Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 18(1): 98–116.
DOI: 10.3934/cam.2026004
Received: 18 December 2024
Revised: 12 October 2025
Accepted: 21 January 2026
Published: 26 January 2026

Research article

Fractional sublinear operators with rough kernels and their commutators on new vanishing generalized Morrey spaces

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Abstract: This study focuses on a class of fractional sublinear operators, denoted as $S_{\Omega,\alpha}$, and their commutators $S_{\Omega,\alpha,b}$ with rough kernels. We establish the boundedness of these operators on newly emerging vanishing generalized Morrey spaces that are characterized by (V_∞) or (V^*). The primary innovation of this paper lies in the novel approach of controlling $S_{\Omega,\alpha}$ via the Riesz potential I_α and the management of $S_{\Omega,\alpha,b}$ through fractional maximal commutators with rough kernels $M_{\Omega,\alpha-\varepsilon,b}$ and $M_{\Omega,\alpha+\varepsilon,b}$ for some $\varepsilon > 0$.

Keywords: Morrey space; vanishing properties; fractional integral; commutators; rough kernel
Mathematics Subject Classification: 47B34, 47B38

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1. Introduction

It is widely known that Morrey spaces hold a significant position in harmonic analysis, especially in the study of partial differential equations (PDEs). These spaces were introduced by Morrey [1] back in 1938 in his work on systems of second order elliptic PDE. Subsequently, Campanato [2] and Peetre [3] extended the theory of Morrey spaces as Banach spaces of functions. Morrey spaces are extensively applied to various equations and analytic theories; for more details, we refer the reader to the literature on potential and harmonic analysis [4–6], the heat kernel approach [7], interpolation theory [8, 9], applications to periodic Schrödinger operators [10], and applications to evolution equations including the Navier–Stokes equations [11–13].

The classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ are equipped with the finite norm

$$\|f\|_{L^{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))},$$

where $1 \le p < \infty$, $0 \le \lambda \le n$, and in the sequel, $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$. Following this, the generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ are defined by replacing r^λ with a general nonnegative function

$\varphi(x,r)$ satisfying certain assumptions. That is, the generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ consist of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with the finite norm

$$\|f\|_{L^{p,\varphi}} := \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{M}_{p,\varphi}(f;x,r)^{\frac{1}{p}}, \tag{1.1}$$

where

$$\mathfrak{M}_{p,\varphi}(f;x,r) := \frac{1}{\varphi(x,r)} \int_{B(x,r)} |f(y)|^p \, dy, \quad x \in \mathbb{R}^n, r > 0. \tag{1.2}$$

The behavior of various classic operators on generalized Morrey spaces has drawn substantial academic attention. For further references and historical notes, one may refer to the survey articles such as [14–17]. It is important to note that Morrey spaces, while capable of characterizing local properties of functions more effectively than Lebesgue spaces, lack the property of being separable and the approximation of Morrey functions by nice functions. In response to this challenge in the approximation problem, appropriate subspaces have been introduced.

Recently, as presented in [14, 18], the authors delineated three distinct Morrey classes. The first two of these classes are related to the behavior of (1.2) at the origin and infinity, respectively. Additionally, they introduced a new subspace that makes use of an additional vanishing property, which involves truncations in large balls. The space $V_0L^{p,\varphi}(\mathbb{R}^n)$ introduced by Samko [19] is constituted by all those functions $f \in L^{p,\varphi}(\mathbb{R}^n)$ satisfying the vanishing property at zero (for short, the vanishing property (V_0)):

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi}(f;x,r) = 0. \tag{1.3}$$

The class $V_0L^{p,\varphi}(\mathbb{R}^n)$ is nontrivial if the function $\varphi(x,r)$ satisfies

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{r^n}{\varphi(x,r)} = 0 \tag{1.4}$$

and

$$\inf_{r>1} \sup_{x\in\mathbb{R}^n} \varphi(x,r) > 0. \tag{1.5}$$

Consequently, it encompasses compactly supported bounded functions.

Another two subspaces of $L^{p,\varphi}(\mathbb{R}^n)$ introduced in [14] are often termed as new vanishing generalized Morrey spaces. The class $V_\infty L^{p,\varphi}(\mathbb{R}^n)$ is constituted by all those functions $f \in L^{p,\varphi}(\mathbb{R}^n)$ satisfying the following vanishing property at infinity (for short, the vanishing property (V_∞)), namely:

$$\lim_{r\to \infty} \sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi}(f;x,r) = 0. \tag{1.6}$$

As shown in [20], to ensure that all bounded functions with compact support belong to the space, we impose the following assumptions on the function φ:

$$\lim_{r\to \infty} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi(x,r)} = 0 \tag{1.7}$$

and

$$\inf_{r>1} \sup_{x\in\mathbb{R}^n} \varphi(x,r) > 0. \tag{1.8}$$

The class $V^{(*)}L^{p,\varphi}(\mathbb{R}^n)$ is constituted by all functions $f \in L^{p,\varphi}(\mathbb{R}^n)$ satisfying the following vanishing property (for short, the vanishing property (V^*)):

$$\lim_{N\to\infty} \mathcal{A}_{N,p}(f) := \lim_{N\to\infty} \sup_{x\in\mathbb{R}^n} \int_{B(x,1)} |f(y)|^p \chi_N(y)\, dy = 0, \tag{1.9}$$

where and in the sequel, for any $N \in \mathbb{N}$,

$$\chi_N := \chi_{\mathbb{R}^n\setminus B(0,N)}. \tag{1.10}$$

The vanishing properties characterized by (V_∞) or (V^*) can be readily verified for all functions in $L^p(\mathbb{R}^n)$. To some extent, the vanishing property (V^*) helps us overcome the challenges of applying such a theorem in cases where the functions in $L^p(\mathbb{R}^n)$ are not necessary. Additionally, it should also be noted that the vanishing property (V^*) itself remains unaffected by the generalized parameter φ. In the work of [14], Almeida et al. remarked that these Morrey functions, having all these vanishing traits, are collected in a closed subspace of the generalized Morrey spaces, which is denoted as $V^{(*)}_{0,\infty}L^{p,\varphi}(\mathbb{R}^n)$. Furthermore, this subspace explicitly describes the closure of $C_0^\infty(\mathbb{R}^n)$.

Over the past few years, many papers have carried out intensive explorations into the behavior of classical operators on vanishing Morrey spaces, including studies on generalized parameters as seen in [18–22]. However, few papers have focused on the mapping properties and boundedness of commutators of operators on newly defined vanishing Morrey spaces $V_\infty L^{p,\varphi}(\mathbb{R}^n)$ and $V^{(*)}L^{p,\varphi}(\mathbb{R}^n)$. To our knowledge, the study of commutators in the new vanishing Morrey spaces is only found in the recent literature [23]. Notably, Almeida [23] studied maximal commutators and commutators related to fractional integral operators on these new vanishing generalized Morrey spaces in 2020.

Inspired by [23], this paper studies the behavior of a class of sublinear operators $S_{\Omega,\alpha}$ with rough kernels (see (2.2)) and their commutators $S_{\Omega,\alpha,b}$ (see (2.6)) in the new vanishing generalized Morrey space. This research covers the fractional integral operator $T_{\Omega,\alpha}$ with rough kernels (see (2.3)) and the corresponding fractional maximal operator $M_{\Omega,\alpha}$ with rough kernels (see (2.4)) and their commutators, but the research scope is not limited to these fields. The boundedness of these operators in the space $V_0L^{p,\varphi}(\mathbb{R}^n)$ has been studied by Gürbüz in [21]. The current research mainly focuses on establishing the conditions for these operators to map from one new vanishing generalized Morrey space to another while maintaining the vanishing properties (V_∞) or (V^*).

The structure of this paper is as follows. Section 2 provides some essential definitions and preliminary information. The main results are presented in Sections 3 and 4. In Section 3, we demonstrate the boundedness properties of the sublinear operators $S_{\Omega,\alpha}$ with the framework of generalized Morrey spaces that exhibit the vanishing properties (V_∞) or (V^*). Section 4 extends this investigation to the commutators $S_{\Omega,\alpha,b}$ associated with these sublinear operators, exploring their implications on the same class of Morrey spaces. The examination of how these operators interact with the closure of $C_0^\infty(\mathbb{R}^n)$ constitutes the focus of Section 5, providing insights into their application to smooth functions with compact support. The main results are presented in Theorems 3.4, 3.9, 4.4, 4.7 , 4.9, and 5.1.

2. Preliminaries

The notations utilized herein are enumerated as follows: $L^p_{\rm loc}(\mathbb{R}^n)$ denotes the space consisting of all functions that are locally p-integrable on $\mathbb{R}^n$. The notation $C_0^\infty(\mathbb{R}^n)$ refers to the set of all complex-valued infinitely differentiable functions on $\mathbb{R}^n$ with compact support. If E is a subset of $\mathbb{R}^n$, then $|E|$ stands for its Lebesgue measure and χ_E represents its characteristic function. The expression $A \lesssim B$ indicates that $A \le cB$ for some positive constant c that is independent of relevant quantities, and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. For all $p \in (1, \infty)$, we will, as usual, write $p' = p/(p - 1)$.

Let $0 < \alpha < n$. Assume that $\Omega \in L^s(\mathbb{S}^{n-1})$ with $s > 1$ is homogeneous of degree zero on $\mathbb{R}^n$, that is,

$$\Omega(\lambda x) = \Omega(x) \tag{2.1}$$

holds for every $\lambda > 0$ and $x \in \mathbb{R}^n$. Suppose that $S_{\Omega,\alpha}$ represents a linear or a sublinear operator, which satisfies, for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin {\rm supp} f$,

$$|S_{\Omega,\alpha} f(x)| \le c \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy, \tag{2.2}$$

where c is not dependent on f and x.

The fractional integral operator with rough kernel $T_{\Omega,\alpha}$ is defined by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y) f(y)}{|x-y|^{n-\alpha}} dy, \tag{2.3}$$

and the rough maximal operator $M_{\Omega,\alpha}$ is defined by

$$M_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|\le r} |\Omega(x-y)||f(y)| dy. \tag{2.4}$$

From the definition in (2.3) and (2.4), it is clear that both operators $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ satisfy condition (2.2). We note that in the case of $\Omega = 1$, a variety of interesting operators meet the condition (2.2). Such operators cover the fractional integral operator (Riesz potential), fractional maximal operator, as well as fractional Marcinkiewicz operator, and so on. If $\alpha = 0$ and Ω has a cancelation condition on $\mathbb{S}^{n-1}$, i.e.,

$$\int_{\mathbb{S}^{n-1}} \Omega(x')\, d\sigma(x') = 0, \tag{2.5}$$

where $x' = x/|x|$ for any $x \neq 0$, then the operators $T_{\Omega,0}$ and $M_{\Omega,0}$ become a Calderón–Zygmund singular integral operator and Hardy–Littlewood maximal operator with rough kernel.

In recent years, many papers have studied the mapping properties of $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ on certain function spaces. In particular, the boundedness of $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ on Lebesgue spaces has been obtained in [24,25]. Furthermore, researchers have established the boundedness of these operators in generalized Morrey spaces and generalized vanishing Morrey spaces with the vanishing property (V_0); see [21,26]. So, one of our main objectives is to investigate the boundedness on new vanishing generalized Morrey spaces with the vanishing properties (V_∞) or (V^*). The corresponding results are given in Section 3.

For a function b, suppose that the commutator operator $S_{\Omega,\alpha,b}$ represents a linear or a sublinear operator, which satisfies, for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin {\rm supp} f$,

$$|S_{\Omega,\alpha,b} f(x)| \le c \int_{\mathbb{R}^n} |b(x) - b(y)| \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy, \tag{2.6}$$

where c is not dependent on f and x.

Assume that $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In 1999, Ding and Lu [27] presented the commutator of the fractional integral operator with rough kernel, denoted as $[b, T_{\Omega,\alpha}]$, and the maximal commutator generated by $M_{\Omega,\alpha}$ and b. They are defined as follows:

$$[b, T_{\Omega,\alpha}]\,(f)(x) = b(x)T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(bf)(x) = \int_{\mathbb{R}^n} \frac{[b(x) - b(y)]}{|x - y|^{n-\alpha}}\Omega(x - y)f(y)\,dy \tag{2.7}$$

and

$$M_{\Omega,\alpha,b}(f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|\le r} |b(x) - b(y)||\Omega(x - y)||f(y)|\,dy. \tag{2.8}$$

It is obvious that the commutators $[b, T_{\Omega,\alpha}]$ and $M_{\Omega,\alpha,b}$ satisfy condition (2.6). In particular, Ding and Lu [27] proved the boundedness of $[b, T_{\Omega,\alpha}]$ and $M_{\Omega,\alpha,b}$ on Lebesgue spaces. The proofs of the boundedness of $[b, T_{\Omega,\alpha}]$ and $M_{\Omega,\alpha,b}$ on generalized Morrey spaces and generalized Morrey spaces with vanishing property (V_0) can be respectively found in [21, 26]. Thus, it is a natural progression to investigate the boundedness on new vanishing generalized Morrey spaces characterized by the vanishing properties (V_∞) or (V^*). This constitutes our second objective, and the corresponding results will be presented in Section 4.

Before presenting our results, it is necessary to introduce the definition of bounded mean oscillation functions (BMO) and several lemmas (see [28]). Assume that f is a locally integrable function on $\mathbb{R}^n$. We then define $\|f\|_{BMO}$ as follows:

$$\|f\|_{BMO} := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy,$$

where $f_{B(x,r)}$ is given by $f_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy$. The space BMO is subsequently defined as the collection of all such functions f for which $\|f\|_{BMO}$ is finite, that is,

$$\mathrm{BMO}(\mathbb{R}^n) = \{f : \|f\|_{BMO} < \infty\}.$$

Lemma 2.1. ([23], [28]) *Assume that* $f \in \mathrm{BMO}(\mathbb{R}^n)$.

(1) For $1 < p < \infty$,

$$\|f\|_{BMO} \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p dy\right)^{\frac{1}{p}}. \tag{2.9}$$

(2) For any balls $B_1 \subset B_2 \subset \mathbb{R}^n$,

$$|f_{B_1} - f_{B_2}| \lesssim \frac{|B_2|}{|B_1|}\|f\|_{BMO}. \tag{2.10}$$

(3) For $0 < 2r < t$,

$$|f_{B(x,r)} - f_{B(x,t)}| \lesssim \|f\|_{BMO} \ln \frac{t}{r}, \tag{2.11}$$

with the implicit constant independent of x, r, t, and f.

3. Fractional sublinear operators with rough kernels on new vanishing generalized Morrey spaces

It is known that the fractional sublinear operators $S_{\Omega,\alpha}$ are bounded on generalized Morrey spaces and also on vanishing generalized Morrey spaces at the origin. In this section, we will establish the boundedness results on new vanishing generalized Morrey spaces with the vanishing property (V_∞) or (V^*) for $0<\alpha<n$. The attainment of the boundedness of $S_{\Omega,\alpha}$ on $V^{(*)}L^{p,\varphi}(\mathbb{R}^n)$ is achieved by virtue of the fact that $S_{\Omega,\alpha}$ can be controlled by the Riesz potential I_α. Moreover, we have also managed to secure the boundedness of operators $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ on the same spaces.

3.1. Known boundedness results of $S_{\Omega,\alpha}$

Before starting one of our main Theorems, let us recall some known results. The statements below have been proved in [26].

Lemma 3.1. ([26]) *Let $\alpha\in(0,n)$, $p\in(1,\frac{n}{\alpha})$, and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Suppose that $\Omega\in L^s(\mathbb{S}^{n-1})$ is homogeneous of degree zero. Moreover, suppose that $S_{\Omega,\alpha}$ is a sublinear operator satisfying condition (2.2) and bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Then, for any ball $B(x_0,r)$ and for all $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, the subsequent estimates are valid. When $s'\le p$,*

$$\|S_{\Omega,\alpha}f\|_{L^q(B(x_0,r))}\lesssim r^{\frac{n}{q}}\int_r^\infty t^{-\frac{n}{q}-1}\|f\|_{L^p(B(x_0,t))}dt, \tag{3.1}$$

and when $q<s$,

$$\|S_{\Omega,\alpha}f\|_{L^q(B(x_0,r))}\lesssim r^{\frac{n}{q}-\frac{n}{s}}\int_r^\infty t^{\frac{n}{s}-\frac{n}{q}-1}\|f\|_{L^p(B(x_0,t))}dt. \tag{3.2}$$

Lemma 3.2. ([26]) *Let α, p, q, and Ω be as in Lemma 3.1. $S_{\Omega,\alpha}$ is a bounded sublinear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Then the operator $S_{\Omega,\alpha}$ is bounded from $L^{p,\varphi_1}(\mathbb{R}^n)$ into $L^{q,\varphi_2}(\mathbb{R}^n)$ if s, p, q, and the pair (φ_1,φ_2) satisfy one of the following conditions:*

(a) $s'\le p$ and the pair (φ_1,φ_2) satisfy

$$\int_r^\infty \frac{\varphi_1(x,t)^{\frac{1}{p}}}{t^{\frac{n}{q}+1}}dt\lesssim\frac{\varphi_2(x,r)^{\frac{1}{q}}}{r^{\frac{n}{q}}}; \tag{3.3}$$

(b) $q<s$ and the pair (φ_1,φ_2) satisfy

$$\int_r^\infty \frac{\varphi_1(x,t)^{\frac{1}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}}dt\lesssim\frac{\varphi_2(x,r)^{\frac{1}{q}}}{r^{\frac{n}{q}-\frac{n}{s}}}; \tag{3.4}$$

with the implicit constant independent of x and r.

Lemma 3.3. ([26]) *Let α, p, q, and Ω be as in Lemma 3.1. Suppose the pair (φ_1,φ_2) satisfies* (3.3) *for $s'\le p$ and* (3.4) *for $q<s$. Then the operators $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ are bounded from $L^{p,\varphi_1}(\mathbb{R}^n)$ into $L^{q,\varphi_2}(\mathbb{R}^n)$.*

3.2. Boundedness of $S_{\Omega,\alpha}$ on $V_\infty L^{q,\varphi}(\mathbb{R}^n)$

Theorem 3.4. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and* $\Omega \in L^s(\mathbb{S}^{n-1})$ *be homogeneous of degree zero. Moreover, suppose that* $S_{\Omega,\alpha}$ *is bounded from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$, *and the pair* (φ_1, φ_2) *satisfies conditions* (1.7), (1.8), *and* (3.3) *for* $s' \leq p$ *or* (3.4) *for* $q < s$. *Then the operator* $S_{\Omega,\alpha}$ *is bounded from* $V_\infty L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V_\infty L^{q,\varphi_2}(\mathbb{R}^n)$.

Proof. For any $f \in V_\infty L^{p,\varphi_1}(\mathbb{R}^n)$, Lemma 3.2 shows that $S_{\Omega,\alpha} f \in L^{q,\varphi_2}(\mathbb{R}^n)$. So, it remains to show that

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathfrak{M}_{q,\varphi_2}(S_{\Omega,\alpha} f; x, r) = 0. \tag{3.5}$$

In fact, for any $\epsilon \in (0,\infty)$, there exists a positive constant $R = R(\epsilon)$, depending on ϵ, such that for every $t \in (R,\infty)$, $\sup\limits_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi_1}(f; x, t) \lesssim \epsilon^{\frac{p}{q}}$.

It is easy to check that $\mathfrak{M}_{q,\varphi_2}(S_{\Omega,\alpha} f; x, r) = \frac{\|S_{\Omega,\alpha} f\|^q_{L^q(B(x,r))}}{\varphi_2(x,r)}$. Then, for $r \in (R,\infty)$, if $s' \leq p$, by Lemma 3.1 and the condition (3.3), we obtain that

$$\begin{aligned}
\mathfrak{M}_{q,\varphi_2}(S_{\Omega,\alpha} f; x, r) \lesssim & \frac{r^n}{\varphi_2(x,r)} \left\{ \int_r^\infty t^{-\frac{n}{q}-1} \|f\|_{L^p(B(x,t))} dt \right\}^q \\
\lesssim & \frac{r^n}{\varphi_2(x,r)} \left\{ \int_r^\infty \frac{\varphi_1(x,t)^{\frac{1}{p}}}{t^{\frac{n}{q}+1}} \left[\mathfrak{M}_{p,\varphi_1}(f; x, t)\right]^{\frac{1}{p}} \, dt \right\}^q \lesssim \epsilon.
\end{aligned}$$

If $q < s$, Lemma 3.1 and the condition (3.4) give that

$$\begin{aligned}
\mathfrak{M}_{q,\varphi_2}(S_{\Omega,\alpha} f; x, r) \lesssim & \frac{r^{(\frac{n}{q}-\frac{n}{s})q}}{\varphi_2(x,r)} \left\{ \int_r^\infty t^{\frac{n}{s}-\frac{n}{q}-1} \|f\|_{L^p(B(x,r))} \, dt \right\}^q \\
\lesssim & \frac{r^{(\frac{n}{q}-\frac{n}{s})q}}{\varphi_2(x,r)} \left\{ \int_r^\infty \frac{\varphi_1(x,t)^{\frac{1}{p}}}{t^{-\frac{n}{s}+\frac{n}{q}+1}} \left[\mathfrak{M}_{p,\varphi_1}(f; x, t)\right]^{\frac{1}{p}} \, dt \right\}^q \lesssim \epsilon.
\end{aligned}$$

Therefore, (3.5) has been proved, that is, $S_{\Omega,\alpha} f \in V_\infty L^{q,\varphi_2}(\mathbb{R}^n)$. □

It is well known that the fractional integral operator $T_{\Omega,\alpha}$ and fractional maximal operator $M_{\Omega,\alpha}$ with rough kernels are bounded on the Lebesgue spaces and weighted Lebesgue spaces; see [24, 25]. Furthermore, these operators satisfy condition (2.2). So, the following result can be easily obtained.

Corollary 3.5. *Let* α, p, q, *and* Ω *be as in Theorem 3.4 and the pair* (φ_1, φ_2) *satisfy conditions* (1.7), (1.8), *and* (3.3) *for* $s' \leq p$ *or* (3.4) *for* $q < s$. *Then the operators* $T_{\Omega,\alpha}$ *and* $M_{\Omega,\alpha}$ *are bounded from* $V_\infty L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V_\infty L^{q,\varphi_2}(\mathbb{R}^n)$.

3.3. Boundedness of $S_{\Omega,\alpha}$ on $V^{(*)}L^{q,\varphi}(\mathbb{R}^n)$

Iida showed that the fractional integral operator with rough kernel $T_{\Omega,\alpha}$ can be dominated by the Riesz potential I_α in a pointwise sense (see [29, Section 3.1, Lemma 1]). From the proof of this result, we find that the pointwise estimate is also true for the sublinear operators $S_{\Omega,\alpha}$. That is,

Lemma 3.6. *If* $\Omega \in L^s(\mathbb{S}^{n-1})$*,* $s > 1$ *is homogeneous of degree zero, then*

$$\left|S_{\Omega,\alpha} f(x)\right| \lesssim \|\Omega\|_{L^s(\mathbb{S}^{n-1})} I_\alpha(M_{s'} f)(x), \tag{3.6}$$

where

$$M_{s'} f(x) = \left(M(|f|^{s'})(x)\right)^{\frac{1}{s'}}.$$

Notice that Fu et al. [20, 22] studied the boundedness of the bilinear maximal function and bilinear fractional integral on vanishing generalized Morrey spaces. We hereby present the results concerning the boundedness of M and I_α on vanishing generalized Morrey spaces with the vanishing property (V^*).

Lemma 3.7. ([22]) *Let* $1 < p < \infty$*, and the pair* (φ_1, φ_2) *satisfies the conditions*

$$\int_r^\infty \frac{\varphi_1(x,t)^{\frac{1}{p}}}{t^{\frac{n}{p}+1}} dt \lesssim \frac{\varphi_2(x,r)^{\frac{1}{p}}}{r^{\frac{n}{p}}} \tag{3.7}$$

and

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\varphi_1(x,r)}{r^n} = 0. \tag{3.8}$$

Then the Hardy–Littlewood maximal function M *is bounded from* $V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V^{(*)}L^{p,\varphi_2}(\mathbb{R}^n)$*.*

Lemma 3.8. ([20]) *Let* α*,* p*, and* q *be as in Lemma 3.1. The pair* (φ_1, φ_2) *satisfies the conditions* (3.3) and (3.8)*. Then the Riesz potential* I_α *is bounded from* $V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V^{(*)}L^{q,\varphi_2}(\mathbb{R}^n)$*.*

With these results, we would like to give the boundedness of fractional sublinear operators $S_{\Omega,\alpha}$ with rough kernel on generalized Morrey spaces with the vanishing property (V^*) as follows.

Theorem 3.9. *Let* α*,* p*,* q*, and* Ω *be as in Theorem 3.4. Moreover, suppose that* $S_{\Omega,\alpha}$ *is bounded from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$*. If* $p > s'$ *and the pair* (φ_1, φ_2) *satisfies conditions* (3.3), (3.8), *and*

$$\int_r^\infty \frac{\varphi_1(x,t)^{\frac{1}{p}}}{t^{\frac{n}{p}+1}} dt \lesssim \frac{\varphi_1(x,r)^{\frac{1}{p}}}{r^{\frac{n}{p}}}, \tag{3.9}$$

then the operator $S_{\Omega,\alpha}$ *is bounded from* $V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V^{(*)}L^{q,\varphi_2}(\mathbb{R}^n)$*.*

Proof. In light of Lemma 3.2, it can be ascertained that $S_{\Omega,\alpha} f \in L^{q,\varphi_2}(\mathbb{R}^n)$. Thus, it remains to verify that

$$\lim_{N\to\infty} \mathcal{A}_{N,q}(S_{\Omega,\alpha} f) = 0 \quad \text{for any} \quad f \in V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n).$$

By making use of (3.6), we can infer

$$\mathcal{A}_{N,q}(S_{\Omega,\alpha} f) \lesssim \|\Omega\|^q_{L^s(\mathbb{S}^{n-1})} \mathcal{A}_{N,q}(I_\alpha(M_{s'} f)).$$

Given that $f \in V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$, it follows that $f^{s'} \in V^{(*)}L^{\frac{p}{s'},\varphi_1}(\mathbb{R}^n)$.

Thanks to Lemma 3.7, and considering that $p > s'$, we can conclude that $M_{s'}(|f|^{s'}) \in V^{(*)}L^{\frac{p}{s'},\varphi_1}(\mathbb{R}^n)$, which further implies that $M_{s'} f \in V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$. By virtue of Lemma 3.8, we are able to determine that $I_\alpha(M_{s'} f) \in V^{(*)}L^{q,\varphi_2}(\mathbb{R}^n)$. As a result, it can be deduced that $S_{\Omega,\alpha} f \in V^{(*)}L^{q,\varphi_2}(\mathbb{R}^n)$. With the above derivations, the proof of Theorem 3.9 is thereby accomplished. □

Corollary 3.10. *Let* α*,* p*,* q*, and* Ω *be as in Theorem 3.4 and the pair* (φ_1, φ_2) *satisfy conditions* (3.3), (3.8), *and* (3.9)*. Then, for* $p > s'$*, the operators* $T_{\Omega,\alpha}$ *and* $M_{\Omega,\alpha}$ *are bounded from* $V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V^{(*)}L^{q,\varphi_2}(\mathbb{R}^n)$*.*

4. Commutators of fractional sublinear operators with rough kernel on new vanishing generalized Morrey spaces

In this section, our objective is to demonstrate the boundedness results of commutators of fractional sublinear operators $S_{\Omega,\alpha,b}$ with rough kernel on new vanishing generalized Morrey spaces with the vanishing property (V_∞) or (V^*). In the case of the vanishing property (V^*), our initial step is to ascertain the boundedness of the fractional maximal commutator $M_{\Omega,\alpha,b}$. Subsequently, the boundedness of $S_{\Omega,\alpha,b}$ can be deduced as it can be effectively controlled by $M_{\Omega,\alpha,b}$. Besides, the commutators $[b, T_{\Omega,\alpha}]$ and $M_{\Omega,\alpha,b}$ hold as well.

4.1. Known boundedness results of $S_{\Omega,\alpha,b}$

The proof of Theorem 4.4 is based on the following lemmas, which are essentially contained in [26].

Lemma 4.1. ([26]) *Let α, p, q, and Ω be as in Lemma 3.1 and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose $S_{\Omega,\alpha,b}$ is a bounded sublinear operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Then, for any ball $B(x_0,r)$ and for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, the subsequent estimates are valid. When $s' \le p$,*

$$\|S_{\Omega,\alpha,b}f\|_{L^q(B(x_0,r))} \lesssim \|b\|_{BMO} r^{\frac{n}{q}} \int_r^\infty \left(1+\ln\frac{t}{r}\right) t^{-\frac{n}{q}-1} \|f\|_{L^p(B(x_0,t))} dt, \tag{4.1}$$

and when $q < s$,

$$\|S_{\Omega,\alpha,b}f\|_{L^q(B(x_0,r))} \lesssim \|b\|_{BMO} r^{\frac{n}{q}-\frac{n}{s}} \int_r^\infty \left(1+\ln\frac{t}{r}\right) t^{\frac{n}{s}-\frac{n}{q}-1} \|f\|_{L^p(B(x_0,t))} dt. \tag{4.2}$$

Lemma 4.2. ([26]) *Let α, p, q, and Ω be as in Lemma 3.1 and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose $S_{\Omega,\alpha,b}$ is a bounded sublinear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Then the operator $S_{\Omega,\alpha,b}$ is bounded from $L^{p,\varphi_1}(\mathbb{R}^n)$ into $L^{q,\varphi_2}(\mathbb{R}^n)$ if s, p, q, and the pair (φ_1, φ_2) satisfy one of the following conditions:*

(a) $s' \le p$ and the pair (φ_1, φ_2) satisfy

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\varphi_1(x,t)^{\frac{1}{p}}}{t^{\frac{n}{q}+1}} dt \le C_0 \frac{\varphi_2(x,r)^{\frac{1}{q}}}{r^{\frac{n}{q}}}; \tag{4.3}$$

(b) $q < s$ and the pair (φ_1, φ_2) satisfy

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\varphi_1(x,t)^{\frac{1}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt \le C_0 \frac{\varphi_2(x,r)^{\frac{1}{q}}}{r^{\frac{n}{q}-\frac{n}{s}}}; \tag{4.4}$$

with the implicit constant independent of x and r.

Lemma 4.3. ([26]) *Let α, p, q, and Ω be as in Lemma 3.1 and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose the pair (φ_1, φ_2) satisfies* (4.3) *for $s' \le p$ and* (4.4) *for $q < s$. Then the operators $[b, T_{\Omega,\alpha}]$ and $M_{\Omega,\alpha,b}$ are bounded from $L^{p,\varphi_1}(\mathbb{R}^n)$ into $L^{q,\varphi_2}(\mathbb{R}^n)$.*

4.2. Boundedness of $S_{\Omega,\alpha,b}$ on $V_\infty L^{q,\varphi}(\mathbb{R}^n)$

Theorem 4.4. *Let* α, p, q, *and* Ω *be as in Theorem 3.4 and* $b \in \mathrm{BMO}(\mathbb{R}^n)$. *Moreover, suppose* $S_{\Omega,\alpha,b}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$. *The pair* (φ_1, φ_2) *satisfies conditions* (1.7), (1.8), *and* (4.3) *for* $s' \leq p$ *or* (4.4) *for* $q < s$. *Then the operator* $S_{\Omega,\alpha,b}$ *is bounded from* $V_\infty L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V_\infty L^{q,\varphi_2}(\mathbb{R}^n)$.

Proof. The idea in the proof is essentially similar to that of Theorem 3.4. Lemma 4.2 shows that $S_{\Omega,\alpha,b} \in L^{q,\varphi_2}(\mathbb{R}^n)$ for any $f \in V_\infty L^{p,\varphi_1}(\mathbb{R}^n)$. So, it suffices to prove that

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathfrak{M}_{q,\varphi_2}(S_{\Omega,\alpha,b}f; x, r) = 0. \tag{4.5}$$

From (1.6), we know that for any $\epsilon > 0$ there exists a positive constant R(depends only on ϵ) such that for all $t \geq R$, $\sup\limits_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi_1}(f; x, t) \lesssim \epsilon^{\frac{p}{q}}$.

It is straightforward to calculate that $\mathfrak{M}_{q,\varphi_2}(S_{\Omega,\alpha,b}f; x, r) = \varphi_2(x, r)^{-1} \|S_{\Omega,\alpha,b}f\|^q_{L^q(B(x,r))}$. Then, for $r \in (R, \infty)$, if $s' \leq p$, by Lemma 4.1 and condition (4.3), we obtain

$$\begin{aligned}
\mathfrak{M}_{q,\varphi_2}(S_{\Omega,\alpha,b}f; x, r) \lesssim & \frac{\|b\|^q_{BMO} r^n}{\varphi_2(x, r)} \left\{ \int_r^\infty \left(1 + \ln\frac{t}{r}\right) t^{-\frac{n}{q}-1} \|f\|_{L^p(B(x,t))}\, dt \right\}^q \\
\lesssim & \frac{\|b\|^q_{BMO} r^n}{\varphi_2(x, r)} \left\{ \int_r^\infty \frac{\varphi_1(x, t)^{\frac{1}{p}}}{t^{\frac{n}{q}+1}} \left(1 + \ln\frac{t}{r}\right) [\mathfrak{M}_{p,\varphi_1}(f; x, t)]^{\frac{1}{p}}\, dt \right\}^q \lesssim \epsilon,
\end{aligned}$$

and for $q < s$, Lemma 4.1 and condition (4.4) give that

$$\begin{aligned}
\mathfrak{M}_{q,\varphi_2}(S_{\Omega,\alpha,b}f; x, r) \lesssim & \frac{\|b\|^q_{BMO} r^{(\frac{n}{q}-\frac{n}{s})q}}{\varphi_2(x, r)} \left\{ \int_r^\infty \left(1 + \ln\frac{t}{r}\right) t^{-\frac{n}{q}+\frac{n}{s}-1} \|f\|_{L^p(B(x,t))}\, dt \right\}^q \\
\lesssim & \frac{\|b\|^q_{BMO} r^{(\frac{n}{q}-\frac{n}{s})q}}{\varphi_2(x, r)} \left\{ \int_r^\infty \frac{\varphi_1(x, t)^{\frac{1}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} \left(1 + \ln\frac{t}{r}\right) [\mathfrak{M}_{p,\varphi_1}(f; x, t)]^{\frac{1}{p}}\, dt \right\}^q \lesssim \epsilon.
\end{aligned}$$

Therefore, (4.5) has been proved, that is, $S_{\Omega,\alpha,b}f \in V_\infty L^{q,\varphi_2}(\mathbb{R}^n)$. □

Notice that the commutators $[b, T_{\Omega,\alpha}]$ and $M_{\Omega,\alpha,b}$ satisfy condition (2.6) and are bounded on both Lebesgue spaces and weighted Lebesgue spaces; see [25, 27]. We also have the following result.

Corollary 4.5. *Let* α, p, q, *and* Ω *be as in Theorem 3.4 and* $b \in \mathrm{BMO}(\mathbb{R}^n)$. *Suppose the pair* (φ_1, φ_2) *satisfies conditions* (1.7), (1.8), *and* (4.3) *for* $s' \leq p$ *or* (4.4) *for* $q < s$. *Then the commutators* $[b, T_{\Omega,\alpha}]$ *and* $M_{\Omega,\alpha,b}$ *are bounded from* $V_\infty L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V_\infty L^{q,\varphi_2}(\mathbb{R}^n)$.

4.3. Boundedness of $S_{\Omega,\alpha,b}$ on $V^{(*)}L^{q,\varphi}(\mathbb{R}^n)$

In this subsection, we will give the boundedness of the commutators $S_{\Omega,\alpha,b}$ on $V^{(*)}L^{p,\varphi}(\mathbb{R}^n)$. First, we study the boundedness of commutators of fractional maximal operators with rough kernels $M_{\Omega,\alpha,b}$.

Almeida et al. [30] showed that the vanishing property (V^*) remains invariant concerning the choice of radius in the ball centered at x for any $f \in L^{p,\lambda}$ (see [30, lemma 3.4]). Following their ideal, we can obtain the following equivalent proposition of property (V^*) for any $f \in L^{p,\varphi}$.

Lemma 4.6. *A function $f \in L^{p,\varphi}$ satisfies the vanishing property* (V^*) (1.9) *if and only if*

$$\lim_{N\to\infty} \sup_{x\in\mathbb{R}^n} \int_{B(x,r)} |f(y)|^p \chi_N(y) dy = 0$$

uniformly in $r \in [0, R_0]$ for any fixed $R_0 > 0$.

Theorem 4.7. *Let α, p, q, and Ω be as in Theorem 3.4 and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that the pair (φ_1, φ_2) satisfies conditions* (4.3) *and*

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\varphi_1(x,r)}{r^{n-\alpha p-\gamma}} = 0 \quad \textit{for some} \quad 0 < \gamma < n. \tag{4.6}$$

Then, for $p > s'$, the operator $M_{\Omega,\alpha,b}$ is bounded from $V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$ to $V^{(*)}L^{q,\varphi_2}(\mathbb{R}^n)$.

Proof. In light of Lemma 4.3, it can be deduced that $M_{\Omega,\alpha,b}f \in L^{q,\varphi_2}(\mathbb{R}^n)$. Thus, it suffices to verify that

$$\lim_{N\to\infty} \sup_{x\in\mathbb{R}^n} \int_{B(x,1)} |M_{\Omega,\alpha,b}f(y)|^q \chi_N(y)\, dy = 0 \quad \text{for any} \quad f \in V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n). \tag{4.7}$$

For any given $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$, we decompose the function f into $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2)\cap B^c(0,N/2)}$ and $f_2 = f - f_1$. Owing to the operator $M_{\Omega,\alpha,b}$ being sublinear,

$$\mathcal{A}_{N,q}(M_{\Omega,\alpha,b}f) \lesssim \mathcal{A}_{N,q}(M_{\Omega,\alpha,b}f_1) + \mathcal{A}_{N,q}(M_{\Omega,\alpha,b}f_2). \tag{4.8}$$

Subsequently, our objective is to prove that both terms on the right side of (4.8) tend to zero as N tends to infinity. By leveraging the (L^p, L^q)-boundedness of $M_{\Omega,\alpha,b}$, we obtain

$$\begin{aligned}\int_{B(x,1)} (M_{\Omega,\alpha,b}f_1(y))^q \chi_N(y)\, dy &\le \int_{\mathbb{R}^n} (M_{\Omega,\alpha,b}f_1(y))^q\, dy \lesssim \left(\int_{\mathbb{R}^n} |f_1(y)|^p\, dy\right)^{\frac{q}{p}} \\ &= \int_{B(x,2)} |f(y)|^p \chi_{N/2}(y)\, dy.\end{aligned}$$

With the aid of Lemma 4.6 and considering that $f \in V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$, we can arrive at the conclusion that

$$\lim_{N\to\infty} \sup_{x\in\mathbb{R}^n} \int_{B(x,1)} (M_{\Omega,\alpha,b}f_1(y))^q \chi_N(y)\, dy = 0.$$

At present, our focus turns to dealing with the term $\mathcal{A}_{N,q}(M_{\Omega,\alpha,b}f_2)$. According to condition (4.6), for any arbitrarily chosen $\epsilon > 0$, there exists a $t_0 > 4$ such that $\sup_{x\in\mathbb{R}^n} t^{\gamma+\alpha p-n}\varphi_1(x,t) < \epsilon^{\frac{p}{q}}$ for all $t \ge t_0$. Once this specific t_0 has been fixed, we are then able to proceed as follows.

$$\begin{aligned}&\int_{B(x,1)} (M_{\Omega,\alpha,b}f_2(y))^q \chi_N(y)\, dy \\ &\lesssim \int_{B(x,1)} \chi_N(y)\Big(\sup_{0<t<t_0} t^{\alpha-n} \int_{B(y,t)} |b(y)-b(z)||\Omega(y-z)||f_2(z)|\, dz\Big)^q\, dy \\ &\quad + \int_{B(x,1)} \chi_N(y)\Big(\sup_{t\ge t_0} t^{\alpha-n} \int_{B(y,t)} |b(y)-b(z)||\Omega(y-z)||f_2(z)|\, dz\Big)^q\, dy\end{aligned}$$

$$:= I(x,N) + II(x,N).$$

We deal with the quantity $II(x,N)$ first. To do so, we have the following decomposition:

$$II(x,N) \lesssim II_1(x,N) + II_2(x,N) + II_3(x,N) + II_4(x,N),$$

with

$$II_1(x,N) = \int_{B(x,1)} \chi_N(y)\Big(\sup_{t\geq t_0} t^{\alpha-n}\int_{B(y,t)} |b(y) - b_{B(x,1)}||\Omega(y-z)||f_2(z)|\,dz\Big)^q\,dy,$$

$$II_2(x,N) = \int_{B(x,1)} \chi_N(y)\Big(\sup_{t\geq t_0} t^{\alpha-n}\int_{B(y,t)} |b_{B(x,1)} - b_{B(y,2)}||\Omega(y-z)||f_2(z)|\,dz\Big)^q\,dy,$$

$$II_3(x,N) = \int_{B(x,1)} \chi_N(y)\Big(\sup_{t\geq t_0} t^{\alpha-n}\int_{B(y,t)} |b_{B(y,2)} - b_{B(y,t)}||\Omega(y-z)||f_2(z)|\,dz\Big)^q\,dy,$$

and

$$II_4(x,N) = \int_{B(x,1)} \chi_N(y)\Big(\sup_{t\geq t_0} t^{\alpha-n}\int_{B(y,t)} |b(z) - b_{B(y,t)}||\Omega(y-z)||f_2(z)|\,dz\Big)^q\,dy.$$

For $II_1(x,N)$, note that

$$\|\Omega(y-\cdot)\|_{L^s(B(y,t))} = \left(\int_0^t \int_{\mathbb{S}^{n-1}} |\Omega(x')|^s\,d\sigma(x')\,r^{n-1}\,dr\right)^{\frac{1}{s}} \lesssim \|\Omega\|_{L^s(\mathbb{S}^{n-1})}|B(0,t)|^{\frac{1}{s}}.$$

By the Hölder inequality and (2.9), we have

$$\begin{aligned} II_1(x,N) &\leq \int_{B(x,1)} |b(y) - b_{B(x,1)}|^q\Big(\sup_{t\geq t_0} t^{\alpha-n}\int_{B(y,t)} |\Omega(y-z)||f(z)|\,dz\Big)^q\,dy \\ &\lesssim \|b\|_{BMO}^q\|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q\|f\|_{L^{p,\varphi_1}}^q \sup_{t\geq t_0}\sup_{y\in\mathbb{R}^n}\left(t^{\alpha p-n}\varphi_1(y,t)\right)^{\frac{q}{p}} \lesssim \epsilon. \end{aligned}$$

For $II_2(x,N)$, since $y\in B(x,1)$, it is easy to verify that $B(x,1)\subset B(y,2)$. Applying the Hölder inequality again and by (2.10), we get

$$II_2(x,N) \lesssim \|b\|_{BMO}^q\|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q\|f\|_{L^{p,\varphi_1}}^q \sup_{t\geq t_0}\sup_{y\in\mathbb{R}^n}\left(t^{\alpha p-n}\varphi_1(y,t)\right)^{\frac{q}{p}} \lesssim \epsilon.$$

As for $II_3(x,N)$, we can give the following estimate analogous to $II_2(x,N)$. It is clear for all $t\geq t_0>4$, by (2.11), we can obtain

$$\begin{aligned} II_3(x,N) &\lesssim \|b\|_{BMO}^q\|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q\|f\|_{L^{p,\varphi_1}}^q \sup_{t\geq t_0}\sup_{y\in\mathbb{R}^n}\left(t^{\alpha p-n}\varphi_1(y,t)\right)^{\frac{q}{p}}\left(\ln\frac{t}{2}\right)^q \\ &\lesssim \sup_{t\geq t_0}\sup_{y\in\mathbb{R}^n}\left(t^{\alpha p-n+\gamma}\varphi_1(y,t)\right)^{\frac{q}{p}} \lesssim \epsilon. \end{aligned}$$

Note that $p > s'$, with similar techniques, using the Hölder inequality with exponents $p, s, r\,(\frac{1}{r}+\frac{1}{p}+\frac{1}{s}=1)$ for f, Ω, $b(\cdot)-b_{B(y,t)}$, respectively, we have

$$\begin{aligned}
II_4(x,N) \leq& \int_{B(x,1)} \chi_N(y)\Big(\sup_{t\geq t_0} t^{\alpha-n}\int_{B(y,t)} |b(z)-b_{B(y,t)}||\Omega(y-z)||f(z)|\,dz\Big)^q\,dy\\
\leq& \|b\|_{BMO}^q \sup_{t\geq t_0} t^{q(\alpha+\frac{n}{r}-n)}\|f\|^q_{L^p(B(y,t))}\|\Omega(y-\cdot)\|^q_{L^s(B(y,t))}\\
\leq& \|b\|_{BMO}^q\|\Omega\|^q_{L^s(\mathbb{S}^{n-1})}\|f\|^q_{L^{p,\varphi_1}} \sup_{t\geq t_0}\sup_{y\in\mathbb{R}^n}\big(t^{\alpha p-n}\varphi_1(y,t)\big)^{\frac{q}{p}} \lesssim \epsilon.
\end{aligned}$$

At this moment, we direct our attention to estimating $I(x,N)$. Here, we need to consider relevant situations according to the different conditions of its location. Specifically, $z\in B(y,t)$ and $z\notin B(x,2)\cap B^c(0,N/2)$. Suppose $z\in B(0,N/2)$. In this situation, we can deduce that $t>|z-y|\geq|y|-|z|>N/2$. This implies that when $N\geq 2t_0$, there will be no impact on the supremum with respect to t within the interval $(0,t_0)$. On the other hand, if $z\notin B(x,2)$, then it follows that $t>|z-y|\geq|z-x|-|y-x|\geq 1$. Consequently, what remains for us to do is to estimate $I(x,N)$ when the supremum is computed over all $t\in(1,t_0)$. Subsequently, for these specific values of t, we can perform the following breakdown:

$$\begin{aligned}
I(x,N) \lesssim& \int_{B(x,1)} \chi_N(y)\Big(\int_{B(y,t_0)} |b(y)-b(z)||\Omega(y-z)||f(z)|\,dz\Big)^q\,dy\\
=:& I_1(x,N)+I_2(x,N)+I_3(x,N),
\end{aligned}$$

with

$$I_1(x,N) = \int_{B(x,1)} \chi_N(y)\Big(\int_{B(y,t_0)} |b(y)-b_{B(x,1)}||\Omega(y-z)||f(z)|\,dz\Big)^q\,dy,$$

$$I_2(x,N) = \int_{B(x,1)} \chi_N(y)\Big(\int_{B(y,t_0)} |b_{B(x,1)}-b_{B(y,t_0)}||\Omega(y-z)||f(z)|\,dz\Big)^q\,dy,$$

and

$$I_3(x,N) = \int_{B(x,1)} \chi_N(y)\Big(\int_{B(y,t_0)} |b(z)-b_{B(y,t_0)}||\Omega(y-z)||f(z)|\,dz\Big)^q\,dy.$$

For the term $I_1(x,N)$, we can show that for every $z\in B(y,t_0)$, the inequality $\chi_N(y)\leq\chi_{N-t_0}(z)$ holds true. Additionally, for any $y\in B(x,1)$, it can be readily verified that $B(y,t_0)\subset B(x,2t_0)$. Also applying the Hölder inequality, we obtain

$$\begin{aligned}
I_1(x,N) \leq& \int_{B(x,1)} |b(y)-b_{B(x,1)}|^q\Big(\int_{B(y,t_0)} \chi_{N-t_0}(z)|\Omega(y-z)||f(z)|\,dz\Big)^q\,dy\\
\lesssim& \|b(\cdot)-b_{B(x,1)}\|^q_{L^q(B(x,1))}\|\Omega\|^q_{L^s(\mathbb{S}^{n-1})}\left(\int_{B(x,2t_0)} \chi_{N-t_0}(z)|f(z)|^p\,dz\right)^{\frac{q}{p}}.
\end{aligned}$$

Lemma 4.6 indicates that

$$\sup_{x\in\mathbb{R}^n} I_1(x,N) \lesssim \|b\|^q_{BMO}\|\Omega\|^q_{L^s(\mathbb{S}^{n-1})}\left[\mathcal{A}_{N-t_0,p}(f)\right]^{\frac{q}{p}}.$$

From this, we can deduce that $I_1(x, N) \to 0$ as $N \to \infty$ uniformly on $x \in \mathbb{R}^n$.

When it comes to the term $I_2(x, N)$, we note that given $y \in B(x, 1)$ and $t_0 \geq 2$, it follows that $B(x, 1) \subset B(y, t_0)$. Moreover, according to (2.10), we have $|b_{B(x,1)} - b_{B(y,t_0)}| \lesssim t_0^n \|b\|_{BMO}$. Note also that $p > s'$, and applying the Hölder inequality once again, we obtain

$$\begin{aligned}
I_2(x, N) \lesssim & \|b\|_{BMO}^q \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q \int_{B(x,1)} \chi_N(y) \left(\int_{B(y,t_0)} |f(z)|^p \, dz \right)^{\frac{q}{p}} dy \\
\lesssim & \|b\|_{BMO}^q \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q \int_{B(x,1)} \chi_N(y) \left(\int_{B(y,t_0)} |f(z)|^p \chi_{N-t_0}(z) \, dz \right)^{\frac{q}{p}} dy \\
\lesssim & \|b\|_{BMO}^q \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q \left(\sup_{y \in \mathbb{R}^n} \int_{B(y,t_0)} |f(z)|^p \chi_{N-t_0}(z) \, dz \right)^{\frac{q}{p}},
\end{aligned}$$

with a modified notation χ_a in (1.10): $\chi_a := 1$ if $a \leq 0$ and $\chi_a := \chi_{\mathbb{R}^n \setminus B(0,a)}$ if $a > 0$. Lemma 4.6 shows that the right hand of the above inequality goes to zero as $N \to \infty$ uniformly on $x \in \mathbb{R}^n$.

Finally, we take into account $I_3(x, N)$. Analogously, applying the Hölder inequality with exponents p, s, r $(\frac{1}{r} + \frac{1}{p} + \frac{1}{s} = 1)$, we get

$$\begin{aligned}
I_3(x, N) \lesssim & \int_{B(x,1)} \chi_N(y) \|b(\cdot) - b_{B(y,t_0)}\|_{L^r(B(y,t_0))}^q \|\Omega(y - \cdot)\|_{L^s(B(y,t_0))}^q \left(\int_{B(y,t_0)} |f(z)|^p \, dz \right)^{\frac{q}{p}} dy \\
\lesssim & \|b\|_{BMO}^q \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q \left(\sup_{y \in \mathbb{R}^n} \int_{B(y,t_0)} |f(z)|^p \chi_{N-t_0}(z) \, dz \right)^{\frac{q}{p}}.
\end{aligned}$$

Thus, we get $I_3(x, N) \to 0$ as $N \to \infty$ uniformly on $x \in \mathbb{R}^n$.

Combining the above estimates, we have $\lim\limits_{N \to \infty} \mathcal{A}_{N,q}(M_{\Omega,\alpha,b} f_2) = 0$. The desirable conclusion can be easily deduced from this now, that is, (4.7) holds. The proof of Theorem 4.7 is thereby accomplished. □

In the research work presented in [27], Ding et al. established a pointwise connection between the commutators $[b, T_{\Omega,\alpha}]$ and $M_{\Omega,\alpha,b}$. Let $\varepsilon > 0$ be a positive real number fulfilling the condition $0 < \alpha - \varepsilon < \alpha + \varepsilon < n$. Then, for every $x \in \mathbb{R}^n$, the subsequent inequality holds true:

$$|[b, T_{\Omega,\alpha}] f(x)| \lesssim \left(M_{\Omega,\alpha+\varepsilon,b} f(x) \right)^{\frac{1}{2}} \left(M_{\Omega,\alpha-\varepsilon,b} f(x) \right)^{\frac{1}{2}}, \tag{4.9}$$

with the implicit constant independent of α, ε, and n. Through an examination of the proof of this outcome, we discern that the pointwise domination for the fractional sublinear operator $S_{\Omega,\alpha,b}$ also holds valid. We omit its proof here for brevity.

Lemma 4.8. *Suppose that $\varepsilon > 0$ satisfies $0 < \alpha - \varepsilon < \alpha + \varepsilon < n$. Then, for any $x \in \mathbb{R}^n$,*

$$|S_{\Omega,\alpha,b} f(x)| \lesssim \left(M_{\Omega,\alpha+\varepsilon,b} f(x) \right)^{\frac{1}{2}} \left(M_{\Omega,\alpha-\varepsilon,b} f(x) \right)^{\frac{1}{2}}. \tag{4.10}$$

Theorem 4.9. *Let α, p, q, Ω, and b be as in Theorem 4.7 and the pair (φ_1, φ_2) satisfy conditions (4.3) and (4.6). Moreover, suppose that $S_{\Omega,\alpha,b}$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Then, for $p > s'$, the operator $S_{\Omega,\alpha,b}$ is bounded from $V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$ to $V^{(*)}L^{q,\varphi_2}(\mathbb{R}^n)$.*

Proof. From Lemma 4.2, it can be concluded that $S_{\Omega,\alpha,b}f \in L^{q,\varphi_2}(\mathbb{R}^n)$. Thus, what remains is to validate that

$$\lim_{N\to\infty} \mathcal{A}_{N,q}(S_{\Omega,\alpha,b}f) = 0 \quad \text{for any} \quad f \in V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n).$$

By (4.10), we obtain

$$\mathcal{A}_{N,q}(S_{\Omega,\alpha,b}f) \lesssim \sup_{x\in\mathbb{R}^n} \int_{B(x,1)} (M_{\Omega,\alpha+\varepsilon,b}f(y))^{\frac{q}{2}} (M_{\Omega,\alpha-\varepsilon,b}f(y))^{\frac{q}{2}} \chi_N(y)dy.$$

Note that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and we define q_1, q_2 such that $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha+\varepsilon}{n}$ and $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha-\varepsilon}{n}$, respectively, which implies that $\frac{2}{q} = \frac{1}{q_1}+\frac{1}{q_2}$. For $f \in V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$, it follows from Theorem 4.7 that $M_{\Omega,\alpha+\varepsilon,b}f \in V^{(*)}L^{q_1,\varphi_2}(\mathbb{R}^n)$ and $M_{\Omega,\alpha-\varepsilon,b}f \in V^{(*)}L^{q_2,\varphi_2}(\mathbb{R}^n)$. Applying the Hölder inequality, we derive

$$\begin{aligned}\mathcal{A}_{N,q}(S_{\Omega,\alpha,b}f) &\lesssim \left(\sup_{x\in\mathbb{R}^n} \int_{B(x,1)} (M_{\Omega,\alpha+\varepsilon,b}f(y))^{q_1} \chi_N(y)dy\right)^{\frac{q}{2q_1}} \\ &\quad\times \left(\sup_{x\in\mathbb{R}^n} \int_{B(x,1)} (M_{\Omega,\alpha-\varepsilon,b}f(y))^{q_2} \chi_N(y)dy\right)^{\frac{q}{2q_2}} \\ &\lesssim \left(\mathcal{A}_{N,q_1}(M_{\Omega,\alpha+\varepsilon,b}f)\right)^{\frac{q}{2q_1}} \left(\mathcal{A}_{N,q_2}(M_{\Omega,\alpha-\varepsilon,b}f)\right)^{\frac{q}{2q_2}}.\end{aligned}$$

Consequently, we conclude that $S_{\Omega,\alpha,b}f \in V^{(*)}L^{q,\varphi_2}(\mathbb{R}^n)$. This completes the proof of Theorem 4.9. □

Corollary 4.10. *Let α, p, q, Ω, b, φ_1, and φ_2 be as in Theorem 4.9. Then, for $p > s'$, the operator $[b, T_{\Omega,\alpha}]$ is bounded from $V^{(*)}L^{p,\varphi_1}(\mathbb{R}^n)$ to $V^{(*)}L^{q,\varphi_2}(\mathbb{R}^n)$.*

5. Additional results

In this section, we will focus on discussing the behavior of sublinear operators $S_{\Omega,\alpha}$ and $S_{\Omega,\alpha,b}$ on specific closed subspaces of generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$. The specific subspace in question, namely,

$$V^{(*)}_{0,\infty}L^{p,\varphi}(\mathbb{R}^n) := V_0L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n) \cap V^{(*)}L^{p,\varphi}(\mathbb{R}^n),$$

is rather distinctive as it is actually obtained by the intersection of three different types of vanishing subspaces. It is worth noting that the function φ satisfies the decay assumptions (1.4) and the additional condition

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\log^p r}{\varphi(x,r)} = 0. \tag{5.1}$$

This subspace precisely coincides with the closure of the class of $C_0^\infty(\mathbb{R}^n)$ functions under the generalized Morrey norm $\|\cdot\|_{L^{p,\varphi}}$. This implies that from the perspectives of topological structure and function approximation, this subspace inherits some crucial characteristics of the class of $C_0^\infty(\mathbb{R}^n)$ functions with the specific norm system, laying a solid foundation for subsequently analyzing the behaviors of sublinear operators on this subspace by leveraging the excellent properties of compactly supported smooth functions. For instance, when exploring the boundedness, continuity, and various convergence properties of the operators, the approximation relationships of function sequences implied by the closure property can be fully utilized.

Theorem 5.1. *Let α, p, q, and Ω be as in Theorem 3.4 and the pair (φ_1, φ_2) satisfy* (1.4), (5.1)*, and* (3.3) *for $s' \leq p$ or* (3.4) *for $q < s$. Suppose that $S_{\Omega,\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Then the operator $S_{\Omega,\alpha}$ is bounded from $V^{(*)}_{0,\infty}L^{p,\varphi_1}(\mathbb{R}^n)$ to $V^{(*)}_{0,\infty}L^{q,\varphi_2}(\mathbb{R}^n)$.*

Proof. Without loss of generality, we may assume that $\operatorname{supp} f \subset \{y : |y| < R\}$. First, we will show that for any bounded function with compact support $f \in C^\infty_0(\mathbb{R}^n)$, $S_{\Omega,\alpha}f \in C^\infty_0(\mathbb{R}^n)$. In fact, if $|x| > 2R$, it implies that

$$|x-y| \geq |x| - |y| > \frac{|x|}{2}.$$

Note that for $p \geq s'$, $\frac{n}{s'} \geq \frac{n}{p} > \alpha$; For $q < s$, $\frac{n}{s'} > \frac{n}{q'} = n\left(\frac{1}{p'} + \frac{\alpha}{n}\right) > \alpha$. Thus, we may choose any $\epsilon \in (0, \frac{n}{s'} - \alpha)$ such that

$$\begin{aligned}
|S_{\Omega,\alpha}f(x)| \leq & \int_{|y|<R} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\, dy \\
\lesssim & \left(\int_{|x-y|>R} \frac{|\Omega(x-y)|^s}{|x-y|^{(n-\alpha-\epsilon)s}}\, dy\right)^{\frac{1}{s}} \left(\int_{|y|<R} \frac{|f(y)|^{s'}}{|x-y|^{\epsilon s'}}\, dy\right)^{\frac{1}{s'}} \\
\lesssim & \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left(\int_R^\infty \frac{1}{r^{(n-\alpha-\epsilon)s-n+1}}\, dr\right)^{\frac{1}{s}} \frac{\|f\|_\infty}{|x|^\epsilon} \\
\lesssim & \frac{1}{|x|^\epsilon} \|f\|_\infty \|\Omega\|_{L^s(\mathbb{S}^{n-1})}
\end{aligned}$$

for sufficiently large $|x|$. That is to say, $S_{\Omega,\alpha}f \in C^\infty_0(\mathbb{R}^n)$.

Now, let $f \in V^{(*)}_{0,\infty}L^{p,\varphi_1}(\mathbb{R}^n)$. We only need to prove $S_{\Omega,\alpha}f \in V^{(*)}_{0,\infty}L^{q,\varphi_2}(\mathbb{R}^n)$. Since $C^\infty_0(\mathbb{R}^n)$ is dense in $(V^{(*)}_{0,\infty}L^{p,\varphi_1}(\mathbb{R}^n))'$, there exists a sequence $\{f_k\} \subset C^\infty_0(\mathbb{R}^n)$ such that $f_k \to f$ as $k \to \infty$. Thus, by the continuity of $S_{\Omega,\alpha}$ from $L^{p,\varphi_1}(\mathbb{R}^n)$ into $L^{q,\varphi_2}(\mathbb{R}^n)$, we get

$$S_{\Omega,\alpha}f = S_{\Omega,\alpha}\left(\lim_{k\to\infty} f_k\right) = \lim_{k\to\infty}(S_{\Omega,\alpha}f_k).$$

This implies that $S_{\Omega,\alpha}f_k \in C^\infty_0(\mathbb{R}^n)$, and then we have $S_{\Omega,\alpha}f \in V^{(*)}_{0,\infty}L^{q,\varphi_2}(\mathbb{R}^n)$. □

Corollary 5.2. *Let α, p, q, and Ω be as in Theorem 3.4 and the pair (φ_1, φ_2) satisfy* (1.4), (5.1)*, and* (3.3) *for $s' \leq p$ or* (3.4) *for $q < s$. Then the operators $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ are bounded from $V^{(*)}_{0,\infty}L^{p,\varphi_1}(\mathbb{R}^n)$ to $V^{(*)}_{0,\infty}L^{q,\varphi_2}(\mathbb{R}^n)$.*

In [21], Gürbüz demonstrated that if $S_{\Omega,\alpha,b}$ is a sublinear operator fulfilling condition (2.6), with p, q, α, and b conforming to Theorem 4.9, and it is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, while the pair (φ_1, φ_2) satisfies (1.4), (1.5), (4.3), and

$$\int_\delta^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\sup\limits_{x\in\mathbb{R}^n} \varphi_1(x,t)^{\frac{1}{p}}}{t^{\frac{n}{q}+1}} dt < \infty \quad \text{for any} \quad \delta > 0, \tag{5.2}$$

then the operator $S_{\Omega,\alpha,b}$ is bounded from $V_0L^{p,\varphi_1}(\mathbb{R}^n)$ to $V_0L^{q,\varphi_2}(\mathbb{R}^n)$. By integrating the findings from Section 4 and the boundedness results of the commutator $S_{\Omega,\alpha,b}$ on the vanishing generalized Morrey spaces at the origin, we can derive the following corollaries regarding the subspaces $V^{(*)}_{0,\infty}L^{p,\varphi}(\mathbb{R}^n)$.

Corollary 5.3. *Let* α, p, q, Ω, *and* b *be as in Theorem 4.9 and the pair* (φ_1, φ_2) *satisfy conditions* (1.4), (1.5), (1.7), (1.8), (4.3), (4.6), *and* (5.2). *Suppose that* $S_{\Omega,\alpha,b}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$. *Then, for* $p > s'$, *the operator* $S_{\Omega,\alpha,b}$ *is bounded from* $V^{(*)}_{0,\infty}L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V^{(*)}_{0,\infty}L^{q,\varphi_2}(\mathbb{R}^n)$.

Corollary 5.4. *Let* α, p, q, Ω, *and* b *be as in Theorem 4.9 and the pair* (φ_1, φ_2) *satisfy conditions* (1.4), (1.5), (1.7), (1.8), (4.3), (4.6), *and* (5.2). *Then, for* $p > s'$, *the operators* $[b, T_{\Omega,\alpha}]$ *and* $M_{\Omega,\alpha,b}$ *are bounded from* $V^{(*)}_{0,\infty}L^{p,\varphi_1}(\mathbb{R}^n)$ *to* $V^{(*)}_{0,\infty}L^{q,\varphi_2}(\mathbb{R}^n)$.

6. Conclusions

We have intensively studied the boundedness of fractional sublinear operators $S_{\Omega,\alpha}$ and their commutators $S_{\Omega,\alpha,b}$ on the newly emerging vanishing generalized Morrey spaces characterized by properties (V_∞) or (V^*). Our research introduces an innovative approach by leveraging the Riesz potential I_α to control $S_{\Omega,\alpha}$ and managing $S_{\Omega,\alpha,b}$ through fractional maximal commutators with rough kernel $M_{\Omega,\alpha-\varepsilon,b}$ and $M_{\Omega,\alpha+\varepsilon,b}$ for some $\varepsilon > 0$. This methodology has not only enhanced our understanding of the behavior of these operators within the specific contexts of these spaces but also provided a robust framework for future investigations into similar operators across a wider spectrum of function spaces. The insights gained from this study are pivotal for tackling associated mathematical challenges and applications.

Author contributions

Qi Wei: Conceptualization, formal analysis, investigation, methodology for the whole paper, writing-original draft; Ting Mei: Conceptualization, formal analysis, investigation, validation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the National Natural Science Foundation of China (No. 11901047).

Conflict of interest

The authors declare there is no conflict of interest.

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