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Research article

Local well-posedness, global existence and blow-up for a heat equation with interior logarithmic source and nonlinear dynamic boundary condition

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Abstract: In this article, a heat equation with an interior logarithmic source and a nonlinear dynamic boundary condition is considered. After establishing the local well-posedness, the global existence and finite time blow-up results of the solutions with different initial energy levels are given. Moreover, the lower bound of the maximal existence time of the weak solution is deduced for the special case $m = 2$.

Keywords: local well-posedness; global existence; blow-up; logarithmic nonlinearity source; nonlinear dynamic boundary condition
Mathematics Subject Classification: 35K20, 35B44

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1 with $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$, where Γ_0 and Γ_1 are measurable over $\partial\Omega$, endowed with $(N-1)$-dimensional surface measure and $\text{mes}\,(\Gamma_0) > 0$. The goals of the present article are to study the local well-posedness, global existence, and blow-up phenomenon of the following parabolic problem with logarithmic reaction term and nonlinear dynamic boundary condition:

$$\begin{cases} u_t - \Delta u = |u|^{p-2}u \ln|u|, & t > 0,\ x \in \Omega, \\ u(t,x) = 0, & t \geq 0,\ x \in \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -|u_t|^{m-2} u_t, & t \geq 0,\ x \in \Gamma_1, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial}{\partial \nu}$ stands for the unit outer normal derivative. The positive parameters m, p satisfy

$$m > 1 \text{ and } 2 < p < 1 + \frac{2^*}{2}, \tag{1.2}$$

where 2^* denotes the Sobolev conjugate of 2, namely,

$$2^* = \begin{cases} \frac{2N}{N-2}, & \text{if} \quad N \geq 3, \\ +\infty, & \text{if} \quad N = 1, 2. \end{cases}$$

The initial datum u_0 belongs to $H^1(\Omega)$ and fulfills the compatibility condition $u_0 \equiv 0$ on Γ_0.

First of all, we briefly introduce the relevant physical background of model (1.1). Let Ω be a solid body surrounded by a fluid denoted by $\mathscr{A}$, with contact Γ_1 and having an internal cavity with contact boundary Γ_0. We suppose that a heat reaction-diffusion process occurs inside Ω such that, if $u = u(t, x)$ represents the temperature at time t and point x, the quantity of heat produced by the reaction is $|u|^{p-2} u \ln |u|$. Thus, the process can be described by the following heat equation with logarithmic reaction term

$$u_t - \rho \Delta u = |u|^{p-2} u \ln |u|, \ t > 0, \ x \in \Omega, \tag{1.3}$$

where the thermal conductivity $\rho > 0$ is taken to be 1 for simplicity. Meanwhile, the surrounding fluid is assumed to be well stirred. That is to say, $\mathscr{A}$ is a perfect conductor of heat such that the temperature in $\mathscr{A}$ is spatially homogeneous and can be described by $v = v(t)$ for any $t \geq 0$. To control the reaction-diffusion process inside the solid Ω, a refrigerating system is installed in the fluid. We assume that the refrigerating system is controlled in such a way that the heat absorbed from the fluid is proportional to a power of the rate of change of the temperature, as $|v'(t)|^{m-2} v'(t)$. Then, the rate of change of the temperature $v'(t)$ can be written as

$$v'(t) = -|v'(t)|^{m-2} v'(t) + \int_{\Gamma_1} j(t, x) \mathrm{d}\Gamma, \tag{1.4}$$

where $j = j(t, x)$ is the heat flux from Ω to $\mathscr{A}$. Since $\rho = 1$, according to the classical conductivity rule, one knows that the heat from Ω to $\mathscr{A}$ is equal to $-\frac{\partial u}{\partial \nu}$. Further, the thermal contact of the fluid at Γ_1 yields the continuity condition

$$u(t, x) = v(t), \ t \geq 0, \ x \in \Gamma_1.$$

This, along with (1.4), yields that

$$\frac{\partial u}{\partial \nu} = -u_t - |u_t|^{m-2} u_t, \ t \geq 0, \ x \in \Gamma_1. \tag{1.5}$$

To simplify from the point of mathematics, we take the modified form of (1.5) as follows,

$$\frac{\partial u}{\partial \nu} = -|u_t|^{m-2} u_t, \ t \geq 0, \ x \in \Gamma_1. \tag{1.6}$$

On the other hand, the temperature on Γ_0 is assumed to be constant (for simplicity, constantly vanishing), that is

$$u(t, x) = 0, \ x \in \Gamma_0, \ t \geq 0, \tag{1.7}$$

which, together with (1.3) and (1.6), leads to model (1.1).

The semilinear heat equation

$$u_t - \Delta u = f(u), \ t > 0, \ x \in \Omega, \tag{1.8}$$

subject to zero Dirichlet (or Neumann) boundary condition and initial datum $u_0(x)$ has been considered extensively and systematically by many mathematicians in the last few decades. For instance, the authors of [1] studied the relation between the global existence of the solution to (1.8) and the existence of the weak solution to the corresponding stationary problem. Levine in [2, 3] proved the finite time blow-up result of the solution to problem (1.8) by using the concavity method. In particular, for $f(u) = |u|^{p-2}u$ in (1.8), the authors of [4] dealt with the behavior of the solution when the initial energy is subcritical, that is, initial energy smaller than the mountain pass level. Gazzola and Weth [5] obtained the global well-posedness by using semigroup theory and potential well theory, and proved the finite time blow-up of the solution with supercritical initial energy by applying the comparison principle and variational method. For $f(u) = u\ln|u|$ in (1.8), under different initial energy levels, the authors of [6–8] proved that the solutions no longer blow up in finite time but blow up in infinite time and claimed that the power nonlinearity is a critical condition of blow-up in finite time for the solutions of semilinear heat equations. For $f(u) = |u|^{p-2}u\ln|u|$ with $p > 2$ in (1.8), Le and Le in [9, 10] showed the finite time blow-up phenomena of the solutions with subcritical and critical energy levels, while Han et al. in [11] claimed that problem admits a finite time blow-up solution for arbitrarily high initial energy by borrowing some ideas from [5, 12]. For other related works on the well-posedness, global existence, and blow-up behavior of parabolic equations, we refer the readers to the references [13–16].

Recently, the authors of [17, 18] considered problem (1.1) by replacing $f(u) = |u|^{p-2}u\ln|u|$ with $f(u) = |u|^{p-2}u$. Fiscella and Vitillaro in [17] obtained the blow-up result of the weak solution with a low initial energy level, while the authors of [18] showed the blow-up phenomena of the solutions with arbitrary high initial energy levels.

To the best of our knowledge, models like (1.1) involving both an interior logarithmic source and a nonlinear dynamic boundary condition have not been studied yet. On the one hand, similar to the power-type source term, the interior logarithmic source will also promote the occurrence of blow-up phenomenon. On the other hand, unlike the zero Neumann boundary condition, the dynamic boundary condition (1.6) may prevent (or delay) the occurrence of the blow-up phenomenon; see [19]. To some extent, there is a competition between the interior logarithmic source and the dynamic boundary condition. So, a natural and interesting question arises, that is, how do they compete with each other? Meanwhile, if the blow-up phenomenon occurs, can we give the lower bound of the blow-up time? The main objectives of this article are the global existence and finite time blow-up of the weak solution (see Definition 2.1) to problem (1.1). More specifically, we aim to investigate the competitive interaction among the interior logarithmic source, diffusion, and dynamic boundary conditions with nonlinear dissipation, and to elucidate the effect of the initial energy level on the blow-up behavior of the weak solution. To answer these questions, one has to develop some new skills and techniques to solve the following two difficulties caused by the concurrent occurrence of the interior logarithmic source and nonlinear dynamic boundary condition.

(1) The structure and property of the corresponding energy functional to problem (1.1) will change radically when the power-type source term is replaced by the logarithmic source, and the potential well method proposed in [20] and used in [4, 5, 17, 18] cannot be directly used to handle our problem. To be exact, the primary difficulty caused by the logarithmic source lies in energy-based estimates. To solve it, one has to establish a positive lower bound of d_0 (see Lemma 3.1) and some other new inequalities and methods (see Lemmas 3.2, 3.3, and 3.4).

(2) Since the homogeneous Dirichlet boundary condition is replaced by the nonlinear dynamic boundary condition, one need to modify and improve some details of the concavity method introduced by H. Levine in [2]. First, inspired by [21], we give a crucial modification by skillfully constructing some new auxiliary functions $Z(t)$ and $Y(t)$ (see (3.23) and (3.26), respectively). Then, the trace embedding for Sobolev space of fractional order (see (3.19)) and the interpolation inequality are used to estimate the $L^m(\Gamma_1)$ norm of $u|_{\Gamma_1}$.

The rest of this article is organized as follows. Section 2 is devoted to the local existence, uniqueness, and continuous dependence on the initial datum of the weak solution to problem (1.1). By making use of the Banach contraction theorem, the local well-posedness result of the weak solution is obtained. Section 3 mainly concerns the global existence and finite-time blow-up behavior of the weak solution to problem (1.1) with different initial energy levels. By virtue of a new family of potential wells, together with the modified differential inequality approach and ω-limit set, the sufficient conditions on the global existence and the occurrence of the finite time blow-up phenomenon are discussed. Moreover, by using the first-order differential inequality technique for a suitably defined auxiliary function and some Sobolev-type inequalities, a lower bound of the maximal existence time $T_{\max}$ (see Definition 3.1) for the special case $m = 2$ is also deduced in section 3.

2. Local well-posedness

In this section, our attention is focused on the local existence, uniqueness, and continuous dependence on the initial datum of the weak solution to problem (1.1). For this purpose, we first introduce some notations, known facts, and the definition of the weak solution that will be used throughout this article.

For given $q \in [1, +\infty)$, we use $\|\cdot\|_q$ and $\|\cdot\|_{q,\Gamma_1}$ to represent the $L^q(\Omega)$ norm and $L^q(\Gamma_1)$ norm, respectively. We denote the Hilbert space $H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) : u = 0 \text{ for } x \in \Gamma_0\right\}$, endowed with the norm

$$\|u\|_{H^1_{\Gamma_0}} = \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}}.$$

The trace theorem indicates the existence of the continuous trace mapping $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^2(\partial\Omega)$. Meanwhile, since $\operatorname{mes}(\Gamma_0) > 0$, a Poincaré-type inequality holds (see [22,23]). Hence $\|\nabla u\|_2$ can be viewed as an equivalent norm to the norm $\|u\|_{H^1_{\Gamma_0}}$. Throughout this article, the symbols $\widetilde{C}$ and C_i with $i = 1, 2, \cdots$, will denote positive constants that may change from line to line even if in the same inequality.

The definition of the weak solution and the local well-posedness result of problem (1.1) are stated as follows:

Definition 2.1 (Weak solution). *Assume that (1.2) holds. Let* $u_0 \in H^1_{\Gamma_0}(\Omega)$ *and* $T > 0$. *A function* $u(t) := u(t, x)$ *defined in* $[0, T] \times \Omega$ *is called a weak solution of problem (1.1) if*

(i) $u \in L^\infty\left(0, T; H^1_{\Gamma_0}(\Omega)\right)$ *and* $u_t \in L^2\left(0, T; L^2(\Omega)\right)$;

(ii) the spatial trace of $u(t)$ *on* $(0, T) \times \partial\Omega$, *which exists by the trace theorem, has a distributional time derivative on* $(0, T) \times \partial\Omega$ *whose restriction to* $(0, T) \times \Gamma_1$ *belongs to* $L^m((0, T) \times \Gamma_1)$;

(iii) for all $\vartheta \in H^1_{\Gamma_0}(\Omega) \cap L^m(\Gamma_1)$ *and* $t \in [0, T]$, *the distribution identity*

$$\int_\Omega (\vartheta u_t + \nabla\vartheta \cdot \nabla u)\,\mathrm{d}x + \int_{\Gamma_1} \vartheta\, |u_t|^{m-2}\, u_t \mathrm{d}\Gamma = \int_\Omega \vartheta |u|^{p-2} u \ln|u| \mathrm{d}x \tag{2.1}$$

holds;

(iv) $u(0,x) = u_0(x)$.

Theorem 2.1 (Local well-posedness). *Suppose that (1.2) holds and* $u_0 \in H^1_{\Gamma_0}(\Omega)$*. Then there exists a* T^* *such that problem (1.1) admits a unique weak solution* $u(t)$ *in* $[0,T^*]\times\Omega$*. Furthermore, one has*

$$u \in C\left([0,T^*];H^1_{\Gamma_0}(\Omega)\right),\ u_t \in L^m((0,T^*)\times\Gamma_1)\cap L^2((0,T^*)\times\Omega), \tag{2.2}$$

$$\frac{1}{2}\|\nabla u\|_2^2\Big|_s^t + \int_s^t\left(\|u_\tau\|_2^2 + \|u_\tau\|_{m,\Gamma_1}^m\right)\mathrm{d}\tau = \int_s^t\int_\Omega |u|^{p-2}u\ln|u|\,u_\tau\mathrm{d}x\mathrm{d}\tau,\ 0\le s\le t\le T^*, \tag{2.3}$$

and

$$\|u\|_{C\left([0,T^*];H^1_{\Gamma_0}(\Omega)\right)} \le \widetilde{C}\,\|u_0\|_{H^1_{\Gamma_0}}. \tag{2.4}$$

In addition, $u(t)$ *depends continuously on the initial datum* u_0*. In other words, for any* $\{u_{0n}\}_{n=1}^\infty \subseteq H^1_{\Gamma_0}(\Omega)$ *and* $u_{0n}\to u_0$ *as* $n\to+\infty$ *in* $H^1_{\Gamma_0}(\Omega)$*, the weak solution* $u^n(t)$ *to problem (1.1) with initial datum* u_{0n} *is defined in* $[0,T^*]\times\Omega$ *and* $u^n(t)\to u(t)$ *as* $n\to+\infty$ *in* $C\left([0,T^*];H^1_{\Gamma_0}(\Omega)\right)$.

To prove the local well-posedness result, we need the following lemma.

Lemma 2.1. *(see [24]) Assume that* ϵ_0 *and* ϵ_1 *are two positive constants. Then one has*

$$\Psi^{\epsilon_0}\ln\Psi \le (e\epsilon_1)^{-1}\Psi^{\epsilon_0+\epsilon_1}\ \textit{for all}\ \Psi\ge 1,$$

and

$$|\Psi^{\epsilon_0}\ln\Psi| \le (e\epsilon_0)^{-1}\ \textit{for all}\ 0<\Psi<1.$$

Proof of Theorem 2.1. For arbitrary $T\in(0,+\infty)$, we set $Y_T = C\left([0,T];H^1_{\Gamma_0}(\Omega)\right)$ endowed with the usual norm $\|u\|_{Y_T} = \|u\|_{L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)}$, and the closed convex set $X_T = \{u\in Y_T : u(0)=u_0\}$. The proof of Theorem 2.1 is divided into three steps as follows:

Step 1: Local existence. Let $u\in X_T$. From (1.2), it follows that there exist some appropriate $\epsilon_2 \in \left(0,\frac{2^*-2(p-1)}{2}\right]$ such that $2 < 2(p-1+\epsilon_2)\le 2^*$. Then, owing to Sobolev embedding theorem, for $t\in[0,T]$, one has

$$\|u\|_{2(p-1+\epsilon_2)} \le C_1\|u\|_{H^1_{\Gamma_0}}. \tag{2.5}$$

With the help of Lemma 2.1 and (2.5), one can see

$$\begin{aligned}\int_\Omega\left(|u|^{p-2}u\ln|u|\right)^2\mathrm{d}x &= \int_{\Omega_1}\left(|u|^{p-2}u\ln|u|\right)^2\mathrm{d}x + \int_{\Omega_2}\left(|u|^{p-2}u\ln|u|\right)^2\mathrm{d}x\\ &\le \frac{|\Omega|}{e^2(p-1)^2} + \frac{1}{e^2\epsilon_2^2}\|u\|_{2(p-1+\epsilon_2)}^{2(p-1+\epsilon_2)}\\ &\le C_2 + C_3\|u\|_{H^1_{\Gamma_0}}^{2(p-1+\epsilon_2)},\end{aligned} \tag{2.6}$$

which implies that $|u|^{p-2}u\ln|u| \in L^\infty\left((0,T);L^2(\Omega)\right)$, where $\Omega_1 = \{x\in\Omega : |u(x,t)|<1\}$, $\Omega_2 = \{x\in\Omega : |u(x,t)|\ge 1\}$, $C_2 = |\Omega|\left[e(p-1)\right]^{-2}$ and $C_3 = (e\epsilon_2)^{-2}C_2^{2(p-1+\epsilon_2)}$. A direct application of Theorem 1.5 in [17] tells us that

$$\begin{cases} v_t - \Delta v = |u|^{p-2}u\ln|u|, & t\in(0,T),\ x\in\Omega,\\ v(x,t) = 0, & t\in[0,T),\ x\in\Gamma_0,\\ \frac{\partial v}{\partial \nu} = -|v_t|^{m-2}v_t, & t\in[0,T),\ x\in\Gamma_1,\\ v(0,x) = u_0(x), & x\in\Omega,\end{cases} \tag{2.7}$$

admits a unique weak solution $v(t) \in Y_T$ with $v_t \in L^m((0,T)\times\Gamma_1) \cap L^2((0,T)\times\Omega)$ and

$$\frac{1}{2}\|\nabla v\|_2^2\Big|_0^t + \int_0^t \left(\|v_\tau\|_2^2 + \|v_\tau\|_{m,\Gamma_1}^m\right)\mathrm{d}\tau = \int_0^t\int_\Omega |u|^{p-2}u\ln|u|\,v_\tau \mathrm{d}x\mathrm{d}\tau,\ t\in[0,T]. \tag{2.8}$$

Define $\Phi : X_T \to X_T$ by $\Phi(u) = v$, where v stands for the solution of problem (2.7) that corresponds to u. Set $B_R = \left\{u \in X_T : \|u\|_{Y_T} \le R\right\}$. It is obvious that B_R is a nonempty set provided that $R_0 = \|u_0\|_{H^1_{\Gamma_0}} \le R$. Now, we are in the position to show that, for sufficiently large R and sufficiently small T, $\Phi : B_R \to B_R$ is a contraction mapping. To this end, we first show that Φ maps B_R into itself. Let $u \in B_R$. From (2.6), (2.8), Hölder's inequality, and Young's inequality, one can infer that, for $t\in[0,T]$,

$$\begin{aligned}
&\frac{1}{2}\|\nabla v\|_2^2 + \int_0^t \|v_\tau\|_2^2\,\mathrm{d}\tau\\
\le&\frac{1}{2}\|\nabla u_0\|_2^2 + \int_0^t\int_\Omega |u|^{p-2}u\ln|u|\,v_\tau\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{1}{2}\|\nabla u_0\|_2^2 + \int_0^t \|v_\tau\|_2\left(\int_\Omega\left(|u|^{p-2}u\ln|u|\right)^2\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}\tau\\
\le&\frac{1}{2}\|\nabla u_0\|_2^2 + \int_0^t \|v_\tau\|_2\left(C_2 + C_3\|u(\tau)\|_{H^1_{\Gamma_0}}^{2(p-1+\epsilon_2)}\right)^{\frac{1}{2}}\mathrm{d}\tau\\
\le&\frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\int_0^t \|v_\tau\|_2^2\,\mathrm{d}\tau + \frac{1}{2}\int_0^t\left(C_2 + C_3\|u(\tau)\|_{H^1_{\Gamma_0}}^{2(p-1+\epsilon_2)}\right)\mathrm{d}\tau\\
\le&\frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\int_0^t \|v_\tau\|_2^2\,\mathrm{d}\tau + \frac{1}{2}\left(C_2 + C_3R^{2(p-1+\epsilon_2)}\right)T,
\end{aligned}$$

which implies that,

$$\|\nabla v\|_2^2 + \int_0^t \|v_\tau\|_2^2\,\mathrm{d}\tau \le R_0^2 + \left(C_2 + C_3R^{2(p-1+\epsilon_2)}\right)T. \tag{2.9}$$

Consequently, one has

$$\|\nabla v\|^2_{L^\infty(0,T;L^2(\Omega))} \le R_0^2 + \left(C_2 + C_3R^{2(p-1+\epsilon_2)}\right)T. \tag{2.10}$$

On the other hand, Hölder's inequality and (2.9) can be used to obtain

$$\begin{aligned}
\|v(t)\|_2^2 = \left\|u_0 + \int_0^t v_\tau\mathrm{d}\tau\right\|_2^2 &\le 2\|u_0\|_2^2 + 2\int_\Omega\left(\int_0^t v_\tau\mathrm{d}\tau\right)^2\mathrm{d}x\\
&\le 2\left(\|u_0\|_2^2 + \int_\Omega t\int_0^t (v_\tau)^2\,\mathrm{d}\tau\mathrm{d}x\right)\\
&\le 2\left[R_0^2 + R_0^2T + \left(C_2 + C_3R^{2(p-1+\epsilon_2)}\right)T^2\right].
\end{aligned} \tag{2.11}$$

Taking

$$R = \max\left\{1,\ \sqrt{10R_0^2 + 6C_2}\right\},\ T \le \min\left\{1,\ \frac{1}{6C_3}R^{2(2-p-\epsilon_2)}\right\}, \tag{2.12}$$

then from (2.10) and (2.11), it follows that

$$\|v\|_{Y_T}^2 \leq 5R_0^2 + 3C_2 + 3C_3R^{2(p-1+\epsilon_2)}T \leq \frac{1}{2}R^2 + \frac{1}{2}R^2 = R^2,$$

which means that Φ maps B_R into itself for sufficiently large R and sufficiently small T.

Next, suppose that u and $\overline{u}$ belong to B_R, denote $v = \Phi(u)$, $\overline{v} = \Phi(\overline{u})$, and $w = u - \overline{u}$. Then it is trivial to check that w is a weak solution of the problem

$$\begin{cases} w_t - \Delta w = |u|^{p-2}u \ln|u| - |\overline{u}|^{p-2}\overline{u}\ln|\overline{u}|, & t \in (0,T),\ x \in \Omega, \\ w(x,t) = 0, & t \in [0,T),\ x \in \Gamma_0, \\ \frac{\partial w}{\partial \nu} = -|v_t|^{m-2} v_t + |\overline{v}_t|^{m-2}\overline{v}_t, & t \in [0,T),\ x \in \Gamma_1, \\ w(0,x) = 0, & x \in \Omega. \end{cases} \tag{2.13}$$

Since v_t, $\overline{v}_t \in L^m((0,T)\times\Gamma_1)$, then by Hölder's inequality, one can deduce that $|v_t|^{m-2} v_t$, $|\overline{v}_t|^{m-2}\overline{v}_t \in L^{\frac{m}{m-1}}((0,T)\times\Gamma_1)$. In view of Lemma 2.2 in [17], one has

$$\begin{aligned} &\frac{1}{2}\|\nabla w\|_2^2\Big|_0^t + \int_0^t \|w_\tau\|_2^2\,\mathrm{d}\tau + \int_0^t\int_{\Gamma_1}\left[|v_t|^{m-2} v_\tau - |\overline{v}_\tau|^{m-2}\overline{v}_t\right] w_\tau \mathrm{d}\Gamma\mathrm{d}\tau \\ =& \int_0^t\int_\Omega \left(|u|^{p-2} u\ln|u| - |\overline{u}|^{p-2}\overline{u}\ln|\overline{u}|\right) w_\tau \mathrm{d}x\mathrm{d}\tau,\ t \in [0,T]. \end{aligned} \tag{2.14}$$

Denote $G(s) = |s|^{p-2} s\ln|s|$. Then $G'(s) = |s|^{p-2}[1 + (p-1)\ln|s|]$. Since $\lim\limits_{|s|\to+\infty}\frac{\ln|s|}{s^{\epsilon_3}} = 0$ holds for any positive constant ϵ_3, one can claim that there exists a sufficiently large constant C_4 such that $\left|\frac{\ln|s|}{s^{\epsilon_3}}\right| < 1$ holds for $|s| \geq C_4$. As a result,

$$|s|^{p-2}|\ln|s|| < |s|^{p-2+\epsilon_3},\ \text{for}\ |s| \geq C_4. \tag{2.15}$$

On the other hand, noticing that $\lim\limits_{|s|\to 0^+}|s|^{p-2}\ln|s| = 0$ holds true for $p > 2$, then one can infer that there exists a positive constant C_5 such that

$$|s|^{p-2}|\ln|s|| \leq C_5,\ \text{for}\ |s| \leq C_4. \tag{2.16}$$

Bearing (2.15) and (2.16) in mind yields that

$$|s|^{p-2}|\ln|s|| \leq C_5 + |s|^{p-2+\epsilon_3},\ p > 2,\ \epsilon_3 > 0. \tag{2.17}$$

By the mean value theorem and (2.17), there exists a constant $\epsilon_4 \in (0,1)$ such that

$$\begin{aligned} |G(u) - G(\overline{u})| &= |G'(\epsilon_4 u + (1-\epsilon_4)\overline{u})||u - \overline{u}| \\ &= |1 + (p-1)\ln|\epsilon_4 u + (1-\epsilon_4)\overline{u}|||\epsilon_4 u + (1-\epsilon_4)\overline{u}|^{p-2}|u - \overline{u}| \\ &\leq \left[C_5(p-1) + (p-1)(|u| + |\overline{u}|)^{p-2+\epsilon_3} + (|u| + |\overline{u}|)^{p-2}\right]|u - \overline{u}|. \end{aligned} \tag{2.18}$$

In light of Hölder's inequality and Young's inequality, one can immediately get

$$\begin{aligned} \int_0^t\int_\Omega C_5(p-1)|u - \overline{u}| w_\tau \mathrm{d}x\mathrm{d}\tau &\leq C_5(p-1)\int_0^t \|u - \overline{u}\|_2\|w_\tau\|_2\,\mathrm{d}\tau \\ &\leq C_6\int_0^t \|u - \overline{u}\|_{H^1_{\Gamma_0}}\|w_\tau\|_2\,\mathrm{d}\tau \\ &\leq C_6\left(\frac{1}{2\epsilon_5}\int_0^t \|u - \overline{u}\|_{H^1_{\Gamma_0}}^2\mathrm{d}\tau + \frac{\epsilon_5}{2}\int_0^t \|w_\tau\|_2^2\,\mathrm{d}\tau\right), \end{aligned} \tag{2.19}$$

where $\epsilon_5 > 0$ will be chosen later. Now, set $r = 2^*$ if $N \in \mathbb{N} \setminus \{2\}$, while $r = 2p$ when $N = 2$, such that $2 < p < 1 + \frac{r}{2} \leq 1 + \frac{2^*}{2}$ and $r > 2$. Fixing ϵ_3 such that $2 < p + \epsilon_3 \leq 1 + \frac{r}{2} \leq 1 + \frac{2^*}{2}$ and taking $s = \frac{2r}{r-2}$ such that $\frac{1}{s} + \frac{1}{r} + \frac{1}{2} = 1$, then by using Hölder's inequality, Minkowski's inequality and Young's inequality, one can claim that

$$\begin{aligned}
&\int_0^t \int_\Omega (p-1)\,(|u| + |\overline{u}|)^{p-2+\epsilon_3}\,|u - \overline{u}|\,w_\tau \mathrm{d}x\mathrm{d}\tau \\
\leq& (p-1) \int_0^t \||u| + |\overline{u}|\|_{s(p-2+\epsilon_3)}^{p-2+\epsilon_3} \|u - \overline{u}\|_r \|w_\tau\|_2 \,\mathrm{d}\tau \\
\leq& (p-1) \int_0^t \left(\|u\|_{s(p-2+\epsilon_3)}^{p-2+\epsilon_3} + \|\overline{u}\|_{s(p-2+\epsilon_3)}^{p-2+\epsilon_3}\right) \|u - \overline{u}\|_r \|w_\tau\|_2 \,\mathrm{d}\tau \\
\leq& C_7 \int_0^t \left(\|u\|_{H^1_{\Gamma_0}}^{p-2+\epsilon_3} + \|\overline{u}\|_{H^1_{\Gamma_0}}^{p-2+\epsilon_3}\right) \|u - \overline{u}\|_{H^1_{\Gamma_0}} \|w_\tau\|_2 \,\mathrm{d}\tau \\
\leq& 2C_7 R^{p-2+\epsilon_3} \left(\frac{1}{2\epsilon_6} \int_0^t \|u - \overline{u}\|_{H^1_{\Gamma_0}}^2 \,\mathrm{d}\tau + \frac{\epsilon_6}{2} \int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau\right),
\end{aligned} \tag{2.20}$$

where $\epsilon_6 > 0$ will be chosen later. Similarly, one has

$$\int_0^t \int_\Omega (|u| + |\overline{u}|)^{p-2}\,|u - \overline{u}|\,w_\tau \mathrm{d}x\mathrm{d}\tau \leq C_8 R^{p-2} \left(\frac{1}{2\epsilon_7} \int_0^t \|u - \overline{u}\|_{H^1_{\Gamma_0}}^2 \,\mathrm{d}\tau + \frac{\epsilon_7}{2} \int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau\right), \tag{2.21}$$

where $\epsilon_7 > 0$ will be chosen later. Recalling (2.12), collecting (2.14), (2.19), (2.20), and (2.21) yields that

$$\begin{aligned}
&\frac{1}{2} \|\nabla w\|_2^2 + \int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau \\
\leq& C_6 \left(\frac{1}{2\epsilon_5} \int_0^t \|u - \overline{u}\|_{H^1_{\Gamma_0}}^2 \,\mathrm{d}\tau + \frac{\epsilon_5}{2} \int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau\right) \\
&+ 2C_7 R^{p-2+\epsilon_3} \left(\frac{1}{2\epsilon_6} \int_0^t \|u - \overline{u}\|_{H^1_{\Gamma_0}}^2 \,\mathrm{d}\tau + \frac{\epsilon_6}{2} \int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau\right) \\
&+ C_8 R^{p-2} \left(\frac{1}{2\epsilon_7} \int_0^t \|u - \overline{u}\|_{H^1_{\Gamma_0}}^2 \,\mathrm{d}\tau + \frac{\epsilon_7}{2} \int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau\right) \\
\leq& C_9 R^{p-2+\epsilon_3} \left[\left(\frac{1}{\epsilon_5} + \frac{1}{\epsilon_7} + \frac{1}{\epsilon_7}\right) \int_0^t \|u - \overline{u}\|_{H^1_{\Gamma_0}}^2 \,\mathrm{d}\tau + (\epsilon_5 + \epsilon_7 + \epsilon_7) \int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau\right].
\end{aligned} \tag{2.22}$$

Choosing suitable ϵ_5, ϵ_6, and ϵ_7 such that $2C_9 R^{p-2+\epsilon_3} (\epsilon_5 + \epsilon_6 + \epsilon_7) = 1$, then from (2.22), one can conclude that

$$\|\nabla w\|_2^2 + \int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau \leq C_{10} R^{p-2+\epsilon_3} T \|u - \overline{u}\|_{L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)}^2,$$

which indicates that

$$\|\nabla w\|_2^2 \leq C_{10} R^{p-2+\epsilon_3} T \|u - \overline{u}\|_{L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)}^2, \tag{2.23}$$

and

$$\int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau \leq C_{10} R^{p-2+\epsilon_3} T \|u - \overline{u}\|_{L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)}^2. \tag{2.24}$$

Noticing that $w(0) = 0$, and combining Hölder's inequality with (2.24) leads us to

$$\|w(t)\|_2^2 = \left\|\int_0^t w_\tau \mathrm{d}\tau\right\|_2^2 \leq T\int_0^t \|w_\tau\|_2^2 \,\mathrm{d}\tau \leq C_{10}R^{p-2+\epsilon_3}T^2 \|u-\overline{u}\|^2_{L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)}. \tag{2.25}$$

Since $T < 1$, then from (2.23) and (2.25), one has

$$\|w(t)\|^2_{H^1_{\Gamma_0}} = \|w(t)\|_2^2 + \|\nabla w\|_2^2 \leq 2C_{10}R^{p-2+\epsilon_3}T\,\|u-\overline{u}\|^2_{L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)}, \tag{2.26}$$

which means that Φ is a contraction provided that $2C_{10}R^{p-2+\epsilon_3}T < 1$. Up to now, putting

$$T^* = \min\left\{1, \frac{1}{6C_3}R^{2(2-p-\epsilon_2)}, \frac{1}{2C_{10}}R^{2-p-\epsilon_3}\right\} \tag{2.27}$$

with $R = \max\left\{1, \sqrt{10R_0^2 + 6C_2}\right\}$, and keeping the Banach contraction theorem in mind, one can claim that problem (1.1) admits a weak solution in $[0, T^*] \times \Omega$ fulfills (2.2), (2.3), and (2.4).

Step 2: Uniqueness. Let $u, \widetilde{u} \in C\left([0, T^*]; H^1_{\Gamma_0}(\Omega)\right)$ be two weak solutions of problem (1.1). Then, by the uniqueness of the weak solution of the linearized problem (2.7) and the fixed point property of the contraction mapping, one can directly claim that $u_1 = \widetilde{u}_1$.

Step 3: Continuous dependence on the initial datum. Put

$$C_{11} = \|u\|_{C\left([0,T^*];H^1_{\Gamma_0}(\Omega)\right)}$$

and suppose that $\{u_{0n}\}_{n=1}^\infty \subseteq H^1_{\Gamma_0}(\Omega)$ and $u_{0n} \to u_0$ in $H^1_{\Gamma_0}(\Omega)$. Then there exists a $n_1 \in \mathbb{N}$ such that $\|u_{0n}\|_{H^1_{\Gamma_0}} \leq \|u_0\|_{H^1_{\Gamma_0}} + 1 \leq C_{11} + 1$ holds for $n > n_1$. Moreover, for any $n \in \mathbb{N}$, one has

$$\|u_{0n}\|_{H^1_{\Gamma_0}} \leq \|u_0\|_{H^1_{\Gamma_0}} + 1 \leq C_{12} := \max\left\{C_{11}+1, \|u_{01}\|_{H^1_{\Gamma_0}}, \cdots, \left\|u_{0n_1}\right\|_{H^1_{\Gamma_0}}\right\}.$$

Step 1 tells us that problem (1.1) with initial datum u_{0n} admits a unique weak u^n in $[0, T^*] \times \Omega$ with $T^* \in (0, 1]$ and $\|u^n\|_{C\left([0,T^*];H^1_{\Gamma_0}(\Omega)\right)} \leq \widetilde{C}C_{12}$. Consequently, $w^n = u^n - u$ is a weak solution of the problem

$$\begin{cases} w_t^n - \Delta w^n = |u^n|^{p-2}u^n \ln|u^n| - |u|^{p-2}u\ln|u|, & t \in (0, T^*),\ x \in \Omega, \\ w^n(x,t) = 0, & t \in [0, T^*),\ x \in \Gamma_0, \\ \frac{\partial w^n}{\partial \nu} = -\left|u_t^n\right|^{m-2} u_t^n + |u_t|^{m-2}u_t, & t \in [0, T^*),\ x \in \Gamma_1, \\ w^n(0,x) = u_{0n} - u_0, & x \in \Omega, \end{cases}$$

and satisfies

$$\begin{aligned} &\frac{1}{2}\|\nabla w^n\|_2^2\Big|_0^t + \int_0^t \left\|w_\tau^n\right\|_2^2 \mathrm{d}\tau + \int_0^t\int_{\Gamma_1}\left[\left|u_t^n\right|^{m-2}u_\tau^n - |u_\tau|^{m-2}u_t\right]w_\tau^n \mathrm{d}\Gamma\mathrm{d}\tau \\ &= \int_0^t\int_\Omega \left(|u^n|^{p-2}u^n\ln|u^n| - |u|^{p-2}u\ln|u|\right)w_\tau^n \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{2.28}$$

Similar to the process of the derivation of (2.22), one has

$$\begin{aligned}&\frac{1}{2}\|\nabla w^n(t)\|_2^2-\frac{1}{2}\|\nabla(u_{0n}-u_0)\|_2^2+\int_0^t\left\|w_\tau^n\right\|_2^2\mathrm{d}\tau\\ \le&C_{13}\left[\left(\frac{1}{\epsilon_8}+\frac{1}{\epsilon_9}+\frac{1}{\epsilon_{10}}\right)\int_0^t\|w^n\|_{H_{\Gamma_0}^1}^2\,\mathrm{d}\tau+(\epsilon_8+\epsilon_9+\epsilon_{10})\int_0^t\left\|w_\tau^n\right\|_2^2\mathrm{d}\tau\right],\end{aligned}\tag{2.29}$$

where $C_{13}=\frac{C_9\left(\widetilde{C}C_{12}\right)^{p-2+\epsilon_3}}{2}$. Selecting suitable ϵ_8, ϵ_9, and ϵ_{10} such that

$$C_{13}(\epsilon_8+\epsilon_9+\epsilon_{10})=\frac{1}{2},$$

then (2.29) means that

$$\|\nabla w^n(t)\|_2^2+\int_0^t\left\|w_\tau^n\right\|_2^2\mathrm{d}\tau\le C_{14}\int_0^t\|w^n\|_{H_{\Gamma_0}^1}^2\,\mathrm{d}\tau+\|\nabla(u_{0n}-u_0)\|_2^2.\tag{2.30}$$

By virtue of Minkowski's inequality and Hölder's inequality, one finds that

$$\begin{aligned}\|w^n(t)\|_2^2&=\left\|\int_0^t w_\tau^n\mathrm{d}\tau+(u_{0n}-u_0)\right\|_2^2\\&\le\left(\left\|\int_0^t w_\tau^n\mathrm{d}\tau\right\|_2+\|u_{0n}-u_0\|_2\right)^2\\&\le2\left(\left\|\int_0^t w_\tau^n\mathrm{d}\tau\right\|_2^2+\|u_{0n}-u_0\|_2^2\right)\\&\le2\left(T\int_0^t\left\|w_\tau^n\right\|_2^2\mathrm{d}\tau+\|u_{0n}-u_0\|_2^2\right),\end{aligned}$$

which, together with (2.30), leads to

$$\|w^n(t)\|_2^2\le2\left(C_{14}\int_0^t\|w^n\|_{H_{\Gamma_0}^1}^2\,\mathrm{d}\tau+\|u_{0n}-u_0\|_{H_{\Gamma_0}^1}^2\right).\tag{2.31}$$

Combining (2.30) with (2.31), one can arrive at

$$\|w^n(t)\|_{H_{\Gamma_0}^1}^2\le3\left(C_{14}\int_0^t\|w^n\|_{H_{\Gamma_0}^1}^2\,\mathrm{d}\tau+\|u_{0n}-u_0\|_{H_{\Gamma_0}^1}^2\right).$$

Gronwall inequality can be used to obtain

$$\|w^n(t)\|_{H_{\Gamma_0}^1}^2\le3\|u_{0n}-u_0\|_{H_{\Gamma_0}^1}^2\left(1+3C_{14}te^{3C_{14}t}\right),\ t\in[0,T^*]\subseteq[0,1].$$

Moreover,

$$\|w^n(t)\|_{H_{\Gamma_0}^1}\le C_{15}\|u_{0n}-u_0\|_{H_{\Gamma_0}^1}e^{C_{16}t},\ t\in[0,T^*].$$

where $C_{15}=\sqrt{6\max\{1,3C_{14}\}}$ and $C_{16}=\frac{3C_{14}}{2}$. As a consequence of this inequality, one has

$$\|u^n(t)-u(t)\|_{H_{\Gamma_0}^1}=\|w^n(t)\|_{H_{\Gamma_0}^1}\le C_{15}\|u_{0n}-u_0\|_{H_{\Gamma_0}^1}e^{C_{16}t}\to0,\ \text{as } n\to+\infty,$$

which tells us that $u(t)$ depends continuously on the initial datum u_0. The proof of Theorem 2.1 is complete. □

3. Global existence and blow-up of the weak solution

In this section, we shall discuss the global existence and finite-time blow-up behavior of the weak solutions to problem (1.1) at both the subcritical and supercritical initial energy levels. To achieve these goals, we first introduce some definitions and notations.

Definition 3.1 (Maximal existence time). *Suppose that $u(t)$ is a weak solution of problem (1.1). We define the maximal existence time $T_{\max}$ of $u(t)$ as follows:*

- *(i) If $u(t)$ exists for all $t \in [0, +\infty)$, then $T_{\max} = +\infty$. In this case, we say that the weak solution $u(t)$ of problem (1.1) is global in time.*
- *(ii) If there exists a $T \in (0, +\infty)$ such that $u(t)$ exists for $t \in [0, T)$, but does not exist at $t = T$, then $T_{\max} = T$.*

Definition 3.2 (ω-limit set). *If the weak solution $u(t)$ of problem (1.1) is global in time, we say that*

$$\omega(u_0) = \bigcap_{t \geq 0} \overline{\{u(s) : s \geq t\}}$$

is the ω-limit set of $u_0 \in H^1_{\Gamma_0}(\Omega)$. Here the closure is taken in $H^1_{\Gamma_0}(\Omega)$.

Definition 3.3 (Finite time blow-up). *Suppose that $u(t)$ is a weak solution of problem (1.1). We say that $u(t)$ blows up in finite time if the maximal existence time $T_{\max}$ is finite and*

$$\lim_{t \to T_{\max}^-} \|u(t)\|_{H^1_{\Gamma_0}} = +\infty.$$

In what follows, we introduce some functionals and sets related to problem (1.1). Let $u(t) \in H^1_{\Gamma_0}(\Omega)$. We define the energy functional

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln|u| \, \mathrm{d}x, \tag{3.1}$$

and the Nehari functional

$$I(u) = \|\nabla u\|_2^2 - \int_\Omega |u|^p \ln|u| \, \mathrm{d}x. \tag{3.2}$$

We call the number

$$d_0 = \inf_{u \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}} \sup_{\lambda > 0} J(\lambda u)$$

as mountain-pass level or potential well depth.

We denote the Nehari manifold relative to $J(u)$

$$\mathcal{N} = \{u \in H^1_{\Gamma_0}(\Omega) : I(u) = 0\} \setminus \{0\},$$

which separates the two unbounded sets

$$\mathcal{N}_+ = \left\{u \in H^1_{\Gamma_0}(\Omega) : I(u) > 0\right\} \text{ and } \mathcal{N}_- = \left\{u \in H^1_{\Gamma_0}(\Omega) : I(u) < 0\right\}.$$

Then d_0 can also be characterized as $d_0 = \inf_{u\in\mathcal{N}} J(u)$. Namely,

$$d_0 = \inf_{u\in H^1_{\Gamma_0}(\Omega)\setminus\{0\}} \sup_{\lambda>0} J(\lambda u) = \inf_{u\in\mathcal{N}} J(u). \tag{3.3}$$

What is worth mentioning is that, although the occurrence of the logarithmic nonlinearity makes it impossible to determine the specific value of d_0, we are able to prove that d_0 has a positive lower bound d_1 (see Lemma 3.1), which can help us to prove the finite time blow-up result for subcritical initial energy level.

Furthermore, for any $a > d_0$, we define

$$J^a = \left\{u \in H^1_{\Gamma_0}(\Omega) : J(u) < a\right\},$$

and

$$\mathcal{N}_a = \mathcal{N} \cap J^a = \left\{u \in \mathcal{N} : \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p^2}\|u\|_p^p < a\right\} \neq \emptyset.$$

We put

$$\lambda_a = \inf_{u\in\mathcal{N}_a}\left(\|u\|_2^2 + \|u\|_{2,\Gamma_1}^2\right),\ \Lambda_a = \sup_{u\in\mathcal{N}_a}\left(\|u\|_2^2 + \|u\|_{2,\Gamma_1}^2\right).$$

For fixed $a > d_0$, by the definitions of infimum and supremum, one knows that $\lambda_a \le \Lambda_a$. Moreover, if $a_1 \ge a_2 > d_0$, then $\mathcal{N}_{a_2} \subseteq \mathcal{N}_{a_1}$. Hence, $\lambda_{a_1} \le \lambda_{a_2}$ and $\Lambda_{a_2} \le \Lambda_{a_1}$. That is to say, λ_a is nonincreasing with respect to a, while Λ_a is nondecreasing.

Now, we give some lemmas, which have great significance in the proofs of our global existence and blow-up results.

Lemma 3.1. *Suppose that (1.2) holds. Let $u(t)$ be a weak solution of problem (1.1). If $I(u) \le 0$, then one has*

$$\|\nabla u\|_2^2 > r_*^2 \tag{3.4}$$

and

$$d_0 \ge d_1 := \frac{(p-2)r_*^2}{2p}, \tag{3.5}$$

where

$$r_* = \begin{cases} \sup\limits_{\epsilon_{11}\in\left(0,\frac{2(N-1)}{N-2}-p\right]} \left(\dfrac{\epsilon_{11}}{B_{\epsilon_{11}}^{p+\epsilon_{11}}}\right)^{\frac{1}{p+\epsilon_{11}-2}}, & \text{if } N \ge 3,\\ \sup\limits_{\epsilon_{11}\in(0,+\infty)} \left(\dfrac{\epsilon_{11}}{B_{\epsilon_{11}}^{p+\epsilon_{11}}}\right)^{\frac{1}{p+\epsilon_{11}-2}}, & \text{if } N = 1,2, \end{cases} \tag{3.6}$$

and $B_{\epsilon_{11}}$ is the optimal embedding constant of $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{p+\epsilon_{11}}(\Omega)$, namely,

$$B_{\epsilon_{11}} = \left(\inf_{u\in H^1_{\Gamma_0}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_2}{\|u\|_{p+\epsilon_{11}}}\right)^{-1}.$$

Proof. Since $\epsilon_{11} > 0$, then by the fact that $\ln|u| < \frac{1}{\epsilon_{11}}|u|^{\epsilon_{11}}$ and the Sobolev embedding theorem, one has

$$\begin{aligned} I(u) &= \|\nabla u\|_2^2 - \int_\Omega |u|^p \ln|u| \,\mathrm{d}x \\ &> \|\nabla u\|_2^2 - \frac{1}{\epsilon_{11}} \|u\|_{p+\epsilon_{11}}^{p+\epsilon_{11}} \\ &\geq \|\nabla u\|_2^2 - \frac{B_{\epsilon_{11}}^{p+\epsilon_{11}}}{\epsilon_{11}} \|\nabla u\|_2^{p+\epsilon_{11}} . \end{aligned} \tag{3.7}$$

Combining $I(u) \leq 0$ with (3.7) leads to

$$\|\nabla u\|_2^2 - \frac{B_{\epsilon_{11}}^{p+\epsilon_{11}}}{\epsilon_{11}} \|\nabla u\|_2^{p+\epsilon_{11}} = \|\nabla u\|_2^2 \left(1 - \frac{B_{\epsilon_{11}}^{p+\epsilon_{11}}}{\epsilon_{11}} \|\nabla u\|_2^{p+\epsilon_{11}-2}\right) < 0,$$

which results in $\|\nabla u\|_2^2 > r_*^2$. On the other hand, for all $u \in \mathcal{N}$, one can find that

$$\begin{aligned} J(u) &= \frac{1}{p} I(u) + \frac{p-2}{2p} \|\nabla u\|_2^2 + \frac{1}{p^2} \|u\|_p^p \\ &= \frac{p-2}{2p} \|\nabla u\|_2^2 + \frac{1}{p^2} \|u\|_p^p \\ &\geq \frac{p-2}{2p} \|\nabla u\|_2^2 \\ &\geq \frac{(p-2)\, r_*^2}{2p}, \end{aligned} \tag{3.8}$$

which tells us that $d_0 \geq d_1$. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Suppose that (1.2) holds. Let $u(t)$ be a weak solution of problem (1.1). Then for any $T \in (0, T_{\max})$, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{p^2} \|u\|_p^p - \frac{1}{p} \int_\Omega |u|^p \ln|u| \,\mathrm{d}x \right) = - \int_\Omega |u|^{p-2} u \ln|u|\, u_t \mathrm{d}x \ \ \textit{a.e. in } (0, T). \tag{3.9}$$

Proof. From (1.2) and Definition 2.1, one can infer that $|u|^p \ln|u| \in L^\infty\left(0, T; L^1(\Omega)\right)$, which implies that

$$\int_\Omega |u|^p \ln|u| \,\mathrm{d}x \in L^\infty(0, T) \subset L^2(0, T).$$

It also follows that $u \in H^1((0, T) \times \Omega)$. Since $u \mapsto |u|^p \ln|u|$ is locally Lipschitz continuous and $t \mapsto |u(t)|^p \ln|u(t)|$ is absolutely continuous for almost all $x \in \Omega$, then by the chain rule in Sobolev spaces, one has

$$\frac{\partial}{\partial t} |u|^p \ln|u| = \left(|u|^{p-2} u + p|u|^{p-2} u \ln|u|\right) u_t \in L^2\left(0, T; L^1(\Omega)\right) \hookrightarrow L^1((0, T) \times \Omega).$$

For any $\varphi \in C_0^\infty(\Omega)$ and $\chi \in C_0^\infty(0, T)$, one has

$$\iint_{(0,T)\times\Omega} |u|^p \ln|u|\, \varphi \chi' \mathrm{d}t\mathrm{d}x = - \iint_{(0,T)\times\Omega} \left(|u|^{p-2} u + p|u|^{p-2} u \ln|u|\right) u_t \varphi \chi \mathrm{d}t\mathrm{d}x.$$

This, along with the arbitrariness of φ and Fubini's theorem, yields that

$$\int_0^T |u|^p \ln|u|\chi' \mathrm{d}t = -\int_0^T \left(|u|^{p-2}u + p|u|^{p-2}u\ln|u|\right)u_t\chi \mathrm{d}t,$$

which, together with

$$\int_\Omega \left(|u|^{p-2}u + p|u|^{p-2}u\ln|u|\right)u_t \mathrm{d}x \in L^2(0,T),$$

tells us that $\int_\Omega |u|^p \ln|u|\,\mathrm{d}x \in H^1(0,T)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^p \ln|u|\,\mathrm{d}x = \int_\Omega \left(|u|^{p-2}u + p|u|^{p-2}u\ln|u|\right)u_t \mathrm{d}x \text{ a.e. in } (0,T). \tag{3.10}$$

By similar arguments, one can derive that $\|u\|_p^p \in H^1(0,T)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p = p\int_\Omega |u|^{p-2}uu_t \mathrm{d}x \text{ a.e. in } (0,T). \tag{3.11}$$

Collecting (3.10) and (3.11), one can see that (3.9) holds. The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Suppose that (1.2) holds. Let $u(t)$ be a weak solution of problem (1.1). If $u_0 \in \mathcal{N}_-$ and $J(u_0) < d_1$, then one has*

$$d_1 < \frac{p-2}{2p}\int_\Omega |u|^p \ln|u|\,\mathrm{d}x,\ t \in [0,T_{\max}). \tag{3.12}$$

Proof. First, we claim that $u(t) \in \mathcal{N}_-$ holds for all $t \in [0,T_{\max})$. In fact, if this statement is not true, then by $u_0 \in \mathcal{N}_-$ and the continuity of $u(t)$ in $[0,T_{\max})$, there exist some intervals $[0,t_1) \subseteq [0,T_{\max})$ such that $u(t) \in \mathcal{N}_-$ for $t \in [0,t_1)$. Denote

$$t_2 = \sup\{t_1 : u(t) \in \mathcal{N}_- \text{ for } t \in [0,t_1)\}.$$

Then $u(t_2) \in \mathcal{N}$, which yields

$$J(u(t_2)) \geq \inf_{u\in\mathcal{N}} J(u(t)) = d_0 \geq d_1. \tag{3.13}$$

On the other hand, by Lemma 3.2, the energy identity (2.3) can be written as

$$J(u)|_s^t = -\int_s^t \left(\|u_\tau\|_2^2 + \|u_\tau\|_{m,\Gamma_1}^m\right)\mathrm{d}\tau,\ 0 \leq s \leq t < T_{\max}. \tag{3.14}$$

With the help of (3.14) and the condition $J(u_0) < d_1$, one can immediately conclude that

$$J(u(t_2)) = J(u_0) - \int_0^{t_2} \left(\|u_\tau\|_2^2 + \|u_\tau\|_{m,\Gamma_1}^m\right)\mathrm{d}\tau \leq J(u_0) < d_1.$$

This is contradictory to (3.13). Hence, $u(t) \in \mathcal{N}_-$ in $[0,T_{\max})$, namely,

$$\|\nabla u\|_2^2 < \int_\Omega |u|^p \ln|u|\,\mathrm{d}x. \tag{3.15}$$

Collecting (3.4) and (3.15) leads to the desired result (3.12). The proof of Lemma 3.3 is complete. □

Lemma 3.4. *One has*

(i) $\mathcal{N}$ *and* $\mathcal{N}_-$ *are away from* 0*, that is to say,*

$$dist\,(0,\mathcal{N}) = \min_{u\in\mathcal{N}} \|u\|_{H^1_{\Gamma_0}} \text{ and } dist\,(0,\mathcal{N}_-) = \min_{u\in\mathcal{N}_-} \|u\|_{H^1_{\Gamma_0}}$$

are positive;

(ii) $J^a \cap \mathcal{N}_+$ *is bounded in* $H^1_{\Gamma_0}(\Omega)$ *for any* $a > d_0$*;*

(iii) $0 < \lambda_a \le \Lambda_a < +\infty$ *for any* $a > d_0$*.*

Proof. (i). For any $u \in \mathcal{N}$, one has

$$\|\nabla u\|_2^2 = \int_\Omega |u|^p \ln|u|\,\mathrm{d}x.$$

Then, in view of the Sobolev embedding inequality, one knows that

$$\begin{aligned} 0 < d_0 \le J(u) &= \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p^2}\|u\|_p^p \\ &\le \frac{p-2}{2p}\|\nabla u\|_2^2 + C_{17}\|\nabla u\|_2^p \\ &\le \begin{cases} C_{18}\|\nabla u\|_2^p, & \text{if } \|\nabla u\|_2^2 \ge 1, \\ C_{19}\|\nabla u\|_2^2, & \text{if } \|\nabla u\|_2^2 < 1, \end{cases} \end{aligned}$$

which means that $\|\nabla u\|_2^2 \ge C_{20} > 0$. Namely, $\mathrm{dist}\,(0,\mathcal{N}) = \min_{u\in\mathcal{N}} \|u\|_{H^1_{\Gamma_0}} > 0$. By similar arguments, $\mathrm{dist}\,(0,\mathcal{N}_-) > 0$ can also be proved.

(ii). For any $u \in J^a \cap \mathcal{N}_+$, one sees that $J(u) < a$ and $I(u) > 0$. Furthermore, one has

$$a > J(u) = \frac{1}{p}I(u) + \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p^2}\|u\|_p^p > \frac{p-2}{2p}\|\nabla u\|_2^2, \tag{3.16}$$

which implies that $\|\nabla u\|_2^2 < \frac{2pa}{p-2}$. That is to say, $\|u\|_{H^1_{\Gamma_0}}$ is bounded.

(iii). It suffices to prove $\lambda_a > 0$ and $\Lambda_a < +\infty$. First, noticing $u \in \mathcal{N}_a$, using Lemma 2.1, and the Gagliardo-Nirenberg interpolation inequality, one gets

$$\|\nabla u\|_2^2 = \int_\Omega |u|^p \ln|u|\,\mathrm{d}x \le C_{21}\|u\|_{p+\epsilon_{12}}^{p+\epsilon_{12}} \le C_{22}\|\nabla u\|_2^{\theta(p+\epsilon_{12})}\|u\|_2^{(1-\theta)(p+\epsilon_{12})}.$$

From this inequality, it follows that

$$\|\nabla u\|_2^{2-\theta(p+\epsilon_{12})} \le C_{23}\|u\|_2^{(1-\theta)(p+\epsilon_{12})}, \tag{3.17}$$

where $\epsilon_{12} \in \left(0, \frac{2^*-2(p-1)}{2}\right)$ and

$$\theta = N\left(\frac{1}{2} - \frac{1}{p+\epsilon_{12}}\right) = \frac{N(p+\epsilon_{12}-2)}{2(p+\epsilon_{12})} \in (0,1).$$

From (i) and (ii) of this lemma, it follows that $\|\nabla u\|_2$ has both positive upper and lower bounds. This fact, along with (3.17), leads to $\|u\|_2^2 \geq C_{24} > 0$. Therefore, $\|u\|_2^2 + \|u\|_{2,\Gamma_1}^2 \geq C_{24} > 0$, which implies that $\lambda_a > 0$.

On the other hand, $u \in \mathcal{N}_a$ indicates that $\|u\|_p^p < ap^2$ and $\|\nabla u\|_2^2 < \frac{2ap}{p-2}$. By Hölder's inequality, one has

$$\|u\|_2^2 \leq |\Omega|^{\frac{p-2}{p}} \|u\|_p^2 \leq |\Omega|^{\frac{p-2}{p}} \left(ap^2\right)^{\frac{2}{p}}. \tag{3.18}$$

We now recall the trace embedding for fractional-order Sobolev space (see Theorem 7.58 of [22]) $H^\eta\left(\mathbb{R}^N\right) \hookrightarrow W^{\beta,l}\left(\mathbb{R}^{N-1}\right)$ when $\eta > 0$, $2 \leq l < +\infty$, and $\beta = \eta - \frac{N}{2} + \frac{N-1}{l} > 0$. Since $W^{\beta,l}\left(\mathbb{R}^{N-1}\right) \hookrightarrow L^l\left(\mathbb{R}^{N-1}\right)$, and Ω is of class C^1, then by a standard partition of the unity, one has the trace embedding

$$H^\eta(\Omega) \hookrightarrow L^l(\partial\Omega), \text{ where } \eta > 0 \text{ and } 2 \leq l < +\infty \text{ satisfy } \eta - \frac{N}{2} + \frac{N-1}{l} > 0. \tag{3.19}$$

Utilizing (3.19) with $l = 2$ and $\eta \in \left(\frac{1}{2}, 1\right)$, together with the interpolation inequality and (3.18), one has

$$\|u\|_{2,\Gamma_1}^2 \leq C_{25} \|u\|_{H^\eta(\Omega)}^2 \leq C_{26} \|u\|_2^{2(1-\eta)} \|\nabla u\|_2^{2\eta} \leq C_{27} \left(|\Omega|^{p-2} a^2 p^4\right)^{\frac{1-\eta}{p}} \left(\frac{2ap}{p-2}\right)^\eta. \tag{3.20}$$

Summing (3.18) and (3.20) indicates that $\Lambda_a < +\infty$. The proof of Lemma 3.4 is complete. □

The main results of this section are stated as follows. Theorems 3.1 and 3.2 concern the global existence and finite time blow-up of the solution at subcritical initial energy levers, while Theorems 3.3 and 3.4 concern that of the solution at supercritical initial energy levels. Theorem 3.5 concerns with the lower bound of the maximal existence time $T_{\max}$.

Theorem 3.1. *Suppose that (1.2) holds. Assume that* $J(u_0) < d_0$, *and* $u_0 \in \mathcal{N}_+$. *Then the weak solution* $u(t)$ *of problem (1.1) exists globally.*

Theorem 3.2. *Suppose that (1.2) holds and* $m < 1 + \frac{2^*}{2}$. *Assume that* $J(u_0) \leq d_1$, *and* $u_0 \in \mathcal{N}_-$. *Then the weak solution* $u(t)$ *of problem (1.1) blows up in finite time.*

Remark 3.1. *From Theorem 3.2, one can find that there is a gap, that is, when* $J(u_0) \in (d_1, d_0)$, *whether the weak solution of problem (1.1) possesses blow-up property? The estimate (3.29) in the proof of Theorem 3.2 is based on Lemma 3.3, which needs the restriction* $J(u_0) \leq d_1$ *rather than* $J(u_0) \leq d_0$. *We hope to be able to fill this gap in the near future.*

Theorem 3.3. *Suppose that* $m = 2$ *and* $2 < p < 1 + \frac{2^*}{2}$. *Assume that* $J(u_0) > d_0$, $u_0 \in \mathcal{N}_+$, *and* $\|u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \leq \lambda_{J(u_0)}$. *Then the solution of problem (1.1) exists globally and vanishes as* $t \to +\infty$.

Theorem 3.4. *Suppose that* $m = 2$ *and* $2 < p < 1 + \frac{2^*}{2}$. *Assume that* $J(u_0) > d_0$, $u_0 \in \mathcal{N}_-$, *and* $\|u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \geq \Lambda_{J(u_0)}$. *Then the solution of problem (1.1) blows up in finite time.*

Theorem 3.5. *Suppose that* $m = 2$ *and* $2 < p < \min\left\{1 + \frac{2^*}{2}, 2\left(1 + \frac{2}{N}\right)\right\}$. *Assume that* $u_0 \in \mathcal{N}_-$ *and* $J(u_0) \leq d_1$ *or* $J(u_0) > d_0$ *with* $\|u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \geq \Lambda_{J(u_0)}$. *Then the maximal existence time* $T_{\max}$ *of* $u(t)$ *satisfies*

$$T_{\max} \geq C_{42} \left(\|u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2\right)^{\frac{2[2-(p+\epsilon_{20})]}{4-N(p+\epsilon_{20}-2)}},$$

where ϵ_{20} *is a constant satisfying*

$$2 < p + \epsilon_{20} < \min\left\{1 + \frac{2^*}{2}, 2\left(1 + \frac{2}{N}\right)\right\},$$

and

$$C_{42} = \frac{p + \epsilon_{20} - 2}{4 - N(p + \epsilon_{20} - 2)} 2^{\frac{(p+\epsilon_{20})(2-N)+2N}{4-N(p+\epsilon_{20}-2)}} \left(\frac{C_{40}^{p+\epsilon_{20}}}{e\epsilon_{20}}\right)^{\frac{4}{4-N(p+\epsilon_{20}-2)}},$$

where C_{40} *is the embedding constant in the following Gagliardo-Nirenberg interpolation embedding inequality*

$$\|u\|_{p+\epsilon_{20}} \le C_{40} \|\nabla u\|_2^{\frac{N(p+\epsilon_{20}-2)}{2(p+\epsilon_{20})}} \|u\|_2^{1-\frac{N(p+\epsilon_{20}-2)}{2(p+\epsilon_{20})}}.$$

We first consider the global existence and nonexistence of the solution at a subcritical initial energy level and give the proofs of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. First, we are going to prove that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T_{\max})$. If this conclusion is not true, then there exists a $t_3 \in (0, T_{\max}]$ such that $J(u(t_3)) = d_0$ or $I(u(t_3)) = 0$. From (3.14) and the assumption $J(u_0) < d_0$, one knows that $J(u(t_3))$ cannot equal d_0. Hence, $u(t_3) \in \mathcal{N}$, which combined with the definition of d_0 in (3.3) yields that $J(u(t_3)) \ge d_0$. This also contradicts (3.14). That is to say, $u(t) \in \mathcal{N}_+$ for all $t \in [0, T_{\max})$.

Second, we shall show that $\|\nabla u(t)\|_2^2 < \frac{2pd_0}{p-2}$ holds for all $t \in [0, T_{\max})$. In fact, from (3.14), the definitions of $J(u(t))$ and $I(u(t))$, one knows immediately

$$\begin{aligned} J(u(t)) &= \frac{1}{p} I(u(t)) + \frac{p-2}{2p} \|\nabla u(t)\|_2^2 + \frac{1}{p^2} \|u(t)\|_p^p \\ &= J(u_0) - \int_0^t \left(\|u_\tau\|_2^2 + \|u_\tau\|_{m,\Gamma_1}^m\right) \mathrm{d}\tau \\ &\le J(u_0) < d_0, \end{aligned}$$

which together with $I(u(t)) > 0$ in $[0, T_{\max})$ tells us that, for all $t \in [0, T_{\max})$,

$$\|\nabla u(t)\|_2^2 < \frac{2pd_0}{p-2}. \tag{3.21}$$

In the end, we suppose, on the contrary, that $T_{\max} < +\infty$ and $\lim\limits_{t \to T_{\max}^-} \|u(t)\|_{H_{\Gamma_0}^1} = +\infty$. Then $\lim\limits_{t \to T_{\max}^-} \|\nabla u(t)\|_2 = +\infty$, which is contradictory with (3.21). This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Suppose, by contradiction, that u exists globally, i.e., $T_{\max} = +\infty$. Then for any $T \in (0, +\infty)$ and $t \in [0, T]$, there exists a positive constant C_{28} such that

$$\|u(t)\|_{H_{\Gamma_0}^1}^2 = \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \le C_{28}. \tag{3.22}$$

Case 1: $J(u_0) < d_1$. Selecting $\alpha \in (0,1)$ and $h \in (J(u_0), d_1)$, defining

$$H(t) = h - J(u(t)),$$

and

$$Z(t) = H^{1-\alpha}(t) + \epsilon_{13} \|u(t)\|_2^2, \tag{3.23}$$

where $\epsilon_{13} > 0$ will be determined later. Then by a series of simple calculations, one has

$$H'(t) = \|u_t\|_{m,\Gamma_1}^m + \|u_t\|_2^2 \geq \|u_t\|_{m,\Gamma_1}^m \geq 0, \tag{3.24}$$

and

$$\begin{aligned} Z'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + 2\epsilon_{13} \int_\Omega u u_t \mathrm{d}x \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + 2\epsilon_{13} \int_\Omega u \left(\Delta u + |u|^{p-2} u \ln |u|\right) \mathrm{d}x \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) - 2\epsilon_{13} \|\nabla u\|_2^2 - 2\epsilon_{13} \int_{\Gamma_1} u |u_t|^{m-2} u_t \mathrm{d}\Gamma + 2\epsilon_{13} \int_\Omega |u|^p \ln |u| \mathrm{d}x \\ . &= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon_{13}(p-2) \|\nabla u\|_2^2 - 2\epsilon_{13} \int_{\Gamma_1} u |u_t|^{m-2} u_t \mathrm{d}\Gamma \\ &\quad + 2\epsilon_{13} p H(t) + \frac{2\epsilon_{13}}{p} \|u(t)\|_p^p - 2\epsilon_{13} p h. \end{aligned} \tag{3.25}$$

Defining a function as the form

$$Y(t) = \epsilon_{14}^{-\frac{m-1}{m}} H^{\frac{\alpha(m-1)}{m}}(t), \tag{3.26}$$

where ϵ_{14} is a positive constant to be determined later, and using Young's inequality, one has

$$\begin{aligned} &\left| \int_{\Gamma_1} u |u_t|^{m-2} u_t \mathrm{d}\Gamma \right| \\ = &\left| \int_{\Gamma_1} (uY)\left(Y^{-1} |u_t|^{m-2} u_t\right) \mathrm{d}\Gamma \right| \\ \leq &\frac{Y^m}{m} \|u(t)\|_{m,\Gamma_1}^m + \frac{m-1}{m} Y^{-m/(m-1)} \|u_t(t)\|_{m,\Gamma_1}^m \\ = &\frac{1}{m\epsilon_{14}^{m-1}} H^{\alpha(m-1)}(t) \|u(t)\|_{m,\Gamma_1}^m + \frac{\epsilon_{14}(m-1)}{m} H^{-\alpha}(t) \|u_t(t)\|_{m,\Gamma_1}^m. \end{aligned} \tag{3.27}$$

Inserting (3.24) and (3.27) into (3.25) leads to

$$\begin{aligned} Z'(t) \geq &\left[1 - \alpha - \frac{2\epsilon_{13}\epsilon_{14}(m-1)}{m}\right] H^{-\alpha}(t) \|u_t(t)\|_{m,\Gamma_1}^m + \frac{2\epsilon_{13}}{p} \|u(t)\|_p^p - 2\epsilon_{13} p h \\ &+ 2\epsilon_{13} p H(t) + \epsilon_{13}(p-2) \|\nabla u(t)\|_2^2 - \frac{2\epsilon_{13}}{m\epsilon_{14}^{m-1}} H^{\alpha(m-1)}(t) \|u(t)\|_{m,\Gamma_1}^m. \end{aligned} \tag{3.28}$$

By (3.12) in Lemma 3.3, one has

$$\begin{aligned} H(t) &= h - J(u(t)) \\ &= h - \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{1}{p^2} \|u(t)\|_p^p + \frac{1}{p} \int_\Omega |u(t)|^p \ln |u(t)| \mathrm{d}x \\ &\leq d_1 + \frac{1}{p} \int_\Omega |u(t)|^p \ln |u(t)| \mathrm{d}x \\ &\leq \int_\Omega |u(t)|^p \ln |u(t)| \mathrm{d}x. \end{aligned} \tag{3.29}$$

On the other hand, by Lemma 2.1, one can arrive at

$$\begin{aligned}\int_{\Omega} |u(t)|^p \ln |u(t)| \,\mathrm{d}x &= \int_{\Omega_1} |u(t)|^p \ln |u(t)| \,\mathrm{d}x + \int_{\Omega_2} |u(t)|^p \ln |u(t)| \,\mathrm{d}x \\ &\leq \frac{|\Omega|}{ep} + \frac{1}{e\epsilon_{15}} \int_{\Omega_2} |u(t)|^{p+\epsilon_{15}} \mathrm{d}x,\end{aligned} \tag{3.30}$$

where ϵ_{15} is an arbitrary positive constant. Since $2 < p < 1 + \frac{2^*}{2}$, one can take ϵ_{15} such that

$$\begin{cases} \epsilon_{15} > \max\{0, 4-p\}, & \text{for } N = 1, 2; \\ \max\{0, 4-p\} < \epsilon_{15} < \frac{4(N-1)}{N-2} - p, & \text{for } N \geq 3. \end{cases} \tag{3.31}$$

In other words, $2 < \frac{p+\epsilon_{15}}{2} \leq 1 + \frac{2^*}{2}$. Using the following Sobolev-type inequality

$$\int_{\Omega} |u(x,t)|^{2\epsilon_{16}} \,\mathrm{d}x \leq C_{29} \left(\int_{\Omega} |\nabla u(x,t)|^2 \,\mathrm{d}x \right)^{\epsilon_{16}},$$

with $\epsilon_{16} = \frac{p+\epsilon_{15}}{2}$, then from (3.22) and (3.30), one can conclude that

$$\int_{\Omega} |u(t)|^p \ln |u(t)| \,\mathrm{d}x \leq \frac{|\Omega|}{ep} + \frac{1}{e\epsilon_{15}} \int_{\Omega_2} |u(t)|^{p+\epsilon_{15}} \mathrm{d}x \leq \frac{|\Omega|}{ep} + \frac{C_{29}}{e\epsilon_{15}} \|\nabla u(t)\|_2^{p+\epsilon_{15}} \leq C_{30},$$

which together with (3.29) yields that $H(t) \leq C_{30}$, where $C_{30} = \frac{|\Omega|}{ep} + \frac{C_{29}}{e\epsilon_{15}} C_{28}^{\frac{p+\epsilon_{15}}{2}}$. Now, we estimate the $L^m(\Gamma_1)$ norm of $u|_{\Gamma_1}$ as follows. If $2 \leq m < 1 + \frac{2^*}{2}$, then one can immediately check that $0 < \frac{N}{2} - \frac{N-1}{m} < 1$. Using (3.19) with $l = m$ and $\eta \in \left(\frac{N}{2} - \frac{N-1}{m}, 1\right)$, the interpolation inequality, and (3.22), one has

$$\|u(t)\|_{m,\Gamma_1}^m \leq C_{31} \|u\|_{H^\eta(\Omega)}^m \leq C_{32} \|u\|_2^{m(1-\eta)} \|\nabla u\|_2^{m\eta} \leq C_{33} C_{28}^{\frac{m}{2}}.$$

If $m \in (1, 2)$, then Hölder's inequality, (3.19) with $l = 2$ and $\frac{1}{2} < \eta < 1$, the interpolation inequality and (3.22) can be used to obtain that

$$\|u(t)\|_{m,\Gamma_1}^m \leq \mathrm{mes}(\Gamma_1)^{1-\frac{m}{2}} \|u(t)\|_{2,\Gamma_1}^m \leq C_{34} \|u\|_{H^\eta(\Omega)}^m \leq C_{35} \|u\|_2^{m(1-\eta)} \|\nabla u\|_2^{m\eta} \leq C_{36} C_{28}^{\frac{m}{2}}.$$

Consequently, one has

$$H^{\alpha(m-1)}(t) \|u(t)\|_{m,\Gamma_1}^m \leq \underbrace{C_{30}^{\alpha(m-1)} C_{28}^{\frac{m}{2}} \max\{C_{33}, C_{36}\}}_{C_{37}}.$$

Inserting this into (3.28), one can deduce that

$$\begin{aligned} Z'(t) \geq{}& \underbrace{\left[1 - \alpha - \frac{2\epsilon_{13}\epsilon_{14}(m-1)}{m}\right]}_{\epsilon_{17}} H^{-\alpha}(t) \|u_t(t)\|_{m,\Gamma_1}^m + \frac{2\epsilon_{13}}{p} \|u(t)\|_p^p \\ &+ 2\epsilon_{13} p H(t) + \underbrace{\epsilon_{13} \left[(p-2) \|\nabla u(t)\|_2^2 - 2ph - \frac{2\epsilon_{14}^{1-m} C_{37}}{m}\right]}_{\epsilon_{18}}. \end{aligned} \tag{3.32}$$

Fixing

$$\epsilon_{14} \in \left(\left(\frac{C_{37}}{mp\left(d_1 - h\right)}\right)^{\frac{1}{m-1}}, +\infty\right),$$

and using condition $I\left(u_0\right) < 0$, one can infer from (3.4) in Lemma 3.1 that

$$\epsilon_{18} = \epsilon_{13}\left[\left(p-2\right)\left\|\nabla u\left(t\right)\right\|_2^2 - 2ph - \frac{2\epsilon_{14}^{1-m}C_{37}}{m}\right] > 2\epsilon_{13}\left(d_1 - h - \frac{\epsilon_{14}^{1-m}C_{37}}{m}\right) > 0.$$

Selecting

$$\epsilon_{13} \in \left(0, \frac{m\left(1-\alpha\right)}{2\epsilon_{14}\left(m-1\right)}\right),$$

to ensure that ϵ_{17} is positive, then it follows from (3.32) that

$$Z'(t) \geq 2p\epsilon_{13}\left(H\left(t\right) + \|u(t)\|_p^p\right). \tag{3.33}$$

On the other hand, since $H\left(t\right)$ is bounded, then from (3.22), one can claim that there exists a positive constant C_{38} such that

$$\begin{aligned} Z^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}\left(t\right) + \epsilon_{13}\left\|u\left(t\right)\right\|_2^2\right)^{\frac{1}{1-\alpha}} \\ &\leq 2^{\frac{\alpha}{1-\alpha}}\left(H\left(t\right) + \epsilon_{13}^{\frac{1}{1-\alpha}}\left\|u\left(t\right)\right\|_2^{\frac{2}{1-\alpha}}\right) \\ &\leq 2^{\frac{\alpha}{1-\alpha}}\left(H\left(t\right) + \left(\epsilon_{13}C_{28}\right)^{\frac{1}{1-\alpha}}\right) \\ &\leq C_{38}H\left(t\right) \\ &\leq C_{38}\left(H\left(t\right) + \|u(t)\|_p^p\right). \end{aligned} \tag{3.34}$$

Combining (3.33) with (3.34) means that

$$Z'(t) \geq C_{39}Z^{\frac{1}{1-\alpha}}(t),$$

where $C_{39} = 2p\epsilon_{13}C_{38}^{-1}$. Integrating this inequality from 0 to t yields that

$$Z^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{Z^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha C_{39}}{1-\alpha}t}.$$

Since $Z\left(0\right) = \left(h - J\left(u_0\right)\right)^{1-\alpha} + \epsilon_{13}\left\|u_0\right\|_2^2 > 0$, then the above inequality implies that $Z(t)$ blows up in a finite time T_1, where

$$T_1 \leq \frac{1-\alpha}{\alpha C_{39}}Z^{-\frac{\alpha}{1-\alpha}}(0).$$

If one takes $T \geq \frac{1-\alpha}{\alpha C_{39}}Z^{-\frac{\alpha}{1-\alpha}}(0)$, then $T_1 \leq T$. This contradicts our assumption.

Case 2: $J\left(u_0\right) = d_1$. Since $I\left(u_0\right) < 0$, one has $I\left(u\left(t\right)\right) < 0$ for all $t \geq 0$. Moreover, it follows from Hölder's inequality that

$$0 < -I\left(u\left(t\right)\right) = \int_\Omega u_t u \mathrm{d}x + \int_{\Gamma_1} u\left|u_t\right|^{m-2}u_t \mathrm{d}\Gamma \leq \|u\|_2\|u_t\|_2 + \|u\|_{m,\Gamma_1}\|u_t\|_{m,\Gamma_1}^{m-1},$$

which implies that, at least, one of $\|u_t\|_2$ and $\|u_t\|_{m,\Gamma_1}$ is positive. This results in

$$\|u_t\|_2^2 + \|u_t\|_{m,\Gamma_1}^m > 0, \text{ for all } t > 0.$$

Therefore, for any fixed sufficiently small positive constant ϵ_{19}, there is a constant $t_{\epsilon_{19}} > 0$ such that

$$J\left(u\left(t_{\epsilon_{19}}\right)\right) = J\left(u_0\right) - \int_0^{t_{\epsilon_{19}}} \left(\|u_\tau\|_2^2 + \|u_\tau\|_{m,\Gamma_1}^m\right) \mathrm{d}\tau = d_1 - \epsilon_{19} < d_1.$$

Up to now, by taking $t_{\epsilon_{19}} > 0$ as the initial time and applying the analogous arguments as in Case 1, one can immediately get the blow-up result. The proof of the Theorem 3.2 is complete. □

Now, we are in the position of considering the case $J\left(u_0\right) > d_0$. Inspired by the ideas of [5,11,12,25], we will give some criteria on the global existence and finite time blow-up of the solutions to problem (1.1) with $m = 2$.

Proof of Theorem 3.3. We divide the proof into two steps as follows.

Step 1: Global existence and uniformly boundedness of the solution. First, we claim that $u\left(t\right) \in \mathcal{N}_+$ for all $t \in [0, T_{\max})$. If it is not true, then there exists a $t_4 \in (0, T_{\max})$ such that $u\left(t\right) \in \mathcal{N}_+$ for $t \in [0, t_4)$ and $u\left(t_4\right) \in \mathcal{N}$. Moreover, it follows from Hölder's inequality that

$$0 < I\left(u\left(t\right)\right) = -\left(\int_\Omega u u_t \mathrm{d}x + \int_{\Gamma_1} u u_t \mathrm{d}\Gamma\right) \leq \|u\|_2 \, \|u_t\|_2 + \|u\|_{2,\Gamma_1} \, \|u_t\|_{2,\Gamma_1} ,$$

which implies that, at least, one of $\|u_t\|_2$ and $\|u_t\|_{2,\Gamma_1}$ is positive. This results in

$$\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 > 0 \text{ for all } t \in [0, t_4).$$

From (3.14), one knows that $J\left(u\left(t_4\right)\right) < J\left(u_0\right)$, which indicates that $u\left(t_4\right) \in J^{J(u_0)}$. Hence, $u\left(t_4\right) \in \mathcal{N}_{J(u_0)}$, which combined with the definition of $\lambda_{J(u(t_4))}$ yields that

$$\|u\left(t_4\right)\|_2^2 + \|u\left(t_4\right)\|_{2,\Gamma_1}^2 \geq \lambda_{J(u_0)}. \tag{3.35}$$

On the other hand, for $t \in [0, t_4)$, one has

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\left(t\right)\|_2^2 + \|u\left(t\right)\|_{2,\Gamma_1}^2\right) &= \int_\Omega u u_t \mathrm{d}x + \int_{\Gamma_1} u u_t \mathrm{d}\Gamma \\ &= \int_\Omega |u\left(t\right)|^p \ln |u\left(t\right)| \, \mathrm{d}x - \|\nabla u\left(t\right)\|_2^2 \\ &= -I\left(u\left(t\right)\right) < 0, \end{aligned} \tag{3.36}$$

which means that $\|u\left(t\right)\|_2^2 + \|u\left(t\right)\|_{2,\Gamma_1}^2$ is strictly decreasing in $[0, t_4)$. Thus,

$$\|u\left(t_4\right)\|_2^2 + \|u\left(t_4\right)\|_{2,\Gamma_1}^2 < \|u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \leq \lambda_{J(u_0)}.$$

This is contradictive with (3.35). Namely, $u\left(t\right) \in \mathcal{N}_+$ for all $t \in [0, T_{\max})$. Furthermore, one has $u\left(t\right) \in J^{J(u_0)} \cap \mathcal{N}_+$ for any $t \in (0, T_{\max})$. This, along with the statement (ii) of Lemma 3.4, indicates that

$T_{\max} = +\infty$. Then, for all $t \in (0, +\infty)$, it follows from (3.16) that $\|\nabla u(t)\|_2^2 < \frac{2p}{p-2} J(u_0)$. In other words, $u(t)$ is uniformly bounded with respect to t in $H^1_{\Gamma_0}(\Omega)$.

Step 2: Vanishing of the solution. To achieve this goal, we first need to show $\omega(u_0)$ is non-empty. We take a monotone increasing sequence $\{t_n\}_{n=1}^{+\infty}$ such that $t_n \to +\infty$ as $n \to +\infty$ and denote $u_n = u(t_n)$. By step 1, one knows the sequence $\{u_n\}_{n=1}^{+\infty}$ is uniformly bounded in $H^1_{\Gamma_0}(\Omega)$. Thus, there exist a function $\chi \in H^1_{\Gamma_0}(\Omega)$ and a subsequence $\{u_{n_k}\}_{k=1}^{+\infty} \subseteq \{u_n\}_{n=1}^{+\infty}$ such that

$$u_{n_k} \to \chi \text{ weakly in } H^1_{\Gamma_0}(\Omega) \text{ and } u_{n_k} \to \chi \text{ a.e. in } \Omega.$$

Moreover, from the compactness of the embeddings $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^2(\Omega)$ and $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^2(\Gamma_1)$, it follows that

$$u_{n_k} \to \chi \text{ strongly in } L^2(\Omega) \text{ and } u_{n_k} \to \chi \text{ strongly in } L^2(\Gamma_1). \tag{3.37}$$

Let us construct a function in the form

$$\phi(x,t) = \begin{cases} \rho(t - t_{n_k})\varphi(x), & (x,t) \in \bar{\Omega} \times (t_{n_k}, +\infty), \\ 0, & (x,t) \in \bar{\Omega} \times [0, t_{n_k}], \end{cases}$$

where φ and ρ are some suitable test functions that satisfy

$$\varphi \in H^1_{\Gamma_0}(\Omega), \ \rho \in C^1_0(0, \tilde{T}), \ \rho \geq 0, \ \int_0^{\tilde{T}} \rho(s)\mathrm{d}s = 1, \quad \text{for } \tilde{T} \in (0, +\infty).$$

In (2.1), taking $\vartheta = \phi$ and integrating it over $\left(t_{n_k}, \tilde{T} + t_{n_k}\right)$ with respect to t, one obtains

$$\int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \left(\int_\Omega (\phi u_t + \nabla\phi \cdot \nabla u)\,\mathrm{d}x + \int_{\Gamma_1} \phi u_t \mathrm{d}\Gamma \right) \mathrm{d}t = \int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_\Omega \phi |u|^{p-2} u \ln|u| \mathrm{d}x\mathrm{d}t. \tag{3.38}$$

Using integration by parts and $\rho(0) = \rho(\tilde{T}) = 0$, one finds that

$$\begin{aligned} \int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_\Omega \phi u_t \mathrm{d}x\mathrm{d}t &= \int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_\Omega u_t \rho(t - t_{n_k})\varphi \mathrm{d}x\mathrm{d}t \\ &= \int_\Omega [u\rho(\tilde{T})\varphi - u\rho(0)\varphi]\mathrm{d}x - \int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_\Omega u\varphi\rho'(t - t_{n_k})\,\mathrm{d}x\mathrm{d}t \\ &= -\int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_\Omega u\varphi\rho'(t - t_{n_k})\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.39}$$

Analogously,

$$\int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_{\Gamma_1} \phi u_t \mathrm{d}\Gamma\mathrm{d}t = -\int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_{\Gamma_1} u\varphi\rho'(t - t_{n_k})\,\mathrm{d}\Gamma\mathrm{d}t. \tag{3.40}$$

Collecting (3.38), (3.39), and (3.40) yields that

$$\begin{aligned} 0 = &\int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_\Omega u\varphi\rho'(t - t_{n_k})\,\mathrm{d}x\mathrm{d}t - \int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_\Omega \rho(t - t_{n_k})\nabla u \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t \\ &+ \int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_{\Gamma_1} u\varphi\rho'(t - t_{n_k})\,\mathrm{d}\Gamma\mathrm{d}t + \int_{t_{n_k}}^{\tilde{T}+t_{n_k}} \int_\Omega \rho(t - t_{n_k})\varphi|u|^{p-2}u \ln|u| \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.41}$$

Putting $\tilde{s} = t - t_{n_k}$, then (3.41) becomes

$$\begin{aligned} 0 =& \int_0^{\tilde{T}} \int_\Omega \rho'(\tilde{s})\, u(t_{n_k} + \tilde{s})\, \varphi \mathrm{d}x \mathrm{d}\tilde{s} + \int_0^{\tilde{T}} \int_{\Gamma_1} \rho'(\tilde{s})\, u(t_{n_k} + \tilde{s})\, \varphi \mathrm{d}\Gamma \mathrm{d}\tilde{s} \\ &- \int_0^{\tilde{T}} \int_\Omega \rho(\tilde{s})\, \nabla u(t_{n_k} + \tilde{s}) \cdot \nabla \varphi \mathrm{d}x \mathrm{d}\tilde{s} \\ &+ \int_0^{\tilde{T}} \int_\Omega \rho(\tilde{s})\, \varphi |u(t_{n_k} + \tilde{s})|^{p-2} u(t_{n_k} + \tilde{s}) \ln |u(t_{n_k} + \tilde{s})| \mathrm{d}x \mathrm{d}\tilde{s}. \end{aligned} \tag{3.42}$$

Noticing that $H^1_{\Gamma_0}(\Omega) \subset\subset L^2(\Omega)$, $H^1_{\Gamma_0}(\Omega) \subset\subset L^2(\Gamma_1)$ and the fact that $\{u(t_{n_k} + \tilde{s})\}_{k=1}^{+\infty}$ is uniformly bounded in $H^1_{\Gamma_0}(\Omega)$ for any $\tilde{s} \in [0, \tilde{T}]$, it follows that there exist a function $\tilde{\chi} \in H^1_{\Gamma_0}(\Omega)$ and a subsequence of $\{u(t_{n_k} + \tilde{s})\}_{k=1}^{+\infty}$ which still denoted by $\{u(t_{n_k} + \tilde{s})\}_{k=1}^{+\infty}$ such that

$$u(t_{n_k} + \tilde{s}) \to \tilde{\chi} \text{ strongly in } L^2(\Omega) \text{ and } u(t_{n_k} + \tilde{s}) \to \tilde{\chi} \text{ strongly in } L^2(\Gamma_1). \tag{3.43}$$

Now, we will prove that $\tilde{\chi} = \chi$ a.e. in $\overline{\Omega}$. Since $u(t) \in \mathcal{N}_+$, one has $I(u(t)) > 0$. Hence

$$J(u) = \frac{1}{p} I(u) + \frac{p-2}{2p} \|\nabla u\|_2^2 + \frac{1}{p^2} \|u\|_p^p > 0,$$

which, together with (3.14), yields that

$$\int_0^t \left(\|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma_1}^2 \right) \mathrm{d}\tau = J(u_0) - J(u(t)) < J(u_0).$$

As a consequence of this inequality, one can claim that

$$\int_{t_{n_k}}^{\tilde{T} + t_{n_k}} \left(\|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma_1}^2 \right) \mathrm{d}\tau \to 0 \text{ as } k \to +\infty.$$

Then

$$\begin{aligned} &\int_\Omega \left| u(t_{n_k} + \tilde{s}) - u(t_{n_k}) \right|^2 \mathrm{d}x + \int_{\Gamma_1} \left| u(t_{n_k} + \tilde{s}) - u(t_{n_k}) \right|^2 \mathrm{d}\Gamma \\ \le& \int_\Omega \left| \int_{t_{n_k}}^{t_{n_k} + \tilde{s}} u_\tau \mathrm{d}\tau \right|^2 \mathrm{d}x + \int_{\Gamma_1} \left| \int_{t_{n_k}}^{t_{n_k} + \tilde{s}} u_\tau \mathrm{d}\tau \right|^2 \mathrm{d}\Gamma \\ \le& \tilde{s} \int_{t_{n_k}}^{t_{n_k} + \tilde{s}} \left(\|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma_1}^2 \right) \mathrm{d}\tau \\ \le& \tilde{T} \int_{t_{n_k}}^{t_{n_k} + \tilde{T}} \left(\|u_\tau\|_2^2 + \|u_\tau\|_{2,\Gamma_1}^2 \right) \mathrm{d}\tau \to 0, \text{ as } k \to +\infty. \end{aligned} \tag{3.44}$$

This tells us that $\tilde{\chi} = \chi$ a.e. in Ω for any fixed $\tilde{T} < +\infty$ and $\tilde{s} \in [0, \tilde{T}]$. Taking $k \to +\infty$ in (3.42), and using dominated convergence theorem, one has

$$\begin{aligned} 0 =& \int_0^{\tilde{T}} \int_\Omega \rho'(\tilde{s}) \chi \varphi \mathrm{d}x \mathrm{d}\tilde{s} + \int_0^{\tilde{T}} \int_{\Gamma_1} \rho'(\tilde{s}) \chi \varphi \mathrm{d}\Gamma \mathrm{d}\tilde{s} - \int_0^{\tilde{T}} \int_\Omega \rho(\tilde{s}) \nabla \chi \cdot \nabla \varphi \mathrm{d}x \mathrm{d}\tilde{s} \\ &+ \int_0^{\tilde{T}} \int_\Omega \rho(\tilde{s}) \varphi |\chi|^{p-2} \chi \ln |\chi| \mathrm{d}x \mathrm{d}\tilde{s}. \end{aligned} \tag{3.45}$$

Integrating by parts and recalling the fact of $\rho(0) = \rho(\tilde{T}) = 0$, one can find that

$$\int_0^{\tilde{T}} \int_\Omega \rho'(\tilde{s})\chi\varphi \mathrm{d}x\mathrm{d}\tilde{s} + \int_0^{\tilde{T}} \int_{\Gamma_1} \rho'(\tilde{s})\chi\varphi \mathrm{d}\Gamma\mathrm{d}\tilde{s} = 0.$$

This, along with (3.45), one knows that

$$\int_0^{\tilde{T}} \int_\Omega \rho(\tilde{s})\varphi|\chi|^{p-2}\chi \ln|\chi| \mathrm{d}x\mathrm{d}\tilde{s} - \int_0^{\tilde{T}} \int_\Omega \rho(\tilde{s})\nabla\chi\cdot\nabla\varphi \mathrm{d}x\mathrm{d}\tilde{s} = 0,$$

which implies that

$$\begin{aligned}\int_\Omega \left(\varphi|\chi|^{p-2}\chi\ln|\chi| - \nabla\chi\cdot\nabla\varphi\right)\mathrm{d}x &= \int_0^{\tilde{T}} \rho(\tilde{s})\mathrm{d}\tilde{s}\int_\Omega \left(\varphi|\chi|^{p-2}\chi\ln|\omega| - \nabla\chi\cdot\nabla\varphi\right)\mathrm{d}x \\ &= \int_0^{\tilde{T}} \int_\Omega \rho(\tilde{s})\left(\varphi|\chi|^{p-2}\chi\ln|\omega| - \nabla\chi\cdot\nabla\varphi\right)\mathrm{d}x\mathrm{d}\tilde{s} \\ &= 0.\end{aligned} \tag{3.46}$$

That is to say, χ is a stationary solution of problem (1.1). In other words, $u(t)$ converges to the stationary solution of problem (1.1) as $t \to +\infty$. Furthermore, one knows that $\omega(u_0)$ is non-empty.

Now, we show that $\omega(u_0) = \{0\}$. Let ω be an arbitrary element in $\omega(u_0)$. From (3.14), one finds that $J(\omega) < J(u_0)$, which implies that $\omega \in J^{J(u_0)}$. Meanwhile, it follows from (3.36) that

$$\|\omega\|_2^2 + \|\omega\|_{2,\Gamma_1}^2 < \|u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \le \lambda_{J(u_0)},$$

which together with the definition of $\lambda_{J(u_0)}$ indicates that $\omega \notin \mathcal{N}_{J(u_0)}$. Using this result and $\omega \in J^{J(u_0)}$, one knows that $\omega \notin \mathcal{N}$. On the other hand, recalling that $u(t) \in \mathcal{N}^+$, one has $J(u(t)) > 0$ for all $t \in [0,+\infty)$. In other words, $J(u(t))$ is bounded below. Therefore, there is a nonnegative constant γ such that $\lim\limits_{t\to+\infty} J(u(t)) = \gamma$. Let $u_\omega(t)$ be the weak solution of problem (1.1) with initial value ω. Then $J(u_\omega(0)) = J(\omega) = \gamma$. Moreover, by the monotonicity of $J(u_\omega(t))$, one has $J(u_\omega(t)) \equiv \gamma$ in $[0,+\infty)$. Substituting $u = u_\omega$ into (3.14) tells us that

$$\int_0^t \left(\|(u_\omega)_\tau\|_2^2 + \|(u_\omega)_\tau\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau = 0,\ t \in [0,+\infty),$$

which means that $u_\omega(t) \equiv \omega$. Hence

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_\omega(t)\|_2^2 + \|u_\omega(t)\|_{2,\Gamma_1}^2\right) = I(u_\omega(t)) = 0.$$

This, along with $\omega \notin \mathcal{N}$, leads to $\omega = 0$. Therefore, $\omega(u_0) = \{0\}$, which contradicts the statement (i) of Lemma 3.4 and implies $u(t) \to 0$ as $t \to +\infty$. This completes the proof of Theorem 3.3. □

Proof of Theorem 3.4. Under the assumptions $u_0 \in \mathcal{N}_-$ and $\|u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \ge \Lambda_{J(u_0)}$, one can also show that $u(t) \in \mathcal{N}_-$ for all $t \in [0, T_{\max})$. Suppose that, on the contrary, $T_{\max} = +\infty$. Combining (3.14) with (3.36) results in, for any $\omega \in \omega(u_0)$,

$$J(\omega) < J(u_0),\ \|\omega\|_2^2 + \|\omega\|_{2,\Gamma_1}^2 > \Lambda_{J(u_0)}.$$

This together with the definition of $\Lambda_{J(u_0)}$ indicates that $\omega(u_0) \cap \mathcal{N} = \emptyset$. By the similar arguments as those in Theorem 3.3, one can obtain that $\omega(u_0) = \{0\}$, which contradicts the statement (i) of Lemma 3.4. The proof of Theorem 3.4 is complete. □

Now, we will deduce a lower bound of the maximum existence time $T_{\max}$ of the solution $u(t)$ as the end of this section.

Proof of Theorem 3.5. As a direct application of Theorems 3.2 and 3.4, one knows that the weak solution $u(t)$ of problem (1.1) will blow up in finite time, namely, $T_{\max} < +\infty$. For arbitrary $T \in (0, T_{\max})$, we define

$$\Phi(u(t)) = \frac{1}{2}\left(\|u(t)\|_2^2 + \|u(t)\|_{2,\Gamma_1}^2\right), \ t \in [0, T]. \tag{3.47}$$

Taking the derivative of $\Phi(t)$ with respect to t, and using Lemma 2.1 and the Gagliardo-Nirenberg interpolation embedding inequality, one gets

$$\begin{aligned}
\Phi'(t) = -I(u(t)) &= \int_\Omega |u(t)|^p \ln|u(t)| \,\mathrm{d}x - \|\nabla u(t)\|_2^2 \\
&\le \int_\Omega |u(t)|^p \ln|u(t)| \,\mathrm{d}x \\
&= \int_{\Omega_1} |u(t)|^p \ln|u(t)| \,\mathrm{d}x + \int_{\Omega_2} |u(t)|^p \ln|u(t)| \,\mathrm{d}x \\
&\le \frac{1}{e\epsilon_{20}} \|u(t)\|_{p+\epsilon_{20}}^{p+\epsilon_{20}} \\
&\le \frac{C_{40}^{p+\epsilon_{20}}}{e\epsilon_{20}} \|\nabla u(t)\|_2^{\theta(p+\epsilon_{20})} \|u(t)\|_2^{(1-\theta)(p+\epsilon_{20})},
\end{aligned} \tag{3.48}$$

where C_{40} is a positive constant, ϵ_{20} satisfies

$$2 < p + \epsilon_{20} < \min\left\{1 + \frac{2^*}{2}, 2\left(1 + \frac{2}{N}\right)\right\}, \tag{3.49}$$

and

$$\theta = N\left(\frac{1}{2} - \frac{1}{p+\epsilon_{20}}\right) = \frac{N(p+\epsilon_{20}-2)}{2(p+\epsilon_{20})} \in (0,1).$$

From the assumption $u_0 \in \mathcal{N}_-$, one has

$$I(u(t)) = \|\nabla u(t)\|_2^2 - \int_\Omega |u(t)|^p \ln|u(t)| \,\mathrm{d}x < 0, \ t \in [0, T_{\max}),$$

which together with (3.48) results in

$$\|\nabla u(t)\|_2^{2-\theta(p+\epsilon_{20})} \le \frac{C_{40}^{p+\epsilon_{20}}}{e\epsilon_{20}} \|u(t)\|_2^{(1-\theta)(p+\epsilon_{20})}. \tag{3.50}$$

Noticing that the expression of θ and the restriction on $p + \epsilon_{20}$, one can check that $2 - \theta(p+\epsilon_{20})$ is greater than zero. Then (3.50) results in

$$\|\nabla u(t)\|_2 \le \left(\frac{C_{40}^{p+\epsilon_{20}}}{e\epsilon_{20}}\right)^{\frac{1}{2-\theta(p+\epsilon_{20})}} \|u(t)\|_2^{\frac{(1-\theta)(p+\epsilon_{20})}{2-\theta(p+\epsilon_{20})}}. \tag{3.51}$$

Collecting (3.48) and (3.51) yields that

$$\Phi'(t) \leq \left(\frac{C_{40}^{p+\epsilon_{20}}}{e\epsilon_{20}}\right)^{\frac{2}{2-\theta(p+\epsilon_{20})}} \|u(t)\|_2^{\frac{2(1-\theta)(p+\epsilon_{20})}{2-\theta(p+\epsilon_{20})}} \leq C_{41}\Phi^{\frac{(1-\theta)(p+\epsilon_{20})}{2-\theta(p+\epsilon_{20})}}(t), \tag{3.52}$$

where

$$C_{41} = 2^{\frac{(1-\theta)(p+\epsilon_{20})}{2-\theta(p+\epsilon_{20})}} \left(\frac{C_{40}^{p+\epsilon_{20}}}{e\epsilon_{20}}\right)^{\frac{2}{2-\theta(p+\epsilon_{20})}}.$$

Since $\frac{(1-\theta)(p+\epsilon_{20})}{2-\theta(p+\epsilon_{20})} > 1$, then one can claim that, by integrating inequality (3.52) from 0 to t,

$$t \geq \frac{C_{41}(p+\epsilon_{20}-2)}{2-\theta(p+\epsilon_{20})}\left[\Phi^{\frac{2-(p+\epsilon_{20})}{2-\theta(p+\epsilon_{20})}}(0) - \Phi^{\frac{2-(p+\epsilon_{20})}{2-\theta(p+\epsilon_{20})}}(t)\right],$$

Letting $t \to T_{\max}$ results in

$$T_{\max} \geq C_{42}\Phi^{\frac{2-(p+\epsilon_{20})}{2-\theta(p+\epsilon_{20})}}(0) = C_{42}\Phi^{\frac{2[2-(p+\epsilon_{20})]}{4-N(p+\epsilon_{20}-2)}}(0) = C_{42}\left(\|u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2\right)^{\frac{2[2-(p+\epsilon_{20})]}{4-N(p+\epsilon_{20}-2)}},$$

where

$$C_{42} = \frac{C_{41}(p+\epsilon_{20}-2)}{4-N(p+\epsilon_{20}-2)}, \tag{3.53}$$

which completes the proof of Theorem 3.5. □

Author contributions

The authors have made the same contributions. All authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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