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Research article

Stability and instability of standing waves for the Schrödinger-Choquard equation with mixed fractional Laplacians

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Abstract: In this paper, we investigate the stability and instability of standing waves for the Schrödinger-Choquard equation with mixed fractional Laplacians. We first establish the existence and stability of normalized standing waves in the L^2-subcritical case. Subsequently, we prove the existence and strong instability of normalized ground state standing waves in the L^2-supercritical case.

Keywords: standing waves; stability; instability; mixed fractional Laplacians; ground state
Mathematics Subject Classification: 35Q55, 35A15

1. Introduction

In this paper, we investigate the stability and instability of standing waves to the following Schrödinger-Choquard equation with mixed fractional Laplacians

$$\begin{cases} i\partial_t\psi - (-\Delta)^{s_1}\psi - a(-\Delta)^{s_2}\psi + (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x) \in H^{s_1}(\mathbb{R}^N), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $N > 2s_1$, $a = 1$, $0 < s_2 < s_1 < 1$ and $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s_1}$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}},$$

$\alpha \in (0,N)$, Γ denotes the Gamma function. The fractional Laplacian $(-\Delta)^s$ is defined by $\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi)$ for $\xi \in \mathbb{R}^N$, where $\mathcal{F}$ represents the Fourier transform.

In the case $a = 0$, equation (1.1) simplifies to the standard Schrödinger-Choquard equation. It describes various physical phenomena, such as Hartree-Fock theory, nonrelativistic quantum theory, and the space-fractional quantum mechanics, see [1–4] for a more comprehensive introduction. In the case $a \neq 0$, equation (1.1) naturally arises in the description of the diffusion of biological populations within an ecological niche. In ecology, the biological population density ultimately satisfies an equation involving mixed fractional Laplacians, see [5] for more related physical backgrounds. Recently, there has been increasing attention in extending the results established for the classical fractional Schrödinger equation to the mixed fractional Schrödinger equation, see [6–8]. Furthermore, there also has been increasing attention in extending the methods developed for Schrödinger equation to other dispersive wave equations, see [9, 10].

The fractional Schrödinger-Choquard equations have been widely studied during the past few decades, see [11–14]. Equation (1.1) enjoys a class of special solutions known as standing waves, namely solutions of the form $\psi(t,x) = e^{i\omega t}u(x)$, where $\omega \in \mathbb{R}$ represents the frequency, and $u \in H^{s_1}(\mathbb{R}^N)$ is a non-trivial solution to the following elliptic equation:

$$(-\Delta)^{s_1}u + (-\Delta)^{s_2}u + \omega u - (I_\alpha * |u|^p)|u|^{p-2}u = 0. \tag{1.2}$$

When studying the elliptic equation (1.2), there are two distinct approaches with respect to the frequency ω. The first approach is to fix the frequency $\omega \in \mathbb{R}$, in which case solutions to equation (1.1) can be obtained by searching critical points of the action functional $S_\omega(u)$ on $H^{s_1}(\mathbb{R}^N)$, where

$$S_\omega(u) = \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx. \tag{1.3}$$

In this case, particular attention should be given to least action solutions.

On the other hand, ones can study solutions of equation (1.2) having prescribed L^2-norm. Namely, for every $c > 0$, solutions satisfy the constraint

$$S_1(c) := \{u \in L^2(\mathbb{R}^N) : \ \|u\|_{L^2}^2 = c\}. \tag{1.4}$$

Physically, solutions obtained in this manner are referred as normalized solutions, which formally can be obtained by searching for the critical points of the energy functional $E(u)$ restricted on $S_1(c)$, where

$$E(u) = \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx. \tag{1.5}$$

In this case, the frequency $\omega \in \mathbb{R}$ can no longer be imposed arbitrarily but instead appears as a Lagrange multiplier associated with the constraint $S_1(c)$. Moreover, this approach proves to be significant even from the purely mathematical aspect, as it provides deeper insights into the properties of the stationary solutions for (1.1), such as their stability and instability, which was already obvious in the seminal works of T. Cazenave and P.-L. Lions in [15].

Definition 1.1. (Ground state) *A function $u_\omega \in \mathcal{A}_\omega$ is called a ground state for* (1.2) *if it is a minimizer of $S_\omega(u)$ over the set of $\mathcal{A}_\omega$, where $\mathcal{A}_\omega$ denote the set of non-trivial solutions of* (1.2). *The set of ground state is dedoted by $\mathcal{G}_\omega$. In particular*

$$\mathcal{G}_\omega := \{u_\omega \in \mathcal{A}_\omega,\ S_\omega(u_\omega) \le S_\omega(v),\ for\ any\ v \in \mathcal{A}_\omega\}.$$

For the evolutional equation (1.1), one of the key problems is to investigate the stability of standing waves, which is defined as follows.

Definition 1.2. *The set $\mathcal{M}$ is said to be orbitally stable, if any $\epsilon > 0$, there exists $\delta > 0$ such that for any initial data ψ_0 satisfying*

$$\inf_{u\in\mathcal{M}} \|\psi_0 - u\|_{H^{s_1}} < \delta,$$

the corresponding solutions $\psi(t)$ of (1.1) satisfy that

$$\inf_{u\in\mathcal{M}} \|\psi(t) - u\|_{H^{s_1}} < \epsilon.$$

Remark 1.3. In view of Definition 1.2, to investigate the orbital stability of standing waves, it is important to show that the solutions of equation (1.1) exist globally, at least for the initial data ψ_0 sufficiently close to the set $\mathcal{M}$.

For equation (1.1), Bahri et al. in [7] investigated the half-wave-Schrödinger equation related to (1.1) with $s_1 = \frac{1}{2}$ and $s_2 = 1$. When $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s_1}$, Chergui in [8] studied the blow-up solutions for equation (1.1). However, to the best of our knowledge, there are no any results regarding the stability or instability of standing waves for equation (1.1). Therefore, the main goal of this paper is to study the stability and instability of standing waves for equation (1.1).

We firstly consider the existence and stability of standing waves in the L^2-subcritical case. When $1 + \frac{\alpha}{N} < p < 1 + \frac{\alpha+2s_2}{N}$, for any $c > 0$, it follows easily that the energy functional $E(u)$ restricted on $S_1(c)$ is bounded from below. Therefore, we can obtain the existence of normalized solutions for equation (1.1) by studying the following global minimization problem:

$$\gamma_1(c) = \inf_{u\in S_1(c)} E(u). \tag{1.6}$$

Let $\mathcal{M}$ denote the set of all minimizers of energy functional $E(u)$ on $S_1(c)$, defined as

$$\mathcal{M} := \{u \in S_1(c) : E(u) = \gamma_1(c)\}.$$

Obviously, for any $u \in \mathcal{M}$, there exists a Lagrange multiplier ω such that $(u, \omega) \in H^{s_1}(\mathbb{R}^N) \times \mathbb{R}$ solves equation (1.2). Based on this, we establish the existence of standing waves for equation (1.1).

Our first results are as follows:

Theorem 1.4. *Let $N > 2s_1$, $0 < s_2 < s_1 < 1$. Assume that $1 + \frac{\alpha}{N} < p < 1 + \frac{\alpha+2s_2}{N}$, then for any $c > 0$, (1.2)-(1.4) possesses a ground state $u_c \in S_1(c)$ satisfying $E(u_c) = \gamma_1(c)$.*

By applying the approach developed by T. Cazenave and P.-L. Lions in [15], we can obtain the orbital stability of standing waves for equation (1.1).

Theorem 1.5. *Let $N \geq 2$, $\frac{N}{2N-1} < s_2 < s_1 < 1$. Assume that $1 + \frac{\alpha}{N} < p < 1 + \frac{\alpha+2s_2}{N}$, then for any $c > 0$, the set $\mathcal{M}$ is orbitally stable.*

Next, we focus on studying the existence of standing waves for (1.1) in the L^2-supercritical case $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$. It follows easily that the energy functional $E(u)$ restricted on $S_1(c)$ is unbounded from below. Therefore, the approach used in Theorem 1.4 cannot be applied to establish the existence of standing waves for equation (1.1). In this case, the existence is typically proven using the mountain pass argument developed by Jeanjean in [16], also see [17] and [18]. However, inspired by the work of Feng et al. in [19], here we adopt a simpler method to establish the existence of standing waves, which avoids applying the mountain pass argument. Instead, we consider the following local minimization problem:

$$\gamma_2(c) = \inf_{u\in S_2(c)} E(u), \tag{1.7}$$

where the $S_2(c)$ is defined as

$$S_2(c) := \{u \in S_1(c) : \ Q(u) = 0\}, \tag{1.8}$$

and

$$\begin{aligned} Q(u) &= \frac{dE(u_\lambda)}{d\lambda}\Big|_{\lambda=1} \\ &= s_1\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + s_2\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 - \frac{N(p-1)-\alpha}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx. \end{aligned} \tag{1.9}$$

Here $Q(u) = 0$ is the Pohozaev identity related to (1.2) and (1.4). Let $\mathcal{M}_c$ denote the set of all minimizers of E on $S_2(c)$, defined as

$$\mathcal{M}_c := \{u \in S_2(c) : E(u) = \gamma_2(c)\}.$$

Our second result is as follows:

Theorem 1.6. *Let* $N > 2s_1$, $0 < s_2 < s_1 < 1$, $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$. *Assume that* $\frac{2ps_2}{N(p-1)-\alpha} > 1$, *then for any* $c > 0$, (1.2)-(1.4) *possesses a ground state* $u_c \in S_1(c)$ *satisfying* $E(u_c) = \gamma_2(c)$.

Remark 1.7. In the proof of this Theorem, it is essential to notice the following two points: (i) scaling function $g(\lambda) = E(u_\lambda)$ has a unique critical point of maximum on $(0,\infty)$; (ii) function $\gamma_2(c)$ is strictly decreasing. The first point guarantees the existence of minimizers, while the second ensures that the minimizers we find indeed satisfy the constraint $S_1(c)$.

Finally, we show that the strong instability of standing waves in the L^2-supercritical case. To the best of our knowledge, the usual method for establishing the strong instability of normalized ground state standing waves of the classical NLS ($a = 0$, $s_1 = 1$) is to obtain the key estimate $Q(\psi(t)) \le 2(E(\psi_0) - E(u_c))$. Subsequently, it follows from the virial identity that

$$\frac{d^2}{dt^2}\|x\psi(t)\|_{L^2}^2 = 8Q(\psi(t)) \le 16(E(\psi_0) - E(u_c)) < 0,$$

where the $Q(\psi(t))$ is defined as (1.9). This implies that the corresponding solution $\psi(t)$ for equation (1.1) with $a = 0$, $s_1 = 1$ blows up in finite time.

However, for equation (1.1), it follows from Lemma 2.3 that

$$\frac{d}{dt}M_{\psi R}[\psi(t)] \le 4Q(\psi(t)) + C\epsilon(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2) + o_R(1),$$

where $\epsilon > 0$, $o_R(1) \to 0$ as $R \to \infty$. Since $\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2$ may be unbounded, there exist some essential difficulties for discussing the strong instability of normalized ground state standing waves between the classical NLS ($a = 0$, $s_1 = 1$) and the equation (1.1). In this paper, we will propose some new ideas to overcome these difficulties.

Our final result is as follows:

Theorem 1.8. *Let $N \geq 2$, $\frac{N}{2N-1} < s_2 < s_1 < 1$ and $c > 0$. For each $u_c \in \mathcal{M}_c$, the standing waves $\psi(t,x) = e^{i\omega t}u_c$ of* (1.1)*, where $\omega \in \mathbb{R}$ is the Lagrange multiplier, is strongly unstable in the following sense: for any $\epsilon > 0$, there exists $\psi_0 \in H^{s_1}$ such that $\|\psi_0 - u\|_{H^{s_1}} < \epsilon$ and the corresponding solution $\psi(t)$ of* (1.1) *with initial data ψ_0 blows up in finite time.*

Remark 1.9. In Theorem 1.8, we show that the strong instability of normalized ground state standing waves. Moreover, by applying the similar approach, ones can establish the strong instability of ground state (least action solutions) standing waves.

This paper is organized as follows. In Section 2, we recall some preliminaries, including the generalized Gagliardo–Nirenberg inequality, the profile decomposition of bounded sequences in $H^s(\mathbb{R}^N)$ and other related results. In Section 3, we prove the existence and orbital stability of standing waves in the L^2-subcritical case. In Section 4, we establish the existence of standing waves in the L^2-supercritical case. Finally, in Section 5, we show that the normalized ground state standing waves are strongly unstable.

Notations: Throughout the paper, we use the following notations. For $0 < s < 1$, the fractional Sobolev space is defined by

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s})|\hat{u}(\xi)|^2 dx < \infty\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \|u\|_{\dot{H}^s(\mathbb{R}^N)},$$

and

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}}.$$

2. Preliminaries

In this section, we recall some preliminaries that will be used later in our proof. Firstly, we review the local well-posedness for the Cauchy problem (1.1), which has been established in [8].

Lemma 2.1. *Assume $N \geq 2$, $\frac{N}{2N-1} < s_2 < s_1 \leq 1$, $max(N - 4s_1, 0) < \alpha < N$ and $2 \leq p < 1 + \frac{\alpha+2s_1}{N-2s_1}$. Let $(q_j, r) := (\frac{4s_j p}{Np-N-\alpha}, \frac{2Np}{N+\alpha})$, $j = 1,2$. Then for all $\psi_0 \in H^{s_1}_{rad}(\mathbb{R}^N)$ there exists $T_{max} := T_{max}(\|\psi_0\|_{H^{s_1}}) > 0$ and a unique maximal solution $\psi \in C([0,T_{max}), H^{s_1}_{rad}(\mathbb{R}^N))$ to the problem* (1.1) *which satisfies the alternative: either $T_{max} = \infty$ or $T_{max} < \infty$ and $\lim_{t\to T^*}(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|^2_{L^2} + \|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|^2_{L^2}) = \infty$. Furthermore, for all $0 \leq t < T_{max}$, the solution satisfies the mass and energy conservation laws:*

$$\|\psi(t)\|^2_{L^2} = \|\psi_0\|^2_{L^2},$$

and

$$E(\psi(t)) = E(\psi_0).$$

In addition, to obtain the strong instability of normalized ground state standing waves related to (1.1), the following Lemmas are essential.

Lemma 2.2. [8] *Let $N \geq 2$, $\frac{1}{2} < s_2 < s_1 < 1$ and $p \geq 1 + \frac{2s_1+\alpha}{N}$. Let $u_0 \in H^{s_1}_{rad}(\mathbb{R}^N)$ be such that $E(u_0) \neq 0$ and $u \in C_{T^*}(H^{s_1}_{rad}(\mathbb{R}^N))$ be the maximal solution of* (1.1) *with initial datum u_0. If there exist $R > 0$, $t_0 > 0$ and $C > 0$ such that*

$$M_{\psi R}[u(t)] \leq -C \int_{t_0}^{t} (\|(-\Delta)^{\frac{s_1}{2}} u(s)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}} u(s)\|_{L^2}^2)^2 ds,$$

holds for any $t \geq t_0$. Then $u(t)$ cannot exist globally in time, equivalently $T^* < \infty$.

Lemma 2.3. [8] *Let $N \geq 2$, $\frac{1}{2} < s_2 < s_1 < 1$, $0 < \alpha < N$ and $1 + \frac{\alpha}{N} < p \leq \frac{N+\alpha}{N-2s_1}$. Assume that $\psi \in C_T(H^{s_1}_{rad}(\mathbb{R}^N))$ is the solution of equation* (1.1) *with initial data ψ_0. Then*

$$\begin{aligned}\frac{d}{dt} M_{\psi R}[\psi(t)] \leq\ & 2(Np - N - \alpha)E(\psi_0) \\ & - 2N(p - 1 - \frac{\alpha + 2s_1}{N})(\|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^2) \\ & + C(R^{-2s_2} + R^{-(N-1-\epsilon_1)(p-1-\frac{\alpha}{N})} \|(-\Delta)^{\frac{s_1}{2}} u(t)\|_{L^2}^{\frac{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{s_1}}),\end{aligned} \tag{2.1}$$

where $R > 0$ and $\epsilon > 0$ small enough.

Furthermore, the following compactness lemma plays a significant role in our discussion.

Lemma 2.4. [20] *Let $N \geq 1$, $0 < s < 1$, $0 < p < \frac{4s}{N-2s}$. Let $\{u_n\}$ be a bounded sequence in $H^s(\mathbb{R}^N)$ and satisfies that*

$$\liminf_{n\to\infty} \|u_n\|_{L^{p+2}} \geq m,$$

for some $m > 0$. Then, there exists a sequence $\{x_n\}$ in $\mathbb{R}^N$ and $u \in H^s(\mathbb{R}^N)\backslash\{0\}$ such that up to a subsequence,

$$u_n(\cdot + x_n) \rightharpoonup u \neq 0 \text{ weakly in } H^s(\mathbb{R}^N).$$

In this paper, we shall frequently use the following generalized Gagliardo-Nirenberg inequality in $H^s(\mathbb{R}^N)$, which has been provided in [21].

Lemma 2.5. *Let $0 < s < 1$ and $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s}$, then for all $u \in H^s(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \leq C_{opt} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^{\frac{NP-N-\alpha}{s}} \|u\|_{L^2}^{\frac{N+\alpha-Np+2sp}{s}},$$

where the optimal constant C is given by

$$C_{opt} = \frac{2sp}{2sp - Np + N + \alpha} \left(\frac{2sp - Np + N + \alpha}{Np - N - \alpha}\right)^{\frac{Np-N-\alpha}{2s}} \|Q\|_{L^2}^{2-2p},$$

where Q is the ground state of the elliptic equation:

$$(-\Delta)^s u + u - (I_\alpha * |u|^p)|u|^{p-2} u = 0. \tag{2.2}$$

Next, we recall the profile decomposition of bounded sequences in $H^s(\mathbb{R}^N)$ that has been established in [21, 22].

Lemma 2.6. *Let $N \geq 3$, $0 < s < 1$ and $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s}$, if $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in $H^s(\mathbb{R}^N)$, then there exists a subsequence of $\{u_n\}_{n=1}^{\infty} \subset H^{s_1}(\mathbb{R}^N)$ (still denoted by $\{u_n\}_{n=1}^{\infty}$), a family $\{x_n^j\}$ of sequence in $\mathbb{R}^N$ and a sequence $\{U^j\}_{j=1}^{\infty}$ in $H^s(\mathbb{R}^N)$ such that*

(*i*) *for every $k \neq j$, $\|x_n^k - x_n^j\| \to \infty$, as $n \to \infty$,*

(*ii*) *for every $l \geq 1$ and every $x \in \mathbb{R}^N$, we have*

$$u_n(x) = \sum_{j=1}^{l} U^j(x - x_n^j) + t_n^l,$$

with $\sup_{n\to\infty} \|t_n^l\|_{L^q} \to 0$ as $l \to \infty$ for every $q \in (2, \frac{2N}{N-2s})$. Moreover,

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|U^j\|_{L^2}^2 + \|t_n^l\|_{L^2}^2 + o(1), \tag{2.3}$$

$$\|(-\Delta)^{\frac{s}{2}} u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|(-\Delta)^{\frac{s}{2}} U^j\|_{L^2}^2 + \|(-\Delta)^{\frac{s}{2}} t_n^l\|_{L^2}^2 + o(1), \tag{2.4}$$

$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * |\sum_{j=1}^{l} U^j(\cdot - x_n^j)|^p)|\sum_{j=1}^{l} U^j(\cdot - x_n^j)|^p dx \\ &= \sum_{j=1}^{l} \int_{\mathbb{R}^N} (I_\alpha * |U^j(\cdot - x_n^j)|^p)|U^j(\cdot - x_n^j)|^p dx + o(1), \end{aligned} \tag{2.5}$$

where $o(1) = o_n(1) \to 0$ as $n \to \infty$.

Next, we also require the following Brézis-Lieb Lemma.

Lemma 2.7. [23] *Let $N \geq 1$, suppose that $f_n \to f$ almost everywhere and $\{f_n\}$ is a bounded sequence in $H^s(\mathbb{R}^N)$, then*

$$\lim_{n\to\infty} (\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p - \|f\|_{L^p}^p) = 0,$$

$$\lim_{n\to\infty} \Big(\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p dx - \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^p)|u_n - u|^p dx - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \Big) = 0.$$

Finally, we recall the Hardy-Littlewood-Sobolev inequality, which also plays an important role in our proofs.

Lemma 2.8. (Hardy-Littlewood-Sobolev inequality [24]) *Let $N \geq 3$, $\alpha \in (0, N)$ and $p, q > 1$ be constants such that*

$$\frac{1}{p} + \frac{N-\alpha}{N} + \frac{1}{q} = 2.$$

Assume that $u \in L^p(\mathbb{R}^N)$ and $v \in L^q(\mathbb{R}^N)$. Then there exists a constant $C(N, \alpha, p)$ independent of u, v such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)v(y)}{|x-y|^{N-\alpha}} dxdy \leq C(N, \alpha, p)\|u\|_{L^p}\|v\|_{L^q}.$$

Remark 2.9. *By the Hardy-Littlewood-Sobolev inequality and Sobolev embedding theorem, we can obtain the following inequality*

$$\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx \le C_1(\int_{\mathbb{R}^N}|u|^{\frac{2Np}{N+\alpha}}dx)^{1+\frac{\alpha}{N}} \le \|u\|_{H^1}^{2p},$$

for any $q \in [1+\frac{\alpha}{N}, \frac{N+\alpha}{N-2}]$, $C_1 > 0$ *is a constant depending only on* N, α, p.

3. L^2-Subcritical case

In this section, we solve the minimization problem (1.6) and prove Theorem 1.4 by utilizing the profile decomposition of bounded sequences in $H^s(\mathbb{R}^N)$. The proof is divided into four steps.

Proof of Theorem 1.4. **Step 1.** Firstly, we prove that the minimization problem (1.6) is well defined, namely, $\gamma_1(c) > -\infty$. Indeed, using the generalized Gagliardo –Nirenberg inequality and (1.5), we derive that

$$\begin{aligned} E(u) &= \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx \\ &\ge \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 - \frac{C_{opt}}{2p}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^{\frac{Np-N-\alpha}{s_1}}\|u\|_{L^2}^{\frac{N+\alpha-Np+2s_1p}{s_1}} \\ &= \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 - C(p,s_1,N,\alpha,c)\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^{\frac{Np-N-\alpha}{s_1}}. \end{aligned} \tag{3.1}$$

When $1+\frac{\alpha}{N} < p < 1+\frac{\alpha+2s_2}{N}$, applying the Young inequality with ϵ, (3.1) implies that

$$E(u) \ge \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 - \epsilon\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 - C_1(p,s_1,N,\alpha,c,\epsilon). \tag{3.2}$$

Thus, $\gamma_1(c) > -\infty$.

It follows from (3.2) that the energy functional $E(u)$ restricted on $S_1(c)$ is bounded from below, which implies that minimization problem (1.6) is well-defined. Moreover, we consider the scaling

$$u_\lambda(x) = \lambda^{\frac{N}{2}}u(\lambda x),$$

where $\lambda \in (0,1)$. Consequently,

$$E(u_\lambda) = \frac{\lambda^{2s_1}}{2}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + \frac{\lambda^{2s_2}}{2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 - \frac{\lambda^{Np-N-\alpha}}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx. \tag{3.3}$$

Combining (3.2) and (3.3), we conclude that $-\infty < \gamma_1(c) < 0$.

Step 2. Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence for (1.6), that is, $\lim_{n\to\infty} E(u_n) = \gamma_1(c)$. From the definition of $\gamma_1(c)$ and (3.2), we obtain

$$(\frac{1}{2}-\epsilon)\|(-\Delta)^{\frac{s_1}{2}}u_n\|_{L^2}^2 \le \gamma_1(c) + 1 + C_1(p,s_1,N,\alpha,\epsilon).$$

which implies that the minimizing sequence $\{u_n\}_{n=1}^\infty$ is bounded in $H^{s_1}(\mathbb{R}^N)$.

Step 3. We claim that

$$\lim_{n\to\infty} \|u_n\|_{L^{p+1}}^{p+1} > 0.$$

If not, assume that $\lim_{n\to\infty} \|u_n\|_{L^{p+1}}^{p+1} = 0$, it follows from Lemma 2.8 and Remark 2.9 that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p dx = 0.$$

Thus,

$$\gamma_1(c) = \lim_{n\to\infty} E(u_n) = \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}} u_n\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}} u_n\|_{L^2}^2 \geq 0,$$

which contradicts $\gamma_1(c) < 0$.

Step 4. From Lemma 2.4 and Lemma 2.6, there exists a weakly convergent subsequence (still denoted by $\{u_n\}$) such that

$$u_n = \sum_{j=1}^{l} U^j(x - x_n^j) + t_n^l, \tag{3.4}$$

and $\lim_{n,l\to\infty} \sup \|t_n^l\|_{L^q} = 0$ for every $q \in (2, \frac{2N}{N-2s_1})$. Then

$$E(u_n) = E(\sum_{j=1}^{l} U^j(x - x_n^j)) + E(t_n^l) + o_{n,l}(1). \tag{3.5}$$

Let $U_{r_j}^j = r_j U^j(x - x_n^j)$, where $r_j = \frac{\sqrt{c}}{\|U^j\|_{L^2}} \neq 0$. Substituting $U_{r_j}^j$ into the energy functional $E(u)$ yields that

$$\begin{aligned}
E(U_{r_j}^j) &= \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}} U_{r_j}^j\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}} U_{r_j}^j\|_{L^2}^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |U_{r_j}^j|^p)|U_{r_j}^j|^p dx \\
&= \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}} r_j U^j(x - x_n^j)\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}} r_j U^j(x - x_n^j)\|_{L^2}^2 \\
&\quad - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |r_j U^j(x - x_n^j)|^p)|r_j U^j(x - x_n^j)|^p dx \\
&= r_j^2(\frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}} U^j\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}} U^j\|_{L^2}^2 \\
&\quad - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |U^j|^p)|U^j|^p dx) + \frac{r_j^2 - r_j^{2p}}{2p}\int_{\mathbb{R}^N} (I_\alpha * |U^j|^p)|U^j|^p dx \\
&= r_j^2 E(U^j) + \frac{r_j^2 - r_j^{2p}}{2p}\int_{\mathbb{R}^N} (I_\alpha * |U^j|^p)|U^j|^p dx,
\end{aligned} \tag{3.6}$$

which implies

$$E(U^j) = \frac{E(U_{r_j}^j)}{r_j^2} + \frac{r_j^{2p-2} - 1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |U^j|^p)|U^j|^p dx. \tag{3.7}$$

Noting that $\|\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}\|_{L^2}^2 = c$, therefore

$$\begin{aligned}
E(\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}) &= \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}}\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}}\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}\|_{L^2}^2 \\
&\quad - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}|^p)|\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}|^p dx \\
&= \frac{c}{\|t_n^l\|_{L^2}^2}(\frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}}t_n^l\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}}t_n^l\|_{L^2}^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |t_n^l|^p)|t_n^l|^p dx) \\
&\quad + (\frac{c}{2p\|t_n^l\|_{L^2}^2} - \frac{(\sqrt{c})^{2p}}{2p\|t_n^l\|_{L^2}^{2p}})\int_{\mathbb{R}^N}(I_\alpha * |t_n^l|^p)|t_n^l|^p dx \\
&= \frac{c}{\|t_n^l\|_{L^2}^2}E(t_n^l) + (\frac{c}{2p\|t_n^l\|_{L^2}^2} - \frac{(\sqrt{c})^{2p}}{2p\|t_n^l\|_{L^2}^{2p}})\int_{\mathbb{R}^N}(I_\alpha * |t_n^l|^p)|t_n^l|^p dx. \qquad (3.8)
\end{aligned}$$

By (3.8), we obtain that

$$\begin{aligned}
E(t_n^l) &= \frac{\|t_n^l\|_{L^2}^2}{c}E(\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}) + (\frac{c^{p-1}}{2p\|t_n^l\|_{L^2}^{2p-2}} - \frac{1}{2p})\int_{\mathbb{R}^N}(I_\alpha * |t_n^l|^p)|t_n^l|^p dx \\
&\geq \frac{\|t_n^l\|_{L^2}^2}{c}E(\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}). \qquad (3.9)
\end{aligned}$$

From the definition of $\gamma_1(c)$, it follows that

$$E(U_{r_j}^j) \geq \gamma_1(c),\ E(\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}) \geq \gamma_1(c).$$

By (3.5) and (3.7), it follows that

$$\begin{aligned}
E(u_n) &= E(\sum_{j=1}^{l}U^j(x - x_n^j)) + E(t_n^l) + o_{n,l}(1) \\
&\geq \sum_{j=1}^{l}\Big(\frac{E(U_{r_j}^j)}{r_j^2} + \frac{r_j^{2p-2}-1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |U^j|^p)|U^j|^p dx\Big) + \frac{\|t_n^l\|_{L^2}^2}{c}E(\frac{\sqrt{c}t_n^l}{\|t_n^l\|_{L^2}}) + o_{n,l}(1). \qquad (3.10)
\end{aligned}$$

Meanwhile, since the series $\sum_{j=1}^{l}\|U^j\|_{L^2}$ is convergent, then there exist a $j_1 > 0$ such that

$$\inf_{j\geq 1} r_j = r_{j_1} = \frac{c^{\frac{1}{2}}}{\|U^{j_1}\|_{L^2}}.$$

Therefore,

$$\inf_{j\geq 1}\frac{r_j^{2p-2}-1}{2p} = \inf_{j\geq 1}(\frac{r_j^{2p-2}}{2p} - \frac{1}{2p}) = \frac{1}{2p}(\frac{c^{p-1}}{\|U^{j_1}\|_{L^2}^{2p-2}} - 1).$$

Let $n, l \to \infty$, then (3.10) can be estimated by

$$\gamma_1(c) \geq \gamma_1(c) + \frac{1}{2p}\Big(\frac{c^{p-1}}{\|U^{j_1}\|_{L^2}^{2p-2}} - 1\Big). \tag{3.11}$$

Meanwhile,

$$\lim_{n\to\infty} E(u_n) = \gamma_1(c). \tag{3.12}$$

Combining (3.11) with (3.12), we obtain

$$\frac{1}{2p}\Big(\frac{c^{p-1}}{\|U^{j_1}\|_{L^2}^{2p-2}} - 1\Big) \leq 0.$$

Since $p > 1 + \frac{\alpha}{N}$, then we have $\|U^{j_1}\|_{L^2}^2 \geq c$, which implies that (3.4) reduces to a single term U^{j_1}. Thus,

$$u_n = U^{j_1}(x - x_n^j) + t_n^l \ \ and \ \ \|U^{j_1}\|_{L^2}^2 = c.$$

Namely

$$\|u_n\|_{L^2}^2 \to \|U^{j_1}\|_{L^2}^2 \ \ and \ \ \|t_n^{j_1}\|_{L^2}^2 \to 0 \ (n \to \infty).$$

In order to show that $\lim\limits_{n\to\infty} \|(-\Delta)^{\frac{s_1}{2}} u_n\|_{L^2}^2 = \|(-\Delta)^{\frac{s_1}{2}} U^{j_1}\|_{L^2}^2$, by (2.4) and (3.5), we note that

$$\begin{aligned}
E(u_n) = &\frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}} U^{j_1}\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}} U^{j_1}\|_{L^2}^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |U^{j_1}|^p)|U^{j_1}|^p dx \\
&+ \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}} t_n^{j_1}\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}} t_n^{j_1}\|_{L^2}^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |t_n^{j_1}|^p)|t_n^{j_1}|^p dx + o_n(1) \\
= &E(U^{j_1}) + \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}} t_n^{j_1}\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{\frac{s_2}{2}} t_n^{j_1}\|_{L^2}^2 + o_n(1).
\end{aligned} \tag{3.13}$$

It follows from the definition of $\gamma_1(c)$ that

$$E(U^{j_1}) = \gamma_1(c) = \lim_{n\to\infty} E(u_n),$$

$$\lim_{n\to\infty} \|(-\Delta)^{\frac{s_1}{2}} t_n^{j_1}\|_{L^2}^2 = \lim_{n\to\infty} \|(-\Delta)^{\frac{s_2}{2}} t_n^{j_1}\|_{L^2}^2 = 0,$$

and

$$\lim_{n\to\infty} \|(-\Delta)^{\frac{s_1}{2}} u_n\|_{L^2}^2 = \|(-\Delta)^{\frac{s_1}{2}} U^{j_1}\|_{L^2}^2.$$

Therefore, from (2.3)-(2.5), it follows that $E(U^{j_1}) = \gamma_1(c)$, which implies that the infimum of the variational problem (1.6) is attained at U^{j_1}. This completes the proof.

Next, we provide the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $\psi(t) = \psi(t, \cdot)$. Firstly, we claim that if $1 + \frac{\alpha}{N} < p < 1 + \frac{\alpha+2s_2}{N}$ for all $c > 0$, the solution $\psi(t)$ of equation (1.1) is bounded in $H^{s_1}(\mathbb{R}^N)$ for all t. From Lemma 2.1, we deduce that the solution $\psi(t)$ of (1.1) exists globally in time.

If $1 + \frac{\alpha}{N} < p < 1 + \frac{\alpha+2s_2}{N}$, it follows from the conservation laws of mass and energy that

$$E(\psi_0) = E(\psi(t)) \geq \frac{1}{2}\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2 - \epsilon\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2 - C_1(p, s_1, N, \alpha, c, \epsilon)$$

for all $c > 0$. Therefore, $\psi(t)$ is bounded in $H^{s_1}(\mathbb{R}^N)$.

Secondly, based on above result, we conclude that the solution $\psi(t)$ of (1.1) exists globally. By contradiction, there exists $\epsilon_0 > 0$ and a sequence of initial data $\{\psi_0^n\}$ such that

$$\inf_{u\in\mathcal{M}} \|\psi_0^n - u\|_{H^{s_1}} < \frac{1}{n},$$

and there exists a sequence $\{t_n\} \subset \mathbb{R}$, the corresponding solutions $\psi_n(t_n)$ of cauchy problem (1.1) satisfy

$$\inf_{u\in\mathcal{M}} \|\psi_n(t_n) - u\|_{H^{s_1}} \geq \epsilon_0. \tag{3.14}$$

It follows from the conservation laws of energy and mass that

$$\|\psi_n(t_n)\|_{L^2}^2 = \|\psi_0^n\|_{L^2}^2 \to \|u\|_{L^2}^2 = c \quad (n \to \infty), \tag{3.15}$$

$$E(\psi_n(t_n)) = E(\psi_0^n) \to E(u) = \gamma_1(c) \quad (n \to \infty). \tag{3.16}$$

By (3.16), it is evident that $\{\psi_n(t_n)\}$ is also a minimizing sequence of (1.6). Therefore, from Theorem 1.4, there exists a minimizer v such that

$$\|\psi_n(t_n) - v\|_{H^{s_1}} \to 0 \quad (n \to \infty),$$

which contradicts (3.14). This completes the proof.

4. L^2-Supercritical case

Let $u \in H^{s_1}(\mathbb{R}^N)$, noting that $p > 1 + \frac{\alpha+2s_1}{N}$ as $Np - N - \alpha > 2s_1$, we conclude from (3.3) that $E(u_\lambda) \to -\infty$ as $\lambda \to \infty$. Therefore, the energy functional $E(u)$ restricted on $S_1(c)$ is unbounded from below in the L^2-supercritical case. In order to obtain the existence of standing waves in the L^2-supercritical case, the following lemmas are required to proceed.

Lemma 4.1. *Let $0 < s_2 < s_1 < 1$, $N > 2s_1$, $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$, then for any $u \in S_1(c)$, there exists a $\lambda_u > 0$, such that $u_{\lambda_u} \in S_2(c)$ and $E(u_{\lambda_u}) = \max_{\lambda>0} E(u_\lambda)$. Moreover, function $\lambda \longmapsto E(u_\lambda)$ is concave on $[\lambda_u, \infty)$.*

Proof. Indeed, from (3.3), it follows that for any $u \in S_1(c)$,

$$\begin{aligned}\frac{dE(u_\lambda)}{d\lambda} &= s_1\lambda^{2s_1-1}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + s_2\lambda^{2s_2-1}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 \\ &\quad - \frac{N(p-1)-\alpha}{2p}\lambda^{N(p-1)-\alpha-1}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx,\end{aligned} \tag{4.1}$$

and

$$Q(u_\lambda) = s_1\lambda^{2s_1}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + s_2\lambda^{2s_2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 - \frac{N(p-1)-\alpha}{2p}\lambda^{N(p-1)-\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx. \tag{4.2}$$

Combining (4.1) with (4.2), we obtain that

$$\frac{dE(u_\lambda)}{d\lambda} = \frac{1}{\lambda}Q(u_\lambda). \tag{4.3}$$

Noting that $p \geq 1 + \frac{\alpha+2s_1}{N}$ as $N(p-1)-\alpha \geq 2s_1$, there exists a $\lambda_u > 0$ such that

$$Q(u_{\lambda_u}) = 0. \tag{4.4}$$

From (4.3) and (4.4), it follows that $u_{\lambda_u} \in S_2(c)$ and

$$\frac{dE(u_\lambda)}{d\lambda} > 0, \lambda < \lambda_u, \quad \frac{dE(u_\lambda)}{d\lambda} < 0, \lambda > \lambda_u.$$

This implies that

$$E(u_{\lambda_u}) = \max_{\lambda>0} E(u_\lambda).$$

Now, we are ready to show that the function $\lambda \longmapsto E(u_\lambda)$ is concave on $[\lambda_u, +\infty)$. It follows from (4.1) that

$$\frac{d^2E(u_\lambda)}{d\lambda^2} = s_1(2s_1-1)\lambda^{2s_1-2}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + s_2(2s_2-1)\lambda^{2s_2-2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 - \frac{(N(p-1)-\alpha)(N(p-1)-\alpha-1)}{2p}\lambda^{N(p-1)-\alpha-2}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx. \tag{4.5}$$

Let $\lambda = w\lambda_u$, then, (4.5) can be rewritten as

$$\begin{aligned}
&\frac{d^2E(u_\lambda)}{d\lambda^2} \\
&= s_1(2s_1-1)\lambda_u^{2s_1-2}w^{2s_1-2}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + s_2(2s_2-1)\lambda_u^{2s_2-2}w^{2s_2-2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 \\
&\quad - \frac{(N(p-1)-\alpha)(N(p-1)-\alpha-1)}{2p}\lambda_u^{N(p-1)-\alpha-2}w^{N(p-1)-\alpha-2}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx \\
&= \frac{1}{w^{2-2s_1}{\lambda_u}^2}(s_1(2s_1-1){\lambda_u}^{2s_1}\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + s_2(2s_2-1)w^{2(s_2-s_1)}{\lambda_u}^{2s_2}\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 \\
&\quad - \frac{(N(p-1)-\alpha)(N(p-1)-\alpha-1)}{2p}\lambda_u^{N(p-1)-\alpha}w^{N(p-1)-\alpha-2s_1}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx).
\end{aligned} \tag{4.6}$$

Finally, using the fact that $Q(u_{\lambda_u}) = 0$, we consequently obtain that

$$\frac{d^2E(u_\lambda)}{d\lambda^2} < 0,$$

for any $\omega \geq 1$. This completes the proof.

Lemma 4.2. *Let* $0 < s_2 < s_1 < 1$, $N > 2s_1$, $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$, *then for any* $u \in S_1(c)$, *function* $c \longmapsto \gamma_2(c)$ *is non-increasing.*

Proof. To prove this Lemma, it suffices to show that $\gamma_2(c_1) \geq \gamma_2(c_2)$ holds for any $0 < c_1 < c_2$. By the definition of $\gamma_2(c)$, there exists $u_1 \in S_2(c_1)$ such that $E(u_1) \leq \gamma_2(c_1) + \frac{\epsilon}{2}$ for any $\epsilon > 0$. Moreover, from Lemma 4.1, we also have $\max\limits_{\lambda>0} E((u_1)_\lambda) = E(u_1)$.

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^{s_1}(\mathbb{R}^N)$, there exists $u_1^k \in C_0^\infty(\mathbb{R}^N)$ and $supp u_1^k \subset B(0, \frac{1}{k})$ such that

$$\|u_1^k - u_1\|_{H^{s_1}} = o(k). \tag{4.7}$$

This implies that

$$\|(-\Delta)^{\frac{s_1}{2}} u_1^k\|_{L^2}^2 = \|(-\Delta)^{\frac{s_1}{2}} u_1\|_{L^2}^2 + o(k), \tag{4.8}$$

$$\|(-\Delta)^{\frac{s_2}{2}} u_1^k\|_{L^2}^2 = \|(-\Delta)^{\frac{s_2}{2}} u_1\|_{L^2}^2 + o(k), \tag{4.9}$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |u_1^k|^p)|u_1^k|^p dx = \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p)|u_1|^p dx + o(k). \tag{4.10}$$

Define $u_0^k := (c_2 - \|u_1^k\|_{L^2}^2)^{\frac{1}{2}} \frac{v^k}{\|v^k\|_{L^2}}$, where $v^k \in C_0^\infty(\mathbb{R}^N)$ and $supp v^k \subset B(0, 1 + \frac{3}{k}) \backslash B(0, \frac{3}{k})$. Let $g_\lambda^k = u_1^k + (u_0^k)_\lambda$, where $\lambda \in (0, 1)$ and $(u_0^k)_\lambda = \lambda^{\frac{N}{2}} u_0^k(\lambda x)$. It follows that

$$\begin{aligned}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s_1}{2}} g_\lambda^k|^2 dx &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s_1}{2}} (u_1^k + (u_0^k)_\lambda)|^2 dx \\ &\to \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s_1}{2}} u_1|^2 dx \ (k, \lambda \to 0^+),\end{aligned} \tag{4.11}$$

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s_2}{2}} g_\lambda^k|^2 dx \to \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s_2}{2}} u_1|^2 dx \ (k, \lambda \to 0^+), \tag{4.12}$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |g_\lambda^k|^p)|g_\lambda^k|^p dx \to \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p)|u_1|^p dx \ (k, \lambda \to 0^+). \tag{4.13}$$

Combining (4.11)-(4.13), we consequently obtain that $E(g_\lambda^k) \to E(u_1)$ as $k, \lambda \to 0^+$.

Therefore, we obtain

$$\begin{aligned}\gamma_2(c_2) &\leq \max_{t>0} E((g_\lambda^k)_t) \\ &\leq \max_{t>0} E((u_1)_t) + \frac{\epsilon}{2} \\ &= E(u_1) + \frac{\epsilon}{2} \\ &= \gamma_2(c_1) + \epsilon.\end{aligned} \tag{4.14}$$

By the arbitrariness of ϵ, we conclude that $\gamma_2(c_2) \leq \gamma_2(c_1)$. This completes the proof.

Let

$$
\begin{aligned}
\widetilde{E}(u) &= E(u) - \frac{1}{N(p-1)-\alpha} Q(u) \\
&= (\frac{1}{2} - \frac{s_1}{N(p-1)-\alpha})\|(-\Delta)^{\frac{s_1}{2}} u\|_{L^2}^2 + (\frac{1}{2} - \frac{s_2}{N(p-1)-\alpha})\|(-\Delta)^{\frac{s_2}{2}} u\|_{L^2}^2 \\
&= \frac{Np-N-\alpha-2s_1}{2(N(p-1)-\alpha)}\|(-\Delta)^{\frac{s_1}{2}} u\|_{L^2}^2 + \frac{Np-N-\alpha-2s_2}{2(N(p-1)-\alpha)}\|(-\Delta)^{\frac{s_2}{2}} u\|_{L^2}^2. \qquad (4.15)
\end{aligned}
$$

Next, to address the minimization problem (1.7), we consider the following related minimization problem:

$$
\widetilde{\gamma_2}(c) := inf\{\widetilde{E}(u) : u \in S_1(c),\ Q(u) \leq 0\}. \qquad (4.16)
$$

Lemma 4.3. *Let* $0 < s_2 < s_1 < 1$, $N > 2s_1$, $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$, *then for any* $c > 0$, $\widetilde{\gamma_2}(c) := inf\{\widetilde{E}(u) : u \in S_1(c),\ Q(u) = 0\} = \gamma_2(c)$.

Proof. Firstly, from the definition of $\widetilde{\gamma_2}(c)$ and $\gamma_2(c)$, it is evident to see that $\widetilde{\gamma_2}(c) \leq \gamma_2(c)$. From (4.2), we observe that $Q(u_\lambda)$ is continuous with respect to λ. In addition, for any $u \in S_1(c)$ $Q(u) \leq 0$. Therefore, there exists a $\lambda_0 \in (0, 1)$ such that $Q(u_{\lambda_0}) = 0$. Meanwhile, by the definition of $\widetilde{E}(u)$, it follows that $\gamma_2(c) \leq \widetilde{E}(u_{\lambda_0}) \leq \widetilde{E}(u)$. Finally, taking the infimum over u, we have $\gamma_2(c) \leq \widetilde{\gamma_2}(c)$. This completes the proof.

Lemma 4.4. *Let* $0 < s_2 < s_1 < 1$, $N > 2s_1$, $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$. *Assume that* $\frac{2ps_2}{N(p-1)-\alpha} > 1$, *if* $u_c \in S_1(c)$ *and satisfies with the equation*

$$
(-\Delta)^{s_1} u + (-\Delta)^{s_2} u + \omega u = (I_\alpha * |u|^p)|u|^{p-2}u, \qquad (4.17)
$$

then $\omega > 0$.

Proof. Indeed, multiplying (4.17) by u_c and $x \cdot \nabla u_c$, respectively, and integrating by parts, we have the following Pohozaev identity

$$
s_1\|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 + s_2\|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2 = \frac{N(p-1)-\alpha}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u_c|^p)|u_c|^p dx. \qquad (4.18)
$$

Moreover, multiplying (4.17) by u_c and integrating by parts yields that

$$
\|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2 + \omega\|u_c\|_{L^2}^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_c|^p)|u_c|^p dx. \qquad (4.19)
$$

From (4.18) and (4.19), it follows that

$$
\begin{aligned}
\omega\|u_c\|_{L^2}^2 &= \int_{\mathbb{R}^N} (I_\alpha * |u_c|^p)|u_c|^p dx - \|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 - \|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2 \\
&= \frac{2ps_1}{N(p-1)-\alpha}\|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 + \frac{2ps_2}{N(p-1)-\alpha}\|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2 \\
&\quad - \|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 - \|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2 \\
&= (\frac{2ps_1}{N(p-1)-\alpha} - 1)\|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 + (\frac{2ps_2}{N(p-1)-\alpha} - 1)\|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2. \qquad (4.20)
\end{aligned}
$$

Therefore, if $\frac{2ps_2}{N(p-1)-\alpha} > 1$, then $\omega > 0$.

Lemma 4.5. *Let* $0 < s_2 < s_1 < 1$, $N > 2s_1$, $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$. *Then for any* $c > 0$, *each critical point of* $E|_{S_2(c)}$ *is a critical point of* $E|_{S_1(c)}$.

Proof. Let u_c is a critical point of $E|_{S_2(c)}$, then there exist ω_1, ω_2 such that $E^{'}(u_c) + \omega_1 Q^{'}(u_c) + \omega_2 u_c = 0$, where

$$E^{'}(u_c) = (-\Delta)^{s_1} u_c + (-\Delta)^{s_2} u_c - (I_\alpha * |u_c|^p)|u_c|^{p-2} u_c, \tag{4.21}$$

$$Q^{'}(u_c) = 2s_1(-\Delta)^{s_1} u_c + 2s_2(-\Delta)^{s_2} u_c - (N(p-1) - \alpha)(I_\alpha * |u_c|^p)|u_c|^{p-2} u_c. \tag{4.22}$$

Then

$$\begin{aligned} E^{'}(u_c) + \omega_1 Q^{'}(u_c) + \omega_2 u_c &= (1 + 2\omega_1 s_1)(-\Delta)^{s_1} u_c + (1 + 2\omega_1 s_2)(-\Delta)^{s_2} u_c \\ &\quad - (1 + \omega_1(N(p-1) - \alpha))(I_\alpha * |u_c|^p)|u_c|^{p-2} u_c + \omega_2 u_c \\ &= 0. \end{aligned} \tag{4.23}$$

Multiplying (4.23) by u_c and integrating by parts yields that

$$E^1(u_c) + \omega_1 Q^1(u_c) + \omega_2 u_c = 0,$$

where

$$E^1(u_c) = \|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2 - \int_{\mathbb{R}^N} (I_\alpha * |u_c|^p)|u_c|^p dx, \tag{4.24}$$

$$\begin{aligned} Q^1(u_c) &= 2s_1\|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 + 2s_2\|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2 \\ &\quad - (N(p-1) - \alpha) \int_{\mathbb{R}^N} (I_\alpha * |u_c|^p)|u_c|^p dx. \end{aligned} \tag{4.25}$$

Multiplying (4.23) by $x \cdot \nabla u_c$ and integrating by parts yields that

$$E^2(u_c) + \omega_1 Q^2(u_c) + \omega_2 u_c = 0,$$

where

$$\begin{aligned} E^2(u_c) = &\frac{N-2s_1}{2}\|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 + \frac{N-2s_2}{2}\|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2 \\ &- \frac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u_c|^p)|u_c|^p dx, \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} Q^2(u_c) = &s_1(N-2s_1)\|(-\Delta)^{\frac{s_1}{2}} u_c\|_{L^2}^2 + s_2(N-2s_2)\|(-\Delta)^{\frac{s_2}{2}} u_c\|_{L^2}^2 \\ &- \frac{(N+\alpha)(N(p-1)-\alpha)}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u_c|^p)|u_c|^p dx. \end{aligned} \tag{4.27}$$

Since $\frac{N}{2}E^1(u_c) - E^2(u_c) = Q(u_c) = 0$, it follows that

$$\omega_1(\frac{N}{2}Q^1(u_c) - Q^2(u_c)) = 0.$$

By contradiction, assume that

$$2s_1^2\|(-\Delta)^{\frac{s_1}{2}}u_c\|_{L^2}^2 + 2s_2^2\|(-\Delta)^{\frac{s_2}{2}}u_c\|_{L^2}^2 - \frac{(N(p-1)-\alpha)^2}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u_c|^p)|u_c|^p dx = 0, \tag{4.28}$$

it follows from $Q(u) = 0$ and (4.28) that

$$s_1(N(p-1)-\alpha)\|(-\Delta)^{\frac{s_1}{2}}u_c\|_{L^2}^2 + s_2(N(p-1)-\alpha)\|(-\Delta)^{\frac{s_2}{2}}u_c\|_{L^2}^2 = \frac{(N(p-1)-\alpha)^2}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u_c|^p)|u_c|^p dx. \tag{4.29}$$

From (4.29), we have

$$0 > (2s_1^2 - s_1(N(p-1)-\alpha))\|(-\Delta)^{\frac{s_1}{2}}u_c\|_{L^2}^2 = (s_2(N(p-1)-\alpha) - 2s_2^2)\|(-\Delta)^{\frac{s_2}{2}}u_c\|_{L^2}^2 > 0,$$

which is a contradiction, namely, $\omega_1 = 0$. Therefore, each critical point of $E|_{S_2(c)}$ is a critical point of $E|_{S_1(c)}$.

Lemma 4.6. *Let* $0 < s_2 < s_1 < 1$, $N > 2s_1$. *Assume that* $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$, *then it follows that*
(i) $\widetilde{\gamma}_2(c) > 0$,
(ii) there exists $u \in S_2(\|u\|_{L^2}^2)$, *such that* $0 < \|u\|_{L^2}^2 \leq c$ *and* $E(u) = \widetilde{E}(u) = \widetilde{\gamma}_2(c) = \gamma_2(c)$.

Proof. (i) It follows from Lemma 2.5 and $Q(u) \leq 0$ that

$$\begin{aligned} s_1\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^2}^2 + s_2\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 &\leq \frac{N(p-1)-\alpha}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx \\ &\leq C_*\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^{\frac{Np-N-\alpha}{s_2}}\|u\|_{L^2}^{\frac{N+\alpha-Np+2s_2p}{s_2}} \\ &= C_*\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^{\frac{Np-N-\alpha}{s_2}}c^{\frac{N+\alpha-Np+2s_2p}{2s_2}}, \end{aligned} \tag{4.30}$$

which implies that

$$s_2\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^2 \leq C_*\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^{\frac{Np-N-\alpha}{s_2}}c^{\frac{N+\alpha-Np+2s_2p}{2s_2}}. \tag{4.31}$$

Namely,

$$\|(-\Delta)^{\frac{s_2}{2}}u\|_{L^2}^{\frac{Np-N-\alpha}{s_2}-2} \geq \frac{s_2c^{\frac{Np-N-\alpha-2s_2p}{2s_2}}}{C_*} > 0. \tag{4.32}$$

From (4.32) and the definition of $\widetilde{\gamma}_2(c)$, we conclude that $\widetilde{\gamma}_2(c) > 0$.

(ii) Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence of (4.16). That is, there exists a sequence $\{u_n\}_{n=1}^\infty \subset S_1(c)$ and $Q(u_n) \leq 0$ such that $\widetilde{E}(u_n) \to \widetilde{\gamma}_2(c) = \gamma_2(c)$ as $n \to \infty$. Moreover, by (4.15), we observe that $\{u_n\}$ is bounded in $H^{s_1}(\mathbb{R}^N)$.

Next, we claim that

$$\lim_{n\to\infty} \|u_n\|_{L^{p+1}}^{p+1} > 0. \tag{4.33}$$

Suppose, by contradiction, that $\lim\limits_{n\to\infty} \|u_n\|_{L^{p+1}}^{p+1} = 0$. From Remark 2.9, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p dx = 0. \tag{4.34}$$

It follows from (4.30) and (4.34) that $\lim\limits_{n\to\infty} \|(-\Delta)^{\frac{s_2}{2}} u_n\|_{L^2} = 0$, which contradicts (4.32).

Thus, we deduce from Lemma 2.4 that there exists a subsequence (still denoted by $\{u_n\}_{n=1}^\infty$) $\{u_n\}_{n=1}^\infty \subset S_1(c)$, $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $u \in H^{s_1}(\mathbb{R}^N)\backslash\{0\}$ such that

$$u_n(\cdot + x_n) \rightharpoonup u \neq 0 \textit{ weakly in } H^{s_1}(\mathbb{R}^N). \tag{4.35}$$

Applying Lemma 2.7, we deduce that

$$Q(u_n) - Q(u_n - u) - Q(u) \to 0, \tag{4.36}$$

$$\widetilde{E}(u_n) - \widetilde{E}(u_n - u) - \widetilde{E}(u) \to 0, \tag{4.37}$$

$$\|u_n\|_{L^2}^2 - \|u_n - u\|_{L^2}^2 - \|u\|_{L^2}^2 \to 0. \tag{4.38}$$

By (4.38), we have $0 < \|u\|_{L^2}^2 \leq c$. We now proceed to prove that $Q(u) \leq 0$ by excluding the other possibilities.

Case 1. If $Q(u) > 0$, $\|u\|_{L^2}^2 < c$, it follows from $Q(u_n) \leq 0$ and (4.36) that $Q(u_n - u) \leq 0$ as $n \to \infty$. Let $c_2 = c - \|u\|_{L^2}^2$. Then, we have $\|u_n - u\|_{L^2}^2 \to c_2$ as $n \to \infty$. Define $v_n = \frac{\sqrt{c_2}(u_n-u)}{\|u_n-u\|_{L^2}^2}$, so that $v_n \in S_1(c_2)$ and

$$\begin{aligned}
Q(v_n) &= s_1\|(-\Delta)^{\frac{s_1}{2}} v_n\|_{L^2}^2 + s_2\|(-\Delta)^{\frac{s_2}{2}} v_n\|_{L^2}^2 - \frac{N(p-1)-\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p dx \\
&= s_1\|(-\Delta)^{\frac{s_1}{2}} \frac{\sqrt{c_2}(u_n - u)}{\|u_n - u\|_{L^2}^2}\|_{L^2}^2 + s_2\|(-\Delta)^{\frac{s_2}{2}} \frac{\sqrt{c_2}(u_n - u)}{\|u_n - u\|_{L^2}^2}\|_{L^2}^2 \\
&\quad - \frac{N(p-1)-\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\frac{\sqrt{c_2}(u_n - u)}{\|u_n - u\|_{L^2}^2}|^p)|\frac{\sqrt{c_2}(u_n - u)}{\|u_n - u\|_{L^2}^2}|^p dx \\
&\leq 0.
\end{aligned} \tag{4.39}$$

Therefore, it follows from the definition of $\widetilde{\gamma}_2(c)$ that $\widetilde{E}(v_n) \geq \widetilde{\gamma}_2(c_2)$ and $\widetilde{E}(u_n - u) \geq \widetilde{\gamma}_2(c_2)$. By the fact that the function $\gamma_2(c)$ is non-increasing and (4.37), we obtain

$$\begin{aligned}
\widetilde{E}(u) &= \widetilde{E}(u_n) - \widetilde{E}(u_n - u) + o(1) \\
&\leq \widetilde{\gamma}_2(c) - \widetilde{\gamma}_2(c_2) + o(1) \\
&= \gamma_2(c) - \gamma_2(c_2) + o(1) \\
&\leq 0,
\end{aligned} \tag{4.40}$$

which leads to a contradiction.

Case 2. If $Q(u) > 0$, $\|u\|_{L^2}^2 = c$. It follows from (4.38) that $u_n \to u$ in $L^2(\mathbb{R}^N)$. By Sobolev embedding, we have $u_n \to u$ in $L^q(\mathbb{R}^N)$ for every $q \in [2, \frac{2N}{N-2})$. From (4.36) and $Q(u) > 0$, it is evident to see that $Q(u_n - u) \leq 0$. Specifically, $s_1\|(-\Delta)^{\frac{s_1}{2}}(u_n - u)\|_{L^2}^2 + s_2\|(-\Delta)^{\frac{s_2}{2}}(u_n - u)\|_{L^2}^2 \leq \frac{N(p-1)-\alpha}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u_n - u|^p)|u_n - u|^p dx \to 0$. Meanwhile, since $u_n \to u$ in $L^q(\mathbb{R}^N)$ for any $q \in [2, \frac{2N}{N-2})$, it follows that $\|(-\Delta)^{\frac{s_1}{2}}(u_n - u)\|_{L^2}^2 \to 0$ as $n \to \infty$. Therefore, $u_n \to u$ in $H^{s_1}(\mathbb{R}^N)$ and $Q(u_n - u) \to 0$ as $n \to \infty$. Furthermore, by (4.36), we obtain $Q(u_n) \geq 0$, which is impossible. Therefore, we have $0 < \|u\|_{L^2}^2 \leq c$ and $Q(u) \leq 0$.

Since $\gamma_2(c)$ is non-increasing, it follows that

$$\widetilde{\gamma}_2(c) \leq \widetilde{\gamma}_2(\|u\|_{L^2}^2) \leq \widetilde{E}(u).$$

By the weak lower semicontinuity of norm, we have

$$\widetilde{E}(u) \leq \lim_{n\to\infty} \widetilde{E}(u_n) = \widetilde{\gamma}_2(c).$$

Then $\widetilde{E}(u) = \widetilde{\gamma}_2(c)$.

Finally, we claim that $Q(u) = 0$. Assume, by contradiction, that $Q(u) < 0$, it follows from the continuity of $Q(u_\lambda)$ that there exists a $\lambda_0 \in (0, 1)$ such that $Q(u_{\lambda_0}) = 0$. Therefore, we can conclude that

$$\widetilde{\gamma}_2(c) \leq \widetilde{\gamma}_2(\|u\|_{L^2}^2) \leq \widetilde{E}(u_{\lambda_0}) < \widetilde{E}(u) = \widetilde{\gamma}_2(c),$$

which is a contradiction. The claim follows.

Lemma 4.7. *Let* $0 < s_2 < s_1 < 1$, $N > 2s_1$, $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$. *Assume that* $\frac{2ps_2}{N(p-1)-\alpha} > 1$, *then function* $c \longmapsto \gamma_2(c)$ *is strictly decreasing on* $(0, +\infty)$.

Proof. Suppose, by contradiction, that there exist c_2, c_3 such that

$$\gamma_2(c) > \gamma_2(c_2),\ c \in (0, c_2)\ \ and\ \ \gamma_2(c) = \gamma_2(c_3),\ c \in (c_2, c_3).$$

Since $c_3 > 0$, it follows from Lemma 4.6 that there exists $v \in S_2(\|v\|_{L^2}^2)$ such that $0 < \|v\|_{L^2}^2 \leq c_3$ and $E(v) = \gamma_2(c_3)$. Obviously, v is a local minimizer on set $\mathcal{H} := \{v \in H^{s_1}(\mathbb{R}^N) : Q(v) = 0\}$. Therefore, there exists a Lagrange multiplier $\omega_0 \in \mathbb{R}$ such that

$$E'(v) + \omega_0 Q'(v) = 0 \quad \text{in } (H^{s_1}(\mathbb{R}^N))^*.$$

From Lemma 4.4, we have $\omega_0 = 0$. This implies that $E'(v) = 0$. Then, multiplying $E'(v) = 0$ by v and integrating by parts yields that

$$\|(-\Delta)^{\frac{s_1}{2}}v\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}}v\|_{L^2}^2 - \int_{\mathbb{R}^N}(I_\alpha * |v|^p)|v|^p dx = 0. \tag{4.41}$$

On the other hand, from Lemma 4.6, v is also a minimizer of the following minimization problem

$$\gamma_2(\|v\|_{L^2}^2) := inf\{E(u) : v \in S_1(\|v\|_{L^2}^2),\ Q(v) = 0\}.$$

Therefore, there exist Lagrange multipliers $\omega_1, \omega_2 \in \mathbb{R}$ such that

$$E'(v) + \omega_1 Q'(v) + \omega_2 v = 0 \quad \text{in } (H^{s_1}(\mathbb{R}^N))^*.$$

Similarly, from Lemma 4.4, we have $\omega_1 = 0$. This implies that $E^{'}(v) + \omega_2 v = 0$. Then, multiplying $E^{'}(v) + \omega_2 v = 0$ by v and integrating by parts yields that

$$\|(-\Delta)^{\frac{s_1}{2}} v\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}} v\|_{L^2}^2 - \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p dx + \omega_2 \|v\|_{L^2}^2 = 0. \tag{4.42}$$

Combining (4.41) with (4.42), we have $\omega_2 \|v\|_{L^2}^2 = 0$. It follows from $\omega_2 > 0$ that $v = 0$, which is a contradiction. This completes the proof.

Lemma 4.8. *Let* $0 < s_2 < s_1 < 1$, $N > 2s_1$, $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$. *Assume that* $\frac{2ps_2}{N(p-1)-\alpha} > 1$ *and function* $c \longmapsto \gamma_2(c)$ *is strictly decreasing, then there exists a minimizer* $u \in S_2(c)$ *such that* $E(u) = \widetilde{E}(u) = \widetilde{\gamma}_2(c) = \gamma_2(c)$.

Proof. Based on the results of Lemma 4.6, it suffices to show that $\|u\|_{L^2}^2 = c$. If not, assume by contradiction that $\|u\|_{L^2}^2 < c$. By (4.37) and the fact that $\gamma_2(c)$ is non-increasing, we obtain that

$$\begin{aligned} \widetilde{E}(u_n - u) &= \widetilde{E}(u_n) - \widetilde{E}(u) + o(1) \\ &= \widetilde{\gamma}_2(c) - \widetilde{E}(u) + o(1) \\ &\le \widetilde{\gamma}_2(c) - \widetilde{\gamma}_2(\|u\|_{L^2}^2) + o(1) \\ &= \gamma_2(c) - \gamma_2(\|u\|_{L^2}^2) + o(1) \\ &< 0, \end{aligned} \tag{4.43}$$

which is a contradiction. Therefore, $\|u\|_{L^2}^2 = c$. Namely, there exists a $u \in S_2(c)$ such that $E(u) = \gamma_2(c)$.

Finally, we prove Theorem 1.5.

Proof of Theorem 1.6. From Lemma 4.8, it follows that there exists a $u \in S_2(c)$ such that $E(u) = \gamma_2(c)$. Moreover, from Lemma 4.5, we deduce that each critical point of $E|_{S_2(c)}$ is a critical point of $E|_{S_1(c)}$. Thus, there exists a $u \in S_1(c)$ such that u is the solution of minimization problem (1.7). Furthermore, there exists a $\omega_c \in \mathbb{R}$ such that (ω_c, u) solves the equation (1.2). This completes the proof.

5. Normalized ground state standing waves

In this section, we mainly consider the strong instability of the normalized ground state standing waves for equation (1.1) in the L^2-supercritical case. Our result is as follows.

Proof of Theorem 1.8. For any $c > 0$, let $u_c \in \mathcal{M}_c$ and define the set

$$\Omega := \{u \in H^{s_1}(\mathbb{R}^N) : E(u) < E(u_c),\ \|u\|_{L^2}^2 = \|u_c\|_{L^2}^2,\ Q(u) < 0\}.$$

Let $\psi(t) = \psi(t, \cdot)$. Firstly, letting $\psi_0 = u_c^\lambda(x) = \lambda^{\frac{N}{2}} u_c(\lambda x)$, with $\lambda < 1$. From Lemma 4.1, it follows that $\psi_0 \in \Omega$ and $\psi_0 \to u_c$ in $H^{s_1}(\mathbb{R}^N)$ as $\lambda \to 1^+$. Therefore, the set Ω contains the elements that are arbitrarily close to u_c in $H^{s_1}(\mathbb{R}^N)$. Next, let $\psi(t)$ is the solution of equation (1.1) with initial data ψ_0, and T_{max} is the maximal time of existence. Now, we prove that $\psi(t) \in \Omega$ for all $t \in [0, T_{max})$. From the conservation laws of energy and mass, we have

$$\|\psi(t)\|_{L^2}^2 = \|\psi_0\|_{L^2}^2 = \|u_c\|_{L^2}^2, \tag{5.1}$$

$$E(\psi(t)) = E(\psi_0) < E(u_c). \tag{5.2}$$

We claim that $Q(\psi(t)) < 0$. Indeed, if not, from the definition of $\gamma_2(c)$, there exists a $t_0 \in [0, T_{max})$ such that $E(\psi(t_0)) \geq E(u_c) = \gamma_2(c)$, which contradicts $E(\psi(t)) < E(u_c)$. Consequently, the solution $\psi(t)$ of (1.1) with initial data ψ_0 belongs to Ω. To complete the proof of the Theorem, it suffices to show that the solution $\psi(t)$ blows up in finite time. We now divide the remaining into three steps.

Step 1. We claim that there exists $\beta > 0$ such that $Q(\psi(t)) \leq -\beta$ for all $t \in [0, T_{max})$. Since $Q(\psi(t) < 0$, it follows from Lemma 4.6 and energy conservation that

$$E(u_c) = \gamma_2(c) = \widetilde{\gamma}_2(c) \leq E(\psi(t)) - \frac{1}{N(p-1)-\alpha} Q(\psi(t)) < E(\psi_0) - \frac{1}{2s_1} Q(\psi(t)),$$

which implies that

$$Q(\psi(t)) \leq 2s_1(E(\psi_0) - E(u_c)).$$

Set $\beta = 2s_1(E(u_c) - E(\psi_0))$, the claim follows.

Step 2. We prove that there exists $C_1 > 0$ such that

$$\frac{d}{dt} M_{\psi R}[\psi(t)] \leq -C_1 (\|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}} \psi(t)\|_{L^2}^2), \tag{5.3}$$

for any $t \in [0, T_{max})$. Before going further studying, we claim that $\|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}} \psi(t)\|_{L^2}^2 > 0$. Suppose, by contradiction, that there exists a sequence $\{t_k\} \subset [0, T_{max})$ such that $\|(-\Delta)^{\frac{s_1}{2}} \psi(t_k)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}} \psi(t_k)\|_{L^2}^2 \to 0$ as $k \to \infty$. Therefore, it follows that $\|(-\Delta)^{\frac{s_1}{2}} \psi(t_k)\|_{L^2}^2 \to 0$ as $k \to \infty$. From Lemma 2.5, we obtain that $\lim_{k\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |\psi(t_k)|^p)|\psi(t_k)|^p dx = 0$, which implies that $E(\psi(t_k)) \to 0$ as $k \to \infty$. However, this is impossible. Therefore, the claim follows. From Lemma 2.3, it follows that

$$\begin{aligned}
\frac{d}{dt} M_{\psi R}[\psi(t)] &\leq 2(Np - N - \alpha) E(\psi_0) \\
&\quad - 2N(p - 1 - \frac{\alpha + 2s_1}{N})(\|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}} \psi(t)\|_{L^2}^2) \\
&\quad + C(R^{-2s_2} + R^{-(N-1-\epsilon_1)(p-1-\frac{\alpha}{N})} \|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^{\frac{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{s_1}}).
\end{aligned} \tag{5.4}$$

Noticing that $1 + \frac{\alpha+2s_1}{N} < p < \frac{N+\alpha}{N-2s_1}$, it follows that

$$\begin{aligned}
&R^{-(N-1-\epsilon_1)(p-1-\frac{\alpha}{N})} \|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^{\frac{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{s_1}} \\
&\leq C_2 \epsilon \|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^2 + \epsilon^{-\frac{2s_1-(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}} R^{-\frac{2s_1(N-1-\epsilon_1)}{2s_1-1-\epsilon_1}}.
\end{aligned} \tag{5.5}$$

Therefore, (5.4) can be rewritten as

$$\begin{aligned}
\frac{d}{dt} M_{\psi R}[\psi(t)] &\leq 2(Np - N - \alpha) E(\psi_0) \\
&\quad - 2N(p - 1 - \frac{\alpha + 2s_1}{N})(\|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}} \psi(t)\|_{L^2}^2) \\
&\quad + C_2 \epsilon \|(-\Delta)^{\frac{s_1}{2}} \psi(t)\|_{L^2}^2 + C(R^{-2s_2} + \epsilon^{-\frac{2s_1-(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}} R^{-\frac{2s_1(N-1-\epsilon_1)}{2s_1-1-\epsilon_1}}) \\
&\leq 2(Np - N - \alpha) E(\psi_0)
\end{aligned}$$

$$
\begin{aligned}
&- 2N(p-1-\frac{\alpha+2s_1}{N})(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2)\\
&+ C_2\epsilon(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2)\\
&+ C(R^{-2s_2}+\epsilon^{-\frac{2s_1-(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}}R^{-\frac{2s_1(N-1-\epsilon_1)}{2s_1-1-\epsilon_1}}).
\end{aligned} \tag{5.6}
$$

Let

$$
\delta=\frac{2(Np-N-\alpha)|E(\psi_0)|+1}{N(p-1-\frac{\alpha+2s_1}{N})}.
$$

We now discuss the issue from two cases:

Case 1. If $\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2\le\delta$, then

$$
\begin{aligned}
\frac{d}{dt}M_{\psi R}[\psi(t)] &\le 4Q(\psi(t))+C_2\epsilon(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2)\\
&\quad + C(R^{-2s_2}+\epsilon^{-\frac{2s_1-(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}}R^{-\frac{2s_1(N-1-\epsilon_1)}{2s_1-1-\epsilon_1}})\\
&\le 4Q(\psi(t))+C_2\epsilon\delta+C(R^{-2s_2}+\epsilon^{-\frac{2s_1-(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}}R^{-\frac{2s_1(N-1-\epsilon_1)}{2s_1-1-\epsilon_1}}).
\end{aligned} \tag{5.7}
$$

Then for sufficiently small ϵ and for sufficiently large $R>1$, (5.7) can be estimated by

$$
\frac{d}{dt}M_{\psi R}[\psi(t)]\le -2\beta\le\frac{-2\beta}{\delta}(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2). \tag{5.8}
$$

Case 2. If $\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2>\delta$, then

$$
\begin{aligned}
\frac{d}{dt}M_{\psi R}[\psi(t)] &\le \delta N(p-1-\frac{\alpha+2s_1}{N})-1-N(p-1-\frac{\alpha+2s_1}{N})\delta\\
&\quad - N(p-1-\frac{\alpha+2s_1}{N})(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2)\\
&\quad + C_2\epsilon(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2)\\
&\quad + C(R^{-2s_2}+\epsilon^{-\frac{2s_1-(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}}R^{-\frac{2s_1(N-1-\epsilon_1)}{2s_1-1-\epsilon_1}})\\
&\le -1-N(p-1-\frac{\alpha+2s_1}{N})(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2)\\
&\quad + C_2\epsilon(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2+\|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2)\\
&\quad + C(R^{-2s_2}+\epsilon^{-\frac{2s_1-(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}}R^{-\frac{2s_1(N-1-\epsilon_1)}{2s_1-1-\epsilon_1}}).
\end{aligned} \tag{5.9}
$$

Then for sufficiently small ϵ and for sufficiently large $R>1$, we obtain that

$$
-1+C(R^{-2s_2}+\epsilon^{-\frac{2s_1-(1+\epsilon_1)(p-1-\frac{\alpha}{N})}{(1+\epsilon_1)(p-1-\frac{\alpha}{N})}}R^{-\frac{2s_1(N-1-\epsilon_1)}{2s_1-1-\epsilon_1}})\le 0,
$$

and there exists a constant $C_3>0$ such that

$$
N(p-1-\frac{\alpha+2s_1}{N})-C_2\epsilon\ge C_3.
$$

Therefore,

$$\frac{d}{dt}M_{\psi R}[\psi(t)] \leq -C_3(\|(-\Delta)^{\frac{s_1}{2}}\psi(t)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}}\psi(t)\|_{L^2}^2). \tag{5.10}$$

It follows from (5.8) and (5.10) that (5.3) holds.

Step 3. Suppose $T_{max} = \infty$. Integrating (5.3) over time, we conclude that there exists a $t^* > 0$ such that $M_{\psi R}[\psi(t)] < 0$ for any $t \geq t^*$. Therefore, we can obtain that

$$M_{\psi R}[\psi(t)] \leq -C_2 \int_{t^*}^{t} (\|(-\Delta)^{\frac{s_1}{2}}\psi(s)\|_{L^2}^2 + \|(-\Delta)^{\frac{s_2}{2}}\psi(s)\|_{L^2}^2)^2 ds.$$

Applying Lemma 2.2, it follows that $\psi(t)$ blows up in finite time. This completes the proof.

Author contributions

Yupeng Li: writing-review & editing, writing-original draft; Binhua Feng: writing-review & editing, conceptualization; Zhiqian He: writing-review & editing, conceptualization.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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