))\,dB_t + \left(P_k f(u_k(t)), (I-\mu\Delta)^{-1}f(u_k(t))\right)dt.$$

Upon a second application of Itô's formula, we derive

$$d\left(\|u_k(t)\|^2 + \mu\,\|u_k(t)\|_V^2\right)^{\frac{p}{2}}$$
$$= -p\,\|u_k(t)\|_V^2\left(\|u_k(t)\|^2 + \mu\,\|u_k(t)\|_V^2\right)^{\frac{p}{2}-1}dt$$

$$+ p(u_k(t)\log|u_k(t)|, u_k(t))\left(\|u_k(t)\|^2 + \mu\|u_k(t)\|_V^2\right)^{\frac{p}{2}-1} dt$$
$$+ p(f(u_k(t)), u_k(t))\left(\|u_k(t)\|^2 + \mu\|u_k(t)\|_V^2\right)^{\frac{p}{2}-1} dB_t$$
$$+ \frac{p}{2}\left(P_k f(u_k(t)), (I-\mu\Delta)^{-1} f(u_k(t))\right)\left(\|u_k(t)\|^2 + \mu\|u_k(t)\|_V^2\right)^{\frac{p}{2}-1} dt$$
$$+ \frac{p(p-2)}{2}(f(u_k(t)), u_k(t))^2\left(\|u_k(t)\|^2 + \mu\|u_k(t)\|_V^2\right)^{\frac{p}{2}-2} dt.$$

Integrating both sides from 0 to t, we obtain

$$\begin{aligned}
&\left(\|u_k(t)\|^2 + \mu\|u_k(t)\|_V^2\right)^{\frac{p}{2}} + p\int_0^t \|u_k(s)\|_V^2\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds \\
=&\left(\|u_0\|^2 + \mu\|u_0\|_V^2\right)^{\frac{p}{2}} + p\int_0^t (f(u_k(s)), u_k(s))\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} dB_s \\
&+ p\int_0^t (u_k(s)\log|u_k(s)|, u_k(s))\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds \\
&+ \frac{p}{2}\int_0^t \left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1}(P_k f(u_k(s)), (I-\mu\Delta)^{-1} f(u_k(s)))ds \\
&+ \frac{p(p-2)}{2}\int_0^t (f(u_k(s)), u_k(s))^2\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-2} ds \\
=:&\left(\|u_0\|^2 + \mu\|u_0\|_V^2\right)^{\frac{p}{2}} + p\int_0^t (f(u_k(s)), u_k(s))\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} dB_s \\
&+ J_1 + J_2 + J_3.
\end{aligned} \tag{3.8}$$

Utilizing the logarithmic Sobolev inequality (1.7) and taking $\epsilon = \frac{1}{2}$ yields

$$\begin{aligned}
J_1 \le& p\int_0^t \left(\frac{1}{2}\|u_k(s)\|_V^2 + \frac{d\log 2}{4}\|u_k(s)\|^2 + \|u_k(s)\|^2\log\|u_k(s)\|\right)\times\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds \\
\le& \frac{p}{2}\int_0^t \|u_k(s)\|_V^2\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds + p\frac{d\log 2}{4}\int_0^t \left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}} ds \\
&+ \frac{p}{2}\int_0^t \|u_k(s)\|^2\log\|u_k(s)\|^2\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds.
\end{aligned}$$

Since the function $y = x\log x$ is monotonically increasing on $[\frac{1}{e}, \infty)$ and has a minimum value $-\frac{1}{e}$ on $(0, \infty)$, it follows that

$$\|u_k(s)\|^2\log\|u_k(s)\|^2 \le \left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)\log\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right) + \frac{1}{e}.$$

Thus,

$$\begin{aligned}
J_1 \le& \frac{p}{2}\int_0^t \|u_k(s)\|_V^2\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds + C\int_0^t \left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}} ds \\
&+ C + \frac{p}{2}\int_0^t \left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}}\log\left(\|u_k(s)\|^2 + \mu\|u_k(s)\|_V^2\right) ds.
\end{aligned} \tag{3.9}$$

Noticing that $\left\|(I-\mu\Delta)^{-1}f\right\| \le \|f\|$ and f satisfies the growth condition (1.5), we have

$$\begin{aligned}J_2 &\le \frac{p}{2}\int_0^t \|P_k f(u_k(s))\| \left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1}\left\|(I-\mu\Delta)^{-1}f(u_k(s))\right\| ds\\&\le \frac{p}{2}\int_0^t \|f(u_k(s))\|^2\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds\\&\le C\int_0^t \left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}} ds. \end{aligned}\tag{3.10}$$

Similarly,

$$\begin{aligned}J_3 &\le \frac{p(p-2)}{2}\int_0^t \|f(u_k(s))\|^2\|u_k(s)\|^2\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-2} ds\\&\le \frac{p(p-2)}{2}\int_0^t \left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}} ds. \end{aligned}\tag{3.11}$$

Combining (3.8), (3.9), (3.10), and (3.11), we conclude that

$$\begin{aligned}&\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}}+\frac{p}{2}\int_0^t \|u_k(s)\|_V^2\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds\\\le& C+\left(\|u_0\|^2+\mu\|u_0\|_V^2\right)^{\frac{p}{2}}+C\int_0^t\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}} ds\\&+p\sup_{r\in[0,t]}\left|\int_0^r (f(u_k(s)),u_k(s))\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} dB_s\right|\\&+\int_0^t\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}}\log\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}} ds\\=:&M(t)+C\int_0^t\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}} ds\\&+\int_0^t\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}}\log\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}} ds, \end{aligned}\tag{3.12}$$

where $C>0$ is independent of k and

$$M(t)=C+\left(\|u_0\|^2+\mu\|u_0\|_V^2\right)^{\frac{p}{2}}+p\sup_{r\in[0,t]}\left|\int_0^r (f(u_k(s)),u_k(s))\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} dB_s\right|.$$

By the logarithmic form of the Grönwall's inequality (see Lemma 7.2 in [15]), we obtain that for any $t\in[0,\tau_M^k]$,

$$\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}}+\frac{p}{2}\int_0^t \|u_k(s)\|_V^2\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds \le (1\vee M(t))^{e^t}\times e^{C(e^t-1)}.$$

Let $X_k(t)=E\left[\sup_{t\in[0,t\wedge\tau_M^k]}\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)\right]^{\frac{p}{2}}$, then for all $\epsilon>0$ and $t\le T_p$, we have

$$X_k(t)+\frac{p}{2}E\int_0^{t\wedge\tau_M^k}\|u_k(s)\|_V^2\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds$$

$$
\begin{aligned}
&\le e^{C(e^{T_p-1})} E\left\{[M(t\wedge\tau_M^k)+1]^{e^{T_p}}\right\}\\
&\le C\left[1+\left(\|u_0\|^2+\mu\|u_0\|_V^2\right)^{\frac{p}{2}}\right]^{\frac{p}{p-1+\theta}}\\
&\quad + p\sup_{r\in[0,t]}\left|\int_0^r (f(u_k(s)),u_k(s))\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} dB_s\right|^{\frac{p}{p-1+\theta}}.
\end{aligned}
$$

Using the Burkholder-Davis-Gundy inequality and Young's inequality $ab\le \epsilon a^{\frac{p-1+\theta}{p-1}}+C_{\epsilon,p,\theta}b^{\frac{p-1+\theta}{\theta}}$, we get

$$
\begin{aligned}
&p\sup_{r\in[0,t]}\left|\int_0^r (f(u_k(s)),u_k(s))\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} dB_s\right|^{\frac{p}{p-1+\theta}}\\
\le& CE\left(\int_0^{t\wedge\tau_M^k}(f(u_k(s)),u_k(s))^2\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{p-2} ds\right)^{\frac{p}{2(p-1+\theta)}}\\
\le& CE\left(\int_0^{t\wedge\tau_M^k}\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\theta}\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{p-1} ds\right)^{\frac{p}{2(p-1+\theta)}}\\
\le& CE\left[\sup_{t\in[0,t\wedge\tau_M^k]}\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p(p-1)}{2(p-1+\theta)}}\times\left(\int_0^{t\wedge\tau_M^k}\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\theta} ds\right)^{\frac{p}{2(p-1+\theta)}}\right]\\
\le& C\epsilon E\sup_{t\in[0,t\wedge\tau_M^k]}\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}}+C_\epsilon E\left(\int_0^{t\wedge\tau_M^k}\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\theta} ds\right)^{\frac{p}{2\theta}}\\
\le& C\epsilon E\sup_{t\in[0,t\wedge\tau_M^k]}\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}}+C_\epsilon E\int_0^{t\wedge\tau_M^k}\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}} ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&X_k(t)+\frac{p}{2}E\int_0^{t\wedge\tau_M^k}\|u_k(s)\|_V^2\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds\\
\le& C\left[1+\left(\|u_0\|^2+\mu\|u_0\|_V^2\right)^{\frac{p}{2}}\right]^{\frac{p}{p-1+\theta}}+C\epsilon X_k(t)+C\int_0^t X_k(s)ds.
\end{aligned}
$$

Observe that the constants C, C_ϵ are independent of M, k. By subtracting ϵX_k from both sides and invoking the Grönwall's inequality, we deduce that

$$
\begin{aligned}
&X_k(t)+\frac{p}{2}E\int_0^{t\wedge\tau_M^k}\|u_k(s)\|_V^2\left(\|u_k(s)\|^2+\mu\|u_k(s)\|_V^2\right)^{\frac{p}{2}-1} ds\\
\le& C_{P,\theta}\left[1+\left(\|u_0\|^2+\mu\|u_0\|_V^2\right)^{\frac{p^2}{2(p-1+\theta)}}\right],\quad \forall k\in\mathbb{N},\forall M>0.
\end{aligned}
$$

Sending M to ∞, then applying the Fatou lemma, the conclusion is obtained. □

For the interval $[0,T_2]$. Let $\beta\in(0,1)$, $p>1$. We define the function space $W^{\beta,p}([0,T_2];V^*)$ as all measurable functions $u(t):[0,T_2]\to V^*$, equipped with the norm

$$
\|u\|^p_{W^{\beta,p}([0,T_2];V^*)}:=\int_0^{T_2}\|u(t)\|_{V^*}^p\,dt+\int_0^{T_2}\int_0^{T_2}\frac{\|u(t)-u(s)\|_{V^*}^p}{|t-s|^{1+\beta p}}dtds.
$$

Lemma 3.4. *Assuming that the assumption (1.5) holds, along with the constraints* $1 < p < 2$ *and* $\beta < \frac{1}{2}$*, it follows that*

$$\sup_k E\left(\|u_k\|^p_{W^{\beta,p}([0,T_2];V^*)}\right) < \infty.$$

Proof. Let's consider the equivalent form of (3.16)

$$\begin{aligned}u_k(t) - u_k(0) &= \int_0^t (I-\mu\Delta)^{-1}\Delta u_k(r)dr + \int_0^t (I-\mu\Delta)^{-1}u_k(r)\log|u_k(r)|\,dr \\ &\quad + \int_0^t (I-\mu\Delta)^{-1} f(u_k(r))dB_r \\ &=: K_1(t) + K_2(t) + K_3(t).\end{aligned}$$

Then we obtain

$$E\|u_k(t)-u_k(s)\|^p_{V^*} \le C\left(E\|K_1(t)-K_1(s)\|^p_{V^*} + E\|K_2(t)-K_2(s)\|^p_{V^*} + E\|K_3(t)-K_3(s)\|^p_{V^*}\right).$$

For simplicity, we suppose $t \ge s$. Applying Hölder's inequality yields

$$\begin{aligned}E\|K_1(t)-K_1(s)\|^p_{V^*} &\le CE\left(\int_s^t \left\|(I-\mu\Delta)^{-1}\Delta u_k(r)\right\|_{V^*} dr\right)^p \\ &\le CE\left(\int_s^t \left\|\frac{1}{\mu}(I-\mu\Delta)^{-1}u_k(r) - \frac{1}{\mu}u_k(r)\right\|_{V^*} dr\right)^p \\ &\le CE\left(\int_s^t \|u_k(r)\|_V\, dr\right)^p \\ &\le CE\left(\int_0^{T_2} \|u_k(r)\|^2_V\, dr + 1\right) \times |t-s|^{\frac{p}{2}}.\end{aligned}$$

Since H^1 is continuously embedded in L^q, we obtain the dual embedding $L^{q^*} \hookrightarrow V^*$, where $q^* \in [\frac{2n}{n+2}, 2)$ when $n > 2$, and $q^* \in (1,2)$ for $n = 1, 2$. Choose $\epsilon > 0$ sufficiently small such that $(1+\epsilon)q^* \le 2$ and $(1+\epsilon)p \le 2$, then

$$\begin{aligned}E\|K_2(t)-K_2(s)\|^p_{V^*} &= E\left\|\int_s^t (I-\mu\Delta)^{-1}(u_k(r)\log|u_k(r)|)dr\right\|^p_{V^*} \\ &\le CE\left(\int_s^t \left\||u_k(r)|^{1+\epsilon}+1\right\|_{L^{q^*}} dr\right)^p \\ &\le C\left(E\sup_{r\in[0,T_2]}\|u_k(r)\|^2 + 1\right) \times |t-s|^p.\end{aligned}$$

Similarly,

$$\begin{aligned}E\|K_3(t)-K_3(s)\|^p_{V^*} &= E\left\|\int_s^t (I-\mu\Delta)^{-1} f(u_k(r))dB_r\right\|^p_{V^*} \\ &\le C\left(E\sup_{r\in[0,T_2]}\|u_k(r)\|^2 + 1\right) \times |t-s|^{\frac{p}{2}}.\end{aligned}$$

When putting together the above results, we can make the following derivation:

$$E\,\|u_k(t)\|_{W^{\beta,p}([0,T_2];V^*)} \leq C\left(E\sup_{r\in[0,T_2]}\|u_k(r)\|^2 + E\int_0^{T_2}\|u_k(r)\|_V^2\,dr + 1\right)\times\left(\int_0^{T_2}\int_0^{T_2}|t-s|^{\frac{p}{2}-1-\beta p}\,dtds + 1\right).$$

Thus the above integral is finite when $\beta < \frac{1}{2}$. □

Step 3 Take weak limits.

Lemma 3.5. *If assumption (1.5) is satisfied, then the sequence* $\{u_k\}$ *is compact in* $L^p([0,T_2];H)$ *with* $1 < p < 2$*, and also in* $C([0,T_2];V^*)$*.*

Proof. By Theorem 2.1 in [21], the embedding $L^p([0,T_2];V) \cap W^{\beta,p}([0,T_2];V^*) \hookrightarrow L^p([0,T_2];H)$ is compact. Consequently, the embedding from

$$K_L := \left\{u \in L^p([0,T_2];H) : \|u\|_{W^{\beta,p}([0,T_2];V^*)} + \|u\|_{L^p([0,T_2];V)} \leq L\right\}$$

with any $L > 0$ to $L^p([0,T_2];H)$ is also compact. Furthermore,

$$\begin{aligned}\lim_{L\to\infty}\sup_k P\left(u_k \notin K_L\right) &= \lim_{L\to\infty}\sup_k P\left(\|u_k\|_{W^{\beta,p}([0,T_2];V^*)} + \|u_k\|_{L^p([0,T_2];V)} > L\right)\\ &\leq \lim_{L\to\infty}\frac{2^{p-1}}{L^p}\sup_k E\left(\|u_k\|^p_{W^{\beta,p}([0,T_2];V^*)} + \|u_k\|^p_{L^p([0,T_2];V)} > L\right)\\ &=0.\end{aligned}$$

Hence, $\{u_k\}$ is compact in $L^p([0,T_2];H)$.

For the compactness of u_k in $C([0,T_2];V^*)$, since the embedding $H \hookrightarrow V^*$ is compact, we first observe that

$$\lim_{L\to\infty}\sup_k P\left(\sup_{t\in[0,T_2]}\|u_k(t)\| > L\right) \leq \lim_{L\to\infty}\frac{1}{L^2}\sup_k E\left[\sup_{t\in[0,T_2]}\|u_k(t)\|^2\right] = 0.$$

Moreover, for any $\xi_k \in [0,T_2]$ and any $\epsilon > 0$, similar to the previous proof, we have

$$\begin{aligned}&\lim_{\delta\to 0}\sup_k P\left(\|u_k(\xi_k+\delta) - u_k(\xi_k)\|_{V^*} > \epsilon\right)\\ \leq&\frac{1}{\epsilon^p}\lim_{\delta\to 0}\sup_k E\,\|u_k(\xi_k+\delta) - u_k(\xi_k)\|^p_{V^*}\\ \leq&\frac{C}{\epsilon^p}\lim_{\delta\to 0}\sup_k\left[\left(E\sup_{r\in[0,T_2]}\|u_k(r)\|^2 + E\int_0^{T_2}\|u_k(r)\|_V^2\,dr + 1\right)\times\delta^{\frac{p}{2}}\right]\\ =&0,\end{aligned}$$

where $\xi_k + \delta := T_2 \wedge (\xi_k + \delta)$. By Aldous' compactness criterion [22], $\{u_k\}$ is compact in $C([0,T_2];V^*)$. □

Theorem 3.2. *Assuming (1.4) and (1.5) hold, then for any* $u_0 \in H_0^1(D)$*, (1.3) admits a global solution (in the probabilistic sense). Furthermore, when* $p \geq 2$*, one also has*

$$E\sup_{t\in[0,T]}\left[\left(\|u(t)\|^2 + \mu\,\|u(t)\|_V^2\right)^{\frac{p}{2}} + \int_0^T\left(\|u(s)\|^2 + \mu\,\|u(s)\|_V^2\right)^{\frac{p}{2}-1}\|u(s)\|_V^2\,ds\right] < \infty, \quad \forall T > 0. \tag{3.13}$$

Proof. First, we prove the existence of a probabilistic weak solution on $[0, T_2]$.

Fix a value r in $[1, 2]$. Define $\Gamma := [L^r([0, T_2]; H) \cap C([0, T_2]; V^*)] \times C([0, T_2]; \mathbb{R})$. In light of Lemma 3.5, the sequence of distributions $\mathcal{L}(u_k, B)$ is tight on Γ. Applying Prokhorov's theorem [17], we can deduce the existence of a subsequence, which we continue to denote as (u_k, B), s.t. there is a weak convergent relationship between $\mathcal{L}(u_k, B)$ and μ, a probability measure on Γ. By invoking the generalized Skorohod embedding theorem (Theorem C.1 in [23]), we are capable of constructing a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ along with a sequence of $\Gamma-$valued random vectors $\left\{(\widetilde{u}_k, \widetilde{B}_k)\right\}$ and $(\widetilde{u}, \widetilde{B})$. In this construction, for each $k \in \mathbb{N}$, we have $\widetilde{B}_k = \widetilde{B}$, almost surely with respect to the probability measure $\widetilde{P}$, and $\mathcal{L}(\widetilde{u}_k, \widetilde{B}_k)$ equals to $\mathcal{L}(u_k, B)$, and $\mathcal{L}(\widetilde{u}, \widetilde{B})$ equals to μ. Furthermore,

$$\|\widetilde{u}_k - \widetilde{u}\|_{L^r([0,T_2];H)} \to 0, \quad \|\widetilde{u}_k - \widetilde{u}\|_{C([0,T_2];V^*)} \to 0, \ \widetilde{P} - a.s. \tag{3.14}$$

We assert that

$$\widetilde{E}\,\|\widetilde{u}\|^2_{L^\infty([0,T_2];V)} \leq C \text{ and } \widetilde{u} \in L^\infty([0, T_2]; V), \ \widetilde{P} - a.s. \tag{3.15}$$

In fact, based on the equations that the random vectors (u_k, B) fulfill, $(\widetilde{u}_k, \widetilde{B})$ satisfies

$$\begin{aligned} \widetilde{u}_k(t) - \mu\Delta\widetilde{u}_k(t) =& P_k u_0 - \mu\Delta P_k u_0 + \int_0^t (\Delta\widetilde{u}_k(s) + P_k[\widetilde{u}_k(s)\log|\widetilde{u}_k(s)|])ds \\ &+ \int_0^t P_k f(\widetilde{u}_k(s))d\widetilde{B}_s, \ t \in [0, T_2] \end{aligned} \tag{3.16}$$

in V^*, where P_k is the projection operators. Therefore, $\{\widetilde{u}_k\}$ also satisfies

$$\sup_k \widetilde{E}\left[\sup_{t\in[0,T_2]} (\|\widetilde{u}_k(s)\|^2 + \mu\,\|\widetilde{u}_k(s)\|^2_V) + \int_0^{T_2} \|\widetilde{u}_k(s)\|^2_V\, ds\right] < \infty. \tag{3.17}$$

Hence, $\widetilde{E}\,\|\widetilde{u}_k\|^2_{L^q([0,T_2];V)} \leq CT_2^{\frac{2}{q}}$, for any $q \in [1, \infty)$, $k \in \mathbb{N}$. In other words, $\{\widetilde{u}_k\}_{k\in\mathbb{N}} \subset_b L^2(\widetilde{\Omega}; L^q([0, T_2]; V))$. Since $L^2(\widetilde{\Omega}; L^q([0, T_2]; V))$ is reflexive, thus, by the Kakutani theorem on reflexive spaces,

$$\widetilde{u}_k \overset{k\uparrow\infty}{\rightharpoonup} v^{(q)} \text{ in } L^2(\widetilde{\Omega}; L^q([0, T_2]; V), \tag{3.18}$$

for $q \in (1, \infty)$, where the limit $v^{(q)}$ depends possibly on q. Besides,

$$\widetilde{E}\left\|v^{(q)}\right\|^2_{L^q([0,T_2];V)} \leq \liminf_{k\to\infty} \widetilde{E}\,\|\widetilde{u}_k\|^2_{L^q([0,T_2];V)} \leq CT_2^{\frac{2}{q}}.$$

The continuity of the embedding

$$L^2(\widetilde{\Omega}; L^{q_2}([0, T_2]; V)) \hookrightarrow L^2(\widetilde{\Omega}; L^{q_1}([0, T_2]; V))$$

for $1 < q_1 < q_2 < \infty$, implies that $v^{(q)}$ does not, in fact, depend on q. Therefore, we will write v instead of $v^{(q)}$ in the following.

By the monotone convergence theorem,

$$\widetilde{E}\lim_{q\to\infty} \|v\|^2_{L^q([0,T_2];V)} \leq C.$$

Since the L_t^q norm depends continuously on the index q for any measurable function $f : [0, T_2] \to V$ for which $\lim_{q\to\infty} \left(\int_0^{T_2} \|f(s)\|_V^q\, ds\right)^{1/q} < \infty$, it follows that

$$\widetilde{E}\,\|v\|^2_{L^\infty([0,T_2];V)} \leq C \text{ and } v \in L^\infty([0, T_2]; V),\ \widetilde{P} - a.s. \tag{3.19}$$

It remains to identify v with the $\widetilde{P}$ almost sure Skorokhod-Jakubowski limit $\widetilde{u}$ in $L^r([0, T_2]; H) \cap C([0, T_2]; V^*)$. Consider the following test functions

$$\phi(\widetilde{w}, x, t) = \psi(\widetilde{w})\vartheta(x, t), \qquad \psi \in L^\infty(\widetilde{\Omega}),\ \vartheta \in L^{r^*}([0, T_2]; L^2(D)), \tag{3.20}$$

where $r^* = \frac{r}{r-1}$. From (3.18),

$$\widetilde{E}\left(\psi \int_0^{T_2} \int_D \vartheta(x, t)(\widetilde{u}_k(x, t) - v(x, t))dxdt\right) \stackrel{k\uparrow\infty}{\longrightarrow} 0.$$

On the other hand, by (3.14),

$$\psi \int_0^{T_2} \int_D \vartheta(x, t)(\widetilde{u}_k(x, t) - \widetilde{u}(x, t))dxdt \stackrel{k\uparrow\infty}{\longrightarrow} 0, \qquad \widetilde{P} - a.s.$$

By (3.17), we have the moment bound

$$\widetilde{E}\left|\psi \int_0^{T_2} \int_D \widetilde{u}_k(\widetilde{w}, x, t)\vartheta(x, t)dxdt\right|^2 \leq \|\psi\|^2_{L^\infty(\widetilde{\Omega})} \|\vartheta\|^2_{L^1([0,T_2];L^2(D))}\, \widetilde{E}\,\|\widetilde{u}_k\|^2_{L^\infty([0,T_2];V)} \leq C(\psi, \vartheta).$$

Hence, relying on Vitali's convergence theorem, we can arrive at

$$\widetilde{E}\left|\psi \int_0^{T_2} \int_D \vartheta(x, t)(\widetilde{u}_k(x, t) - \widetilde{u}(x, t))dxdt\right|^r \stackrel{k\uparrow\infty}{\longrightarrow} 0,$$

for any $1 \leq r < 2$. Consequently,

$$\widetilde{E}\left(\psi \int_0^{T_2} \int_D \vartheta(x, t)(\widetilde{u}(x, t) - v(x, t))dxdt\right) = 0, \tag{3.21}$$

for ψ, ϑ as in (3.20). We use $I_z(\vartheta)$ as short-hand for $\int_0^{T_2} \int_D \vartheta(x, t)z(x, t)dxdt$, where $z = \widetilde{u}, v$. Clearly, by(3.19), $I_v(\vartheta) \in L^r(\widetilde{\Omega})$, for any $1 \leq r < 2$. Since (3.21) implies that $I_{\widetilde{u}}(\vartheta) = I_v(\vartheta)$, almost surely, it follows that also $I_{\widetilde{u}}(\vartheta) \in L^p(\widetilde{\Omega})$ for each fixed ϑ. We conclude that for any $\vartheta \in L^{r^*}([0, T_2]; H)$, with $2 < r^* < \infty$, there exists a full $\widetilde{P}$−measure set $\widetilde{\Omega}_\vartheta$ on which $I_{\widetilde{u}}(\vartheta) = I_v(\vartheta)$. By separability of $L^{r^*}([0, T_2]; H)$, we deduce that for any $2 < r^* < \infty$ there exists a full $\widetilde{P}$−measure set on which the identity $I_{\widetilde{u}}(\vartheta) = I_v(\vartheta)$ holds for all $\vartheta \in L^{r^*}([0, T_2]; H)$. We can take this set to be the countable intersection of $\widetilde{\Omega}_\vartheta$ associated with a countable dense subset of $\vartheta \in L^{r^*}([0, T_2]; H)$. This shows that $\widetilde{u} = v$, $\widetilde{P} \otimes dt \otimes dx$−almost everywhere. We also have (3.19) for $\widetilde{u} = v$.

Combining (3.14), (3.17), and (1.5), it becomes readily apparent that a subsequence $\{\widetilde{u}_k\}$ exists, s.t. as $k \to \infty$,
(1) $\widetilde{u}_k(\widetilde{w}, t, x)$ converges to $\widetilde{u}(\widetilde{w}, t, x)$, almost everywhere in $\widetilde{\Omega} \times [0, T_2] \times D$;

(2) $\widetilde{u}_k$ converges to $\widetilde{u}$ in $L^r(\widetilde{\Omega}; L^r([0,T_2]; H))$, weakly in $L^2(\widetilde{\Omega}; L^2([0,T_2]; V))$, and weakly star in $L^\infty([0,T_2]; L^2(\widetilde{\Omega}; V))$;
(3) $\Delta\widetilde{u}_k \xrightarrow{w} \Delta\widetilde{u}$ in $L^2(\widetilde{\Omega}; L^2([0,T_2]; V^*))$;
(4) $P_k[\widetilde{u}_k \log|\widetilde{u}_k|] \longrightarrow \widetilde{u}\log|\widetilde{u}|$ in $L^r(\widetilde{\Omega}; L^r([0,T_2]; V^*))$;
(5) $\int_0^\cdot P_k f(\widetilde{u}_k(s))d\widetilde{B}_s \longrightarrow \int_0^\cdot f(\widetilde{u}(s))d\widetilde{B}_s$ in $L^\infty([0,T_2]; L^2(\widetilde{\Omega}; V^*))$.
By sending $k \to \infty$ in (3.16), we get for any $t \in [0,T_2]$, $\widetilde{u}$ satisfies

$$\widetilde{u}(t) - \mu\Delta\widetilde{u}(t) = u_0 - \mu\Delta u_0 + \int_0^t (\Delta\widetilde{u}(s) + \widetilde{u}(s)\log|\widetilde{u}(s)|)ds + \int_0^t f(\widetilde{u}(s))d\widetilde{B}_s$$

in V^*, $\widetilde{P}$−a.s. Consequently, a probabilistic weak solution exists on $[0,T_2]$.

Next, we fix $T > 0$ and $p \geq 2$, and establish a probabilistic weak solution over the interval $[0,T]$. For $z \geq 2$, define

$$h^0(z) = z, \quad h(z) = \frac{z^2}{z-1+\theta}, \quad h^2(z) = h(h(z)),$$
$$h^k(z) := h(h(\ldots h(z)\ldots)), \quad k \in \mathbb{N}.$$

It follows that $h(z) > z + 1 - \theta$. Consequently, $h^k(z) \to \infty$ when $k \to \infty$, $z \geq 2$. Furthermore, $h(z) \leq z+1$ when $z \geq \frac{1}{\theta} - 1$. Set

$$i_p := \min\left\{i \geq 0 : h^i(p) \geq \frac{1}{\theta} - 1\right\}.$$

For $i \geq i_p$, by iteratively applying the inequality, we obtain

$$h^i(p) \leq h^{i-1}(p) + 1 \leq \cdots \leq h^{i_p}(p) + i - i_p.$$

For simplicity, define $T(i) = T_{h^i(p)}$ for $i \in \mathbb{N}$. Based on the definition of T_p and its monotone decreasing nature, we can readily derive

$$\begin{aligned}\sum_{i=0}^{\infty} T(i) &\geq \sum_{i=i_p+1}^{\infty} T_{h^i(p)} \geq \sum_{i=i_p+1}^{\infty} T_{h^{i_p}(p)+i-i_p} \\ &= \sum_{j=1}^{\infty} \log\frac{h^{i_p}(p)+j}{h^{i_p}(p)+j+\theta-1} \geq \delta\sum_{j=1}^{\infty}\frac{1-\theta}{h^{i_p}(p)+j+\theta-1} = \infty,\end{aligned} \tag{3.22}$$

where $\delta > 0$ represents a sufficiently small constant. Let

$$\kappa = \min\left\{k \geq 0 : \sum_{i=0}^{k} T(i) \geq T\right\},$$

and

$$\Theta(i+1) = \Theta(i) + T(\kappa - i), \; i = 0, 1, \ldots, \kappa$$

with $\Theta(0) = 0$. From (3.22), we see that $\kappa < \infty$. Therefore, it logically follows that $\forall T > 0$, and we can find a nonnegative integer κ s.t. $\Theta(\kappa) < T \leq \Theta(\kappa+1)$.

Given any $\Theta \geq 0$, we set

$$B_t^\Theta = B(t+\Theta) - B(\Theta), \quad \mathcal{F}_t^\Theta = \mathcal{F}_{t+\Theta}, \quad t > 0.$$

Let's consider

$$u^{\Theta,\xi}(t)-\mu\Delta u^{\Theta,\xi}(t)=\xi-\mu\Delta\xi+\int_0^t(\Delta u^{\Theta,\xi}(s)+u^{\Theta,\xi}(s)\log\left|u^{\Theta,\xi}(s)\right|)ds+\int_0^t f(u^{\Theta,\xi}(s))dB_s^{\Theta},\quad t\in[0,T_2].$$

Then, based on the proof in the previous step, we conclude that the above equation admits a unique probabilistic weak solution $\left\{u^{\Theta,\xi}(t)\right\}_{t\in[0,T_2]}$. Moreover, $u^{\Theta,\xi}$ constitutes a measurable mapping with respect to the initial condition $\xi\in H$.

For $t\in[\Theta(i),\Theta(i+1)]$ and $i=0,1,\ldots,\kappa$, define

$$u(t):=u^{\Theta(i),u(\Theta(i))}(t-\Theta(i)),$$

which fulfills

$$u(t)-\mu\Delta u(t)=u(\Theta(i))-\mu\Delta u(\Theta(i))+\int_{\Theta(i)}^t(\Delta u(s)+u(s)\log|u(s)|)ds+\int_{\Theta(i)}^t f(u(s))dB_s.$$

It implies that u satisfies (1.3) within the interval $[0,\Theta(\kappa+1)]$. Noticing that $T\leq\Theta(\kappa+1)$, together with the arbitrariness of T, it follows that u is global. Furthermore, combining the uniqueness of the solution with the Yamada-Watanabe theorem, the strong solution exists uniquely.

Ultimately, we present an estimate for u. Since the estimate in (3.15) holds for probabilistic weak solutions over $[0,T_2]$, it can be similarly extended to hold for probabilistic strong solutions. Analogously, for every pair of indices $i,j\in\{0,\ldots,\kappa\}$ with $j\leq i$, we have

$$\begin{aligned}&E\left(\sup_{t\in[0,T(i)]}\left(\left\|u^{0,\xi}(t)\right\|^2+\mu\left\|u^{0,\xi}(t)\right\|_V^2\right)^{\frac{q(j)}{2}}+\int_0^{T(i)}\left(\left\|u^{0,\xi}(t)\right\|^2+\mu\left\|u^{0,\xi}(t)\right\|_V^2\right)^{\frac{q(j)}{2}-1}\left\|u^{0,\xi}(t)\right\|_V^2dt\right)\\&\leq C_{p,\theta}\left[1+\left(\|\xi\|^2+\mu\|\xi\|_V^2\right)^{\frac{q(j+1)}{2}}\right].\end{aligned}$$

Define

$$\begin{aligned}\Phi_i(\xi):=&E\left(\sup_{t\in[0,T(i)]}\left(\left\|u^{0,\xi}(t)\right\|^2+\mu\left\|u^{0,\xi}(t)\right\|_V^2\right)^{\frac{q(i)}{2}}+\int_0^{T(i)}\left(\left\|u^{0,\xi}(t)\right\|^2+\mu\left\|u^{0,\xi}(t)\right\|_V^2\right)^{\frac{p}{2}-1}\left\|u^{0,\xi}(t)\right\|_V^2dt\right)\\\leq&C_{p,\theta}\left[1+\left(\|\xi\|^2+\mu\|\xi\|_V^2\right)^{\frac{q(i+1)}{2}}\right].\end{aligned}$$

Since $u(\Theta(i))$ does not depend on the Brownian motion $(B_t^{\Theta(i)})_{t\geq0}$, and $u^{\Theta,\xi}$ and $u^{0,\xi}$ have the same distribution, it follows that

$$\begin{aligned}&E\left(\sup_{t\in[\Theta(i),\Theta(i+1)]}\left(\|u(t)\|^2+\mu\|u(t)\|^2\right)^{\frac{q(\kappa-i)}{2}}+\int_{\Theta(i)}^{\Theta(i+1)}\left(\|u(t)\|^2+\mu\|u(t)\|_V^2\right)^{\frac{p}{2}-1}\|u(t)\|_V^2dt\right)\\=&E\left(\sup_{t\in[0,T(\kappa-i)]}\left(\left\|u^{\Theta(i),u(\Theta(i))}(t)\right\|^2+\mu\left\|u^{\Theta(i),u(\Theta(i))}(t)\right\|^2\right)^{\frac{q(\kappa-i)}{2}}\right.\\&\left.+\int_0^{T(\kappa-i)}\left(\left\|u^{\Theta(i),u(\Theta(i))}(t)\right\|^2+\mu\left\|u^{\Theta(i),u(\Theta(i))}(t)\right\|_V^2\right)^{\frac{p}{2}-1}\left\|u^{\Theta(i),u(\Theta(i))}(t)\right\|_V^2dt\right)\end{aligned}$$

$$=E\Phi_{\kappa-i}(u(\Theta(i))) \leq C_{p,\theta}\left[E\left(\|u(\Theta(i))\|^2+\mu\,\|u(\Theta(i))\|^2\right)^{\frac{q(\kappa-i+1)}{2}}+1\right].$$

Thus, through the application of mathematical induction, there holds

$$E\left(\sup_{t\in[0,\Theta(\kappa+1)]}\left(\|u(t)\|^2+\mu\,\|u(t)\|_V^2\right)^{\frac{p}{2}}+\int_0^{\Theta(\kappa+1)}\left(\|u(t)\|^2+\mu\,\|u(t)\|_V^2\right)^{\frac{p}{2}-1}\|u(t)\|_V^2\,ds\right)$$
$$\leq C_{p,\theta}\left[1+\left(\|u_0\|^2+\mu\,\|u_0\|_V^2\right)^{\frac{q(\kappa+1)}{2}}\right]<\infty.$$

Observe that $\Theta(\kappa) < T \leq \Theta(\kappa+1)$, thus (3.13) can be deduced. □

Remark 1. *For the stochastic pseudo-parabolic equation with a logarithmic source term, the spatial regularity of the solution is consistent with the initial data, which aligns with the results in the deterministic pseudo-parabolic equation [1]. However, it is worth noting that compared to the stochastic heat equation, the temporal regularity of the solution to the stochastic pseudo-parabolic equation is enhanced, from the original L^2 to L^∞.*

Remark 2. *Compared to the study of the stochastic heat equation [15], the pseudo-parabolic term $\mu\Delta u_t$ proves to be more beneficial for ensuring the global existence. However, it is worth noting that regarding the growth condition of the noise coefficient $f(\cdot)$, the stochastic pseudo-parabolic equation has relaxed the restrictive conditions used in the stochastic heat equation*

$$|f(u)| \leq C_1 + C_2\,|u|^\theta.$$

Specifically, the stochastic pseudo-parabolic equation allows a more relaxed condition, namely

$$|f(u)| \leq C_1 + C_2(1+\sqrt{\mu})\,|u|^\theta,$$

and as μ increases, the restrictive conditions become more relaxed.

4. Existence of global solutions: Part II

Provided that condition (1.6) holds in this section, we shall adapt the approaches presented in Section 3 to verify the existence of solutions.

According to Theorem 3.1, when the coefficient f of the noise term satisfies condition (1.6), the approximate equation (3.5) still admits a global solution u_k. Due to the different growth constraints on the coefficient of the noise term in this case, in order to establish the compactness of the sequence $\{u_k, k \geq 1\}$, we require some new estimates. To this end, we define

$$\gamma(x) := \begin{cases}\log x, & x \geq e,\\ \frac{x}{e}, & 0 \leq x < e,\end{cases}$$

and

$$\Psi(z) := \exp\left(\int_0^z \frac{1}{1+x+x\gamma(x)}dx\right), \qquad z > 0.$$

Then, we can derive

$$\Psi'(z) = \Psi(z) \times \frac{1}{1 + z + z\gamma(z)}, \qquad \Psi''(z) \le 0. \tag{4.1}$$

For an arbitrary positive real number M, we introduce the stopping times

$$\tau_M^k := \inf\left\{t > 0 : \|u_k\|^2 + \mu \|u_k\|_V^2 > M\right\}.$$

Set $u_k^M(u) = u_k(t \wedge \tau_M^k)$, then the following lemma is valid.

Lemma 4.1. *If the assumption (1.6) is satisfied, then a constant C can be found, s.t.*

$$\begin{aligned}
&\sup_k E\left[\Psi\left(\left\|u_k^M(T)\right\|^2 + \mu \left\|u_k^M(T)\right\|_V^2\right) + \int_0^{T\wedge\tau_M^k} \Psi'\left(\left\|u_k^M(s)\right\|^2 + \mu \left\|u_k^M(s)\right\|_V^2\right)\left\|u_k^M(s)\right\|_V^2 ds\right] \\
\le&\Psi\left(\|u_0\|^2 + \mu \|u_0\|_V^2\right) e^{CT}, \quad \forall T > 0, \quad \forall M > 0.
\end{aligned} \tag{4.2}$$

Proof. Similar to the proof of Lemma 3.3, we have

$$\begin{aligned}
&\left\|u_k^M(t)\right\|^2 + \mu \left\|u_k^M(t)\right\|_V^2 \\
=& \|u_0\|^2 + \mu \|u_0\|_V^2 - 2\int_0^{t\wedge\tau_M^k} \left\|u_k^M(s)\right\|_V^2 ds + 2\int_0^{t\wedge\tau_M^k} (u_k^M(s) \log |u_k^M(s)|, u_k^M(s))ds \\
&+ \int_0^{t\wedge\tau_M^k} \left(P_k f(u_k^M(s)), (I - \mu\Delta)^{-1} f(u_k^M(s))\right) ds + 2\int_0^{t\wedge\tau_M^k} \left(f(u_k^M(s)), u_k^M(s)\right) dB_s.
\end{aligned}$$

By employing Ito's formula on $\left\|u_M^k\right\|^2 + \mu \left\|u_M^k\right\|_V^2$, one can get

$$\begin{aligned}
&\Psi\left(\left\|u_k^M(t)\right\|^2 + \mu \left\|u_k^M(t)\right\|_V^2\right) \\
\le&\Psi\left(\|u_0\|^2 + \mu \|u_0\|_V^2\right) - 2\int_0^{t\wedge\tau_M^k} \Psi'\left(\left\|u_k^M(s)\right\|^2 + \mu \left\|u_k^M(s)\right\|_V^2\right)\left\|u_k^M(s)\right\|_V^2 ds \\
&+ 2\int_0^{t\wedge\tau_M^k} \Psi'\left(\left\|u_k^M(s)\right\|^2 + \mu \left\|u_k^M(s)\right\|_V^2\right)\left(u_k^M(s) \log |u_k^M(s)|, u_k^M(s)\right) ds \\
&+ \int_0^{t\wedge\tau_M^k} \Psi'\left(\left\|u_k^M(s)\right\|^2 + \mu \left\|u_k^M(s)\right\|_V^2\right)\left(P_k f(u_k^M(s)), (I - \mu\Delta)^{-1} f(u_k^M(s))\right) ds \\
&+ 2\int_0^{t\wedge\tau_M^k} \Psi'\left(\left\|u_k^M(s)\right\|^2 + \mu \left\|u_k^M(s)\right\|_V^2\right)\left(f(u_k^M(s)), u_k^M(s)\right) dB_s \\
&+ 2\int_0^{t\wedge\tau_M^k} \Psi''\left(\left\|u_k^M(s)\right\|^2 + \mu \left\|u_k^M(s)\right\|_V^2\right)\left(f(u_k^M(s)), u_k^M(s)\right)^2 ds.
\end{aligned}$$

Based on the logarithmic Sobolev inequality (1.7) when setting $\epsilon = \frac{1}{4}$ and combining with assumption (1.6) , we can derive

$$\Psi\left(\left\|u_k^M(t)\right\|^2 + \mu \left\|u_k^M(t)\right\|_V^2\right)$$

$$
\begin{aligned}
\leq&\Psi\left(\|u_0\|^2+\mu\|u_0\|_V^2\right)-2\int_0^{t\wedge\tau_M^k}\Psi'\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\left\|u_k^M(s)\right\|_V^2ds\\
&+2\int_0^{t\wedge\tau_M^k}\Psi'\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\left(\frac{1}{4}\left\|u_k^M(s)\right\|_V^2+\frac{d\log 4}{4}\left\|u_k^M(s)\right\|^2+\left\|u_k^M(s)\right\|^2\log\left\|u_k^M(s)\right\|\right)ds\\
&+\int_0^{t\wedge\tau_M^k}\Psi'\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\int_D\left(2C_3^2+2C_4^2\left|u_k^M\right|^2\log_+\left|u_k^M\right|\right)dxds\\
&+2\int_0^{t\wedge\tau_M^k}\Psi'\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\left(f(u_k^M(s)),u_k^M(s)\right)dB_s\\
&+2\int_0^{t\wedge\tau_M^k}\Psi''\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\left(f(u_k^M(s)),u_k^M(s)\right)^2ds.
\end{aligned}\tag{4.3}
$$

Noticing that

$$
\begin{aligned}
\int_D\left|u_k^M\right|^2\log_+\left|u_k^M\right|dx=&\int_D\left|u_k^M\right|^2\log\left|u_k^M\right|dx+\int_D\left|u_k^M\right|^2\log\frac{1}{\left|u_k^M\right|}\mathbf{1}_{\{0\leq|u_k^M|\leq 1\}}dx\\
\leq&\int_D\left|u_k^M\right|^2\log\left|u_k^M\right|dx+\frac{1}{2e}m(D)\\
\leq&\epsilon\left\|u_k^M\right\|_V^2+\left(\frac{d}{4}\log\frac{1}{\epsilon}\right)\left\|u_k^M\right\|^2+\left\|u_k^M\right\|^2\log\left\|u_k^M\right\|+\frac{1}{2e}\mathrm{mes}(D),
\end{aligned}
$$

then set $\epsilon=\frac{1}{4C_4^2}$ in the above inequality and substituting it into (4.3), we obtain

$$
\begin{aligned}
&\Psi\left(\left\|u_k^M(t)\right\|^2+\mu\left\|u_k^M(t)\right\|_V^2\right)+\int_0^{t\wedge\tau_M^k}\Psi'\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\left\|u_k^M(s)\right\|_V^2ds\\
\leq&\Psi\left(\|u_0\|^2+\mu\|u_0\|_V^2\right)\\
&+C\int_0^{t\wedge\tau_M^k}\Psi'\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\left(1+\left\|u_k^M(s)\right\|^2+\left\|u_k^M(s)\right\|^2\log\left\|u_k^M(s)\right\|^2\right)ds\\
&+2\int_0^{t\wedge\tau_M^k}\Psi'\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\left(f(u_k^M(s)),u_k^M(s)\right)dB_s\\
&+2\int_0^{t\wedge\tau_M^k}\Psi''\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\left(f(u_k^M(s)),u_k^M(s)\right)^2ds\\
=:&\,\Psi\left(\|u_0\|^2+\mu\|u_0\|_V^2\right)+I_1^k(t)+I_2^k(t)+I_3^k(t).
\end{aligned}\tag{4.4}
$$

From (4.1), it can be noted that

$$
\begin{aligned}
I_1^k(t)=&C\int_0^{t\wedge\tau_M^k}\Psi\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\left(1+\left\|u_k^M(s)\right\|^2+\left\|u_k^M(s)\right\|^2\log\left\|u_k^M(s)\right\|^2\right)\\
&\times\frac{1}{1+\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2+\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)\log\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)}ds\\
\leq&C\int_0^t\Psi\left(\left\|u_k^M(s)\right\|^2+\mu\left\|u_k^M(s)\right\|_V^2\right)ds.
\end{aligned}\tag{4.5}
$$

Since $\Psi''(z) \leq 0$ holds for all $z \geq 0$, it directly follows that $I_3^k(t) \leq 0$. Substituting (4.5) into (4.4) and taking expectations on both sides yields

$$\begin{aligned}
&E\Psi\left(\left\|u_k^M(t)\right\|^2 + \mu\left\|u_k^M(t)\right\|_V^2\right) + 2E\int_0^{t\wedge\tau_M^k} \Psi'\left(\left\|u_k^M(s)\right\|^2 + \mu\left\|u_k^M(s)\right\|_V^2\right)\left\|u_k^M(s)\right\|_V^2 ds \\
\leq&\Psi\left(\|u_0\|^2 + \mu\|u_0\|_V^2\right) + CE\int_0^t \Psi\left(\left\|u_k^M(s)\right\|^2 + \mu\left\|u_k^M(s)\right\|_V^2\right)ds.
\end{aligned}$$

Then, (4.2) follows directly from Grönwall's inequality. □

Lemma 4.2. *Under the assumption (1.6) with the constraints $\beta < \frac{1}{2}$ and $1 < p < 2$, there exists $K_{\{T,M,\beta,p\}} > 0$, s.t.*

$$\sup_k E\left(\left\|u_k^M\right\|_{W^{\beta,p}([0,T];V^*)}^p \mathbf{1}_{\{\tau_M^k\leq T\}}\right) \leq K_{\{T,M,\beta,p\}}, \quad \forall T > 0, \quad \forall M > 0.$$

Proof. Similar to Lemma 3.4, for $t \in [0, T \wedge \tau_M^k]$, consider equation

$$\begin{aligned}
u_k(t) - u_k(s) =& \int_s^t (I-\mu\Delta)^{-1}\Delta u_k(r)dr + \int_s^t (I-\mu\Delta)^{-1}u_k(r)\log|u_k(r)|\,dr \\
&+ \int_s^t (I-\mu\Delta)^{-1}f(u_k(r))dB_r \\
=:& K_1(t) + K_2(t) + K_3'(t).
\end{aligned}$$

Then, we have

$$E\|u_k(t)-u_k(s)\|_{V^*}^p \leq 3^{p-1}\times\left(E\|K_1(t)-K_1(s)\|_{V^*}^p + E\|K_2(t)-K_2(s)\|_{V^*}^p + E\left\|K_3'(t)-K_3'(s)\right\|_{V^*}^p\right),$$

where $0 < s < t < T\wedge\tau_M^k$. The estimations for the two terms $E\|K_1(t)-K_1(s)\|_{V^*}^p$ and $E\|K_2(t)-K_2(s)\|_{V^*}^p$ are the same as that in Lemma 3.4. Direct calculation shows that

$$\begin{aligned}
E\left\|K_3'(t)-K_3'(s)\right\|_{V^*}^p \leq& CE\left(\int_s^t \left\|(I-\mu\Delta)^{-1}f(u_k(r))\right\|_{V^*}^2 dr\right)^{\frac{p}{2}} \\
\leq& CE\left(\int_s^t \|f(u_k(r))\|_{L^{q^*}}^2 dr\right)^{\frac{p}{2}} \\
\leq& CE\left(\int_s^t \left\|C_3 + C_4|u_k(r)|(\log_+|u_k(r)|)^{\frac{1}{2}}\right\|_{L^{q^*}}^2 dr\right)^{\frac{p}{2}} \\
\leq& C\left(1 + E\sup_{r\in[0,T\wedge\tau_M^k]}\|u_k(r)\|^2\right)\times|t-s|^{\frac{p}{2}}.
\end{aligned}$$

From the definition of u_k^M, it is evident that

$$\sup_k \sup_{t\in[0,T]}\left\|u_k^M\right\|^2 < \infty, \quad P-a.s., \quad \forall T > 0, \quad \forall M > 0. \tag{4.6}$$

Then, through discussions analogous to those in Lemma 3.4, the conclusion is proven. □

The following lemma asserts the tightness property of the sequence $\{u_k\}$.

Lemma 4.3. *If the assumption (1.6) holds, then* $\{u_k\}$ *exhibits tightness in* $L^p([0,T];H)$ *with* $1 < p < 2$*, and also in* $C([0,T];V^*)$*, for arbitrary* $T > 0$*.*

Proof. Following analogous reasoning to Lemma 3.5, we only need to prove the boundedness of

$$K_L = \left\{u_k \in L^p([0,T];H) : \|u_k\|_{L^p([0,T];V)} + \|u_k\|_{W^{\beta,p}([0,T];V^*)} \le L\right\}.$$

Applying Lemma 4.1, for any positive constant M, we obtain

$$\Psi(M)P(\tau_M^k \le T) \le E\left[\Psi\left(\left\|u_k^M(t)\right\|^2 + \mu\left\|u_k^M(t)\right\|_V^2\right)\right] \le \Psi\left(\|u_0\|^2 + \mu\|u_0\|_V^2\right)e^{CT}.$$

Thus,

$$P(\tau_M^k \le T) \le \frac{\Psi\left(\|u_0\|^2 + \mu\|u_0\|_V^2\right)e^{CT}}{\Psi(M)}. \tag{4.7}$$

For any $\epsilon > 0$, when $\Psi(M)$ approaches ∞, it is possible to select a sufficiently large value of M, s.t.

$$P(\tau_M^k \le T) \le \frac{\epsilon}{2}, \quad k \ge 1.$$

Therefore,

$$\begin{aligned}
P(u_k \notin K_L) \le& P(u_k \notin K_L, \tau_M^k \ge T) + P(\tau_M^k \le T)\\
\le& \sup_k P\left(\|u_k\|_{L^p([0,T];V)} + \|u_k\|_{W^{\beta,p}([0,T];V^*)} > L, \tau_M^k \ge T\right) + \frac{\epsilon}{2}\\
\le& \frac{2^{p-1}}{L^p}\sup_k E\left[\left(\|u_k\|_{L^p([0,T];V)} + \|u_k\|_{W^{\beta,p}([0,T];V^*)}\right)\mathbf{1}_{\{\tau_\mathbf{M}^\mathbf{k} \ge \mathbf{T}\}}\right] + \frac{\epsilon}{2}\\
\le& \frac{2^{p-1}}{L^p}C_M + \frac{\epsilon}{2},
\end{aligned}$$

where C_M is independent of L, k. By selecting a sufficiently large value L, we get

$$P(u_k \notin K_L) \le \epsilon, \quad k \ge 1.$$

Since ϵ can be chosen arbitrarily small, it follows that $\{u_k\}$ is tight in $L^p([0,T];H)$.

Next, to establish the compactness of the sequence $\{u_k\}$ in $C([0,T];V^*)$, we first observe that the embedding $H \hookrightarrow V^*$ is compact. For one thing, for any given $\epsilon > 0$, we can select a constant $M > 0$ that is sufficiently large to ensure that

$$\begin{aligned}
&\lim_{L\to\infty}\sup_k P\left(\sup_{t\in[0,T]}\|u_k(s)\| > L\right)\\
\le& \lim_{L\to\infty}\sup_k P\left(\sup_{t\in[0,T]}\|u_k(s)\| > L, \tau_M^k \ge T\right) + \lim_{L\to\infty}\sup_k P\left(\sup_{t\in[0,T]}\|u_k(s)\| > L, \tau_M^k < T\right)\\
\le& \lim_{L\to\infty}\frac{1}{L^2}\sup_k E\left[\sup_{t\in[0,T\wedge\tau_M^k]}\|u_k(s)\|^2\right] + P\left(\tau_M^k < T\right)\\
\le& \epsilon.
\end{aligned}$$

For another, given any stopping time $\xi_k \in [0,T]$ and for every $\epsilon > 0$, similar to the previous proof, we have

$$\begin{aligned}
&\lim_{\delta\to 0}\sup_k P(\|u_k(\xi_k+\delta)-u_k(\xi_k)\|_{V^*}>\epsilon)\\
\le&\lim_{\delta\to 0}\sup_k P(\|u_k(\xi_k+\delta)-u_k(\xi_k)\|_{V^*}>\epsilon,\tau_M^k\ge T)+\sup_k P(\tau_M^k<T)\\
\le&\frac{1}{\epsilon^p}\lim_{\delta\to 0}\sup_k E\,\|u_k(\xi_k+\delta)-u_k(\xi_k)\|_{V^*}^p+\epsilon\\
\le&\frac{C}{\epsilon^p}\lim_{\delta\to 0}\sup_k\left[\left(1+E\sup_{r\in[0,T\wedge\tau_M^k]}\|u_k\|^2+E\int_0^{T\wedge\tau_M^k}\|u_k\|_V^2\,dr\right)\times\delta^{\frac{p}{2}}\right]+\epsilon\\
\le&\epsilon,
\end{aligned}$$

where $\xi_k+\delta := T\wedge(\xi_k+\delta)$. By Aldous' compactness criterion [22], $\{u_k\}$ is compact in $C([0,T];V^*)$. □

Based on the above lemmas, we can easily obtain the following theorem.

Theorem 4.1. *Assume that hypotheses(1.4) and (1.6) are satisfied. For any initial value $u_0 \in H_0^1(D)$, (1.3) admits a unique strong solution globally in the probabilistic sense.*

Proof. Similar to Theorem 3.2, $\mathcal{L}(u_k, B)$ are tight in Γ. Via Prokhorov's theorem [17], we have weak convergent relationship between $\mathcal{L}(u_k, B)$ and μ^*, a probability measure on Γ. Applying the modified Skorohod theorem (Theorem C.1 in [23]), there exists $(\overline{\overline{\Omega}},\overline{\overline{\mathcal{F}}},\overline{\overline{P}})$ along with a sequence of $\Gamma-$valued random vectors $\left\{(\overline{\overline{u}}_k,\overline{\overline{B}}_k)\right\}$ and (u^*, B^*), s.t. for each $k\in\mathbb{N}$, $\overline{\overline{B}}_k = B^*$, almost surely with respect to the probability measure $\overline{\overline{P}}$, and $\mathcal{L}(\overline{\overline{u}}_k,\overline{\overline{B}}_k)$ equals to $\mathcal{L}(u_k, B)$, and $\mathcal{L}(u^*, B^*)$ equals to μ^*. Furthermore,

$$\left\|\overline{\overline{u}}_k-u^*\right\|_{L^r([0,T];H)}\to 0,\quad \left\|\overline{\overline{u}}_k-u^*\right\|_{C([0,T];V^*)}\to 0.$$

Similar to Theorem 3.2, we can derive that $(\overline{\overline{u}}_k, B^*)$ satisfies

$$\begin{aligned}
\overline{\overline{u}}_k(t)-\mu\Delta\overline{\overline{u}}_k(t)=&P_ku_0-\mu\Delta P_ku_0+\int_0^t(\Delta\overline{\overline{u}}_k(s)+P_k[\overline{\overline{u}}_k(s)\log\left|\overline{\overline{u}}_k(s)\right|])ds\\
&+\int_0^t P_kf(\overline{\overline{u}}_k(s))dB_s^*,\quad t\in[0,T],
\end{aligned}\tag{4.8}$$

in V^* and

$$\sup_k\sup_{t\in[0,T]}\left\|\overline{\overline{u}}_k\right\|_V^2\le M,\quad P-a.s.$$

Consequently, we may extract a subsequence (without relabeling) $\left\{\overline{\overline{u}}_k\right\}$, s.t. as $k\to\infty$,

(1) $\overline{\overline{u}}_k(\overline{\overline{w}},t,x)$ converges to $u^*(\overline{\overline{w}},t,x)$, almost everywhere in $\overline{\overline{\Omega}}\times[0,T]\times D$;

(2) $\overline{\overline{u}}_k$ converges to u^* in $L^r(\overline{\overline{\Omega}};L^r([0,T];H))$, weakly in $L^2(\overline{\overline{\Omega}};L^2([0,T];V))$, and weakly star in $L^\infty([0,T];L^2(\overline{\overline{\Omega}};V))$;

(3) $\Delta\overline{\overline{u}}_k \overset{w}{\longrightarrow} \Delta u^*$ in $L^2(\overline{\overline{\Omega}};L^2([0,T];V^*))$;

(4) $P_k[\overline{\overline{u}}_k\log\left|\overline{\overline{u}}_k\right|]\longrightarrow u^*\log|u^*|$ in $L^r(\overline{\overline{\Omega}};L^r([0,T];V^*))$;

(5) $\int_0^{\cdot} P_k f(\overline{\overline{u}}_k(s))dB_s^* \longrightarrow \int_0^{\cdot} f(u^*(s))dB_s^*$ in $L^\infty([0,T]; L^2(\overline{\overline{\Omega}}; V^*))$.

Taking the limit as $k \to \infty$ in (4.8), we conclude that for all $t \in [0,T]$, u^* satisfies

$$u^*(t) - \mu\Delta u^*(t) = u_0 - \mu\Delta u_0 + \int_0^t (\Delta u^*(s) + u^*(s)\log|u^*(s)|)ds + \int_0^t f(u^*(s))dB_s^*,$$

in V^*, $\overline{\overline{P}}$-a.s. Consequently, a weak solution exists in the probabilistic sense over $[0,T]$. Furthermore, combining the uniqueness property with the Yamada-Watanabe theorem, the existence and uniqueness of the strong solution are established. □

Author contributions

Yang Cao: Drafting the original manuscript, formulating the core research question, designing the research methodology, and participating in the revision and refinement of the manuscript; Chengyuan Qu: Drafting the original manuscript, formulating the core research question, and serving as the primary contact person for the journal throughout the submission and peer-review process; Benhui Wang: Drafting the original manuscript, designing the research methodology, preparing the initial draft of the manuscript, and overseeing the validation of research results; All authors: Approved the final version of the manuscript and agreed to be accountable for all aspects of the work, including the integrity of the research.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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