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Research article

Global existence, general decay, and blow-up of solutions for a fourth-order viscoelastic equation with variable exponents and logarithmic nonlinearities

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Abstract: This paper investigates the existence, energy decay, and finite-time blow-up of solutions for a class of fourth-order viscoelastic evolution equations involving variable exponent nonlinearities, logarithmic terms, and strong damping effects. Such models arise naturally in the mathematical description of heterogeneous viscoelastic media and nonlinear plate equations with memory. By applying the Nehari manifold method and suitable energy estimates, we establish the global existence of weak solutions under appropriate assumptions on the initial data and the variable exponents. Moreover, using a perturbed energy method combined with a carefully constructed Lyapunov functional, we prove that the solutions exhibit general energy decay rates depending on the properties of the relaxation kernel and the spatially variable nonlinearities. Finally, we derive sufficient conditions ensuring finite-time blow-up for solutions with negative initial energy, thereby highlighting the competing effects of strong damping, viscoelastic memory, and logarithmic nonlinearities.

Keywords: fourth-order viscoelastic equation; global existence; general energy decay; nonlinear equations; finite-time blow-up; logarithmic nonlinearity; variable exponent spaces; strong damping
Mathematics Subject Classification: 35A01, 35B35, 35B44, 35L75, 74D10

1. Introduction

This paper is devoted to the qualitative analysis for a class of nonlinear fourth-order viscoelastic evolution equations with variable exponent nonlinearities, logarithmic source terms, and strong damping effects. More precisely, we consider the following initial–boundary value problem:

$$|u_t|^{\rho(x)-2}u_{tt} + \Delta^2 u - \Delta u_{tt} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u \, \ln|\nabla u|) + g * \Delta u - \varepsilon\Delta u_t$$

$$= |u|^{q(x)-2}u\ln|u|, \qquad x\in\Omega,\ t>0, \tag{1.1}$$

$$u(x,t)=\frac{\partial u}{\partial \nu}(x,t)=0, \qquad x\in\partial\Omega,\ t>0, \tag{1.2}$$

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \qquad x\in\Omega. \tag{1.3}$$

Let Ω be a bounded open subset of $\mathbb{R}^n$ $(n\geq 1)$ with a smooth boundary $\partial\Omega$, and let ν denote the unit outward normal vector on $\partial\Omega$. The functions $u_0(x)$ and $u_1(x)$ represent the prescribed initial displacement and velocity, respectively, and $g*\Delta u=\int_0^t g(t-s)\Delta u(s)\,ds$.

Here, ε is a positive constant and the term $-\varepsilon\Delta u_t$ represents a strong (Kelvin–Voigt type) damping mechanism, while the convolution term $g*\Delta u$ accounts for viscoelastic memory effects. Moreover, the presence of the variable exponent $\rho(x)$ in the inertial term allows the model to describe spatially heterogeneous media.

The kernel of the memory function g and the variable exponents are assumed to satisfy the following hypotheses.

$(A1)$ The functions $\rho(\cdot)$, $p(\cdot)$, and $q(\cdot)$ are measurable exponents defined on $\overline{\Omega}$ such that

$$2<\rho_1\leq\rho(x)\leq\rho_2<\begin{cases}\infty, & n=1,2,\\ \dfrac{2n}{n-2}, & n\geq 3,\end{cases}$$

$$2<p_1\leq p(x)\leq p_2<\begin{cases}\infty, & n<5,\\ \dfrac{2(n-2)}{n-4}, & n\geq 5,\end{cases}$$

$$2\leq q_1\leq q(x)\leq q_2<\begin{cases}\infty, & n<5,\\ \dfrac{2(n-2)}{n-4}, & n\geq 5.\end{cases}$$

These conditions ensure the validity of the required Sobolev embeddings in variable exponent spaces. Here,

$$\rho_1:=\operatorname*{ess\,inf}_{x\in\overline{\Omega}}\rho(x), \quad \rho_2:=\operatorname*{ess\,sup}_{x\in\overline{\Omega}}\rho(x),$$
$$p_1:=\operatorname*{ess\,inf}_{x\in\overline{\Omega}}p(x), \quad p_2:=\operatorname*{ess\,sup}_{x\in\overline{\Omega}}p(x),$$
$$q_1:=\operatorname*{ess\,inf}_{x\in\overline{\Omega}}q(x), \quad q_2:=\operatorname*{ess\,sup}_{x\in\overline{\Omega}}q(x).$$

$(A2)$ The relaxation kernel $g:\mathbb{R}^+\to\mathbb{R}^+$ is a bounded and continuously differentiable function satisfying

$$g(0)\geq 0, \qquad g'(t)\leq -g(t), \qquad 1-\gamma\int_0^\infty g(t)\,dt=\ell>0,$$

where $\gamma>0$ denotes the smallest constant such that

$$\|\nabla u\|^2\leq\gamma\|\Delta u\|^2, \qquad \forall u\in H_0^2(\Omega). \tag{1.4}$$

Remark 1.1. The following functions are examples of admissible variable exponent functions satisfying assumptions (A1)–(A2):

$$p(x) = \alpha_1 + \alpha_2 \sin(|x|), \quad q(x) = \beta_0 + \beta_1 |x|^2,$$

for some positive constants $\alpha_1, \alpha_2, \beta_0$, and β_1. Moreover, relaxation kernels of the form

$$g(t) = g_0 e^{-\omega t}, \quad \omega > 0,$$

where g_0 is a non-negative constant.

The main objective of this work is to investigate the global existence, general energy decay, and finite-time blow-up of solutions for a fourth-order viscoelastic equation with variable exponents and logarithmic nonlinearities. This study lies at the intersection of nonlinear partial differential equations (PDEs), viscoelasticity theory, and variable exponent analysis.

Logarithmic nonlinearities play a fundamental role in mathematical physics, as they naturally arise in a wide range of models describing complex phenomena. In inflationary cosmology, logarithmic potentials were introduced and analyzed by Barrow and Parsons [1] to model the early universe's dynamics. In quantum mechanics and wave propagation, logarithmic nonlinearities were first rigorously studied by Bialynicki-Birula and Mycielski [2], who proposed logarithmic wave equations to preserve important physical principles such as separability and the additivity of energy. Further analytical and physical investigations of logarithmic wave and Klein–Gordon equations were carried out by Bartkowski and Górka [3] and by Górka [4], highlighting their relevance in nonlinear optics and quantum field theory. Applications to supersymmetric field theories and related particle physics models were discussed by Enqvist and McDonald [5] and by Hiramatsu *et al.* [6], where logarithmic terms arise naturally in the study of Q-balls and baryogenesis mechanisms. From the mathematical analysis perspective, foundational results on evolution equations with logarithmic nonlinearities were established by Cazenave and Haraux [7], who developed the existence, regularity, and stability results for logarithmic evolution equations. More recently, logarithmic nonlinear effects in plate and viscoelastic-type equations have been rigorously investigated by Al-Gharabli *et al.* [8], where the existence and stability results were obtained for strongly damped plate equations with logarithmic source terms.

From the mathematical point of view, logarithmic nonlinearities introduce serious analytical difficulties because of their non-polynomial growth and lack of homogeneity, which require refined functional settings and delicate variational arguments. The pioneering analytical treatment of logarithmic nonlinear equations was initiated by Bialynicki-Birula and Mycielski [2], where the logarithmic Schrödinger equation was introduced and its fundamental properties were analyzed. Further stability and qualitative results for logarithmic evolution equations were developed by Cazenave and Haraux [7], who provided a rigorous functional framework and decay analysis. They studied the logarithmic wave equation in $\mathbb{R}^3$

$$u_{tt} - \Delta u = u \ln |u|^k, \tag{1.5}$$

and proved the existence and uniqueness of solutions for the associated Cauchy problem. Later, Górka [4] established the global existence of weak solutions to (1.5) using compactness methods. Moreover, Hiramatsu *et al.* [6] introduced the logarithmic wave equation

$$v_{tt} = \Delta v + v \ln |v|^2,$$

to describe the dynamics of Q-balls in theoretical physics. Wu *et al.* [9] considered the following singular parabolic p-Laplacian equation

$$|x|^{-s}u_t - \Delta_p u = |u|^{q-2}u \ln |u|,$$

and proved the local solvability by the technique of cut-off combined with the method of Faedo-Galerkin approximation. By employing the potential well method and Hardy–Sobolev inequality, they established the global existence and decay estimates of solutions. The blow-up phenomenon was analyzed for solutions corresponding to various indicator ranges. In addition, blow-up under arbitrary initial energy levels and the conditions leading to extinction were also investigated.
In another study, Pang *et al.* [10] investigated a class of generalized nonlinear wave equations with doubly dispersive effects on the real line. By employing potential well theory, they classified the initial profiles that lead to either blow-up or the global existence of solutions. In this regard, see also [11, 12].

In the context of viscoelastic and plate-type equations, the presence of logarithmic source terms has been studied more recently. Al-Gharabli *et al.* [13] investigated existence and general decay for a viscoelastic plate equation with logarithmic nonlinearity, highlighting the impact of logarithmic terms on long-time behavior. Kakumani and Yadav [14] extended this analysis by proving the global existence and asymptotic behavior for viscoelastic plate equations with nonlinear damping and logarithmic sources. Blow-up and instability phenomena induced by logarithmic nonlinearities were addressed by Ferreira *et al.* [15], where finite-time blow-up results were established for Petrovsky-type equations. For more information, see also [16–18].

The presence of variable exponents reflects the nonhomogeneous and anisotropic nature of many modern materials, including electrorheological fluids and composite structures. The functional framework of Lebesgue and Sobolev spaces with variable exponents has been rigorously developed over the past few decades. The monograph by Diening *et al.* [19] provides a comprehensive and systematic treatment of variable-exponent function spaces, including density results, embedding theorems, and modular inequalities. Further analytical developments concerning evolution equations with nonstandard growth conditions were established by Antontsev and Shmarev [20], where the existence, uniqueness, localization, and blow-up phenomena for PDEs with variable exponents are thoroughly analyzed. In the context of viscoelastic and fourth-order equations, Shahrouzi [21] investigated the behavior of solutions for nonlinear viscoelastic equations involving variable-exponent nonlinearities, providing important tools that motivated the present analysis.

Viscoelastic effects with memory terms play a crucial role in the mathematical modeling of realistic materials, as they account for hereditary and relaxation phenomena. Early investigations on the decay and stability of viscoelastic equations were carried out by Ferreira and Messaoudi [22], where general decay results were established for nonlinear viscoelastic plate equations with strong damping. Subsequently, the study of decay properties was extended to more general settings involving variable source terms by Messaoudi *et al.* [23].

To establish global existence and stability, we rely on the potential well method and Nehari manifold techniques, which were originally introduced in [24, 25] and later adapted to viscoelastic and variable exponent problems in [22].

In addition, the blow-up analysis with negative initial energy is inspired by classical works on nonlinear instability and finite-time blow-up for evolution equations. Foundational analytical approaches to blow-up phenomena and instability mechanisms are presented in [26], which constitute the theoretical basis for many subsequent developments in nonlinear PDEs. In the framework of viscoelastic and plate-type models, blow-up results under negative initial energy have been rigorously established by Antontsev and Ferreira [27], where sufficient conditions for finite-time blow-up were derived for nonlinear viscoelastic plate equations. Related blow-up phenomena for viscoelastic wave equations involving variable exponents were investigated by Park and Kang [28], emphasizing the influence of nonstandard growth on the solutions' instability. More recent studies incorporating logarithmic nonlinearities and strong damping effects include the work of Pereira *et al.* [29], where blow-up criteria were obtained for viscoelastic beam equations with logarithmic source terms. Furthermore, Talahmeh *et al.* [30] analyzed finite-time blow-up in nonlinear viscoelastic problems with variable exponents and arbitrary initial energy, providing additional insight into the combined effects of memory, damping, and nonlinear sources.

Memory damping arises naturally in the mathematical modeling of viscoelastic materials, which possess the intrinsic ability to store and dissipate energy through their past deformation history. This memory effect plays a crucial role in the stabilization of solutions and has attracted considerable attention in recent years.

Al-Gharabli *et al.* [13] studied the following viscoelastic plate equation with logarithmic nonlinearity:

$$u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(s)\,ds = ku\ln|u|.$$

By using the Galerkin method together with suitable multiplier techniques, they established the global existence of solutions and derived explicit general decay rate results.

Ferreira *et al.* [15] investigated the following viscoelastic Petrovsky-type equation with logarithmic nonlinearity of the form

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds + |u_t|^{m-2}u_t = |u|^{p-2}u\ln|u|,$$

and proved finite-time blow-up results by combining the perturbation energy method, the concavity method, and differential–integral inequality techniques.

Kakumani and Yadav [14] investigated a viscoelastic plate equation with logarithmic nonlinearity and nonlinear frictional damping:

$$|u_t|^\rho u_{tt} + \Delta^2 u + \Delta^2 u_{tt} + u - \int_0^t b(t-s)\Delta^2 u(s)\,ds + h(u_t) = ku\ln|u|.$$

Using the Galerkin approximation method, they proved the existence of solutions and obtained partial general decay results under additional assumptions on the damping term.

Despite these significant contributions, to the best of our knowledge, there are very few results addressing fourth-order viscoelastic equations with the *simultaneous* presence of variable-exponent nonlinearities, logarithmic source terms, strong damping, and memory effects. This observation strongly motivates the present study.

On a different note, it is well known that several physical and engineering models—such as those arising in electrorheological fluid flows, nonlinear viscoelasticity, and image processing—lead naturally to PDEs with nonstandard growth conditions, particularly those involving variable exponent nonlinearities. These models reflect spatial heterogeneity and adaptive material responses, and have therefore attracted significant mathematical interest.

In this direction, Antontsev and Ferreira [31] studied a class of nonlinear viscoelastic equations with lower-order perturbations of the $\overrightarrow{p}(x,t)$-Laplacian type and a memory term, given by

$$u_{tt} + \Delta^2 u - \Delta_{\overrightarrow{p}(x,t)} u + \int_0^t g(t-s)\Delta u(s)\,ds - \varepsilon\Delta u_t + f(u) = 0. \tag{1.6}$$

By incorporating a strong damping term of the form $-\varepsilon\Delta u_t$, and assuming that the memory kernel decays exponentially while the nonlinear term $f(u)$ acts as a lower-order perturbation, they established the local and global existence as well as the uniqueness of weak solutions.

Subsequently, Ferreira and Messaoudi [22] revisited Eq (1.6) and derived a general decay result under suitable assumptions on the memory function g, the nonlinearity f, and the variable exponent associated with the $\overrightarrow{p}(x,t)$-Laplacian operator. Moreover, Antontsev and Ferreira [27] proved the finite-time blow-up of solutions to (1.6) in the case of negative initial energy.

These studies highlight the rich and delicate interplay among memory effects, variable exponent nonlinearities, and higher-order operators, and also emphasize the mathematical challenges associated with such models. For further studies related to PDEs with nonstandard growth conditions, including variable-exponent and logarithmic nonlinearities, we refer the reader to [32, 33].

Motivated by the aforementioned works, the main objective of this paper is to investigate the global existence, general energy decay, and finite-time blow-up of solutions to problem (1.1)–(1.3). To the best of our knowledge, there are very few results in the literature addressing fourth-order viscoelastic equations with the simultaneous presence of variable exponent nonlinearities, logarithmic terms, strong damping, and memory effects. The present work aims to fill this gap and extend several known results in the literature.

More precisely, the main contributions of this paper can be summarized as follows. We investigate a nonlinear fourth-order viscoelastic evolution equation of the form

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \nabla\cdot\left(|\nabla u|^{p(x)-2}\nabla u \ln|\nabla u|\right) + \int_0^t g(t-s)\Delta u(s)\,ds + \varepsilon\Delta u_t = |u|^{q(x)-2}u\ln|u|,$$

where the nonlinearities involve variable exponents and logarithmic terms. A distinctive feature of our analysis is that the logarithmic nonlinearities may change their sign, depending on whether $|u|$ and $|\nabla u|$ are smaller or larger than one, which naturally leads to a decomposition of the spatial domain Ω into several regions. By carefully analyzing each region separately, we capture the precise influence of the sign-changing logarithmic terms on the qualitative behavior of the solutions. Furthermore, we establish a general decay result for global weak solutions by constructing a suitable Lyapunov functional based on a perturbed energy method. The obtained decay estimate is of the form

$$E(t) \le K\exp\left(-k\int_0^t \xi(s)\,ds\right), \qquad t \ge 0,$$

where the decay rate explicitly depends on the variable exponents $p(x)$, $\rho(x)$, and $q(x)$, as well as on the memory kernel g through the function ξ. This result generalizes several known decay results by allowing nonstandard growth conditions and logarithmic damping. In addition to the stability and decay properties, we prove the finite-time blow-up of solutions with negative initial energy. More precisely, under suitable assumptions on the variable exponents and for a sufficiently small strong damping parameter ε, we show that if $E(0) < 0$, then the corresponding solution cannot exist globally in time. The blow-up analysis combines the potential well method with a carefully designed auxiliary functional and reveals how the interaction among variable-exponent logarithmic damping, strong damping, and viscoelastic memory effects leads to finite-time instability.

The remainder of the paper is organized as follows. In Section 2, we recall some preliminaries on variable exponent Sobolev spaces and collect several auxiliary lemmas needed in the analysis. In Section 3, we establish the global existence of the solutions. Section 4 is devoted to the proof of the general energy decay result. Finally, in Section 5, we investigate the finite-time blow-up of solutions with negative initial energy.

2. Preliminaries

In this section, we recall some basic definitions, notations, and auxiliary results concerning variable-exponent Lebesgue and Sobolev spaces that will be used throughout the paper. For further details, we refer the reader to [34, 35].

Let Ω be a bounded domain in $\mathbb{R}^n$, and let $p : \Omega \to [1, \infty)$ be a measurable function. We introduce the standard notations

$$p_1 := \operatorname*{ess\,inf}_{x \in \bar{\Omega}} p(x), \qquad p_2 := \operatorname*{ess\,sup}_{x \in \bar{\Omega}} p(x).$$

Throughout the paper, we assume that

$$2 < p_1 \le p(x) \le p_2 < \infty \quad \text{for a.e. } x \in \Omega.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |\lambda u(x)|^{p(x)}\, dx < \infty \text{ for some } \lambda > 0\Big\}.$$

The space $L^{p(\cdot)}(\Omega)$ is endowed with the Luxemburg norm

$$\|u\|_{p(x)} := \inf\Big\{\lambda > 0 \ : \ \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} dx \le 1\Big\}.$$

Lemma 2.1. *[19] Let Ω be a bounded domain in $\mathbb{R}^n$. We then have the following.*

(i) $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable and reflexive Banach space, and its dual space is $L^{q(\cdot)}(\Omega)$, where

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \quad \textit{a.e. in } \Omega.$$

(ii) For any $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, the generalized Hölder inequality holds:

$$\Big|\int_\Omega f(x)g(x)\, dx\Big| \le \Big(\frac{1}{p_1} + \frac{1}{q_1}\Big)\|f\|_{p(\cdot)}\|g\|_{q(\cdot)} \le 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

The relationship between the modular and the Luxemburg norm is given by

$$\min\left(\|f\|_{p(\cdot)}^{p_1}, \|f\|_{p(\cdot)}^{p_2}\right) \leq \int_\Omega |f(x)|^{p(x)}\,dx \leq \max\left(\|f\|_{p(\cdot)}^{p_1}, \|f\|_{p(\cdot)}^{p_2}\right).$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists in the weak sense and } |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

Equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},$$

$W^{1,p(\cdot)}(\Omega)$ is a Banach space.

Let $W_0^{1,p(\cdot)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, the seminorm

$$\|u\|_{1,p(\cdot)} := \|\nabla u\|_{p(\cdot)}$$

is equivalent to the full norm of $W^{1,p(\cdot)}(\Omega)$.

Throughout this paper, we assume that the exponent function $p(\cdot)$ satisfies the following log-Hölder continuity condition:

$$|p(x) - p(y)| \leq \frac{A}{-\log|x-y|} \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < \alpha,$$

where $A > 0$ and $0 < \alpha < 1$.

Lemma 2.2 (Sobolev–Poincaré inequality)**.** *[20, 34] Let Ω be a bounded domain in $\mathbb{R}^n$ and assume that the variable exponent $p(\cdot)$ satisfies the log-Hölder continuity condition. Then the continuous embeddings $H_0^2(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow W_0^{1,p(\cdot)}(\Omega)$ hold true, and there are positive constants C_p, $C_{*,p}$ (depending only on the variable exponent bounds and the measure of Ω) such that*

$$\begin{aligned} \|u\|_{p(\cdot)} &\leq C_p \|\Delta u\|_{p(\cdot)}, \\ \|\nabla u\|_{p(\cdot)} &\leq C_{*,p} \|\Delta u\|_{p(\cdot)}, \end{aligned} \tag{2.1}$$

Moreover, in the particular case $p(\cdot) \equiv 2$, we recover

$$C_{*,p} = \sqrt{\gamma},$$

where γ is the embedding constant defined in (1.4)*.*

For the sake of completeness, we recall the local existence result associated with the problem (1.1)–(1.3). Its proof follows from the Faedo–Galerkin approximation method combined with compactness arguments and standard monotonicity techniques, as in [29].

Theorem 2.3 (Local existence)**.** *Let*

$$(u_0, u_1) \in (H_0^2(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)) \times (H_0^1(\Omega) \cap L^{\rho(\cdot)}(\Omega)).$$

Assuming that hypotheses $(A1)$ *and* $(A2)$ *hold, it follows that the problem* (1.1)–(1.3) *admits at least one weak solution $u = u(x,t)$ satisfying*

$$\begin{aligned} &u \in L^\infty((0,T); H_0^2(\Omega)) \cap L^\infty((0,T); W_0^{1,p(\cdot)}(\Omega)), \\ &u_t \in L^\infty((0,T); H_0^1(\Omega)) \cap L^{\rho(\cdot)}(\Omega \times (0,T)). \end{aligned}$$

Moreover, the solution u can be extended to a maximal interval of existence $[0, T_{\max})$, where $T_{\max}$ is the maximal existence time. If $T_{\max} = +\infty$, then the solution is said to be global, and the solution blows up in finite time, when $T_{\max} < +\infty$.

We now introduce the total energy functional associated with the problem (1.1)–(1.3). Define

$$\begin{aligned} E(t) = &\int_\Omega \frac{1}{\rho(x)}|u_t|^{\rho(x)}dx + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} \ln|\nabla u| dx \\ &+ \frac{1}{2}(g \diamond \nabla u)(t) + \int_\Omega \frac{1}{q^2(x)}|u|^{q(x)}dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} \ln|u| dx \\ &- \int_\Omega \frac{1}{p^2(x)}|\nabla u|^{p(x)}dx - \frac{1}{2}\Big(\int_0^t g(s)\, ds\Big)\|\nabla u\|^2, \end{aligned} \tag{2.2}$$

where

$$(g \diamond \nabla u)(t) := \int_0^t g(t-\tau)\|\nabla u(t) - \nabla u(\tau)\|^2 d\tau.$$

Lemma 2.4 (Monotonicity of the energy)**.** *Let $u(x,t)$ be a local weak solution of the problem* (1.1)–(1.3). *Then the energy functional $E(t)$ defined in* (2.2) *is nonincreasing along the trajectories of the solution, and satisfies*

$$E'(t) = -\varepsilon\|\nabla u_t\|^2 + \frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2 \leq 0. \tag{2.3}$$

Proof. By multiplying Eq (1.1) by u_t and integrating over the domain Ω, and utilizing the assumptions on the variable exponents along with the properties of the memory kernel, the required result can be derived for any weak solution. □

3. Global existence

In this section, we establish the global existence of weak solutions to the problem (1.1)–(1.3). The analysis is based on the potential well method and the construction of suitable Nehari-type functionals. More precisely, we prove Theorem 3.2.

To this end, we introduce the following auxiliary functionals, which play a fundamental role in distinguishing the stable and unstable sets. Define

$$\begin{aligned} I(t) = &\|\Delta u\|^2 + \int_\Omega |\nabla u|^{p(x)} \ln|\nabla u|\, dx + (g \diamond \nabla u)(t) - \int_\Omega |\nabla u|^{p(x)}\, dx \\ &- \Big(\int_0^t g(s)\, ds\Big)\|\nabla u\|^2 - \int_\Omega |u|^{q(x)} \ln|u|\, dx, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} J(t) = &\frac{1}{2}\|\Delta u\|^2 + \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} \ln|\nabla u|\, dx + \frac{1}{2}(g \diamond \nabla u)(t) \\ &+ \int_\Omega \frac{1}{q^2(x)}|u|^{q(x)}\, dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} \ln|u|\, dx \\ &- \int_\Omega \frac{1}{p^2(x)}|\nabla u|^{p(x)}\, dx - \frac{1}{2}\Big(\int_0^t g(s)\, ds\Big)\|\nabla u\|^2. \end{aligned} \tag{3.2}$$

By combining the definition of the total energy $E(t)$ given in (2.2) with (3.2), we obtain the decomposition

$$E(t) = \int_\Omega \frac{1}{\rho(x)}|u_t|^{\rho(x)}\,dx + \frac{1}{2}\|\nabla u_t\|^2 + J(t).$$

Using the monotonicity of the energy functional established in Lemma 2.4, together with Assumption $(A1)$, we infer that

$$E(0) \geq E(t) \geq \frac{1}{\rho_2}\int_\Omega |u_t|^{\rho(x)}\,dx + \frac{1}{2}\|\nabla u_t\|^2 + J(t), \qquad \forall\, t \in (0, T_{\max}). \tag{3.3}$$

Lemma 3.1. *Assume that the hypotheses of Theorem 2.3 are satisfied. Let the variable exponents fulfill*

$$p_1 < p_2 < q_1 < q_2 < p_1^2.$$

Moreover, suppose further that $I(0) > 0$ *and*

$$\tilde{\eta}_0 + \tilde{\eta}_1 + \hat{\eta}_2 < \ell,$$

where $\tilde{\eta}_0 := \max\{\hat{\eta}_0, \eta_0\}$ *and* $\tilde{\eta}_1 := \max\{\hat{\eta}_0, \hat{\eta}_1, \eta_1\}$ *such that the constants below depend only on the initial energy* $E(0)$*, the embedding constants, and the variable exponents:*

$$\begin{aligned}
\hat{\eta}_0 &:= \frac{\gamma}{e(p_1-2)}, \quad \eta_1 := \frac{C_2^2}{e(q_1-2)},\\
\eta_0 &:= \frac{1}{2e}\max\Big\{C_{*,p+2}^{p_1+2}\Big(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell}\Big)^{\frac{p_1}{2}},\ C_{*,p+2}^{p_2+2}\Big(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell}\Big)^{\frac{p_2}{2}}\Big\},\\
\hat{\eta}_1 &:= \frac{1}{2e}\max\Big\{C_{q+2}^{q_1+2}\Big(\frac{2p_1(E(0)+\delta)}{(p_1-2)\ell}\Big)^{\frac{q_1}{2}},\ C_{q+2}^{q_2+2}\Big(\frac{2p_1(E(0)+\delta)}{(p_1-2)\ell}\Big)^{\frac{q_2}{2}}\Big\},\\
\hat{\eta}_2 &:= \max\Big\{C_{*,p}^{p_1}\Big(\frac{2p_1(E(0)+\delta)}{(p_1-2)\ell}\Big)^{\frac{p_1-2}{2}},\ C_{*,p}^{p_2}\Big(\frac{2p_1(E(0)+\delta)}{(p_1-2)\ell}\Big)^{\frac{p_2-2}{2}}\Big\},
\end{aligned} \tag{3.4}$$

where $\gamma > 0$ *is the embedding constant defined in* (1.4)*;* $C_2, C_{*,p}, C_{*,p+2}$*, and* $C_{q+2} > 0$ *are Sobolev embedding constants defined in* (2.1)*; and* $\delta > 0$ *is a sufficiently small constant. Then, the Nehari functional remains positive along the trajectory, namely*

$$I(t) > 0, \qquad \forall\, t \in [0, T_{\max}).$$

Proof. Let u be the local weak solution of the problem (1.1)–(1.3) defined on the maximal interval $(0, T_{\max})$. Since $I(t)$ is continuous with respect to t and $I(0) > 0$, a time $t_0 \in (0, T_{\max})$ exists such that $I(t) \geq 0$, for all $t \in (0, t_0]$. Arguing by contradiction, suppose that $I(t)$ does not remain non-negative on $(0, T_{\max})$. Then for any sufficiently small $\delta > 0$, $t_\delta > t_0$ exists satisfying

$$I(t_\delta) = -\delta p_1 < 0. \tag{3.5}$$

In order to prove this result, we decompose the domain Ω according to the magnitude of $|u|$ and $|\nabla u|$ at time t_δ. Define

$$\Omega_1 := \{x \in \Omega : \ |u(t_\delta)| \geq 1 \text{ and } |\nabla u(t_\delta)| \geq 1\},$$

$$\Omega_2 := \{x \in \Omega : \; |u(t_\delta)| > 1 \text{ and } 0 < |\nabla u(t_\delta)| < 1\},$$
$$\Omega_3 := \{x \in \Omega : \; 0 < |u(t_\delta)| < 1 \text{ and } |\nabla u(t_\delta)| > 1\},$$
$$\Omega_4 := \{x \in \Omega : \; 0 < |u(t_\delta)| < 1 \text{ and } 0 < |\nabla u(t_\delta)| < 1\}.$$

In what follows, on the basis of the definitions above, we show that $I(t_\delta) > 0$ on each region Ω_i, $i = 1, \ldots, 4$. We begin with the case $x \in \Omega_1$.

Case (i): $x \in \Omega_1$. Since $|u(t_\delta)| \geq 1$ and $|\nabla u(t_\delta)| \geq 1$, and using Assumption $(A1)$, we have

$$\int_{\Omega_1} \frac{1}{q(x)} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| \, dx \leq \frac{1}{q_1} \int_{\Omega_1} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| \, dx,$$
$$\int_{\Omega_1} \frac{1}{p(x)} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| \, dx \geq \frac{1}{p_2} \int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| \, dx.$$

Substituting these estimates into the definition of $J(t)$ given in (3.2) and again using $(A1)$, we obtain

$$\begin{aligned} J(t_\delta) \geq & \frac{1}{2} \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx + \frac{1}{p_2} \int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx + \frac{1}{2} (g \diamond \nabla u)(t_\delta) \\ & + \frac{1}{q_2^2} \int_{\Omega_1} |u(t_\delta)|^{q(x)} dx - \frac{1}{2} \Big(\int_0^{t_\delta} g(s) ds \Big) \int_{\Omega_1} |\nabla u(t_\delta)|^2 dx \\ & - \frac{1}{p_1^2} \int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} dx - \frac{1}{q_1} \int_{\Omega_1} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx. \end{aligned} \tag{3.6}$$

Using the exponent condition $p_1 < p_2 < q_1 < q_2 < p_1^2$ and the definition of the Nehari functional $I(t)$ in (3.1), we rewrite (3.6) as

$$\begin{aligned} J(t_\delta) \geq & \frac{1}{2} \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx + \frac{1}{2} (g \diamond \nabla u)(t_\delta) - \frac{1}{2} \Big(\int_0^{t_\delta} g(s) ds \Big) \int_{\Omega_1} |\nabla u(t_\delta)|^2 dx \\ & - \frac{1}{p_1^2} \int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} dx + \frac{1}{q_2^2} \int_{\Omega_1} |u(t_\delta)|^{q(x)} dx \\ & + \frac{1}{q_1} \Bigg(\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx - \int_{\Omega_1} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \Bigg). \end{aligned} \tag{3.7}$$

Recalling the definition of $I(t)$, we also obtain

$$\begin{aligned} & \frac{1}{q_1} \Bigg(\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx - \int_{\Omega_1} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \Bigg) \\ = & \frac{1}{q_1} I(t_\delta) - \frac{1}{q_1} \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx - \frac{1}{q_1} (g \diamond \nabla u)(t_\delta) \\ & + \frac{1}{q_1} \Big(\int_0^{t_\delta} g(s) ds \Big) \int_{\Omega_1} |\nabla u(t_\delta)|^2 dx + \frac{1}{q_1} \int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} dx. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we deduce

$$J(t_\delta) \geq \frac{q_1 - 2}{2q_1} \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx + \frac{q_1 - 2}{2q_1} (g \diamond \nabla u)(t_\delta)$$

$$
\begin{aligned}
&- \frac{q_1-2}{2q_1}\left(\int_0^{t_\delta} g(s)ds\right)\int_{\Omega_1} |\nabla u(t_\delta)|^2 dx + \frac{1}{q_2^2}\int_{\Omega_1} |u(t_\delta)|^{q(x)}dx \\
&+ \frac{p_1^2-q_1}{q_1 p_1^2}\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)}dx + \frac{1}{q_1}I(t_\delta) \\
\geq& \frac{q_1-2}{2q_1}\left(1-\gamma\int_0^{t_\delta} g(s)ds\right)\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx + \frac{q_1-2}{2q_1}(g\diamond\nabla u)(t_\delta) \\
&+ \frac{1}{q_2^2}\int_{\Omega_1} |u(t_\delta)|^{q(x)}dx + \frac{p_1^2-q_1}{q_1 p_1^2}\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)}dx + \frac{1}{q_1}I(t_\delta),
\end{aligned}
$$

where (1.4) has been used.
Using (3.5) and Assumption (A2), we obtain

$$
\begin{aligned}
J(t_\delta) \geq& \frac{(q_1-2)\ell}{2q_1}\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx + \frac{q_1-2}{2q_1}(g\diamond\nabla u)(t_\delta) + \frac{1}{q_2^2}\int_{\Omega_1} |u(t_\delta)|^{q(x)}dx \\
&+ \frac{p_1^2-q_1}{q_1 p_1^2}\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)}dx - \frac{p_1\delta}{q_1} \\
\geq& \frac{(q_1-2)\ell}{2q_1}\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx - \delta,
\end{aligned}
$$

where we used the assumption of Lemma 3.1 on the exponents, i.e., $\frac{p_1}{q_1} < 1$.
Since, by virtue of (3.3), we have $J(t_\delta) \leq E(t_\delta) \leq E(0)$ and we deduce

$$
\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \leq \frac{2q_1}{(q_1-2)\ell}(E(0)+\delta). \tag{3.9}
$$

At this point, using (3.9), we estimate the logarithmic terms. Since $\phi^{-2}\ln\phi \leq \frac{1}{2e}$ for any $\phi \geq 1$, we have

$$
\begin{aligned}
&\Big|\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} \ln|\nabla u(t_\delta)|dx\Big| \\
\leq& \frac{1}{2e}\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)+2}dx \\
\leq& \frac{1}{2e}\max\{\left(\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)+2}dx\right)^{p_1+2}, \left(\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)+2}dx\right)^{p_2+2}\} \\
\leq& \frac{1}{2e}\max\{C_{*,p+2}^{p_1+2}\left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx\right)^{\frac{p_1+2}{2}}, C_{*,p+2}^{p_2+2}\left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx\right)^{\frac{p_2+2}{2}}\} \\
=& \frac{1}{2e}\max\{C_{*,p+2}^{p_1+2}\left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx\right)^{\frac{p_1}{2}}, C_{*,p+2}^{p_2+2}\left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx\right)^{\frac{p_2}{2}}\}\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \\
\leq& \frac{1}{2e}\max\{C_{*,p+2}^{p_1+2}\left(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell}\right)^{\frac{p_1}{2}}, C_{*,p+2}^{p_2+2}\left(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell}\right)^{\frac{p_2}{2}}\}\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \\
=& \eta_0\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx,
\end{aligned} \tag{3.10}
$$

where η_0 satisfies (3.4) and $C_{*,p+2}$ denotes the embedding constant of $H_0^2(\Omega_1) \hookrightarrow W_0^{1,p(x)+2}(\Omega_1)$. Similarly, since $p_1 < q_1$, we obtain

$$\begin{aligned}
&\int_{\Omega_1} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \\
\leq & \frac{1}{2e} \int_{\Omega_1} |u(t_\delta)|^{q(x)+2} dx \\
\leq & \frac{1}{2e} \max\{ \left(\int_{\Omega_1} |u(t_\delta)|^{q(x)+2} dx \right)^{q_1+2}, \left(\int_{\Omega_1} |u(t_\delta)|^{q(x)+2} dx \right)^{q_2+2} \} \\
\leq & \frac{1}{2e} \max\{ C_{q+2}^{q_1+2} \left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \right)^{\frac{q_1+2}{2}}, C_{q+2}^{q_2+2} \left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \right)^{\frac{q_2+2}{2}} \} \\
\leq & \frac{1}{2e} \max\{ C_{q+2}^{q_1+2} \left(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell} \right)^{\frac{q_1}{2}}, C_{q+2}^{q_2+2} \left(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell} \right)^{\frac{q_2}{2}} \} \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \\
< & \hat{\eta}_1 \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx,
\end{aligned} \tag{3.11}$$

where $\hat{\eta}_1$ satisfies (3.4) and C_{q+2} is the embedding constant of $H_0^2(\Omega_1) \hookrightarrow L^{q(x)+2}(\Omega_1)$.
Moreover, by using the embedding $H_0^2(\Omega_1) \hookrightarrow W_0^{1,p(x)}(\Omega_1)$ with the embedding constant $C_{*,p}$ and since $p_1 < q_1$, we obtain

$$\begin{aligned}
\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} dx \leq & \max\left\{ \left(\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} dx \right)^{p_1}, \left(\int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} dx \right)^{p_2} \right\} \\
\leq & \max\left\{ C_{*,p}^{p_1} \left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \right)^{\frac{p_1}{2}}, C_{*,p}^{p_2} \left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \right)^{\frac{p_2}{2}} \right\} \\
= & \max\left\{ C_{*,p}^{p_1} \left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \right)^{\frac{p_1-2}{2}}, C_{*,p}^{p_2} \left(\int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \right)^{\frac{p_2-2}{2}} \right\} \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \\
\leq & \max\left\{ C_{*,p}^{p_1} \left(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell} \right)^{\frac{p_1-2}{2}}, C_{*,p}^{p_2} \left(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell} \right)^{\frac{p_2-2}{2}} \right\} \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx \\
< & \hat{\eta}_2 \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx,
\end{aligned} \tag{3.12}$$

where $\hat{\eta}_2$ satisfies (3.4).
Now, from the definition of the Nehari functional (3.1), Assumption (A2), and the estimates (3.10)–(3.12), we deduce

$$\begin{aligned}
I(t_\delta) = & \int_{\Omega_1} |\Delta u(t_\delta)|^2 dx + \int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx + (g \diamond \nabla u)(t_\delta) \\
& - \int_{\Omega_1} |\nabla u(t_\delta)|^{p(x)} dx - \left(\int_0^{t_\delta} g(s) ds \right) \int_{\Omega_1} |\nabla u(t_\delta)|^2 dx \\
& - \int_{\Omega_1} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx
\end{aligned}$$

$$
\begin{aligned}
&\geq\left(1-\gamma\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_1}|\Delta u(t_\delta)|^2dx+\int_{\Omega_1}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx\\
&\quad-\int_{\Omega_1}|\nabla u(t_\delta)|^{p(x)}dx-\int_{\Omega_1}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\\
&\geq\ell\int_{\Omega_1}|\Delta u(t_\delta)|^2dx+\int_{\Omega_1}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx-\int_{\Omega_1}|\nabla u(t_\delta)|^{p(x)}dx\\
&\quad-\int_{\Omega_1}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\\
&\geq(\ell-\eta_0-\hat{\eta}_1-\hat{\eta}_2)\int_{\Omega_1}|\Delta u(t_\delta)|^2dx.
\end{aligned}
$$

Thanks to the hypothesis of Lemma 3.1, namely $\eta_0+\hat{\eta}_1+\hat{\eta}_2<\ell$, we conclude that

$$
I(t_\delta)>0\quad\text{for all } x\in\Omega_1.
$$

(ii) Case $x\in\Omega_2$. By applying Assumption (A1), we observe that

$$
\begin{aligned}
&\int_{\Omega_2}\frac{1}{q(x)}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\leq\frac{1}{q_1}\int_{\Omega_2}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx,\\
&\int_{\Omega_2}\frac{1}{p(x)}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx\geq\frac{1}{p_1}\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx.
\end{aligned}
$$

Thus, by using the potential functional $J(t)$ defined in (3.2), Assumption (A1), and the estimates above, we obtain

$$
\begin{aligned}
J(t_\delta)\geq&\frac{1}{2}\int_{\Omega_2}|\Delta u(t_\delta)|^2dx+\frac{1}{p_1}\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx+\frac{1}{2}(g\diamond\nabla u)(t_\delta)\\
&+\frac{1}{q_2^2}\int_{\Omega_2}|u(t_\delta)|^{q(x)}dx-\frac{1}{2}\left(\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_2}|\nabla u(t_\delta)|^2dx\\
&-\frac{1}{p_1^2}\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}dx-\frac{1}{q_1}\int_{\Omega_2}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx.
\end{aligned}\tag{3.13}
$$

Thanks to the assumption $p_1<p_2<q_1<p_1^2$, and recalling the definition of the Nehari functional (3.1), we deduce from (3.13) that

$$
\begin{aligned}
J(t_\delta)\geq&\frac{1}{2}\int_{\Omega_2}|\Delta u(t_\delta)|^2dx+\frac{1}{2}(g\diamond\nabla u)(t_\delta)-\frac{1}{2}\left(\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_2}|\nabla u(t_\delta)|^2dx\\
&-\frac{1}{p_1^2}\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}dx+\frac{1}{q_2^2}\int_{\Omega_2}|u(t_\delta)|^{q(x)}dx\\
&+\frac{1}{p_1}\left(\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx-\int_{\Omega_2}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\right).
\end{aligned}\tag{3.14}
$$

By using the definition of the Nehari functional $I(t)$, we have

$$
\frac{1}{p_1}\left(\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx-\int_{\Omega_2}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\right)
$$

$$= \frac{1}{p_1}I(t_\delta) - \frac{1}{p_1}\int_{\Omega_2}|\Delta u(t_\delta)|^2dx - \frac{1}{p_1}(g\diamond\nabla u)(t_\delta)$$
$$+ \frac{1}{p_1}\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}dx + \frac{1}{p_1}\left(\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_2}|\nabla u(t_\delta)|^2dx. \qquad (3.15)$$

Applying (3.15) into (3.14), we deduce

$$\begin{aligned}
J(t_\delta) \geq & \frac{p_1-2}{2p_1}\int_{\Omega_2}|\Delta u(t_\delta)|^2dx + \frac{p_1-2}{2p_1}(g\diamond\nabla u)(t_\delta)\\
& - \frac{p_1-2}{2p_1}\left(\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_2}|\nabla u(t_\delta)|^2dx + \frac{1}{q_2^2}\int_{\Omega_2}|u(t_\delta)|^{q(x)}dx\\
& + \frac{p_1-1}{p_1^2}\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}dx + \frac{1}{p_1}I(t_\delta)\\
\geq & \frac{p_1-2}{2p_1}\left(1-\gamma\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_2}|\Delta u(t_\delta)|^2dx + \frac{p_1-2}{2p_1}(g\diamond\nabla u)(t_\delta)\\
& + \frac{1}{q_2^2}\int_{\Omega_2}|u(t_\delta)|^{q(x)}dx + \frac{p_1-1}{p_1^2}\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}dx + \frac{1}{p_1}I(t_\delta),
\end{aligned}$$

where Inequality (1.4) has been used.
Applying (3.5) and (A2), we obtain

$$\begin{aligned}
J(t_\delta) \geq & \frac{(p_1-2)\ell}{2p_1}\int_{\Omega_2}|\Delta u(t_\delta)|^2dx + \frac{p_1-2}{2p_1}(g\diamond\nabla u)(t_\delta) + \frac{1}{q_2^2}\int_{\Omega_2}|u(t_\delta)|^{q(x)}dx\\
& + \frac{p_1-1}{p_1^2}\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}dx - \delta\\
\geq & \frac{(p_1-2)\ell}{2p_1}\int_{\Omega_2}|\Delta u(t_\delta)|^2dx - \delta.
\end{aligned}$$

Since by virtue of (3.3), we have $J(t_\delta) \leq E(t_\delta) \leq E(0)$, thus, by using the fact that $p_1 > 2$, we deduce

$$\int_{\Omega_2}|\Delta u(t_\delta)|^2dx \leq \frac{2p_1}{(p_1-2)\ell}(E(0)+\delta). \qquad (3.16)$$

At this point, by using (3.16), we estimate the following terms. From the fact that $\phi^{p(x)-2}\ln\phi \leq \frac{1}{e(p(x)-2)}$ for any $0 < \phi < 1$, we obtain

$$\begin{aligned}
\left|\int_{\Omega_2}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx\right| \leq & \int_{\Omega_2}\left||\nabla u(t_\delta)|^2|\nabla u(t_\delta)|^{p(x)-2}\ln|\nabla u(t_\delta)|\right|dx\\
\leq & \int_{\Omega_2}\frac{1}{e(p(x)-2)}|\nabla u(t_\delta)|^2dx\\
\leq & \frac{1}{e(p_1-2)}\int_{\Omega_2}|\nabla u(t_\delta)|^2dx\\
\leq & \hat{\eta}_0\int_{\Omega_2}|\Delta u(t_\delta)|^2dx, \qquad (3.17)
\end{aligned}$$

where $\hat{\eta}_0$ satisfies (3.4).
Additionally, by using the same approach as in (3.11), we obtain

$$\begin{aligned}
&\int_{\Omega_2} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \\
&\leq \frac{1}{2e} \max\{C_{q+2}^{q_1+2} \left(\frac{2p_1(E(0)+\delta)}{(p_1-2)\ell}\right)^{\frac{q_1}{2}}, C_{q+2}^{q_2+2} \left(\frac{2p_1(E(0)+\delta)}{(p_1-2)\ell}\right)^{\frac{q_2}{2}}\} \int_{\Omega_2} |\Delta u(t_\delta)|^2 dx \\
&= \hat{\eta}_1 \int_{\Omega_2} |\Delta u(t_\delta)|^2 dx, \qquad (3.18)
\end{aligned}$$

Moreover, by using the embedding $H_0^2(\Omega_2) \hookrightarrow W_0^{1,p(x)}(\Omega_2)$ with the constant $C_{*,p}$, we get

$$\begin{aligned}
&\int_{\Omega_2} |\nabla u(t_\delta)|^{p(x)} dx \\
&\leq \max\{C_p^{p_1} \left(\frac{2p_1(E(0)+\delta)}{(p_1-2)\ell}\right)^{\frac{p_1-2}{2}}, C_p^{p_2} \left(\frac{2p_1(E(0)+\delta)}{(p_1-2)\ell}\right)^{\frac{p_2-2}{2}}\} \int_{\Omega_2} |\Delta u(t_\delta)|^2 dx \\
&= \hat{\eta}_2 \int_{\Omega_2} |\Delta u(t_\delta)|^2 dx. \qquad (3.19)
\end{aligned}$$

Now, from definition of the Nehari functional (3.1), (A2), and the estimations in (3.17), (3.18) and (3.19), we deduce

$$\begin{aligned}
I(t_\delta) = &\int_{\Omega_2} |\Delta u(t_\delta)|^2 dx + \int_{\Omega_2} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx + (g \diamond \nabla u)(t_\delta) \\
&- \int_{\Omega_2} |\nabla u(t_\delta)|^{p(x)} dx - \left(\int_0^{t_\delta} g(s) ds\right) \int_{\Omega_2} |\nabla u(t_\delta)|^2 dx \\
&- \int_{\Omega_2} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \\
\geq &\left(1 - \gamma \int_0^{t_\delta} g(s) ds\right) \int_{\Omega_2} |\Delta u(t_\delta)|^2 dx + \int_{\Omega_2} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx \\
&- \int_{\Omega_2} |\nabla u(t_\delta)|^{p(x)} dx - \int_{\Omega_2} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \\
\geq &\ell \int_{\Omega_2} |\Delta u(t_\delta)|^2 dx + \int_{\Omega_2} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx - \int_{\Omega_2} |\nabla u(t_\delta)|^{p(x)} dx \\
&- \int_{\Omega_2} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \\
\geq &(\ell - \hat{\eta}_0 - \hat{\eta}_1 - \hat{\eta}_2) \int_{\Omega_2} |\Delta u(t_\delta)|^2 dx.
\end{aligned}$$

Thanks to the hypothesis of Lemma 3.1, i.e., $\hat{\eta}_0 + \hat{\eta}_1 + \hat{\eta}_2 < \ell$, we deduce that

$$I(t_\delta) > 0 \quad \text{for all } x \in \Omega_2.$$

(iii) Case $x \in \Omega_3$**.** By using (A1), we have

$$\int_{\Omega_3} \frac{1}{q(x)}|u(t_\delta)|^{q(x)} \ln |u(t_\delta)|dx \leq \frac{1}{q_2}\int_{\Omega_3} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)|dx,$$
$$\int_{\Omega_3} \frac{1}{p(x)}|\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)|dx \geq \frac{1}{p_2}\int_{\Omega_3} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)|dx.$$

Inserting the previous estimates into the inequality for $J(t_\delta)$, we obtain

$$\begin{aligned}
J(t_\delta) \geq & \frac{q_2-2}{2q_2}\int_{\Omega_3} |\Delta u(t_\delta)|^2 dx + \frac{q_2-2}{2q_2}(g \diamond \nabla u)(t_\delta) \\
& - \frac{q_2-2}{2q_2}\left(\int_0^{t_\delta} g(s)ds\right)\int_{\Omega_3} |\nabla u(t_\delta)|^2 dx + \frac{1}{q_2^2}\int_{\Omega_3} |u(t_\delta)|^{q(x)}dx \\
& + \frac{p_1^2-q_2}{q_2 p_1^2}\int_{\Omega_3} |\nabla u(t_\delta)|^{p(x)}dx + \frac{1}{q_2}I(t_\delta) \\
\geq & \frac{q_2-2}{2q_2}\left(1-\gamma\int_0^{t_\delta} g(s)ds\right)\int_{\Omega_3} |\Delta u(t_\delta)|^2 dx + \frac{q_2-2}{2q_2}(g \diamond \nabla u)(t_\delta) \\
& + \frac{1}{q_2^2}\int_{\Omega_3} |u(t_\delta)|^{q(x)}dx + \frac{p_1^2-q_2}{q_2 p_1^2}\int_{\Omega_3} |\nabla u(t_\delta)|^{p(x)}dx + \frac{1}{q_2}I(t_\delta),
\end{aligned}$$

Using (3.5) and (A2), we obtain

$$\begin{aligned}
J(t_\delta) \geq & \frac{(q_2-2)\ell}{2q_2}\int_{\Omega_3} |\Delta u(t_\delta)|^2 dx + \frac{q_2-2}{2q_2}(g \diamond \nabla u)(t_\delta) + \frac{1}{q_2^2}\int_{\Omega_3} |u(t_\delta)|^{q(x)}dx \\
& + \frac{p_1^2-q_2}{q_2 p_1^2}\int_{\Omega_3} |\nabla u(t_\delta)|^{p(x)}dx - \frac{p_1\delta}{q_2} \\
\geq & \frac{(q_2-2)\ell}{2q_2}\int_{\Omega_3} |\Delta u(t_\delta)|^2 dx - \delta,
\end{aligned}$$

where we used the assumption of Lemma 3.1 on the exponents, i.e., $\frac{p_1}{q_2} < 1$.
Since by virtue of (3.3), we have $J(t_\delta) \leq E(t_\delta) \leq E(0)$, and thus we deduce

$$\int_{\Omega_3} |\Delta u(t_\delta)|^2 dx \leq \frac{2q_2}{(q_2-2)\ell}(E(0)+\delta). \tag{3.20}$$

At this point, by using (3.20), we estimate the following terms. From the fact that $\phi^{-2}\ln\phi \leq \frac{1}{2e}$ for any $\phi > 1$, we obtain

$$\begin{aligned}
& \left|\int_{\Omega_3} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)|dx\right| \\
\leq & \frac{1}{2e}\int_{\Omega_3} |\nabla u(t_\delta)|^{p(x)+2}dx \\
\leq & \frac{1}{2e}\max\left\{\left(\int_{\Omega_3} |\nabla u(t_\delta)|^{p(x)+2}dx\right)^{p_1+2}, \left(\int_{\Omega_3} |\nabla u(t_\delta)|^{p(x)+2}dx\right)^{p_2+2}\right\} \\
\leq & \frac{1}{2e}\max\left\{C_{*,p+2}^{p_1+2}\left(\int_{\Omega_3} |\Delta u(t_\delta)|^2 dx\right)^{\frac{p_1+2}{2}}, C_{*,p+2}^{p_2+2}\left(\int_{\Omega_3} |\Delta u(t_\delta)|^2 dx\right)^{\frac{p_2+2}{2}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{2e}\max\left\{C_{*,p+2}^{p_1+2}\left(\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\right)^{\frac{p_1}{2}},C_{*,p+2}^{p_2+2}\left(\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\right)^{\frac{p_2}{2}}\right\}\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\\
&\leq\frac{1}{2e}\max\left\{C_{*,p+2}^{p_1+2}\left(\frac{2q_2(E(0)+\delta)}{(q_2-2)\ell}\right)^{\frac{p_1}{2}},C_{*,p+2}^{p_2+2}\left(\frac{2q_2(E(0)+\delta)}{(q_2-2)\ell}\right)^{\frac{p_2}{2}}\right\}\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\\
&<\eta_0\int_{\Omega_3}|\Delta u(t_\delta)|^2dx, \qquad (3.21)
\end{aligned}
$$

In addition, from the fact that $\phi^{q(x)-2}\ln\phi\leq\frac{1}{e(q(x)-2)}$ for any $0<\phi<1$, we obtain

$$
\begin{aligned}
\left|\int_{\Omega_3}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\right|&\leq\int_{\Omega_3}\left||u(t_\delta)|^2|u(t_\delta)|^{q(x)-2}\ln|u(t_\delta)|\right|dx\\
&\leq\int_{\Omega_3}\frac{1}{e(q(x)-2)}|u(t_\delta)|^2dx\\
&\leq\frac{1}{e(q_1-2)}\int_{\Omega_3}|u(t_\delta)|^2dx\\
&\leq\frac{\gamma}{e(q_1-2)}\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\\
&<\hat{\eta}_0\int_{\Omega_3}|\Delta u(t_\delta)|^2dx, \qquad (3.22)
\end{aligned}
$$

where $\hat{\eta}_0$ satisfies (3.4) and γ satisfies (1.4).
By using the embedding $H_0^2(\Omega_3)\hookrightarrow W_0^{1,p(x)}(\Omega_3)$ with the constant $C_{*,p}$, we get

$$
\begin{aligned}
\int_{\Omega_3}|\nabla u(t_\delta)|^{p(x)}dx&\leq\max\{\left(\int_{\Omega_3}|\nabla u(t_\delta)|^{p(x)}dx\right)^{p_1},\left(\int_{\Omega_3}|\nabla u(t_\delta)|^{p(x)}dx\right)^{p_2}\}\\
&\leq\max\{C_{*,p}^{p_1}\left(\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\right)^{\frac{p_1}{2}},\ C_{*,p}^{p_2}\left(\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\right)^{\frac{p_2}{2}}\}\\
&=\max\{C_{*,p}^{p_1}\left(\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\right)^{\frac{p_1-2}{2}},\ C_{*,p}^{p_2}\left(\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\right)^{\frac{p_2-2}{2}}\}\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\\
&\leq\max\{C_{*,p}^{p_1}\left(\frac{2q_2(E(0)+\delta)}{(q_2-2)\ell}\right)^{\frac{p_1-2}{2}},\ C_{*,p}^{p_2}\left(\frac{2q_2(E(0)+\delta)}{(q_2-2)\ell}\right)^{\frac{p_2-2}{2}}\}\int_{\Omega_3}|\Delta u(t_\delta)|^2dx\\
&<\hat{\eta}_2\int_{\Omega_3}|\Delta u(t_\delta)|^2dx, \qquad (3.23)
\end{aligned}
$$

where the assumption $p_1<q_2$ has been used.
Now, from the definition of the Nehari functional (3.1), (A2), and the estimates (3.21)–(3.29), we deduce

$$
\begin{aligned}
I(t_\delta)=&\int_{\Omega_3}|\Delta u(t_\delta)|^2dx+\int_{\Omega_3}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx+(g\diamond\nabla u)(t_\delta)\\
&-\int_{\Omega_3}|\nabla u(t_\delta)|^{p(x)}dx-\left(\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_3}|\nabla u(t_\delta)|^2dx\\
&-\int_{\Omega_3}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx
\end{aligned}
$$

$$
\begin{aligned}
&\geq\left(1-\gamma\int_0^{t_\delta} g(s)ds\right)\int_{\Omega_3}|\Delta u(t_\delta)|^2dx+\int_{\Omega_3}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx\\
&\quad-\int_{\Omega_3}|\nabla u(t_\delta)|^{p(x)}dx-\int_{\Omega_3}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\\
&\geq\ell\int_{\Omega_3}|\Delta u(t_\delta)|^2dx+\int_{\Omega_3}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx-\int_{\Omega_3}|\nabla u(t_\delta)|^{p(x)}dx\\
&\quad-\int_{\Omega_3}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\\
&\geq(\ell-\eta_0-\hat{\eta}_0-\hat{\eta}_2)\int_{\Omega_3}|\Delta u(t_\delta)|^2dx.
\end{aligned}
$$

Thanks to the hypothesis of Lemma 3.1, i.e., $\eta_0+\hat{\eta}_0+\hat{\eta}_2<\ell$, we deduce that

$$I(t_\delta)>0 \quad \text{for all } x\in\Omega_3.$$

(iv) Case $x\in\Omega_4$**.** By using (A1), it is easy to see that

$$
\begin{aligned}
&\int_{\Omega_4}\frac{1}{q(x)}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\leq\frac{1}{q_2}\int_{\Omega_4}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx,\\
&\int_{\Omega_4}\frac{1}{p(x)}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx\geq\frac{1}{p_1}\int_{\Omega_4}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx.
\end{aligned}
$$

Thus, by using the functional $J(t)$ (i.e., (3.2)), (A1), and estimates above, we obtain

$$
\begin{aligned}
J(t_\delta)\geq&\frac{1}{2}\int_{\Omega_4}|\Delta u(t_\delta)|^2dx+\frac{1}{p_1}\int_{\Omega_4}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx+\frac{1}{2}(g\diamond\nabla u)(t_\delta)\\
&+\frac{1}{q_2^2}\int_{\Omega_4}|u(t_\delta)|^{q(x)}dx-\frac{1}{2}\left(\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_4}|\nabla u(t_\delta)|^2dx\\
&-\frac{1}{p_1^2}\int_{\Omega_4}|\nabla u(t_\delta)|^{p(x)}dx-\frac{1}{q_2}\int_{\Omega_4}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx.
\end{aligned}\tag{3.24}
$$

Recalling the definition of Nehari functional $I(t)$, we get

$$
\begin{aligned}
&-\frac{1}{q_2}\int_{\Omega_4}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\\
=&\frac{1}{q_2}I(t_\delta)-\frac{1}{q_2}\int_{\Omega_4}|\Delta u(t_\delta)|^2dx-\frac{1}{q_2}(g\diamond\nabla u)(t_\delta)+\frac{1}{q_2}\int_{\Omega_4}|\nabla u(t_\delta)|^{p(x)}dx\\
&+\frac{1}{q_2}\left(\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_4}|\nabla u(t_\delta)|^2dx-\frac{1}{q_2}\int_{\Omega_4}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx.
\end{aligned}\tag{3.25}
$$

Applying (3.25) into (3.13), we deduce

$$
\begin{aligned}
J(t_\delta)\geq&\frac{q_2-2}{2q_2}\int_{\Omega_4}|\Delta u(t_\delta)|^2dx-\frac{q_2-2}{2q_2}\left(\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_4}|\Delta u(t_\delta)|^2dx\\
&+\frac{q_2-2}{2q_2}(g\diamond\nabla u)(t_\delta)+\frac{1}{q_2^2}\int_{\Omega_4}|u(t_\delta)|^{q(x)}dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{p_1^2-q_2}{q_2p_1^2}\int_{\Omega_4}|\nabla u(t_\delta)|^{p(x)}dx+\frac{1}{q_2}I(t_\delta)\\
\geq&\frac{q_2-2}{2q_2}\left(1-\gamma\int_0^{t_\delta}g(s)ds\right)\int_{\Omega_4}|\Delta u(t_\delta)|^2dx+\frac{q_2-2}{2q_2}(g\diamond\nabla u)(t_\delta)\\
&+\frac{1}{q_2^2}\int_{\Omega_4}|u(t_\delta)|^{q(x)}dx+\frac{p_1^2-q_2}{q_2p_1^2}\int_{\Omega_4}|\nabla u(t_\delta)|^{p(x)}dx+\frac{1}{q_2}I(t_\delta),
\end{aligned}
$$

where (1.4) has been used.
Applying (3.5) and (A2), we obtain

$$
\begin{aligned}
J(t_\delta)\geq&\frac{(q_2-2)\ell}{2q_2}\int_{\Omega_4}|\Delta u(t_\delta)|^2dx+\frac{q_2-2}{2q_2}(g\diamond\nabla u)(t_\delta)+\frac{1}{q_2^2}\int_{\Omega_4}|u(t_\delta)|^{q(x)}dx\\
&+\frac{p_1^2-q_2}{q_2p_1^2}\int_{\Omega_4}|\nabla u(t_\delta)|^{p(x)}dx-\frac{p_1\delta}{q_2}\\
\geq&\frac{(q_2-2)\ell}{2q_2}\int_{\Omega_4}|\Delta u(t_\delta)|^2dx-\delta,
\end{aligned}
$$

where we used the assumption of Lemma 3.1 on the exponents, i.e., $\frac{p_1}{q_2^2}<1$.
Since by virtue of (3.3), we have $J(t_\delta)\leq E(t_\delta)\leq E(0)$, and thus, we deduce

$$
\int_{\Omega_4}|\Delta u(t_\delta)|^2dx\leq\frac{2q_2}{(q_2-2)\ell}(E(0)+\delta). \tag{3.26}
$$

At this point, by using (3.26), we estimate the following terms. From the fact that $\phi^{p(x)-2}\ln\phi\leq\frac{1}{e(p(x)-2)}$ for any $0<\phi<1$, we obtain

$$
\begin{aligned}
\left|\int_{\Omega_4}|\nabla u(t_\delta)|^{p(x)}\ln|\nabla u(t_\delta)|dx\right|\leq&\int_{\Omega_4}\left||\nabla u(t_\delta)|^2|\nabla u(t_\delta)|^{p(x)-2}\ln|\nabla u(t_\delta)|\right|dx\\
\leq&\int_{\Omega_4}\frac{1}{e(p(x)-2)}|\nabla u(t_\delta)|^2dx\\
\leq&\frac{1}{e(p_1-2)}\int_{\Omega_4}|\nabla u(t_\delta)|^2dx\\
\leq&\hat{\eta}_0\int_{\Omega_4}|\Delta u(t_\delta)|^2dx,
\end{aligned}
\tag{3.27}
$$

where $\hat{\eta}_0$ satisfies (3.4). By using the same approach and since $p_1<q_1$, we have

$$
\begin{aligned}
\left|\int_{\Omega_4}|u(t_\delta)|^{q(x)}\ln|u(t_\delta)|dx\right|\leq&\int_{\Omega_4}\left||u(t_\delta)|^2|u(t_\delta)|^{q(x)-2}\ln|u(t_\delta)|\right|dx\\
\leq&\int_{\Omega_4}\frac{1}{e(q(x)-2)}|u(t_\delta)|^2dx\\
\leq&\frac{1}{e(q_1-2)}\int_{\Omega_4}|u(t_\delta)|^2dx\\
\leq&\frac{C_2^2}{e(q_1-2)}\int_{\Omega_4}|\Delta u(t_\delta)|^2dx
\end{aligned}
$$

$$<\eta_1 \int_{\Omega_4} |\Delta u(t_\delta)|^2 dx, \tag{3.28}$$

where η_1 satisfies (3.4). By virtue of the embedding $H_0^2(\Omega_4) \hookrightarrow W_0^{1,p(x)}(\Omega_4)$ with the constant $C_{*,p}$ and since $p_1 < q_1$, we get

$$\begin{aligned}
\int_{\Omega_4} |\nabla u(t_\delta)|^{p(x)} dx \leq & \max\{\left(\int_{\Omega_4} |\nabla u(t_\delta)|^{p(x)} dx\right)^{p_1}, \left(\int_{\Omega_4} |\nabla u(t_\delta)|^{p(x)} dx\right)^{p_2}\} \\
\leq & \max\{C_{*,p}^{p_1}\left(\int_{\Omega_4} |\Delta u(t_\delta)|^2 dx\right)^{\frac{p_1-2}{2}}, C_{*,p}^{p_2}\left(\int_{\Omega_4} |\Delta u(t_\delta)|^2 dx\right)^{\frac{p_2-2}{2}}\} \int_{\Omega_4} |\Delta u(t_\delta)|^2 dx \\
\leq & \max\{C_{*,p}^{p_1}\left(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell}\right)^{\frac{p_1-2}{2}}, C_{*,p}^{p_2}\left(\frac{2q_1(E(0)+\delta)}{(q_1-2)\ell}\right)^{\frac{p_2-2}{2}}\} \int_{\Omega_4} |\Delta u(t_\delta)|^2 dx \\
< & \hat{\eta}_2 \int_{\Omega_4} |\Delta u(t_\delta)|^2 dx,
\end{aligned} \tag{3.29}$$

where $\hat{\eta}_2$ satisfies (3.4). Now, from definition of the Nehari functional (3.1), (A2), and the estimates (3.27)–(3.29), we deduce

$$\begin{aligned}
I(t_\delta) = & \int_{\Omega_4} |\Delta u(t_\delta)|^2 dx + \int_{\Omega_4} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx + (g \diamond \nabla u)(t_\delta) \\
& - \int_{\Omega_4} |\nabla u(t_\delta)|^{p(x)} dx - \left(\int_0^{t_\delta} g(s) ds\right) \int_{\Omega_4} |\nabla u(t_\delta)|^2 dx \\
& - \int_{\Omega_4} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \\
\geq & \left(1 - \gamma \int_0^{t_\delta} g(s) ds\right) \int_{\Omega_4} |\Delta u(t_\delta)|^2 dx + \int_{\Omega_4} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx \\
& - \int_{\Omega_4} |\nabla u(t_\delta)|^{p(x)} dx - \int_{\Omega_4} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \\
\geq & \ell \int_{\Omega_4} |\Delta u(t_\delta)|^2 dx + \int_{\Omega_4} |\nabla u(t_\delta)|^{p(x)} \ln |\nabla u(t_\delta)| dx - \int_{\Omega_4} |\nabla u(t_\delta)|^{p(x)} dx \\
& - \int_{\Omega_4} |u(t_\delta)|^{q(x)} \ln |u(t_\delta)| dx \\
\geq & (\ell - \hat{\eta}_0 - \eta_1 - \hat{\eta}_2) \int_{\Omega_4} |\Delta u(t_\delta)|^2 dx.
\end{aligned}$$

Thanks to the hypothesis of Lemma 3.1, i.e., $\hat{\eta}_0 + \eta_1 + \hat{\eta}_2 < \ell$, we therefore deduce that

$$I(t_\delta) > 0 \quad \text{for all } x \in \Omega_4.$$

Therefore, we have proved that for any $x \in \Omega$, $I(t_\delta) > 0$, which contradicts (3.5). Thus, $I(t) > 0$ for all $t \in [0, T_{\max})$, and the proof of Lemma 3.1 is completed. □

At this stage, we present and establish the global existence result, which is formulated and proven as follows.

Theorem 3.2. *Under the assumptions of Lemma 3.1, the solution of the problem* (1.1)–(1.3) *is global in time.*

Proof. Using the estimations derived in the proof of Lemma 3.1, we obtain the following lower bounds for the functional $J(t)$:

$$\begin{aligned}
\forall x \in \Omega_1 : \quad J(t) \geq & \frac{(q_1-2)\ell}{2q_1}\int_{\Omega_1}|\Delta u|^2 dx + \frac{q_1-2}{2q_1}(g \diamond \nabla u)(t) \\
& + \frac{1}{q_2^2}\int_{\Omega_1}|u|^{q(x)}dx + \frac{p_1^2-q_1}{q_1p_1^2}\int_{\Omega_1}|\nabla u(t)|^{p(x)}dx + \frac{1}{q_1}I(t), \\
\forall x \in \Omega_2 : \quad J(t) \geq & \frac{(p_1-2)\ell}{2p_1}\int_{\Omega_2}|\Delta u|^2 dx + \frac{p_1-2}{2p_1}(g \diamond \nabla u)(t) \\
& + \frac{1}{q_2^2}\int_{\Omega_2}|u|^{q(x)}dx + \frac{p_1-1}{p_1^2}\int_{\Omega_2}|\nabla u(t)|^{p(x)}dx + \frac{1}{p_1}I(t), \\
\forall x \in \Omega_3 \cup \Omega_4 : \quad J(t) \geq & \frac{(q_2-2)\ell}{2q_2}\int_{\Omega_3\cup\Omega_4}|\Delta u|^2 dx + \frac{q_2-2}{2q_2}(g \diamond \nabla u)(t) \\
& + \frac{1}{q_2^2}\int_{\Omega_3\cup\Omega_4}|u|^{q(x)}dx + \frac{p_1^2-q_2}{q_2p_1^2}\int_{\Omega_3\cup\Omega_4}|\nabla u(t)|^{p(x)}dx + \frac{1}{q_2}I(t).
\end{aligned}$$

Therefore, by invoking the hypothesis of Lemma 3.1 on the variable exponents, namely $p_1 < p_2 < q_1 < q_2 < p_1^2$, we infer the uniform estimate

$$\forall x \in \Omega : \quad J(t) \geq \gamma_0\Big(I(t) + \ell\|\Delta u\|^2 + (g \diamond \nabla u)(t) + \int_\Omega |u|^{q(x)}dx + \int_\Omega |\nabla u|^{p(x)}dx\Big), \tag{3.30}$$

where $\gamma_0 := \min\left\{\frac{1}{p_1}, \frac{q_2-2}{2q_2}\right\} > 0$.

Next, using (3.1), (3.3), Assumption (A1), inequality (3.30), and Lemma 3.1, we obtain

$$\begin{aligned}
E(0) \geq & \frac{1}{\rho_2}\int_\Omega |u_t|^{\rho(x)}dx + \frac{1}{2}\|\nabla u_t\|^2 + J(t) \\
\geq & \frac{1}{\rho_2}\int_\Omega |u_t|^{\rho(x)}dx + \frac{1}{2}\|\nabla u_t\|^2 \\
& + \gamma_0\Big(I(t) + \ell\|\Delta u\|^2 + (g \diamond \nabla u)(t) + \int_\Omega |u|^{q(x)}dx + \int_\Omega |\nabla u|^{p(x)}dx\Big) \\
\geq & \frac{1}{\rho_2}\int_\Omega |u_t|^{\rho(x)}dx + \frac{1}{2}\|\nabla u_t\|^2 \\
& + \gamma_0\Big(\ell\|\Delta u\|^2 + (g \diamond \nabla u)(t) + \int_\Omega |u|^{q(x)}dx + \int_\Omega |\nabla u|^{p(x)}dx\Big) \\
\geq & \gamma_1\ell\Big(\int_\Omega |u_t|^{\rho(x)}dx + \|\nabla u_t\|^2 + \|\Delta u\|^2 + (g \diamond \nabla u)(t) + \int_\Omega |u|^{q(x)}dx + \int_\Omega |\nabla u|^{p(x)}dx\Big),
\end{aligned}$$

where $\gamma_1 := \min\left\{\frac{1}{\rho_2}, \gamma_0\right\} > 0$.
Consequently, it follows that

$$\int_\Omega |u_t|^{\rho(x)}dx + \|\nabla u_t\|^2 + \|\Delta u\|^2 + (g \diamond \nabla u)(t) + \int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |u|^{q(x)}dx \leq \frac{E(0)}{\gamma_1\ell}. \tag{3.31}$$

Inequality (3.31) ensures that no finite-time blow-up can occur; therefore the solution of the problem (1.1)–(1.3) exists for all $t \geq 0$. This completes the proof of Theorem 3.2. □

4. General decay

In this section, we investigate the asymptotic stability and general decay behavior of global solutions to the problem (1.1)–(1.3). More precisely, under suitable assumptions on the relaxation function and the nonlinear terms, we establish a general (non-uniform) decay rate for the energy functional by using the energy perturbation method combined with weighted integral inequalities. This section complements the global existence result proved in Theorem 3.2 by showing that the global solutions are not only bounded but also asymptotically stable.
To prove the decay result, we require the following additional assumption on the memory kernel:
$(B1)$ There is a non–increasing, positive, and differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$g'(t) \leq -\xi(t)g(t), \quad \forall t \geq 0, \qquad \text{and} \qquad \int_0^\infty \xi(s)\, ds = +\infty.$$

Assumption (B1) is standard in the theory of viscoelastic equations and ensures that the memory kernel dissipates energy at a rate governed by $\xi(t)$, allowing for both exponential and polynomial decay as particular cases.

Our main result in this section can now be stated as follows.

Theorem 4.1. *Suppose that the assumptions of Theorem 3.2 and* (B1) *hold. For a sufficiently small* $E(0)$*, the global solution of the problem* (1.1)–(1.3) *decays in a general sense. More precisely, there are positive constants K and k, which are independent of t, such that the energy functional satisfies*

$$E(t) \leq K \exp\left(-k \int_0^t \xi(s)\, ds\right), \qquad \forall t \geq 0.$$

Proof. We begin by considering the case $x \in \Omega_1$. The analysis of the remaining subdomains Ω_2, Ω_3, and Ω_4 follows by analogous arguments and is therefore omitted for brevity.

The perturbed energy functional, denoted as $L(t)$, is defined for sufficiently small parameters satisfying $0 < \varepsilon_0 < \varepsilon$, as follows:

$$L(t) = E(t) + \varepsilon_0 \int_{\Omega_1} \frac{1}{\rho(x) - 1}\, u\, |u_t|^{\rho(x)-1}\, dx - \varepsilon_0 \int_{\Omega_1} u\, \Delta u_t\, dx. \tag{4.1}$$

The functional $L(t)$ is a standard Lyapunov-type modification of the energy, introduced to compensate for the lack of direct dissipation in some nonlinear terms. Using Young's inequality, Poincaré's inequality, and the boundedness of the variable exponents, one can verify that a positive constant C_0 exists such that $L(t) \geq C_0 E(t)$, for all $t \geq 0$.
Differentiating $L(t)$ and by using Eq (2.3), we obtain

$$\begin{aligned} L'(t) = E'(t) &+ \varepsilon_0 \int_{\Omega_1} \frac{1}{\rho(x) - 1} |u_t|^{\rho(x)} dx - \varepsilon_0 \int_{\Omega_1} |\Delta u|^2 dx \\ &- \varepsilon_0 \int_{\Omega_1} |\nabla u|^{p(x)} \ln |\nabla u| dx + \varepsilon_0 \int_{\Omega_1} \nabla u \int_0^t g(t-s) \nabla u(s)\, ds\, dx \\ &+ \varepsilon_0 \int_{\Omega_1} |\nabla u_t|^2 dx - \varepsilon_0 \varepsilon \int_{\Omega_1} \nabla u \cdot \nabla u_t dx + \varepsilon_0 \int_{\Omega_1} |u|^{q(x)} \ln |u| dx \end{aligned}$$

$$
\begin{aligned}
\leq & -(\varepsilon-\varepsilon_0)\int_{\Omega_1}|\nabla u_t|^2dx+\varepsilon_0\int_{\Omega_1}\frac{1}{\rho(x)-1}|u_t|^{\rho(x)}dx-\varepsilon_0\int_{\Omega_1}|\Delta u|^2dx\\
&-\varepsilon_0\int_{\Omega_1}|\nabla u|^{p(x)}\ln|\nabla u|dx+\varepsilon_0\int_{\Omega_1}\nabla u\int_0^t g(t-s)\nabla u(s)\,ds\,dx\\
&-\varepsilon_0\varepsilon\int_{\Omega_1}\nabla u\cdot\nabla u_t dx+\varepsilon_0\int_{\Omega_1}|u|^{q(x)}\ln|u|dx.
\end{aligned}
$$

Now, by adding $\varepsilon_0 L(t)$ to both sides and using the definition in (4.1) and Assumption (A1), we obtain

$$
\begin{aligned}
L'(t)+\varepsilon_0 L(t)\leq &\frac{\varepsilon_0(2\rho_1-1)}{\rho_1(\rho_1-1)}\int_{\Omega_1}|u_t|^{\rho(x)}dx-\frac{\varepsilon_0}{2}\int_{\Omega_1}|\Delta u|^2dx+\frac{\varepsilon_0}{2}(g\diamond\nabla u)(t)\\
&-\frac{\varepsilon_0(p_1-1)}{p_1}\int_{\Omega_1}|\nabla u|^{p(x)}\ln|\nabla u|dx-\frac{\varepsilon_0}{2}\Big(\int_0^t g(s)ds\Big)\int_{\Omega_1}|\nabla u|^2dx\\
&+\frac{\varepsilon_0}{q_1^2}\int_{\Omega_1}|u|^{q(x)}dx-\Big(\varepsilon-\frac{3\varepsilon_0}{2}\Big)\int_{\Omega_1}|\nabla u_t|^2dx\\
&+\frac{\varepsilon_0(q_2-1)}{q_2}\int_{\Omega_1}|u|^{q(x)}\ln|u|dx+\frac{\varepsilon_0^2}{\rho_1-1}\int_{\Omega_1}u|u_t|^{\rho(x)-1}dx\\
&+\varepsilon_0\int_{\Omega_1}\nabla u\int_0^t g(t-s)\nabla u(s)dsdx-\varepsilon_0(\varepsilon-\varepsilon_0)\int_{\Omega_1}\nabla u\cdot\nabla u_t dx. \qquad (4.2)
\end{aligned}
$$

At this stage, we estimate each term appearing on the right-hand side of (4.2). All estimates are carried out carefully so as to obtain a closed differential inequality for the Lyapunov functional $L(t)$.
First, we consider the memory term. By adding and subtracting $\nabla u(t)$, we write

$$
\begin{aligned}
&\varepsilon_0\int_{\Omega_1}\nabla u\int_0^t g(t-s)\nabla u(s)\,ds\,dx\\
=&\,\varepsilon_0\Big(\int_0^t g(s)\,ds\Big)\int_{\Omega_1}|\nabla u|^2dx+\varepsilon_0\int_{\Omega_1}\nabla u\int_0^t g(t-s)(\nabla u(s)-\nabla u)\,ds\,dx.
\end{aligned}
$$

Using Young's inequality together with Assumption (A2), we infer

$$
\begin{aligned}
\varepsilon_0\int_{\Omega_1}\nabla u\int_0^t g(t-s)(\nabla u(s)-\nabla u)\,ds\,dx\leq&\frac{\varepsilon_0\ell}{4}\Big(\int_0^t g(s)\,ds\Big)\int_{\Omega_1}|\nabla u|^2dx\\
&+\frac{\varepsilon_0}{\ell}\Big(\int_0^t g(s)\,ds\Big)(g\diamond\nabla u)(t).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\varepsilon_0\int_{\Omega_1}\nabla u\int_0^t g(t-s)\nabla u(s)\,ds\,dx\\
\leq&\,\varepsilon_0\Big(\int_0^t g(s)\,ds\Big)\int_{\Omega_1}|\nabla u|^2dx+\frac{\varepsilon_0\ell}{4}\Big(\int_0^t g(s)\,ds\Big)\int_{\Omega_1}|\nabla u|^2dx\\
&+\frac{\varepsilon_0}{\ell}\Big(\int_0^t g(s)\,ds\Big)(g\diamond\nabla u)(t).
\end{aligned}
$$

Recalling that $\gamma\int_0^\infty g(s)\,ds<1$ and using (1.4), we finally obtain

$$\begin{aligned}\varepsilon_0\int_{\Omega_1}\nabla u\int_0^t g(t-s)\nabla u(s)\,ds\,dx \leq&\varepsilon_0\Big(\int_0^t g(s)\,ds\Big)\int_{\Omega_1}|\nabla u|^2dx\\&+\frac{\varepsilon_0\ell}{4}\int_{\Omega_1}|\Delta u|^2dx+\frac{\varepsilon_0}{\gamma\ell}(g\diamond\nabla u)(t).\end{aligned}\tag{4.3}$$

Next, by using the Young's inequality, for any $\theta>0$, we get

$$\begin{aligned}\int_{\Omega_1}u|u_t|^{\rho(x)-1}dx \leq&\theta\int_{\Omega_1}|u_t|^{\rho(x)}dx+\int_{\Omega_1}\frac{1}{\rho(x)}\Big(\frac{\theta\rho(x)}{\rho(x)-1}\Big)^{1-\rho(x)}|u|^{\rho(x)}dx\\ \leq&\theta\int_{\Omega_1}|u_t|^{\rho(x)}dx+C_\theta(\rho)\int_{\Omega_1}|u|^{\rho(x)}dx,\end{aligned}\tag{4.4}$$

where

$$C_\theta(\rho)=\max_{x\in\overline{\Omega}_1}\Big\{\frac{1}{\rho(x)}\Big(\frac{\theta\rho(x)}{\rho(x)-1}\Big)^{1-\rho(x)}\Big\}.$$

To estimate the last term, we use the inequality (3.31) together with the embedding $H_0^2(\Omega_1)\hookrightarrow L^{\rho(x)}(\Omega_1)$. Thus, we have

$$\begin{aligned}\int_{\Omega_1}|u|^{\rho(x)}dx \leq&\max\{\Big(\int_{\Omega_1}|u|^{\rho(x)}dx\Big)^{\rho_1},\Big(\int_{\Omega_1}|u|^{\rho(x)}dx\Big)^{\rho_2}\}\\ \leq&\max\{C_\rho^{\rho_1}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{\rho_1}{2}},C_\rho^{\rho_2}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{\rho_2}{2}}\}\\ =&\max\{C_\rho^{\rho_1}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{\rho_1-2}{2}},C_\rho^{\rho_2}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{\rho_2-2}{2}}\}\int_{\Omega_1}|\Delta u|^2dx\\ \leq&\max\{C_\rho^{\rho_1}\left(\frac{E(0)}{\gamma_1\ell}\right)^{\frac{\rho_1-2}{2}},C_\rho^{\rho_2}\left(\frac{E(0)}{\gamma_1\ell}\right)^{\frac{\rho_2-2}{2}}\}\int_{\Omega_1}|\Delta u|^2dx\\ :=&\eta_3\int_{\Omega_1}|\Delta u|^2dx,\end{aligned}\tag{4.5}$$

where C_ρ denotes the embedding constant. Therefore, by combining (4.4) and (4.5), we deduce

$$\int_{\Omega_1}u|u_t|^{\rho(x)-1}dx\leq\theta\int_{\Omega_1}|u_t|^{\rho(x)}dx+\eta_4\int_{\Omega_1}|\Delta u|^2dx.\tag{4.6}$$

By using the embedding $H_0^2(\Omega_1)\hookrightarrow L^{q(x)}(\Omega_1)$ with the constant C_q and the inequality (3.31), we obtain

$$\begin{aligned}\int_{\Omega_1}|u|^{q(x)}dx \leq&\max\{\Big(\int_{\Omega_1}|u|^{q(x)}dx\Big)^{q_1},\Big(\int_{\Omega_1}|u|^{q(x)}dx\Big)^{q_2}\}\\ \leq&\max\{C_q^{q_1}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{q_1}{2}},C_q^{q_2}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{q_2}{2}}\}\\ =&\max\{C_q^{q_1}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{q_1-2}{2}},C_q^{q_2}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{q_2-2}{2}}\}\int_{\Omega_1}|\Delta u|^2dx\end{aligned}$$

$$\leq \max\{C_q^{q_1}\left(\frac{E(0)}{\gamma_1\ell}\right)^{\frac{q_1-2}{2}}, C_q^{q_2}\left(\frac{E(0)}{\gamma_1\ell}\right)^{\frac{q_2-2}{2}}\}\int_{\Omega_1}|\Delta u|^2dx$$
$$:=\eta_5\int_{\Omega_1}|\Delta u|^2dx. \tag{4.7}$$

Finally, by applying Young's inequality to the coupling term $\int_{\Omega_1}\nabla u\cdot\nabla u_t dx$, and using (1.4), we obtain

$$\int_{\Omega_1}\nabla u\cdot\nabla u_t dx \leq \frac{2\varepsilon-3\varepsilon_0}{4\varepsilon_0(\varepsilon-\varepsilon_0)}\int_{\Omega_1}|\nabla u_t|^2dx + \frac{\gamma\varepsilon_0(\varepsilon-\varepsilon_0)}{2\varepsilon-3\varepsilon_0}\int_{\Omega_1}|\Delta u|^2dx, \tag{4.8}$$

where γ satisfies (1.4). This estimate allows us to control the mixed term by the dissipation and elastic energy components.
By using embedding $H_0^2(\Omega_1)\hookrightarrow L^{q(x)+2}(\Omega_1)$, we have

$$\begin{aligned}\int_{\Omega_1}|u|^{q(x)}\ln|u|dx \leq&\frac{1}{2e}\int_{\Omega_1}|u|^{q(x)+2}dx\\ \leq&\frac{1}{2e}\max\{\left(\int_{\Omega_1}|u|^{q(x)+2}dx\right)^{q_1+2}, \left(\int_{\Omega_1}|u|^{q(x)+2}dx\right)^{q_2+2}\}\\ \leq&\frac{1}{2e}\max\{C_{q+2}^{q_1+2}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{q_1+2}{2}}, C_{q+2}^{q_2+2}\left(\int_{\Omega_1}|\Delta u|^2dx\right)^{\frac{q_2+2}{2}}\}\\ \leq&\frac{1}{2e}\max\{C_{q+2}^{q_1+2}\left(\frac{E(0)}{\gamma_1\ell}\right)^{\frac{q_1}{2}}, C_{q+2}^{q_2+2}\left(\frac{E(0)}{\gamma_1\ell}\right)^{\frac{q_2}{2}}\}\int_{\Omega_1}|\Delta u|^2dx\\ <&\eta_2\int_{\Omega_1}|\Delta u|^2dx,\end{aligned} \tag{4.9}$$

By combining (4.3)–(4.9) with (4.2), and using the fact that

$$\int_{\Omega_1}|\nabla u|^{p(x)}\ln|\nabla u|dx \geq 0,$$

we arrive at

$$\begin{aligned}L'(t)+\varepsilon_0L(t) \leq& \varepsilon_0\left(\frac{2\rho_1-1}{\rho_1(\rho_1-1)}+\frac{\theta\varepsilon_0}{\rho_1-1}\right)\int_{\Omega_1}|u_t|^{\rho(x)}dx + \varepsilon_0\left(\frac{1}{2}+\frac{1}{\gamma\ell}\right)(g\diamond\nabla u)(t)\\ &-\left[\frac{\varepsilon_0\ell}{4}-\frac{\varepsilon_0\eta_5}{q_1^2}-\frac{\varepsilon_0(q_1-1)\eta_2}{q_1}-\frac{\varepsilon_0^2\eta_4}{\rho_1-1}-\frac{\gamma\varepsilon_0^2(\varepsilon-\varepsilon_0)^2}{2\varepsilon-3\varepsilon_0}\right]\int_{\Omega_1}|\Delta u|^2dx\\ &-\frac{2\varepsilon-3\varepsilon_0}{4}\int_{\Omega_1}|\nabla u_t|^2dx.\end{aligned} \tag{4.10}$$

Next, we estimate the nonlinear velocity term. Using the embedding $H_0^1(\Omega_1)\hookrightarrow L^{\rho(x)}(\Omega_1)$ with the constant $\hat{C}_\rho$ and the inequality (3.31), we deduce

$$\int_{\Omega_1}|u_t|^{\rho(x)}dx \leq \max\{\left(\int_{\Omega_1}|u_t|^{\rho}dx\right)^{\rho_1}, \left(\int_{\Omega_1}|u_t|^{\rho}dx\right)^{\rho_2}\}$$

$$
\begin{aligned}
&\le \max\{\hat{C}_\rho^{\rho_1}\left(\int_{\Omega_1}|\nabla u_t|^2dx\right)^{\frac{\rho_1}{2}}, \hat{C}_\rho^{\rho_2}\left(\int_{\Omega_1}|\nabla u_t|^2dx\right)^{\frac{\rho_2}{2}}\}\\
&= \max\{\hat{C}_\rho^{\rho_1}\left(\int_{\Omega_1}|\nabla u_t|^2dx\right)^{\frac{\rho_1-1}{2}}, \hat{C}_\rho^{\rho_2}\left(\int_{\Omega_1}|\nabla u_t|^2dx\right)^{\frac{\rho_2-2}{2}}\}\int_{\Omega_1}|\nabla u_t|^2dx\\
&\le \max\{\hat{C}_\rho^{\rho_1}\left(\frac{E(0)}{\gamma_1\ell}\right)^{\frac{\rho_1-1}{2}}, \hat{C}_\rho^{\rho_2}\left(\frac{E(0)}{\gamma_1\ell}\right)^{\frac{\rho_2-2}{2}}\}\int_{\Omega_1}|\nabla u_t|^2dx\\
&=\hat{\eta}_3\int_{\Omega_1}|\nabla u_t|^2dx.
\end{aligned}
$$

This estimate shows that the velocity term can be absorbed by the dissipation, provided that $E(0)$ is sufficiently small.

Substituting this bound into (4.10), we obtain

$$
\begin{aligned}
L'(t)+\varepsilon_0 L(t) \le &-\left[\frac{2\varepsilon-3\varepsilon_0}{4}-\varepsilon_0\hat{\eta}_3\left(\frac{2\rho_1-1}{\rho_1(\rho_1-1)}+\frac{\theta\varepsilon_0}{\rho_1-1}\right)\right]\int_{\Omega_1}|\nabla u_t|^2dx\\
&-\varepsilon_0\left[\frac{\ell}{4}-\frac{\eta_5}{q_1^2}-\frac{(q_1-1)\eta_2}{q_1}-\frac{\varepsilon_0\eta_4}{\rho_1-1}-\frac{\gamma\varepsilon_0(\varepsilon-\varepsilon_0)^2}{2\varepsilon-3\varepsilon_0}\right]\int_{\Omega_1}|\Delta u|^2dx\\
&+\varepsilon_0\left(\frac{1}{2}+\frac{1}{\gamma\ell}\right)(g\diamond\nabla u)(t).
\end{aligned}
$$

Now, by choosing ε_0 and an $E(0)$ that are sufficiently small such that

$$
\begin{aligned}
&\frac{2\varepsilon-3\varepsilon_0}{4}-\varepsilon_0\hat{\eta}_3\left(\frac{2\rho_1-1}{\rho_1(\rho_1-1)}+\frac{\theta\varepsilon_0}{\rho_1-1}\right)\ge 0,\\
&\frac{\ell}{4}-\frac{\eta_5}{q_1^2}-\frac{(q_1-1)\eta_2}{q_1}-\frac{\varepsilon_0\eta_4}{\rho_1-1}-\frac{\gamma\varepsilon_0(\varepsilon-\varepsilon_0)^2}{2\varepsilon-3\varepsilon_0}\ge 0,
\end{aligned}
$$

we finally deduce

$$
L'(t)+\varepsilon_0 L(t)\le \varepsilon_0\left(\frac{1}{2}+\frac{1}{\gamma\ell}\right)(g\diamond\nabla u)(t). \tag{4.11}
$$

This inequality is the key differential estimate from which the general decay rate of the energy follows. Multiplying Inequality (4.11) by $\xi(t)$, we obtain

$$
\begin{aligned}
\xi(t)L'(t) &\le -\varepsilon_0\xi(t)L(t)+\varepsilon_0\xi(t)\left(\frac{1}{2}+\frac{1}{\gamma\ell}\right)(g\diamond\nabla u)(t)\\
&\le -\varepsilon_0\xi(t)L(t)-\varepsilon_0\left(\frac{1}{2}+\frac{1}{\gamma\ell}\right)(g'\diamond\nabla u)(t)\\
&\le -\varepsilon_0\xi(t)L(t)-\varepsilon_0\left(\frac{1}{2}+\frac{1}{\gamma\ell}\right)E'(t),
\end{aligned} \tag{4.12}
$$

where we have used Assumption $(B1)$, namely $g'(t)\le -\xi(t)g(t)$, together with the definition of the energy functional. This step is crucial, as it connects the dissipation induced by the memory kernel directly to the decay mechanism.

Next, we introduce the auxiliary Lyapunov functional

$$F(t) = \xi(t)L(t) + \varepsilon_0\left(\frac{1}{2} + \frac{1}{\gamma\ell}\right)E(t).$$

Since $\xi(t)$ is non-increasing and positive, a constant $k_0 > 0$ exists such that

$$0 \leq F(t) \leq k_0 L(t),$$

which shows that $F(t)$ is equivalent to $L(t)$ and hence to the energy $E(t)$.

Differentiating $F(t)$ and using (4.12), we obtain

$$\begin{aligned} F'(t) =& \xi'(t)L(t) + \xi(t)L'(t) + \varepsilon_0\left(\frac{1}{2} + \frac{1}{\gamma\ell}\right)E'(t) \\ \leq& \xi'(t)L(t) - \varepsilon_0\xi(t)L(t) \\ \leq& -\frac{\varepsilon_0}{k_0}\xi(t)F(t), \end{aligned} \tag{4.13}$$

where the monotonicity of $\xi(t)$, i.e., $\xi'(t) \leq 0$ has been used.

Integrating Inequality (4.13) over the interval (t_0, t) yields

$$F(t) \leq F(t_0)\exp\left(-\frac{\varepsilon_0}{k_0}\int_{t_0}^{t} \xi(s)\, ds\right).$$

Consequently, positive constants K and k exist such that

$$F(t) \leq K e^{-k\int_0^t \xi(s)ds}, \quad \forall t \geq 0.$$

Since $E(t)$ and $F(t)$ are equivalent, we finally obtain

$$E(t) \leq K e^{-k\int_0^t \xi(s)ds}, \quad \forall t \geq 0.$$

This proves the general decay of the energy and completes the proof of Theorem 4.1. Consequently, the global solutions are asymptotically stable and decay to zero with a rate dictated by the function $\xi(t)$. □

5. Blow-up

In this section, we investigate the finite-time blow-up behavior of solutions to the problem (1.1)–(1.3) when the initial energy is negative. The analysis relies on a suitable auxiliary functional and a concavity-type argument, which allows us to derive an explicit upper bound for the blow-up time.

We present and prove our main blow-up result as follows.

Theorem 5.1. *Suppose that the conditions of Theorem 2.3 are satisfied and that $\varepsilon > 0$ is small enough to ensure the dominance of the source terms over the damping effects. Assume that*

$$\max\left\{p^*,\ \frac{\rho_2}{\rho_2 - 1},\ 2 + \frac{4q_1 C_p}{p_1^2\ell}\right\} < p_2 < q_1,$$

where p^* denotes the largest root of the quadratic equation

$$p^2 - 2p - \frac{4}{\gamma\ell} = 0.$$

This condition guarantees the compatibility between the variable exponents and the nonlinear source terms, which is essential for the blow-up mechanism.

In this case, any solution of the problem (1.1)–(1.3) *with negative initial energy* $E(0) < 0$ *blows up in finite time* T^**. Moreover, the blow-up time satisfies the estimate*

$$T^* \leq \frac{1-\sigma}{\eta\,\sigma\,\psi^{\frac{\sigma}{1-\sigma}}(0)},$$

where $0 < \sigma < 1$*, and the auxiliary functional* $\psi(t)$ *is defined in* (5.3)*. In particular, this estimate provides an explicit upper bound on the lifespan of the solutions in terms of the initial data and the structural parameters of the problem.*

Proof. We restrict ourselves to the case $x \in \Omega_1$ for simplicity. The arguments for the remaining subdomains are analogous and are therefore omitted to avoid repetition.

Define $H(t) = -E(t)$. By using (2.3), we obtain

$$H'(t) = -E'(t) \geq \varepsilon \int_{\Omega_1} |\nabla u_t|^2 dx - \frac{1}{2}(g' \diamond \nabla u)(t). \tag{5.1}$$

Since the initial energy is assumed to be negative, that is $E(0) < 0$, it follows immediately from (5.1) that $H(t) \geq H(0) > 0$ for all t in the interval of existence. Moreover, from the definition of $H(t)$, we have

$$\begin{aligned} H(t) \leq& \int_{\Omega_1} \frac{1}{p^2(x)} |\nabla u|^{p(x)} dx + \int_{\Omega_1} \frac{1}{q(x)} |u|^{q(x)} \ln |u| dx \\ \leq& \frac{1}{p_1^2} \int_{\Omega_1} |\nabla u|^{p(x)} dx + \frac{1}{q_1} \int_{\Omega_1} |u|^{q(x)} \ln |u| dx, \end{aligned} \tag{5.2}$$

where the bounds $p(x) \geq p_1$ and $q(x) \geq q_1$ on $\overline{\Omega}_1$ have been used.

Let $0 < \sigma < 1$ and define the auxiliary functional

$$\psi(t) = H^{1-\sigma}(t) + \varepsilon_1 \left(\int_{\Omega_1} \frac{1}{\rho(x)-1} u |u_t|^{\rho(x)-1} dx - \int_{\Omega_1} u \Delta u_t dx \right), \tag{5.3}$$

where $\varepsilon_1 > 0$ is a sufficiently small constant to be fixed later so that all coefficients remain non–negative.

Differentiating (5.3) with respect to t and using Eq (1.1), we get

$$\begin{aligned} \psi'(t) =& (1-\sigma) H'(t) H^{-\sigma}(t) + \varepsilon_1 \int_{\Omega_1} \frac{1}{\rho(x)-1} u_t |u_t|^{\rho(x)-1} dx \\ &+ \varepsilon_1 \int_{\Omega_1} |\nabla u_t|^2 dx + \varepsilon_1 \int_{\Omega_1} u \left(|u_t|^{\rho(x)-2} u_{tt} - \Delta u_{tt} \right) dx \\ =& (1-\sigma) H'(t) H^{-\sigma}(t) + \varepsilon_1 \int_{\Omega_1} \frac{1}{\rho(x)-1} u_t |u_t|^{\rho(x)-1} dx \end{aligned}$$

$$+\varepsilon_1\int_{\Omega_1}|\nabla u_t|^2dx-\varepsilon_1\int_{\Omega_1}|\Delta u|^2dx-\varepsilon_1\int_{\Omega_1}|\nabla u|^{p(x)}\ln|\nabla u|dx$$
$$-\varepsilon_1\int_{\Omega_1}u\int_0^t g(t-s)\Delta u(s)dsdx-\varepsilon\varepsilon_1\int_{\Omega_1}\nabla u\cdot\nabla u_t dx$$
$$+\varepsilon_1\int_{\Omega_1}|u|^{q(x)}\ln|u|dx.$$

By virtue of (A1), it is easy to get

$$\left|\int_{\Omega_1}\frac{1}{\rho(x)-1}u_t|u_t|^{\rho(x)-1}dx\right|\leq\frac{1}{\rho_2-1}\int_{\Omega_1}|u_t|^{\rho(x)}dx. \tag{5.4}$$

Therefore,

$$\begin{aligned}\psi'(t)\geq&(1-\sigma)H'(t)H^{-\sigma}(t)-\frac{\varepsilon_1}{\rho_2-1}\int_{\Omega_1}|u_t|^{\rho(x)}dx+\varepsilon_1\int_{\Omega_1}|\nabla u_t|^2dx\\&-\varepsilon_1\int_{\Omega_1}|\Delta u|^2dx+\varepsilon_1\left(\int_0^t g(s)ds\right)\int_{\Omega_1}|\nabla u|^2dx\\&-\varepsilon_1\int_{\Omega_1}|\nabla u|^{p(x)}\ln|\nabla u|dx+\varepsilon_1\int_{\Omega_1}\nabla u\int_0^t g(t-s)(\nabla u(s)-\nabla u)dsdx\\&-\varepsilon\varepsilon_1\int_{\Omega_1}\nabla u\cdot\nabla u_t dx+\varepsilon_1\int_{\Omega_1}|u|^{q(x)}\ln|u|dx,\end{aligned} \tag{5.5}$$

where Assumption (A1) has been used.
Now, using (A1), (2.2), and the definition of $H(t)$, for some $\beta>0$ to be chosen later, we add and subtract $\beta H(t)$ on the right-hand side of (5.5). This yields

$$\begin{aligned}\psi'(t)\geq&(1-\sigma)H'(t)H^{-\sigma}(t)+\beta H(t)+\left(\frac{\beta}{\rho_2}-\frac{\varepsilon_1}{\rho_2-1}\right)\int_{\Omega_1}|u_t|^{\rho(x)}dx\\&+\left(\varepsilon_1+\frac{\beta}{2}\right)\int_{\Omega_1}|\nabla u_t|^2dx+\left(\frac{\beta}{2}-\varepsilon_1\right)\int_{\Omega_1}|\Delta u|^2dx+\frac{\beta}{2}(g\diamond\nabla u)(t)\\&-\frac{\beta}{p_1^2}\int_{\Omega_1}|\nabla u|^{p(x)}dx+\left(\frac{\beta}{p_2}-\varepsilon_1\right)\int_{\Omega_1}|\nabla u|^{p(x)}\ln|\nabla u|dx\\&+\left(\varepsilon_1-\frac{\beta}{2}\right)\left(\int_0^t g(s)ds\right)\int_{\Omega_1}|\nabla u|^2dx+\frac{\beta}{q_2^2}\int_{\Omega_1}|u|^{q(x)}dx\\&+\left(\varepsilon_1-\frac{\beta}{q_1}\right)\int_{\Omega_1}|u|^{q(x)}\ln|u|dx+\varepsilon_1\int_{\Omega_1}\nabla u\int_0^t g(t-s)(\nabla u(s)-\nabla u)dsdx\\&-\varepsilon\varepsilon_1\int_{\Omega_1}\nabla u\cdot\nabla u_t dx.\end{aligned} \tag{5.6}$$

At this stage, we specify the free parameter β introduced in (5.6). We set

$$\beta:=\varepsilon_1 p_2,$$

which is admissible, since $p_2>2$ by assumption. This choice allows us to balance the nonlinear terms and to obtain positive coefficients in the subsequent estimates.

With this choice, the inequality (5.6) becomes

$$\begin{aligned}\psi'(t) \geq &(1-\sigma)H'(t)H^{-\sigma}(t) + \varepsilon_1 p_2 H(t) + \varepsilon_1\left(\frac{p_2}{\rho_2} - \frac{1}{\rho_2 - 1}\right)\int_{\Omega_1} |u_t|^{\rho(x)}dx \\ &+\frac{\varepsilon_1(p_2+2)}{2}\int_{\Omega_1} |\nabla u_t|^2 dx + \frac{\varepsilon_1(p_2-2)}{2}\int_{\Omega_1} |\Delta u|^2 dx + \frac{\varepsilon_1 p_2}{2}(g \diamond \nabla u)(t) \\ &-\frac{\varepsilon_1 p_2}{p_1^2}\int_{\Omega_1} |\nabla u|^{p(x)}dx + \frac{\varepsilon_1(q_1-p_2)}{q_1}\int_{\Omega_1} |u|^{q(x)}\ln|u|dx \\ &-\frac{\varepsilon_1(p_2-2)}{2}\left(\int_0^t g(s)ds\right)\int_{\Omega_1} |\nabla u|^2 dx + \frac{\varepsilon_1 p_2}{q_2^2}\int_{\Omega_1} |u|^{q(x)}dx \\ &+\varepsilon_1\int_{\Omega_1} \nabla u \int_0^t g(t-s)(\nabla u(s) - \nabla u)dsdx - \varepsilon\varepsilon_1\int_{\Omega_1} \nabla u \cdot \nabla u_t dx. \end{aligned} \tag{5.7}$$

Using Assumption (A2), which ensures the coercivity of the elastic operator, we also obtain

$$\begin{aligned}\psi'(t) \geq &(1-\sigma)H'(t)H^{-\sigma}(t) + \varepsilon_1 p_2 H(t) + \varepsilon_1\left(\frac{p_2}{\rho_2} - \frac{1}{\rho_2 - 1}\right)\int_{\Omega_1} |u_t|^{\rho(x)}dx \\ &+\frac{\varepsilon_1(p_2+2)}{2}\int_{\Omega_1} |\nabla u_t|^2 dx + \frac{\varepsilon_1\ell(p_2-2)}{2}\int_{\Omega_1} |\Delta u|^2 dx + \frac{\varepsilon_1 p_2}{2}(g \diamond \nabla u)(t) \\ &-\frac{\varepsilon_1 p_2}{p_1^2}\int_{\Omega_1} |\nabla u|^{p(x)}dx + \frac{\varepsilon_1(q_1-p_2)}{q_1}\int_{\Omega_1} |u|^{q(x)}\ln|u|dx + \frac{\varepsilon_1 p_2}{q_2^2}\int_{\Omega_1} |u|^{q(x)}dx \\ &+\varepsilon_1\int_{\Omega_1} \nabla u \int_0^t g(t-s)(\nabla u(s) - \nabla u)dsdx - \varepsilon\varepsilon_1\int_{\Omega_1} \nabla u \cdot \nabla u_t dx. \end{aligned}$$

We now estimate the last two terms on the right-hand side using Young's inequality. For the mixed gradient term, we obtain

$$\begin{aligned}\varepsilon\varepsilon_1\left|\int_{\Omega_1} \nabla u \cdot \nabla u_t dx\right| \leq &\frac{\varepsilon_1\ell(p_2-2)}{8\gamma}\int_{\Omega_1} |\nabla u|^2 dx + \frac{2\gamma\varepsilon_1\varepsilon^2}{\ell(p_2-2)}\int_{\Omega_1} |\nabla u_t|^2 dx \\ \leq &\frac{\varepsilon_1\ell(p_2-2)}{8}\int_{\Omega_1} |\Delta u|^2 dx + \frac{2\gamma\varepsilon_1\varepsilon^2}{\ell(p_2-2)}\int_{\Omega_1} |\nabla u_t|^2 dx, \end{aligned} \tag{5.8}$$

where the second inequality follows from (1.4).
Similarly, arguing as in (4.3), we estimate the memory term as

$$\varepsilon_1\left|\int_{\Omega_1} \nabla u \int_0^t g(t-s)(\nabla u(s) - \nabla u)dsdx\right| \leq \frac{2\varepsilon_1}{\gamma\ell(p_2-2)}(g \diamond \nabla u)(t) + \frac{\varepsilon_1\ell(p_2-2)}{8}\int_{\Omega_1} |\Delta u|^2 dx, \tag{5.9}$$

where the Cauchy–Schwarz inequality, the definition of the memory functional $(g \diamond \nabla u)(t)$, and the assumption $\gamma\int_0^\infty g(s)ds < 1$ have been used.
Combining (5.2), (5.8), and (5.9) with (5.7), and using the hypothesis $p_2 < q_1$, we obtain

$$\begin{aligned}\psi'(t) \geq &(1-\sigma)H'(t)H^{-\sigma}(t) + \varepsilon_1 q_1 H(t) + \varepsilon_1\Big(\frac{p_2}{\rho_2} - \frac{1}{\rho_2 - 1}\Big)\int_{\Omega_1} |u_t|^{\rho(x)}dx \\ &+ \frac{\varepsilon_1\ell(p_2-2)}{4}\int_{\Omega_1} |\Delta u|^2 dx + \varepsilon_1\Big(\frac{p_2+2}{2} - \frac{2\gamma\varepsilon^2}{\ell(p_2-2)}\Big)\int_{\Omega_1} |\nabla u_t|^2 dx \end{aligned}$$

$$+\varepsilon_1\Big(\frac{p_2}{2}-\frac{2}{\gamma\ell(p_2-2)}\Big)(g\diamond\nabla u)(t)-\frac{\varepsilon_1 q_1}{p_1^2}\int_{\Omega_1}|\nabla u|^{p(x)}dx. \tag{5.10}$$

Next, we control the last negative term in (5.10) by means of the continuous embedding

$$H_0^2(\Omega_1)\hookrightarrow W_0^{1,p(\cdot)}(\Omega_1).$$

That is, $C_{*,p}>0$ exists such that

$$\int_{\Omega_1}|\nabla u|^{p(x)}dx\le C_{*,p}\int_{\Omega_1}|\Delta u|^2dx.$$

Substituting this into (5.10) yields

$$\begin{aligned}\psi'(t)\ge&(1-\sigma)H'(t)H^{-\sigma}(t)+\varepsilon_1 q_1 H(t)+\varepsilon_1\Big(\frac{p_2}{\rho_2}-\frac{1}{\rho_2-1}\Big)\int_{\Omega_1}|u_t|^{\rho(x)}dx\\&+\varepsilon_1\Big(\frac{\ell(p_2-2)}{4}-\frac{q_1C_{*,p}}{p_1^2}\Big)\int_{\Omega_1}|\Delta u|^2dx+\varepsilon_1\Big(\frac{p_2+2}{2}-\frac{2\gamma\varepsilon^2}{\ell(p_2-2)}\Big)\int_{\Omega_1}|\nabla u_t|^2dx\\&+\varepsilon_1\Big(\frac{p_2}{2}-\frac{2}{\gamma\ell(p_2-2)}\Big)(g\diamond\nabla u)(t).\end{aligned}$$

Now, by choosing $\varepsilon>0$ to be sufficiently small and using the exponent restrictions in Theorem 5.1, all coefficients in front of the terms on the right-hand side become strictly positive. Hence $\Lambda>0$ exists such that

$$\psi'(t)\ge\varepsilon_1\Lambda\Big(H(t)+\int_{\Omega_1}|u_t|^{\rho(x)}dx+\int_{\Omega_1}|\Delta u|^2dx+\int_{\Omega_1}|\nabla u_t|^2dx+(g\diamond\nabla u)(t)\Big),$$

where

$$\Lambda:=\min\Big\{\frac{p_2}{\rho_2}-\frac{1}{\rho_2-1},\ \frac{\ell(p_2-2)}{4}-\frac{q_1C_{*,p}}{p_1^2},\ \frac{p_2+2}{2}-\frac{2\gamma\varepsilon^2}{\ell(p_2-2)},\ \frac{p_2}{2}-\frac{2}{\gamma\ell(p_2-2)}\Big\}>0.$$

In particular, $\psi'(t)\ge0$ and thus $\psi(t)\ge\psi(0)>0$ for all $t\ge0$. Moreover, using the definition in (5.3) and applying Young's inequality to the cross-terms, we deduce that positive constants Λ_0,Λ_1 exist such that

$$\begin{aligned}\psi'(t)\ge&\varepsilon_1\Lambda\Big(H(t)+\int_{\Omega_1}|u_t|^{\rho(x)}dx+\int_{\Omega_1}|\Delta u|^2dx+\int_{\Omega_1}|\nabla u_t|^2dx\Big)\\\ge&\varepsilon_1\Lambda_0\Big(H(t)+\int_{\Omega_1}\frac{1}{\rho(x)-1}u|u_t|^{\rho(x)-1}dx-\int_{\Omega_1}u\Delta u_tdx\Big)\\\ge&\varepsilon_1\Lambda_1\Big(H^{1-\sigma}(t)+\varepsilon_1\int_{\Omega_1}\frac{1}{\rho(x)-1}u|u_t|^{\rho(x)-1}dx-\varepsilon_1\int_{\Omega_1}u\Delta u_tdx\Big)^{\frac{1}{1-\sigma}}\\\ge&\eta\,\psi^{\frac{1}{1-\sigma}}(t),\end{aligned}\tag{5.11}$$

for some $\eta>0$.
Integrating (5.11) from 0 to t, we obtain

$$\psi^{-\frac{\sigma}{1-\sigma}}(t)\le\psi^{-\frac{\sigma}{1-\sigma}}(0)-\frac{\eta\sigma}{1-\sigma}t,$$

which implies that $\psi(t) \to +\infty$ at a finite time T^* satisfying

$$T^* \leq \frac{1-\sigma}{\eta\sigma\,\psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

Consequently, the solution blows up in finite time. This completes the proof of Theorem 5.1. □

Remark 5.1. In the proofs of the general decay and blow-up results, namely Theorems 4.1 and 5.1, we presented the detailed arguments by fixing $x \in \Omega_1$ in order to simplify the exposition and avoid unnecessary repetitions. However, as already emphasized in Section 3 and in the proof of the global existence result, the domain Ω is decomposed into the disjoint subsets Ω_i, $i = 1, \ldots, 4$, according to the behavior of the variable exponents. For each of these subsets, the same analytical techniques and estimates apply with only minor modifications in the constants. Therefore, by repeating the arguments developed in Section 3 for each Ω_i, we can rigorously extend the decay and blow-up results to hold for all $x \in \Omega$.

6. Conclusion and perspective

In this paper, we have analyzed a fourth-order viscoelastic problem with variable exponents, logarithmic nonlinearities, and a strong damping term, which models a broad class of nonhomogeneous and nonlinear materials. Using the Nehari manifold approach, we established the global existence of the solutions. Furthermore, by applying a perturbed energy method combined with an appropriate Lyapunov functional, we proved that the global solutions exhibit a general decay behavior under suitable assumptions on the initial data, the relaxation function, and the variable exponents. Additionally, we derived sufficient conditions ensuring the finite-time blow-up of solutions starting from negative initial energy, thereby revealing the delicate balance between the dissipative effects induced by damping and memory terms and the destabilizing influence of nonlinear logarithmic sources.

The results obtained here extend and unify several existing works by allowing variable exponents and logarithmic terms simultaneously in both the damping and source components. As a perspective for future research, one may investigate the existence and stability of multiple solutions, study alternative damping mechanisms or different memory kernels, or extend the present analysis to coupled systems and other classes of higher-order viscoelastic equations. Moreover, numerical simulations could be developed to illustrate the theoretical decay and blow-up behaviors and to provide further insight into potential applications in material science and engineering.

Author contributions

Mohammad Shahrouzi: writing-original draft, writing-review and editing, methodology; Faramarz Tahamtani and Salah Boulaaras: formal Analysis, review and editing.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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