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Research article

Prescribing the best decay rate of the wave equation using internal delayed feedback damping

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Abstract: In this paper, we investigated the stabilization of the damped wave equation through the use of an internal feedback mechanism incorporating time delay. This work builds upon the partial pole placement paradigm, a recent theoretical framework originally developed for functional differential equations, which enables the selective assignment of eigenvalues within a prescribed region of the complex plane. Using this approach, we designed an internal delayed feedback law that guarantees the exponential stabilization of the resulting closed-loop system. A distinctive feature of our control strategy lies in its ability to prescribe the optimal exponential decay rate for each modal cluster, thereby achieving the fastest possible stabilization consistent with the system's spectral limitations. This can be achieved regardless of the stabilizability domain being delay-independent or delay-dependent. This allows for a highly efficient control mechanism tailored to the specific dynamical behavior of the wave equation. To illustrate the practical relevance of our theoretical findings, we applied the proposed method to the control of transverse vibrations in a taut string. Numerical simulations confirmed the robustness and effectiveness of the feedback design, underscoring its potential for broader applications in the control of distributed parameter systems with delay effects.

Keywords: internal stabilization; internal delay; wave equations
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1. Introduction

As Hazen pointed out in developing the theory of servomechanisms [1], damping and delays, although fundamentally different in nature, both play a vital role in shaping the dynamic behavior of physical systems. Although damping is an inherent and often unavoidable phenomenon present in almost all physical systems, it refers to the mechanisms by which mechanical energy, particularly vibrational and/or oscillatory energy, gradually dissipates over time. A striking example is the internal friction of elastic components such as springs or flexures. With each oscillation cycle, part of the system's mechanical energy is irreversibly transformed into thermal energy through microscopic interactions between the internal layers of the material. This internal friction, though sometimes subtle, plays a significant role in reducing the amplitude of motion over time. Another prevalent source of damping arises from the interaction between the system and the surrounding fluid environment, such as air or liquid. The viscosity of these fluids resists motion, exerting a drag force that continuously leads to substantial energy loss. Together, these and other forms of damping serve to stabilize systems by naturally limiting excessive vibrations, but they can also pose challenges in precision applications where energy conservation and sustained oscillations are desired. Understanding and accurately modeling these damping mechanisms is therefore essential for the effective design and control of dynamic systems.

Delays also naturally arise in the modeling of a wide range of standard processes such as transport, propagation, communication, and heredity, where time delays are intrinsic to the system dynamics. For fundamental concepts and introductory theory, the reader is referred to [2–4], while a broader collection of examples and methodological developments—both spectral and time-domain—can be found in [5–9]. In the linear time-invariant (LTI) case, the dynamics of a delay equation are governed by a characteristic *quasipolynomial*, whose infinite set of roots (the characteristic roots) determine the system's qualitative properties such as stability and oscillation. Furthermore, the location of these characteristic roots depends critically on system parameters, including time-delay, where understanding the behavior of the corresponding spectral abscissa—the real part of the rightmost root—as a function of these parameters is essential for stability analysis and control design, see, for instance, [8, 10, 11]. Since the early developments of control theory [1], it has been observed that time delays can degrade closed-loop performance or induce instability. Moreover, classical finite-dimensional techniques such as pole placement [12] are not directly applicable to systems with delays due to their infinite-dimensional nature, see, for instance, [13]. Nonetheless, as highlighted in [14] and related studies, time delays can, in some scenarios, contribute positively to stability.

A central concept in the analysis of delay systems is the *degree* of the quasipolynomial, which is closely linked to the structure and distribution of its characteristic roots. As demonstrated in [15, 16], this degree is intimately connected to the maximum allowable multiplicity of a characteristic root. While assigning the entire spectrum of a quasipolynomial is generally unfeasible, it was established in [17, 18]—using tools from degenerate hypergeometric functions [19]—that, in generic quasipolynomials, the root with the highest multiplicity necessarily governs the system's spectral behavior. This phenomenon is known as *generic multiplicity-induced dominancy* (GMID). Moreover, [20] showed that, under appropriate conditions, spectral values with intermediate—though not necessarily maximal—multiplicities can also dominate the remaining spectrum. This extension is referred to as *intermediate multiplicity-induced dominancy* (IMID). These MID properties provide a foundation for extending classical pole placement strategies to time-delay systems, enabling partial pole assignment and, equivalently, the achievement of prescribed exponential stabilization.

Interestingly, beyond their traditional role in the qualitative analysis of functional differential equations, the properties of quasipolynomials can also be effectively utilized in the analysis and control of certain classes of partial differential equations (PDEs); see, for instance, [18, 21, 22] where the boundary control of the transport equation has been investigated in [18, 21, 22], while the boundary control of the wave equation is addressed in [23] and [24], with the latter focusing on the suppression of axial/torsional vibrations in drilling systems. Additionally, pointwise control of wave equation solutions is studied in [25].

The concept of prescribed exponential stabilization, central to these works, forms the core focus of the present paper. More precisely, we study the internal prescribed exponential stabilization of a wave equation in $(0, \ell) \subset \mathbb{R}$ using the action of the damping coupled with the delay effect seen as a control parameter inspired from the *proportional-minus-delay* (PMD) controller introduced in [26]. The system under study is given by:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \mathbf{u}(x,t) = 0, x \in (0,\ell),\ t > 0 \quad \text{where} \\ \mathbf{u}(x,t) = \alpha\, u_t(x,t) + \beta\, u_t(x, t-\tau), \\ u(0,t) = 0 = u(\ell,t),\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in (0,\ell), \\ u_t(x,t-\tau) = f_0(x,t-\tau),\ x \in (0,\ell),\ t \in (0,\tau). \end{cases} \tag{$\mathcal{P}$}$$

The constant $\tau > 0$ denotes the time-delay and $\ell > 0$, α, β, are real numbers such that $\beta \neq 0$ and the initial data u_0, u_1, and f_0 are given functions belonging to suitable spaces that will be specified later.

In this paper, we aim to contrast the effect of intentional time delays by incorporating a dissipative feedback strategy—specifically, by expressing the control in feedback form $\alpha\, u_t(x,t) + \beta\, u_t(x,t-\tau)$, $x \in (0,\ell)$, $t > 0$, where the delay τ is considered as a control parameter enabling exploitation of the so-called *delay stabilizing effect*, see, for instance, [6] (and the references therein). In their seminal work [27], Cox and Zuazua investigated the energy decay of a vibrating string subjected to a positive viscous damping term. Assuming the damping coefficient is of bounded variation, they identified the optimal exponential decay rate as the spectral abscissa of the associated non-self-adjoint operator. Their analysis included a thorough spectral study, yielding explicit bounds on the real eigenvalues and a precise characterization of the high-frequency asymptotic behavior. These insights allowed them to distinguish clearly between over-damped and under-damped regimes. Moreover, they showed that this decay rate is optimal among all damping profiles with a fixed total variation, thus establishing a variational framework for damping optimization. In contrast to this spatially based variational approach, the present work explores an alternative stabilization strategy based on time-delay feedback requiring the analysis of quasipolynomials rather than polynomials as the one established in [27]. Notice that problem ($\mathcal{P}$) with localized damping was studied in the noteworthy work [28] where, using an approach based on an appropriate observability estimate, Nicaise and Pignotti considered the case $\beta < \alpha$ and successfully demonstrated that the energy decays exponentially to zero. It appears that the emphasized condition $\beta < \alpha$ is precisely what is known as *delay-independent stability*, a well-established concept in the theory of functional differential equations.

In addition, Ammari et al. [29] investigated the problem of boundary stabilization for a multi-dimensional wave equation subject to internal time-delay feedback. The authors established the exponential stability of the system in the multi-dimensional setting, assuming that the *geometric control condition* (GCC) introduced by Lebeau [30] satisfied.

Furthermore, the study considers a related wave system that can be viewed as a multi-dimensional extension of the delayed model originally analyzed by Datko et al. [31]. In this extended framework, the delayed and instantaneous dissipations act on different regions of the domain (internal versus boundary). The findings offer valuable insights into the complex interaction between time delays, geometric constraints, and damping mechanisms in multi-dimensional wave dynamics.

The contributions of the present paper can be summarized as follows. We consider a feedback law characterized by the parameters (α, β, τ), where $\alpha, \beta \in \mathbb{R}$ are control gains, and $\tau > 0$ is a delay that is also treated as a control parameter. We first show (see Proposition 5.4) that the optimal decay rate corresponds to the spectral abscissa. We then demonstrate that, even beyond the delay-independent stability region established in [28], it is possible—under suitable initial conditions and an appropriate tuning of these parameters—to prescribe an admissible desired exponential decay rate, thus actively shaping the system's stability properties (see Theorems 6.1–6.4). This methodology offers a novel perspective on stabilization design—one that harnesses the effects of delay rather than counteracting delayed damping with instantaneous damping, as demonstrated in Theorem 6.1. Moreover, in Section 8, we provide an in-depth parametric analysis that offers further insight into the *delay-dependent stability* behavior. In particular, we revisit the result from [28], which showed that arbitrarily small delays can destabilize the system, and we refine the understanding of this phenomenon.

The remainder of the paper is organized as follows: Section 2 reviews some recent results on prescribed stabilization and optimal exponential decay of second-order ordinary differential equations using a delayed *proportional derivative* controller. These findings provide essential groundwork for the main result presented in this study. Next, we address the internal delayed damping of PMD-type control for the wave equation. For the sake of self-containment, the existence and uniqueness of the solution of problem $(\mathcal{P})$ is addressed in Section 3. Section 4 examines the asymptotic behavior of the closed-loop system, establishing conditions under which the semigroup $\mathrm{e}^{t\mathscr{A}}$ exhibits exponential decay to the null steady state. A spectral analysis is presented in Section 5, where an estimate of the resolvent is derived. Proposition 5.4 shows that the eigenvectors and generalized eigenvectors of the operator $\mathscr{A}$ form a Riesz basis in the Hilbert state space. Furthermore, Proposition 5.5 demonstrates that each value $\gamma \in (s_n, 0)$ corresponds to an admissible exponential decay rate that can be prescribed through an appropriate choice of design parameters (α, β, τ). In Section 6, we present the main results of this paper. The proofs of the main results are detailed in Section 7, and additional insights into parametric stability are discussed in Section 8. Finally, to demonstrate the effectiveness and practical relevance of the proposed approach, Section 9 applies the results to the control of transverse vibrations in a string in order to illustrate the main results.

2. Spectral multiplicity and prescribed stabilization in second-order delay systems

Second-order linear systems model the dynamic behavior of many natural phenomena and have extensive applications in fields such as vibration analysis and structural engineering, see, for instance, [32, 33]. Stabilizing solutions for these reduced-order models is a standard benchmark for evaluating new paradigms and methodologies in control design, see, for instance, [34]. In [33], the design of delayed feedback control was investigated for an open-loop system modeled by a second-order ordinary differential equation. The objective was to develop a delayed state-feedback controller of the

form
$$u(t) := -k_0\,\xi(t-\tau) - k_1\,\dot{\xi}(t-\tau), \quad \text{where } k_0,\ k_1 \in \mathbb{R},\ \tau > 0, \tag{1}$$
which not only stabilizes the solutions of the given control system but also guarantees a prescribed exponential decay rate for the closed-loop system's response:
$$\ddot{\xi}(t) + a_1\,\dot{\xi}(t) + a_0\,\xi(t) = u(t) \quad \text{where } a_0,\ a_1 \in \mathbb{R}. \tag{2}$$

In the Laplace domain, the latter standard control problem yields the following characteristic quasipolynomial function:
$$\begin{cases} \Delta(s) := P_0(s) + P_1(s)\,\mathrm{e}^{-\tau s}, \quad \text{where} \\ P_0(s) := s^2 + a_1\,s + a_0 \quad \text{and} \quad P_1(s) := k_1\,s + k_0. \end{cases} \tag{3}$$
The following two theorems, each of which is based on a variant of the MID property, provide explicit design guaranteeing a prescribed stabilization. Namely, it has been emphasized that the nature of the characteristic roots (two real roots or a pair of complex conjugates) of the autonomous equation $\ddot{\xi}(t) + a_1\dot{\xi}(t) + a_0\xi(t) = 0$ defines a natural way to select the appropriate MID property. Obviously, the roots' natures depend on the sign of $\delta := \operatorname{discrim}(P_0) = a_1^2 - 4a_0$.

Theorem 2.1 ([33]). *Considering equation* (3)*, the following assertions hold.*

i) The multiplicity of any given root of the quasipolynomial function (3) *is bounded by* 4*, and it can only be achieved on the real axis and under negativity of* δ*.*

ii) The quasipolynomial (3) *admits a real spectral value at* $s = s_0$ *with algebraic multiplicity* 4 *if, and only if,*
$$s_0 = -\frac{a_1 + \sqrt{-2\,\delta}}{2}, \tag{4}$$
and the system parameters satisfy
$$\tau = 2\,\sqrt{-\frac{2}{\delta}},\ k_0 = \frac{\left(5\,\delta - a_1\,\sqrt{-2\,\delta}\right)}{4}\,\mathrm{e}^{s_0\tau},\ k_1 = -\frac{\sqrt{-2\,\delta}}{2}\,\mathrm{e}^{s_0\,\tau}. \tag{5}$$

iii) If (5) *is satisfied, then* $s = s_0$ *is the spectral abscissa corresponding to* (3)*.*

iv) If (5) *is satisfied, then the trivial solution of the closed-loop equation* (2) *is asymptotically stable if, and only if, either* $\left(a_1 \geq 0 \text{ and } a_0 > \frac{a_1^2}{4}\right)$ *or* $\left(a_1 < 0 \text{ and } a_0 > \frac{3\,a_1^2}{8}\right)$.

While the above theorem can only be exploited if the discriminant δ is negative (which corresponds to an open-loop equation with a pair of complex-conjugate characteristic roots), the next one also enables one to handle autonomous open-loop equations with two real characteristic roots.

Theorem 2.2 ([33]). *Considering equation* (3)*, the following assertions hold.*

i) For an arbitrary positive delay τ *satisfying* $8 + \tau^2\delta \geq 0$*, the quasipolynomial* (3) *admits a real spectral value at* $s = s_\pm$ *with algebraic multiplicity* 3 *if, and only if,*
$$s_\pm = \frac{-\tau\,a_1 - 4 \pm \sqrt{8 + \tau^2\delta}}{2\,\tau}, \tag{6}$$

and the system parameters satisfy:

$$\begin{cases} k_0 = \left(a_1 s_\pm + \dfrac{{a_1}^2}{2} - \dfrac{\delta}{2} + \dfrac{6\,a_1 + 10\,s_\pm}{\tau} + \dfrac{6}{\tau^2}\right) \mathrm{e}^{s_\pm \tau}, \\ k_1 = \left(2\,s_0 + a_1 + \dfrac{2}{\tau}\right) \mathrm{e}^{s_\pm \tau}. \end{cases} \tag{$\star_\pm$}$$

ii) If ($\star_+$) *(respectively* ($\star_-$)*) is satisfied, then* $s = s_+$ *is the spectral abscissa corresponding to* (3) *(respectively* s_- *cannot be the spectral abscissa corresponding to* (3)*). Furthermore, for an arbitrary delay* τ*, the multiple spectral value at* s_- *is always dominated by a single real root* s_0*.*

iii) If ($\star_+$) *is satisfied, then the trivial solution is asymptotically stable if, and only if,* τ *satisfies the following conditions*

$$\begin{cases} \tau \in \,]0,\, \tau_-[\ \ \textit{when} \ \ a_0 < 0, \\ \textit{or} \\ \tau \in \,]0,\, \tau_-[\cup]\tau_+,\, \infty[\ \ \textit{when} \ \ a_0 > 0 \ \textit{and} \ a_1 < 0, \\ \textit{or} \\ \tau \in \,]0, 2/\sqrt{a_0}[\ \textit{when} \ \ a_0 > 0 \ \textit{and} \ a_1 > 0, \\ \textit{or} \\ \tau \in \mathbb{R}_+^* \ \textit{when} \ \ a_0 = 0, \\ \textit{or} \\ \tau \in \,]0,\, \sqrt{2}/\sqrt{|a_0|}[\ \textit{when} \ \ a_0 \neq 0 \ \textit{and} \ a_1 = 0, \end{cases} \tag{7}$$

where $\tau_\pm = \dfrac{-a_1 \pm \sqrt{\delta + 2\,a_0}}{a_0}$.

Remark 2.3. *Theorem 2.2 extends the original result from [33, Th. 4.2] by incorporating the last three cases of admissible delays in* (7)*. While the initial formulation focused solely on unstable open-loop plants with a positive discriminant (*$\delta > 0$*) and such that* $a_0\, a_1 \neq 0$*, the updated result extends to any open-loop plant satisfying* $\delta > -8/\tau^2$*. These additional cases, which will be useful in the sequel, arise from a standard analysis of the constraint* $a_0\,\tau^2 + 2\,a_1\,\tau + 2 > 0$*, which follows directly from the negativity of the triple root* s_+*.*

Although the previous two results facilitate the assignment of negative spectral abscissas, certain applications demand the placement of complex roots (non-real) with negative real parts. The following result from [35], which is also based on the MID property, provides a means to achieve this goal.

Theorem 2.4 ([35])**.** *Consider the quasipolynomial* Δ *given by* (3) *and let* $s_0 \in \mathbb{C}$*,* $\sigma_0 = \mathfrak{R}(s_0)$*, and* $\theta_0 = \mathfrak{I}(s_0)$*. Assume that* $\theta_0 \neq 0$*. Then* s_0 *and* $\overline{s}_0$ *are roots of multiplicity* 2 *of* Δ *if, and only if, the* *coefficients a_0, a_1, k_0, k_1, the real and imaginary parts σ_0 and θ_0, and the delay τ satisfy the relations*

$$\begin{cases} a_1 = -2\sigma_0 - 2\theta_0 \dfrac{\tau\theta_0 - \sin(\tau\theta_0)\cos(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)}, \\ a_0 = \sigma_0^2 + 2\sigma_0\theta_0 \dfrac{\tau\theta_0 - \sin(\tau\theta_0)\cos(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)} + \theta_0^2 \dfrac{\tau^2\theta_0^2 + \sin^2(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)}, \\ k_1 = 2\theta_0 \mathrm{e}^{\sigma_0\tau} \dfrac{\tau\theta_0\cos(\tau\theta_0) - \sin(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)}, \\ k_0 = 2\theta_0 \mathrm{e}^{\sigma_0\tau}\left(\sigma_0 \dfrac{\sin(\tau\theta_0) - \tau\theta_0\cos(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)} - \dfrac{\tau\theta_0^2\sin(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)}\right). \end{cases} \tag{8}$$

Furthermore, if (8) *is satisfied, then s_0 and $\overline{s}_0$ are a pair of strictly dominant roots of Δ.*

Finally, in the recent study [10], the relations between the spectral abscissa optimization of linear second-order time-delay systems and the dominance of multiple rightmost characteristic roots has been emphasized. As a matter of fact, the following optimization problem has been considered in [10]: The spectral abscissa associated with Δ is the function

$$(k_0, k_1) \in \mathbb{R} \times \mathbb{R} \mapsto \gamma(k_0, k_1) := \max_{s \in \mathbb{C}} \{\Re(s) : \ \Delta(s;\ k_0, k_1) = 0\}. \tag{9}$$

Since the closed-loop system, resulting from the combination of equations (2) and (1), describes a time-delay system of retarded type, the maximum in (9) is well-defined, since the supremum of the real parts of the roots of $\Delta(\cdot;\ k_0, k_1)$ is attained by some root $s \in \mathbb{C}$ (see, e.g., [8, Section 1.1.5]).

The following result provides a characterization of γ_0, the global minimum of the spectral abscissa of the quasipolynomial Δ given by (3), $\gamma_0 := \min_{(k_0,k_1)\in\mathbb{R}\times\mathbb{R}} \gamma(k_0, k_1)$, depending on the sign of the discriminant of the polynomial P_0.

Theorem 2.5 ([10])**.** *Consider equation* (2)*, controlled with static state feedback* (1)*. Then the spectral abscissa function is quasi-convex. Moreover, the following results hold:*

- *If $\delta < -\frac{8}{\tau}$, then γ_0, the global minimum of the spectral abscissa function, is characterized by a complex conjugate pair of rightmost characteristic roots of multiplicity two;*
- *if $\delta > -\frac{8}{\tau}$, then γ_0, the global minimum of the spectral abscissa function, is characterized by a rightmost real root of multiplicity three;*
- *if $\delta = -\frac{8}{\tau}$, then γ_0, the global minimum of the spectral abscissa function, is characterized by a rightmost real root of multiplicity four.*

Remark 2.6. *Theorem 2.5 states that the global minimizer of the spectral abscissa corresponds to certain spectral values whose multiplicity depends on the sign of the discriminant of the open-loop characteristic function δ, but it does not specify which of the multiple roots to consider. To identify the appropriate multiple root, a finer analysis in the spirit of Theorem 2.2 needs to be performed.*

3. Well-posedness of problem ($\mathcal{P}$)

To ensure that the paper is self-contained, we explicitly recall the well-posedness of the problem under study. Although these results are available in the literature, see [36, 37], we include them here for the reader's convenience and to provide a complete understanding of the theoretical setting.

First, we transform the delay terms by adding new unknowns. Then, we use a semigroup approach and the Lumer-Phillips theorem to prove the existence and uniqueness of the solution of the problem $(\mathcal{P})$. To do so, let us introduce the new variables, as in [37]:

$$z\,(x,\rho,t) = u_t\,(x,t-\tau\rho)\,,\ x\in(0,\ell),\rho\in(0,1)\,,\ t>0. \tag{10}$$

Then, we have

$$\tau\, z_t\,(x,\rho,t) + z_\rho\,(x,\rho,t) = 0,\ \text{in } (0,\ell)\times(0,1)\times(0,+\infty)\,. \tag{11}$$

Hence, we show that sufficiently smooth solutions of problem $(\mathcal{P})$ can equivalently be characterized as solutions of:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha\, u_t(x,t) + \beta\, z\,(x,1,t) = 0, \quad x\in(0,\ell),\ t>0,\\ \tau\, z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x\in(0,\ell),\ \rho\in(0,1),\ t>0,\\ z(x,0,t) = u_t(x,t), \quad u(0,t) = u(\ell,t) = 0, \quad x\in(0,\ell),\ t>0,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in(0,\ell),\\ z(x,\rho,0) = f_0(x,-\tau\rho), \quad x\in(0,\ell),\ \rho\in(0,1). \end{cases} \tag{12}$$

In this section, we provide sufficient conditions to guarantee that the above problem is well-posed. For this purpose, we will use a semigroup formulation of the initial-boundary value problem (12).

First, we define the energy space:

$$\mathscr{H} = H_0^1(0,\ell)\times L^2(0,\ell)\times L^2((0,\ell)\times(0,1)).$$

Clearly, $\mathscr{H}$ is a Hilbert space with respect to the inner product

$$\langle V_1, V_2\rangle_{\mathscr{H}} = \int_0^\ell u_x^1\,\overline{v_x^1}dx + \int_0^\ell u^2\,\overline{v^2}dx + \tau|\beta|\int_0^1\int_0^\ell z\overline{w}\,dxd\rho \tag{13}$$

for $V_1 = (u^1,u^2,z)^T$, $V_2 = (v^1,v^2,w)^T$.

Let $\mathscr{A}$ be defined as follows:

$$\mathscr{A}\begin{pmatrix} u\\ v\\ z\end{pmatrix} = \begin{pmatrix} v\\ u_{xx} - \alpha\, v - \beta\, z(.,1)\\ -\tau^{-1}\, z_\rho\end{pmatrix}.$$

and with domain $\mathscr{D}\,(\mathscr{A})$ defined as the set of $V = (u,v,z)^T$ satisfying

$$(u,v,z)^T \in (H^2(0,\ell)\cap H_0^1(0,\ell))\times H_0^1(0,\ell)\times L^2((0,\ell);H^1(0,1)), \tag{14}$$

$$v(x) = z(x,0). \tag{15}$$

Therefore, if $V_0, V\in\mathscr{D}\,(\mathscr{A})$, problem (12) is formally equivalent to the following abstract evolution equation in the Hilbert space $\mathscr{H}$:

$$\begin{cases} V'(t) = \mathscr{A}V(t), \quad t>0,\\ V\,(0) = V_0, \end{cases} \tag{16}$$

where $'$ denotes the derivative with respect to time t, $V_0 := (u_0, u_1, f_0(-\tau))^T$.

Now, if (α, β) satisfies the following condition:

$$\alpha > |\beta|, \tag{17}$$

then the well-posedness of problem (12) is ensured by:

Theorem 3.1. *Let $V_0 \in \mathscr{H}$. Then, there exists a unique solution $V \in C(\mathbb{R}_+; \mathscr{H})$ of problem (16) and the first component u of V is the mild solution of $(\mathcal{P})$. In particular, for $V_0 \in \mathscr{D}(\mathscr{A})$, the problem (16) admits a unique classical solution*

$$V \in C(\mathbb{R}_+; \mathscr{D}(\mathscr{A})) \cap C^1(\mathbb{R}_+; \mathscr{H})$$

and thus problem $(\mathcal{P})$ admits a unique classical solution $u \in C([0, +\infty); H^2(0, \ell) \cap H_0^1(0, \ell)) \cap C^1([-\tau, +\infty); H_0^1(0, \ell))$.

Proof. In order to prove the existence and uniqueness of the solution of problem (16), we use the semigroup approach and the Lumer-Phillips theorem.

Indeed, let $V = (u, v, z)^T \in \mathscr{D}(\mathscr{A})$. By definition of the operator $\mathscr{A}$ and the scalar product of $\mathscr{H}$, we have

$$\begin{aligned}\langle \mathscr{A}V, V\rangle_{\mathscr{H}} = &\int_0^\ell v_x(x)\overline{u_x(x)}\,dx + \int_0^\ell (u_{xx}(x) - \alpha\, v(x) - \beta\, z(x,1))\,\overline{v(x)}dx \\ &- \tau|\beta| \int_0^1 \int_0^\ell \tau^{-1}\, z_\rho(x,\rho)\overline{z(x,\rho)}dx\,d\rho.\end{aligned}$$

We obtain

$$\begin{aligned}\Re\,\langle \mathscr{A}V, V\rangle_{\mathscr{H}} &= \left(-\alpha + \frac{|\beta|}{2}\right)\int_0^\ell |z(x,0)|^2 dx - \beta\Re \int_0^\ell z(x,1)\overline{z(x,0)}dx - \frac{|\beta|}{2}\int_0^\ell |z(x,1)|^2 dx, \\ &= \int_0^\ell \left[\begin{pmatrix} z(x,0) & z(x,1)\end{pmatrix} D_{\alpha,\beta} \begin{pmatrix}\overline{z(x,0)} \\ \overline{z(x,1)}\end{pmatrix}\right] dx,\end{aligned} \tag{18}$$

where

$$D_{\alpha,\beta} := \begin{pmatrix} -\alpha + \frac{|\beta|}{2} & -\frac{\beta}{2} \\ -\frac{\beta}{2} & -\frac{|\beta|}{2}\end{pmatrix}. \tag{19}$$

One can notice that if (α, β) satisfies condition (17), then $D_{\alpha,\beta}$ given by (19) is a definite negative matrix. Hence, one can explicitly compute its eigenvalues given by $\lambda_1 = -\frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 - 2\alpha|\beta| + 2\beta^2}$, $\lambda_2 = -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - 2\alpha|\beta| + 2\beta^2}$. As a result, we get

$$\begin{aligned}\Re\,\langle \mathscr{A}V, V\rangle_{\mathscr{H}} = &\lambda_1\beta^2 \int_0^\ell \left|\frac{(-2\lambda_1 - |\beta|)}{\sqrt{(2\lambda_1 + |\beta|)^2 + \beta^2}}\, z(x,0) + \frac{1}{\sqrt{(2\lambda_1 + |\beta|)^2 + \beta^2}}\, z(x,1)\right|^2 dx \\ &+ \lambda_2\beta^2 \int_0^\ell \left|\frac{(-2\lambda_2 - |\beta|)}{\sqrt{(2\lambda_2 + |\beta|)^2 + \beta^2}}\, z(x,0) + \frac{1}{\sqrt{(2\lambda_2 + |\beta|)^2 + \beta^2}}\, z(x,1)\right|^2 dx\end{aligned}$$

$$\leq 0. \tag{20}$$

Thus, the operator $\mathscr{A}$ is dissipative.

Next, let us show that $\mathscr{A}$ is invertible.

For $F = (f_1, f_2, f_3)^T \in \mathscr{H}$, let $V = (u, v, z)^T \in \mathscr{D}(\mathscr{A})$ be a solution of

$$\mathscr{A} V = F,$$

so that

$$\begin{aligned} v &= f_1, & (21)\\ u_{xx} - \alpha f_1 - \beta z(.,1) &= f_2, & (22)\\ -\frac{1}{\tau} z_\rho &= f_3. & (23) \end{aligned}$$

Thus, from (21) and (23), z is given by

$$z(x, \rho) = f_1(x) - \tau \int_0^\rho f_3(x, s)\, ds\,,\ \rho \in (0,1), x \in (0, \ell). \tag{24}$$

Using the preceding expression and assumption (17), we have

$$u(x) = -\frac{x}{\ell} \int_0^\ell \int_0^y \left\{ \alpha f_1(z) + f_2(z) + \beta f_1(z) - \tau\beta \int_0^1 f_3(z, s)\, ds \right\} dz dy + \int_0^x \int_0^y \left\{ \alpha f_1(z) + f_2(z) + \beta f_1(z) - \tau\beta \int_0^1 f_3(z, s)\, ds \right\} dz dy. \tag{25}$$

Hence, we have found $V = (u, v, z)^T \in \mathscr{D}(\mathscr{A})$, the unique solution of $\mathscr{A} V = F$. The operator $\mathscr{A}$ generates a C_0 semigroup of contractions $\mathrm{e}^{t\mathscr{A}}$ on $\mathscr{H}$. Thus, by the Lumer-Phillips theorem, there exists a unique solution $V \in C(\mathbb{R}_+; \mathscr{H})$ of problem (16). This completes the proof of Theorem 3.1. □

Since $\mathscr{D}(\mathscr{A}) \subset \left[H^2(0, \ell) \cap H_0^1(0, \ell))\right] \times H_0^1(0, 1) \times L^2(0, \ell; H^1(0, 1))$, Sobolev's embedding implies that $\mathscr{A}^{-1}$ is a compact operator on $\mathscr{H}$. Consequently, we have the following corollary.

Corollary 3.2. *The spectrum of $\mathscr{A}$, $\sigma(\mathscr{A})$, only consists on eigenvalues of finite multiplicity.*

4. Asymptotic behavior

In line with the objective of keeping this paper self-contained, we provide a detailed overview of the exponential stability of the system under consideration. Specifically, in this section, we demonstrate that, under condition (17), the semigroup $\mathrm{e}^{t\mathscr{A}}$ decays exponentially to the null steady state.

To establish this result, we employ a frequency domain approach. The method combines a contradiction argument with the classical multiplier technique, allowing us to perform a refined analysis of the resolvent operator associated with the system.

Theorem 4.1. *Suppose that condition* (17) *holds. Then there exist constants $C, \omega > 0$ such that, for all $V_0 \in \mathcal{H}$, the semigroup $\mathrm{e}^{t\mathscr{A}}$ satisfies the following estimate:*

$$\left\| \mathrm{e}^{t\mathscr{A}} V_0 \right\|_{\mathscr{H}} \leq C\, \mathrm{e}^{-\omega t} \left\| V_0 \right\|_{\mathscr{H}}, \forall\ t > 0. \tag{26}$$

Proof of Theorem 4.1. We will use the following frequency domain theorem for uniform stability from Huang-Prüss [38, 39] of a C_0 semigroup of contractions on a Hilbert space:

Lemma 4.2. *A C_0 semigroup $\mathrm{e}^{t\mathcal{L}}$ of contractions on a Hilbert space $\mathcal{H}$ satisfies*

$$\|\mathrm{e}^{t\mathcal{L}}U_0\|_{\mathcal{H}} \leq C\,\mathrm{e}^{-\theta t}\|U_0\|_{\mathcal{H}}$$

for some constant $C > 0$ and for $\theta > 0$ if, and only if,

$$\rho(\mathcal{L}) \supset \{i\delta \mid \delta \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{27}$$

and

$$\limsup_{\delta\in\mathbb{R}, |\delta|\to\infty} \|(i\delta I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{28}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

First we look at the point spectrum of $\mathscr{A}$.

Lemma 4.3. *Suppose that condition* (17) *holds. The spectrum of $\mathscr{A}$ contains no point on the imaginary axis.*

Proof. Since $\mathscr{A}$ has a compact resolvent, its spectrum $\sigma(\mathscr{A})$ only consists of eigenvalues of $\mathscr{A}$. We will show that the equation

$$\mathscr{A}Z = i\delta Z, \tag{29}$$

with $Z = (u, v, z)^T \in \mathscr{D}(\mathscr{A})$ and $\delta \in \mathbb{R}$, admits only the trivial solution.
Equation (29) reads

$$i\delta u = v, \tag{30}$$

$$u_{xx} + \delta^2 u - i\alpha\delta u + \beta z(., 1) = 0, \tag{31}$$

$$i\delta z + \tau^{-1} z_\rho = 0. \tag{32}$$

By taking the inner product of (29) with Z and using (18), we get

$$0 = \Re\,\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \leq (-\alpha + |\beta|)\int_0^\ell |z(x,0)|^2 dx,$$

which infers, under condition (17), that

$$z(., 0) = v = 0.$$

Thus, we have $z(x,\rho) = v(x)\mathrm{e}^{-i\delta\tau\rho} = 0$ and we obtain also $u = 0$. So, the only solution of (29) is the trivial one. □

The following lemma shows that (28) holds with $\mathcal{L} = \mathscr{A}$.

Lemma 4.4. *Suppose that condition* (17) *holds. The resolvent operator of $\mathscr{A}$ satisfies condition* (28).

Proof. Suppose that condition (28) is false. By the Banach-Steinhaus theorem (see [40]), there exist a sequence of real numbers $\delta_n \to +\infty$ and a sequence of vectors $Z_n = (u_n, v_n, z_n)^t \in \mathscr{D}(\mathscr{A})$ with $\|Z_n\|_{\mathscr{H}} = 1$ such that

$$\|(i\delta_n I - \mathscr{A})Z_n\|_{\mathscr{H}} \to 0 \quad \text{as} \quad n \to \infty, \tag{33}$$

i.e.,

$$i\delta_n u_n - v_n \equiv f_n^1 \to 0 \quad \text{in} \quad H_0^1(0,\ell), \tag{34}$$

$$i\delta_n v_n - u_n'' + \alpha v_n + \beta z_n(.,1) \equiv f_n^2 \to 0 \quad \text{in} \quad L^2(0,\ell), \tag{35}$$

$$i\delta_n z_n + \frac{1}{\tau}\partial_\rho z_n \equiv f_n^3 \to 0 \quad \text{in} \quad L^2(0,\ell;L^2(0,1)). \tag{36}$$

Our goal is to derive from (33) that $\|Z_n\|_{\mathscr{H}}$ converges to zero, thereby establishing a contradiction.

First step. Notice that we have

$$\|(i\delta_n I - \mathscr{A})Z_n\|_{\mathscr{H}} \geq |\Re\left(\langle(i\delta_n I - \mathscr{A})Z_n, Z_n\rangle_{\mathscr{H}}\right)| \geq (\alpha - |\beta|)\int_0^\ell |z_n(x,0)|^2 dx. \tag{37}$$

Then, owing to (17) and (33), we get

$$z_n(.,0) = v_n \to 0 \text{ in } L^2(0,\ell), \tag{38}$$

which implies according to (34) that $\delta_n u_n \longrightarrow 0$ in $L^2(0,\ell)$.

Thus, taking into account (38) and the expression of z_n, we write

$$z_n(x,\rho) = v_n(x)\,\mathrm{e}^{-i\delta_n\tau\rho} + \tau\int_0^\rho \mathrm{e}^{-i\delta_n\tau(\rho-s)}\, f_n^3(x,s)\,ds,$$

which leads to

$$z_n \longrightarrow 0 \text{ in } L^2(0,\ell;L^2(0,1)),$$

$$z_n(x,1) = v_n(x)\,\mathrm{e}^{-i\delta_n\tau} + \tau\int_0^1 \mathrm{e}^{-i\delta_n\tau(1-s)}\, f_n^3(x,s)\,ds \longrightarrow 0 \text{ in } L^2(0,\ell). \tag{39}$$

Next, since we have

$$\left\|u_n'\right\|^2_{L^2(0,\ell)} = \|\delta_n u_n\|^2_{L^2(0,\ell)} - i\,\alpha\,\delta_n\,\|u_n\|^2_{L^2(0,\ell)} + < f_n^1, (-i\delta_n + \alpha)\,u_n >_{L^2(0,\ell)} +$$

$$< f_n^2, u_n >_{L^2(0,\ell)} -\beta < z_n(.,1), u_n >_{L^2(0,\ell)},$$

we deduce that

$$\left\|u_n'\right\|^2_{L^2(0,\ell)} = \|\delta_n u_n\|^2_{L^2(0,\ell)} + \Re\left(< f_n^1, (-i\delta_n + \alpha)\,u_n >_{L^2(0,\ell)}\right) +$$

$$\Re\left(< f_n^2, u_n >_{L^2(0,\ell)} -\beta < z_n(.,1), u_n >_{L^2(0,\ell)}\right) \to 0. \tag{40}$$

To conclude, the above limit (40) clearly contradicts $\|Z_n\|_{\mathscr{H}} = 1$. □

The two hypotheses of Lemma 4.2 are proved by Lemma 4.4, so that (26) holds. The proof of Theorem 4.1 is therefore finished. □

5. Spectral analysis and decay rate

In order to state the result on the prescribed stabilization, we define the decay rate, depending on α, β, τ, as

$$\omega(\alpha,\beta,\tau) := \inf\left\{\omega \,|\, \text{there exists } C = C(\omega) > 0 \text{ such that } \left\|e^{t\mathscr{A}}\right\|_{\mathscr{L}(\mathscr{H})} \leq C(\omega)e^{\omega t}\right\}, \tag{41}$$

and the spectral abscissa of $\mathscr{A}$ as

$$\mu(\mathscr{A}) := \sup\left\{\Re(\lambda) \,:\, \lambda \in \sigma(\mathscr{A})\right\}, \tag{42}$$

where $\sigma(\mathscr{A})$ denotes the spectrum of $\mathscr{A}$. It follows easily that

$$\mu(\mathscr{A}) \leq \omega(\alpha,\beta,\tau). \tag{43}$$

According to (26), we have that $\omega(\alpha,\beta,\tau) < 0$.

We give the following characterization of the eigenvalues and eigenvectors of $\mathscr{A}$.

Lemma 5.1. *A complex number $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathscr{A}$ if, and only if,*

$$p(\lambda) = \lambda^2 + \alpha\lambda + \beta\lambda e^{-\lambda\tau} = -\frac{n^2\pi^2}{\ell^2},\ n \in \mathbb{N}^*, \tag{44}$$

where the corresponding eigenfunction F_λ is given by

$$F_\lambda(x,\rho) = \frac{\sin\left(\frac{n\pi}{\ell}x\right)}{\sqrt{\left(\frac{n^2\pi^2}{\ell^2} + |\lambda|^2\right)\frac{\ell}{2} + |\lambda|^2\frac{\ell}{2}\frac{1-e^{-2\Re(\lambda)\tau}}{2\Re(\lambda)\tau}}}\left[1, \lambda, \lambda e^{-\lambda\tau\rho}\right]. \tag{45}$$

Proof. Assume that $(u_\lambda, v_\lambda, z_\lambda)^T$ is an eigenvector associated to the eigenvalue λ of $\mathscr{A}$. Then $v_\lambda = \lambda\, u_\lambda$ and (u_λ, z_λ) satisfy

$$\begin{cases} \left(\lambda^2 + \lambda\left(\alpha + \beta e^{-\lambda\tau}\right)\right)u_\lambda - \dfrac{d^2u_\lambda}{dx^2}(x) = 0, & x \in (0,\ell), \\ \dfrac{dz_\lambda}{d\rho}(x,\rho) = -\lambda\tau z_\lambda(x,\rho), & (x,\rho) \in (0,\ell)\times(0,1), \end{cases} \tag{46}$$

with boundary conditions

$$u_\lambda(0) = u_\lambda(\ell) = 0. \tag{47}$$

Equations (46) and (47) infer that

$$\begin{cases} u_\lambda(x) = A\, \sinh\left(\sqrt{p(\lambda)}\, x\right), & x \in (0,\ell), \\ z_\lambda(x,\rho) = \lambda u_\lambda(x)\, e^{-\lambda\tau\rho}, & x \in (0,\ell),\ \rho \in (0,1), \end{cases}$$

where $p(\lambda) := \lambda^2 + \lambda\left(\alpha + \beta e^{-\lambda\tau}\right)$.

The system (46) has a non-trivial solution if, and only if, λ satisfies

$$\sinh\left(\sqrt{p(\lambda)}\,\ell\right) = 0, \tag{48}$$

i.e.,

$$p(\lambda) + \frac{n^2\pi^2}{\ell^2} = 0,\ n \in \mathbb{N}^*.$$

Then, the solutions of (46) have the following form:

$$\begin{cases} u_\lambda(x) = \sin\left(\dfrac{n\pi}{\ell}x\right),\ x \in (0,\ell), \\ v_\lambda(x) = \lambda\,\sin\left(\dfrac{n\pi}{\ell}x\right),\ x \in (0,\ell), \\ z_\lambda(x,\rho) = \lambda\,\sin\left(\dfrac{n\pi}{\ell}x\right)\mathrm{e}^{-\lambda\tau\rho},\ x \in (0,\ell),\ \rho \in (0,1). \end{cases}$$

□

According to Corollary 3.2, $\sigma(\mathscr{A})$ is given by the eigenvalues of $\mathscr{A}$ and under condition (17), it is localized in $\mathbb{C}_- := \{\lambda \in \mathbb{C} \,|\, \Re\lambda < 0\}$. More precisely, we have the following corollary.

Corollary 5.2.

$$\sigma(\mathscr{A}) = \bigcup_{n\geq 1}\sigma_n(\mathscr{A}) = \bigcup_{n\geq 1}\left\{\lambda \in \mathbb{C}_- \,|\, p(\lambda) + \frac{n^2\pi^2}{\ell^2} = 0\right\}$$

and is symmetrically distributed with respect to the real axis.

A straightforward computation yields the expression for the adjoint of the operator $\mathscr{A}$, given by:

$$\mathscr{A}^*\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} -v \\ -u_{xx} - \alpha v + |\beta| z(.,0) \\ \tau^{-1} z_\rho \end{pmatrix}.$$

The domain of $\mathscr{A}^*$ is the set of $V = (u,v,z)^T$ such that:

$$(u,v,z)^T \in (H^2(0,\ell) \cap H_0^1(0,\ell)) \times H_0^1(0,\ell) \times L^2((0,\ell); H^1(0,1)), \tag{49}$$

$$\beta v(x) = -|\beta| z(x,1). \tag{50}$$

To establish that the generalized eigenvectors of the operator $\mathscr{A}$ form a Riesz basis in the Hilbert space $\mathscr{H}$, we employ the criterion introduced by Xu and Yung in [41]. In this context, we assume that the parameter $\beta > 0$. As a first step in this approach, we proceed to compute the resolvent of the adjoint operator $\mathscr{A}^*$.

$$(\lambda - \mathscr{A}^*)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, (u,v,w)^T \in \mathscr{D}(\mathscr{A}^*), (f,g,h)^T \in \mathscr{H}, \lambda \in \rho(\mathscr{A}^*).$$

So, $U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = (\lambda - \mathscr{A}^*)^{-1} \begin{pmatrix} f \\ g \\ h \end{pmatrix} = (\lambda - \mathscr{A}^*)^{-1} F$, where

$$\begin{aligned}
u(x) = &-\frac{1}{\sqrt{p(\lambda)}} \frac{\sinh(\sqrt{p(\lambda)}x)}{\sinh(\sqrt{p(\lambda)}\ell)} \int_0^\ell \sinh(\sqrt{p(\lambda)}(\ell - y))F(y)dy + \\
&\frac{1}{\sqrt{p(\lambda)}} \int_0^x \sinh(\sqrt{p(\lambda)}(x-y))\, F(y)dy,\ x \in (0,\ell), \\
v(x) = &\ f(x) - \lambda \left\{ -\frac{1}{\sqrt{p(\lambda)}} \frac{\sinh(\sqrt{p(\lambda)}x)}{\sinh(\sqrt{p(\lambda)}\ell)} \int_0^\ell \sinh(\sqrt{p(\lambda)}(\ell - y))F(y)dy + \right. \\
&\left. \frac{1}{\sqrt{p(\lambda)}} \int_0^x \sinh(\sqrt{p(\lambda)}(x-y))\, F(y)dy \right\},\ x \in (0,\ell), \\
w(x,\rho) = &-\frac{\beta}{|\beta|} f(x)\, \mathrm{e}^{\lambda\tau(\rho-1)} + \tau \int_\rho^1 \mathrm{e}^{\lambda\tau(\rho-s)} h(x,s)ds + \\
&\frac{\beta}{|\beta|} \lambda\, \mathrm{e}^{\lambda\tau(\rho-1)} \left\{ -\frac{1}{\sqrt{p(\lambda)}} \frac{\sinh(\sqrt{p(\lambda)}x)}{\sinh(\sqrt{p(\lambda)}\ell)} \int_0^\ell \sinh(\sqrt{p(\lambda)}(\ell - y))F(y)dy + \right. \\
&\left. \frac{1}{\sqrt{p(\lambda)}} \int_0^x \sinh(\sqrt{p(\lambda)}(x-y))\, F(y)dy \right\},\ x \in (0,\ell), \rho \in (0,1),
\end{aligned}$$

where

$$F(x) = |\beta|\, \tau \int_0^1 \mathrm{e}^{-\lambda\tau s} h(x,s)ds + g(x) + (-\lambda - \beta \mathrm{e}^{-\lambda\tau} - \alpha) f(x),\ x \in (0,\ell).$$

We observe that

$$\sqrt{p(\lambda)}\ \sinh(\sqrt{p(\lambda)}\ell)(\lambda - \mathscr{A}^*)^{-1} F$$

is a $\mathscr{H}$-valued entire function of finite exponential type and order 1. The scalar function $\sqrt{p(\lambda)}\ \sinh(\sqrt{p(\lambda)}\ell)$ is also an entire function of order 1.

Moreover, along each of the rays

$$\gamma_0 = -D + i\mathbb{R}_+, \gamma_1 = -D - \mathbb{R}_+, \gamma_2 = -D - i\mathbb{R}_+,$$

for a sufficiently large positive constant D, $(\lambda - \mathscr{A}^*)^{-1} F$ is uniformly bounded. Hence, by applying Lemma 3.1 in [42] (see also [43]), it follows that

$$Sp(\mathscr{A}) = \mathscr{H},$$

where $Sp(\mathscr{A})$ denotes the closure of linear span of the generalized eigenvectors of $\mathscr{A}$.

Proposition 5.3. *The system of generalized eigenvectors of $\mathscr{A}$ is complete in $\mathscr{H}$.*

Furthermore, according to Theorem 3.4 in [42] (see also [41]) for

$$\sigma_1(\mathscr{A}) = \{-\infty\} \text{ and } \sigma_2(\mathscr{A}) = \sigma(\mathscr{A}),$$

the system of generalized eigenvectors of $\mathscr{A}$ forms a Riesz basis of $Sp(\mathscr{A})$. Thus, by Proposition 5.3, this system forms a Riesz basis of $\mathscr{H}$.

We therefore state the following.

Proposition 5.4. *The system of generalized eigenvectors of $\mathscr{A}$ forms a Riesz basis of $\mathscr{H}$. Consequently, the system (16) satisfies the spectrum-determined growth condition, i.e.,*

$$\mu(\mathscr{A}) = \omega(\alpha, \beta, \tau). \tag{51}$$

On the other hand, the decay rate is determined by the spectral abscissa.

Owing to the continuous dependence of the spectrum on the system parameters [8]—particularly the spectral abscissa—it becomes possible to design the control parameters to assign the spectral abscissa at a desired admissible negative value.

Proposition 5.5. *For all $\gamma \in (s_n, 0)$, where s_n is defined in Theorem 6.4, there exists a triplet $(\alpha, \beta, \tau) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}_+$ such that $\gamma = \omega_n(\alpha, \beta, \tau)$, where*

$$\omega_n(\alpha, \beta, \tau) := \inf \left\{ \omega \middle| \text{ there exists } C = C(\omega) > 0 \text{ such that } \left\| \mathrm{e}^{t\mathscr{A}_n} \right\|_{\mathscr{L}(\mathscr{H}_n)} \leq C(\omega)\mathrm{e}^{\omega t} \right\} \tag{52}$$

$$\mathscr{H}_n := Sp(\mathscr{A}_n), \mathscr{A}_n := \mathscr{A} : \mathscr{D}(\mathscr{A}_n) = \mathscr{D}(\mathscr{A}) \cap \mathscr{H}_n \to \mathscr{H}_n,$$

and $Sp(\mathscr{A}_n)$ is the closure of linear span of the generalized eigenvectors F_λ, $\lambda \in \sigma_n(\mathscr{A})$.

6. Main results

In this section, we state the main results of this paper. First, in the case of purely delayed damping control, that is $\alpha = 0$, and under a quasimode-type initial conditions, the following theorem provides the global optimal decay of the wave equations' solution. Here, the parameters β and τ are the only control parameters.

Theorem 6.1. *Under the initial conditions*

$$u_0(x) = \varsigma_0 \sin\left(\frac{n\pi}{\ell} x\right) \quad \text{and} \quad u_1(x) = \varsigma_1 \sin\left(\frac{n\pi}{\ell} x\right)$$

with $(\varsigma_0, \varsigma_1, n) \in \mathbb{R} \times \mathbb{R} \times \mathbb{N}^*$ and for arbitrary $(\beta, \tau) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, the negative real number

$$s_n = -\frac{n\pi \sqrt{-11 + 5\sqrt{5}}\, \sqrt{2}\left(3\sqrt{5} + 7\right)}{4\ell} \tag{53}$$

is the global minimum of the exponential decay of the solution of $(\mathcal{P})$ which is achieved if, and only if,

$$\begin{cases} \tau = \dfrac{\sqrt{-22 + 10\sqrt{5}}\,\ell}{2n\pi} \\ \beta = \dfrac{n\pi\, \mathrm{e}^{\frac{1}{2} - \frac{\sqrt{5}}{2}} \sqrt{-11 + 5\sqrt{5}}\, \sqrt{2}\left(\sqrt{5} + 2\right)}{\ell}. \end{cases} \tag{54}$$

In other words, the fastest decay rate of the solutions of $(\mathcal{P})$ is obtained if (β, τ) satisfies (54). *Furthermore, for all $\gamma \in (s_n, 0)$, there exists a couple $(\beta, \tau) \in \mathbb{R} \times \mathbb{R}_+^*$ such that the decay rate of the solutions of $(\mathcal{P})$ is exactly $\gamma = \omega_n(0, \beta, \tau)$ (see* (52) *for the definition).*

Remark 6.2. *It is important to highlight the beneficial role played by the delay. As a matter of fact, when the delay τ is treated as a control parameter and chosen to satisfy condition* (54)*, the resulting decay rate specified in* (53) *outperforms the one obtained in the delay-free setting, as reported in [27], for all $(n, \ell) \in \mathbb{N}^* \times \mathbb{R}_+^*$. Indeed, in the absence of delay (i.e., when $\tau = 0$), the associated characteristic equation reduces to a second-order polynomial whose roots are given by $s_{1,2} = -\beta/2 \pm \sqrt{\beta^2/4 - n^2\pi^2}$.*

Remark 6.3. *Interestingly, the global optimal decay given in* (53) *is achieved in the region $0 = \alpha < \beta$, where, as demonstrated in [28], one can construct an explicit sequence of arbitrarily small delays that destabilize the system, a result that may seem surprising and counterintuitive. However, this phenomenon is consistent as the optimal decay does not necessarily lie in the domain of delay-independent stability, where the delay has no beneficial effect on the system's performance. In fact, in our setting, the delay parameter is not arbitrarily chosen but is precisely tuned according to the MID property as shown in* (54).

Although the previous result allows the assignment of the exponential decay rate of the closed-loop solution at the global optimal value, the delay provided by Theorem 6.1 is precisely determined in (54). This lack of parametric freedom in the control structure can be problematic, especially when robustness is a major concern. In practical applications, it is often desirable to have greater control over spectral placement to account for uncertainties, perturbations, or parameter variations. Consequently, greater parametric freedom is required to guarantee stability and performance over a wider range of conditions.

We are now able to consider the problem of the prescribed stabilization of the wave equation $(\mathcal{P})$ by means of PMD control. In fact, this can be seen as an option for increasing the parametric freedom on the controller structure. In the next result, although the parameters α, β, and τ are considered as control parameters, the delay is kept free in order to remedy to potential parameter uncertainties.

Theorem 6.4. *Under the initial conditions*

$$u_0(x) = \varsigma_0 \sin\left(\frac{n\pi}{\ell}x\right) \quad \text{and} \quad u_1(x) = \varsigma_1 \sin\left(\frac{n\pi}{\ell}x\right)$$

with $(\varsigma_0, \varsigma_1, n) \in \mathbb{R} \times \mathbb{R} \times \mathbb{N}^*$ and for arbitrary $(\alpha, \beta, \tau) \in \mathbb{R} \times \mathbb{R}^* \times \mathbb{R}_+^*$, the negative real number

$$s_n = -\frac{\alpha^*}{2} - \frac{2}{\tau} + \frac{\sqrt{(\tau^2\alpha^{*2} + 8)\,\ell^2 - 4\tau^2 n^2\pi^2}}{2\ell\tau} \tag{55}$$

defines the global optimum exponential decay rate for the solution of $(\mathcal{P})$ which is achieved if, and only if,

$$\beta = \frac{\left(-2\ell + \sqrt{(\tau^2\alpha^{*2} + 8)\,\ell^2 - 4\tau^2 n^2\pi^2}\right) \mathrm{e}^{\frac{-\alpha^*\tau\ell - 4\ell + \sqrt{(\tau^2\alpha^{*2} + 8)\,\ell^2 - 4\tau^2 n^2\pi^2}}{2\ell}}}{\ell\tau} \tag{56}$$

where $\alpha^* := \alpha^*(n, \ell, \tau)$ denotes the permanent real root of the polynomial

$$Q(\alpha) = 4\alpha^3\tau^3 + \left(\frac{n^2\pi^2\tau^4}{\ell^2} + 12\tau^2\right)\alpha^2 + \left(12\tau - \frac{16n^2\pi^2\tau^3}{\ell^2}\right)\alpha - \frac{4\tau^4 n^4\pi^4}{\ell^4} - \frac{44\tau^2 n^2\pi^2}{\ell^2} + 4. \tag{57}$$

In other words, the fastest decay rate of the solutions of $(\mathcal{P})$ is obtained if (α, β) satisfies (56)–(57) *for $\tau > 0$. Furthermore, for all $\gamma \in (s_n, 0)$ and $\tau > 0$, there exists a couple $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that the decay rate of the solutions of $(\mathcal{P})$ is exactly $\gamma = \omega_n(\alpha, \beta, \tau)$ (see* (52) *for the definition).*

Under the assumption of delay-independent stability of the corresponding quasipolynomial, the following result, which is a direct consequence of the preceding theorem, ensures not only the prescribed stabilization of the n-th mode but also guarantees the exponential stability of the infinitely many remaining modes, without any restrictive initial conditions. See, for instance, [28] for a further discussion about the delay-independent stability condition for our problem.

Corollary 6.5. *For all initial conditions* $(u_0, u_1) \in H_0^1(0,\ell) \times L^2(0,\ell)$ *and an appropriate choice of* τ *such that* $\alpha^* > \beta$ *where* α^* *is the permanent real root of* (57) *and* β *is given by* (56)*, the solution of* $(\mathcal{P})$ *is exponentially stable and the corresponding n-th mode decays with a decay rate* $\omega_n(\alpha^*, \beta, \tau) = s_n$ *given by* (55).

7. Quasipolynomials spectral abscissa assignment and proofs of the main results

Building on the previous section, where it was shown that the eigenvectors and generalized eigenvectors form a Riesz basis in the state Hilbert space, it follows that the optimal decay rate is determined by the spectral abscissa of the corresponding quasipolynomial's characteristic function. Consequently, the proofs of the main results—Theorems 6.1 and 6.4—with respect to the global optimality of the decay rate reduce to establishing the global optimum of the spectral abscissa associated with these quasipolynomial functions.

7.1. Prescribed stabilization of the wave equation via internal delayed damping

To assign the decay rate of the solution of the wave equation $(\mathcal{P})$ via delayed damping only (that is $\alpha = 0$), one needs to assign the eigenvalues of the corresponding operator $\mathscr{A}$, which amounts to assigning the rightmost zeros of the quasipolynomial family

$$\Delta_n(s) = P_0(s) + P_\tau(s)\,\mathrm{e}^{-s\tau}, \tag{58}$$

where P_0, P_τ are given by

$$\begin{cases} P_0(s) := s^2 + \dfrac{n^2\pi^2}{\ell^2} \\ P_\tau(s) := \beta s. \end{cases} \tag{59}$$

In the next result, parameters β and τ are considered as control parameters.

Lemma 7.1. *For any* $(\beta, \tau) \in \mathbb{R} \times \mathbb{R}_+^*$*, the triple negative real root of* (58)–(59)

$$s_n = -\frac{n\pi\sqrt{-11 + 5\sqrt{5}}\,\sqrt{2}\left(3\sqrt{5} + 7\right)}{4\,\ell} \tag{60}$$

which is achieved if, and only if,

$$\begin{cases} \tau = \dfrac{\sqrt{-22 + 10\sqrt{5}}\,\ell}{2n\pi} \\ \beta = \dfrac{n\pi\,\mathrm{e}^{\frac{1}{2} - \frac{\sqrt{5}}{2}}\sqrt{-11 + 5\sqrt{5}}\,\sqrt{2}\left(\sqrt{5} + 2\right)}{\ell} \end{cases} \tag{61}$$

defines the global minimum of the spectral abscissa of Δ_n *given in* (58)–(59).

Remark 7.2. *Notice that the delay value provided in* (61) *satisfies, as expected, the admissible delay region asserted in equation* (7) *of Theorem 2.2. As a matter of fact, in our case, $a_1 = 0$, and $a_0 > 0$, and $\tau \in (0, \sqrt{2}\,\ell/(n\,\pi))$, which is consistent with the last item of* (7).

Remark 7.3. *Thanks to the linear dependency of the assigned triple root* (60) *in the parameter n, we immediately see that s_1 is the maximum of the sequence $(s_n)_{n\in\mathbb{N}^*}$ corresponding respectively to the spectral abscissae of the family $(\Delta_n)_{n\in\mathbb{N}^*}$.*

Proof of Lemma 7.1. It is obvious that assigning the closed-loop abscissa to a quadruple root of (58)–(59) is not feasible, as the conditions outlined in Theorem 2.1 require the open-loop discriminant δ to vanish, a direct consequence of the sparsity of P_τ. This condition is indeed incompatible with our setup, where $\delta = -4\,n\,\pi/\ell < 0$. So, one exploits Theorem 2.2. To do so, we assign the spectral abscissa s_n of (58)–(59) at the dominant triple root s_+ given by (6) and the parameter values given in ($\star_+$). Furthermore, the vanishing of k_0 from ($\star_+$) leads to the delay value given in (61). Finally, one easily checks that $\delta > -8/\tau^2$, which, thanks to Theorem 2.5, proves that s_n given by (60) is a global minimizer of the spectral abscissa of Δ_n given by (58)–(59). □

Proof of Theorem 6.1. Given the initial conditions

$$u_0(x) = \varsigma_0 \sin\left(\frac{n\pi}{\ell}x\right) \quad \text{and} \quad u_1(x) = \varsigma_1 \sin\left(\frac{n\pi}{\ell}x\right)$$

with $n \in \mathbb{N}^*$, the only relevant quasipolynomial is Δ_n. Owing to Proposition 5.4, the result follows directly from Lemma 7.1. □

7.2. Robustifying the design by the additional action of instantaneous damping

We are now able to consider the problem of the prescribed stabilization of the wave equation ($\mathcal{P}$) by means of both instantaneous and delayed damping. In fact, this can be seen as an option for increasing the parametric freedom on the controller structure. Namely, one assigns the eigenvalues of the corresponding operator $\mathscr{A}$, which amounts to assigning the rightmost zeros of the quasipolynomial family

$$\Delta_n(s) = P_0(s) + P_\tau(s)\,\mathrm{e}^{-s\,\tau} \tag{62}$$

where P_0, P_τ are given by

$$\begin{cases} P_0(s) := s^2 + \alpha\, s + \dfrac{n^2\,\pi^2}{\ell^2} \\ P_\tau(s) := \beta s. \end{cases} \tag{63}$$

Also in this configuration, the results of Section 2 can be exploited. In the next result, although the parameters α, β, and τ are considered as control parameters, the delay remains free in order to remedy potential parameter uncertainties.

Lemma 7.4. *For any $(\alpha,\beta,\tau) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^*$, the triple negative real number*

$$s_n = -\frac{\alpha^*}{2} - \frac{2}{\tau} + \frac{\sqrt{(\tau^2\alpha^{*2} + 8)\,\ell^2 - 4\tau^2 n^2\pi^2}}{2\ell\tau} \tag{64}$$

which is achieved if, and only if,

$$\beta = \frac{\left(-2\ell + \sqrt{(\tau^2\alpha^{*2} + 8)\,\ell^2 - 4\tau^2 n^2\pi^2}\right) \mathrm{e}^{\dfrac{-\alpha^*\tau\ell - 4\ell + \sqrt{(\tau^2\alpha^{*2} + 8)\,\ell^2 - 4\tau^2 n^2\pi^2}}{2\ell}}}{\ell\tau}, \tag{65}$$

where $\alpha^* := \alpha^*(n, \ell, \tau)$ *denotes the permanent real root of*

$$Q(\alpha) = 4\alpha^3\tau^3 + \left(\frac{n^2\pi^2\tau^4}{\ell^2} + 12\tau^2\right)\alpha^2 + \left(-\frac{16n^2\pi^2\tau^3}{\ell^2} + 12\tau\right)\alpha - \frac{4\tau^4 n^4\pi^4}{\ell^4} - \frac{44\tau^2 n^2\pi^2}{\ell^2} + 4, \tag{66}$$

defines the global minimum of the spectral abscissa of Δ_n *given in* (62)–(63).

Remark 7.5. *One easily sees that the assigned triple root* (64) *is decreasing in the parameter n, so that* s_1 *is the minimum of the sequence* $(s_n)_{n\in\mathbb{N}^*}$.

Proof of Lemma 7.4. Since the open-loop characteristic function admits a negative discriminant $\delta = -4\,n\,\pi/\ell < 0$, then we use the same argument as in the previous proof by exploiting Theorem 2.2. To do so, we assign the spectral abscissa s_n of (62)–(63) at the dominant triple root s_+ given by (6) and the parameter values given in ($\star_+$). Furthermore, the vanishing of k_0 from ($\star_+$) leads to the gain β value given in (65) as well as the characteristic polynomial for α admitting the discriminant $(\frac{\tau^2 n^2\pi^2}{\ell^2} - 3^3)^3$. Consequently, if $\frac{\tau^2 n^2\pi^2}{\ell^2} < 27$, then α^* is the unique real root of (66). Otherwise, the polynomial Q given in (66) admits three real roots. In that case, one selects the lower branch, itself guaranteeing the negativity of the assigned spectral abscissa. Finally, one easily checks that $\delta > -8/\tau^2$, which owing to Theorem 2.5, proves that s_n given by (60) is a global minimizer of the spectral abscissa of Δ_n given by (62)–(63). □

As can be observed in the expression of Q given in (66), the change of variable $\tau\alpha \mapsto \alpha$ and the change of parametrization $\tau n\pi/\ell \mapsto n\pi/\ell$ allow one to conclude that the number of real roots of Q exclusively depends on the sign of the difference $n^2\pi^2/\ell^2 - 3^3$ as illustrated in Figure 1 where the real and imaginary parts of the three branches of roots of the third-order polynomial Q defined in (66) are presented and the critical value $n\pi/\ell = 3\sqrt{3}$ is recovered. Furthermore, Figure 2 shows that the branch of interest is nothing but the permanent real branch (plotted with a solid line), i.e., independently of the discriminant sign, which is nothing but the least real branch. As a matter of fact, this is the only solution branch guaranteeing the negativity of the spectral abscissa and, consequently, the closed-loop solution decay. Then, this branch can be written explicitly as:

$$\begin{aligned}\alpha^* = &\frac{\tau^{\frac{2}{3}} n^{\frac{2}{3}} \left(-n^4\pi^6\tau^4 + 540\ell^2 n^2\pi^4\tau^2 + 5832\ell^4\pi^2 + 24\sqrt{\pi^4\left(-\tau^2 n^2\pi^2 + 27\ell^2\right)^3}\,\sqrt{3}\,\ell\right)^{\frac{1}{3}}}{12\ell^2} - \frac{\tau^2 n^2\pi^2}{12\ell^2} \\ &+ \frac{\tau^{\frac{4}{3}} n^{\frac{4}{3}} \pi^2 \left(\tau^2 n^2\pi^2 + 216\ell^2\right)}{12\ell^2\left(-n^4\pi^6\tau^4 + 540\ell^2 n^2\pi^4\tau^2 + 5832\ell^4\pi^2 + 24\sqrt{\pi^4\left(-\tau^2 n^2\pi^2 + 27\ell^2\right)^3}\,\sqrt{3}\,\ell\right)^{\frac{1}{3}}} - 1\end{aligned} \tag{67}$$

Finally, the concepts of delay-dependent and delay-independent stability are illustrated in Figure 3. The blue curve, lying below the red one, corresponds to the delay-dependent case, while the delay-independent case is represented when the red curve lies below the blue. The latter configuration lies within the scope of Corollary 6.5.

Proof of Theorem 6.4. As in the proof of Theorem 6.1, under initial conditions

$$u_0(x) = \varsigma_0 \sin\left(\frac{n\pi}{\ell}x\right) \quad \text{and} \quad u_1(x) = \varsigma_1 \sin\left(\frac{n\pi}{\ell}x\right)$$

with $n \in \mathbb{N}^*$, the only relevant quasipolynomial is Δ_n. Owing to Proposition 5.4, the result follows directly from Lemma 7.4. $\square$

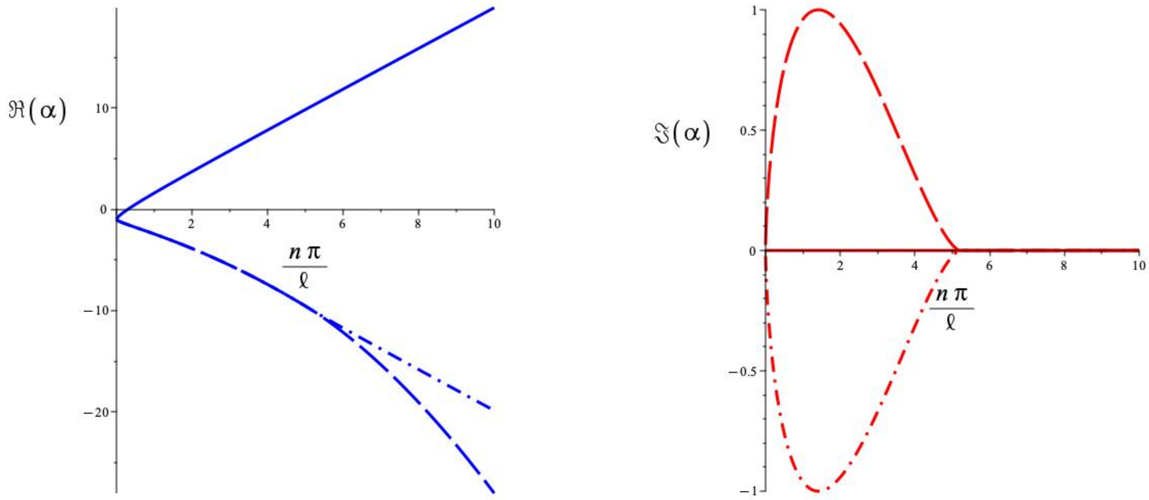


Figure 1. Real and imaginary parts of the three solution branches of the polynomial Q given by (66). The solid curve represents the real branch, which remains real regardless of the sign of the discriminant of Q, and is the only branch of interest.

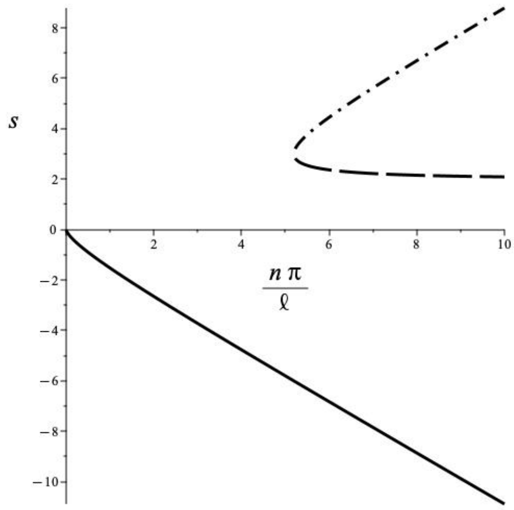


Figure 2. The spectral abscissa given by (64) as a function of $n\pi/\ell$, where α^* corresponds to the three admissible real branches of the solution to Q in (66). The solid curve, which remains real regardless of the sign of the discriminant, is the branch of interest, as it is the only one that guarantees the negativity of the spectral abscissa.

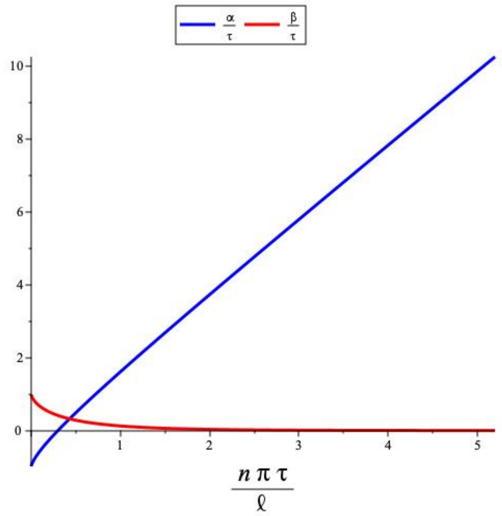


Figure 3. The delay-dependent vs. delay-independent stability. Delay-dependent versus delay-independent stability: the quantities $(\alpha/\tau, \beta/\tau)$, given respectively by (67) and (65), are plotted, respectively, in blue and red, as functions of $n\pi\tau/\ell$.

8. Further insights on parametric stability analysis

In this section, we provide further stability analysis in the parameter space defined by $(n, \ell, \alpha, \beta, \tau) \in \mathbb{Z}^* \times \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R}_+$, which gives valuable insights into the choice of control parameters with a particular emphasis on the delay-induced stability/instability. Let us go back to the family of quasipolynomials (63)–(62) derived from (44) and seen as parameter-dependent quasipolynomials $\Delta_n : \mathbb{C} \times \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{C}$ given by:

$$\Delta_n(s; \ell, \alpha, \beta, \tau) = P_0(s; \ell, \alpha) + P_\tau(s; \beta)\, \mathrm{e}^{-s\tau} \tag{68}$$

where P_0, P_τ are given by

$$\begin{cases} P_0(s; \ell, \alpha) := s^2 + \alpha s + \dfrac{n^2 \pi^2}{\ell^2} \\ \quad P_\tau(s; \beta) := \beta s, \end{cases} \tag{69}$$

where the parameter-dependence is explicitly indicated.

In the sequel, for the sake of brevity, when using Δ_n, P_0, or P_τ, we will conserve only the complex variable s, or indicate its explicit dependence with respect to the delay parameter τ. Beyond the addressed problem of prescribed stabilization by using the MID property and in order to make the paper self-contained, we believe that the characterization of the stability regions in the parameter-space defined by $(n, \ell, \alpha, \beta, \tau) \in \mathbb{Z}^* \times \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R}_+$ will allow a complete understanding of the dynamical behavior. As explained in [8] (see also [44] for the classification of DDEs), Δ_n is a quasipolynomial of *retarded type* (since $\deg(P_0) > \deg(P_\tau)$) with an infinite number of roots but with only *point spectrum*. Furthermore, by using the properties of analytic functions, it follows that we will have a finite number of roots in any compact set of the complex plane $\mathbb{C}$, and that the number of roots located to the right of any vertical line of $\mathbb{C}$, parallel to the imaginary axis $i\mathbb{R}$, is also finite. In other words, in particular, if unstable roots exist, their number is finite. Finally, similar to the polynomial case, the roots of a quasipolynomial are continuously dependent with respect to its coefficients (see, e.g., [8] and the references therein) and, as a consequence, any changes in the number of unstable roots corresponds to the case when roots cross the imaginary axis $i\mathbb{R}$ (toward stability/instability).

Accordingly, and drawing inspiration from the classical parameter-based τ- and D-decomposition methods explained and detailed in [8] (see also the references therein), it is natural to investigate the presence of crossing frequencies ω, defined by the condition $\Delta(i\omega) = 0$. The last condition implies that

$$|P_0(i\omega)|^2 - |P_1(i\omega)|^2 = 0$$

must hold. Applying basic algebraic manipulations, we infer that

$$-\omega^4 + \left(\beta^2 - \alpha^2 + \frac{2n^2\pi^2}{\ell^2}\right)\omega^2 - \frac{\pi^4 n^4}{\ell^4} = 0,$$

where only even powers of ω appear. Introducing the substitution $\omega^2 = \Omega$, we define the quadratic function

$$\mathcal{Z}_n(\Omega) = -\Omega^2 + \left(\beta^2 - \alpha^2 + \frac{2n^2\pi^2}{\ell^2}\right)\Omega - \frac{\pi^4 n^4}{\ell^4}. \tag{70}$$

Thus, the existence of a real crossing frequency ω for Δ_n requires the existence of a positive solution $\Omega > 0$ of $\mathcal{Z}_n$. This is only possible if the polynomial is not Hurwitz and admits a positive discriminant. Using Descartes' Rule of Signs, we find that a necessary condition for the existence of such a root is

$$\beta^2 - \alpha^2 + \frac{2n^2\pi^2}{\ell^2} > 0. \tag{71}$$

Furthermore, by performing standard computation of the discriminant of $\mathcal{Z}_n$, one gets the second necessary condition

$$\left(\left(\alpha^2 - \beta^2\right)\ell^2 - 4n^2\pi^2\right)\left(\alpha^2 - \beta^2\right) > 0. \tag{72}$$

The current state of the analysis may now be concisely summarized.

(i) If $\beta^2 > \alpha^2$, then both conditions (71)–(72) are satisfied, and the sign of the constant term of $\mathcal{Z}_n$ allows us to conclude that $\mathcal{Z}_n$ admits two positive roots $\Omega_\pm$. Thus, Δ_n has two pairs of (simple) complex conjugate roots on the imaginary axis that we will denote as $\pm i\omega_+$ and $\pm i\omega_-$, with $0 < \omega_- < \omega_+$. Even though the crossing frequency values $\omega_\pm$ do not depend on the delay parameter, each corresponds to a unique positive delay $\tau_\pm$ within each interval of length 2π for the product $\tau_\pm\omega_\pm$. On the other hand, although the number of the (critical) crossing frequencies is finite, it is worth mentioning that, at each critical frequency, as long as the delay is increased, we will have an infinite number of roots crossing the imaginary axis.

(ii) If $\alpha^2 > \beta^2$, condition (71) and the negativity of the first factor of (72) are inconsistent. Thus, there are no roots on the imaginary axis, and the closed-loop system is *hyperbolic*, independently of the delay value.

Let us discuss the "limit" case when $|\alpha| = |\beta|$. For instance, if $\alpha = \beta > 0$, there exists only one root on the imaginary axis at the frequency $\omega = n^2\pi^2/\ell^2$. It should be mentioned that, for a fixed positive integer $n \in \mathbb{N}^*$, the critical delays associated to ω have the form:

$$\tau_k := (2k+1)\frac{\ell^2}{\pi n^2}, \quad k \in \mathbb{N}.$$

This root is simple for all $\alpha \neq 2n^2\pi/\ell^2(2k+1)$, and double if $\alpha = 2n^2\pi/\ell^2(2k+1)$. It is easy to observe that if $\tau = 0$, the quasipolynomial Δ_n reduces to a Hurwitz polynomial, and the roots $s := s(\tau)$ located in $\mathbb{C}_-$ are not crossing the imaginary axis; they are simply "touching"* the imaginary axis turning back in $\mathbb{C}_-$. Furthermore, there are no roots located in $\mathbb{C}_+$. In fact, we are in the so-called *marginally hyperbolic* system.

Remark 8.1. *For a given $k \in \mathbb{N}$, the choice $\alpha = 2n^2\pi/\ell^2(2k+1)$ and $\tau = (2k+1)\frac{\ell^2}{\pi n^2}$ leads to a quasipolynomial Δ_n for which the generic MID property holds. In fact, since* $\deg(\Delta_n) = 4$, *it is easy to observe that the maximum multiplicity of a non-zero root located on the imaginary axis is* 2. *To the best of the authors' knowledge, the GMID for second-order DDEs with one delay was discussed in [35] but using a different argument to prove the dominancy property.*

A similar situation occurs when $\alpha = \beta < 0$ with the same frequency $\omega = n^2\pi^2/\ell^2$, but with the difference that the corresponding critical delays are:

$$\tau_k := 2k\frac{\ell^2}{\pi n^2}, \quad k \in \mathbb{N}.$$

This root is simple for all $\alpha \neq n^2\pi/(k\ell^2)$, and double if $\alpha = n^2\pi/(k\ell^2)$. It is easy to observe that if $\tau = 0$, the quasipolynomial Δ_n reduces to a(n) (unstable) polynomial with 2 roots located in $\mathbb{C}_+$. In such a configuration, we are still in the so-called *marginally hyperbolic* case, but, depending on the location of the roots $s := s(\tau)$ at $\tau := 0_+$, we will have two behaviors: (i) all the roots located in $\mathbb{C}_-$ (infinitely many) will stay in $\mathbb{C}_-$ without "touching" the imaginary axis; (ii) the two roots located in $\mathbb{C}_+$ at $\tau = 0$ will stay in $\mathbb{C}_+$ but they will "touch" the imaginary axis when the delay is increased and "turn" back in $\mathbb{C}_+$.

Condition (ii) above simply says that the system is *hyperbolic* with respect to the delay parameter in the sense that its stability/instability property at $\tau = 0$ is *independent* of the delay value. For a deeper discussion of the hyperbolicity property and criteria to test it, we refer to [45]. In other words, in this case, if the polynomial $P_0(s) + P_\tau(s)$ is Hurwitz, that is, $\alpha + \beta > 0$ and $\alpha > |\beta|$, the roots of Δ_n will be located in $\mathbb{C}_-$ for all delay values $\tau \in \mathbb{R}_+$ and for all $n \in \mathbb{Z}^*$ and $\ell \in \mathbb{R}_+$. Now if $\alpha + \beta < 0$, and $|\alpha| > \beta$, there exists one unstable root for all $\tau \in \mathbb{R}_+$ independent of the parameters $n \in \mathbb{Z}^*$ and $\ell \in \mathbb{R}_+$.

Condition (i) corresponds to the situation when there are roots of Δ_n crossing the imaginary axis when the delay τ, seen as a parameter, is increased. In fact, depending on the critical delay values associated to the frequencies $\omega_\pm$, we will have roots crossing the imaginary axis toward stability/instability. As explained in [8] (see also the references therein and, in particular, [46]), the roots will cross the imaginary axis toward instability (stability) at ω_+ (ω_-).

With the remarks and observations above, depending on the behavior in the delay-free case, we will have two distinct cases.

(i.1) If $\alpha + \beta > 0$, then the quasipolynomial Δ_n reduces to a Hurwitz polynomial when $\tau = 0$ and we will have a sequence switching from stability (instability) to instability (stability) when the delay is increased with the first switch toward instability leading to a partition of the (positive) delay axis in several intervals such that the number of unstable roots is invariant within each delay interval.

(i.2) If $\alpha + \beta < 0$, then Δ_n reduces to a polynomial with 2 roots located in $\mathbb{C}_+$ when $\tau = 0$, and similar to the previous case, we will have a sequence of switches with a similar partition of the (positive) delay axis.

*The curve $s = s(\tau)$ is tangent to the imaginary axis ($i\mathbb{R}$) when $\tau = \tau_k$, for all $k \in \mathbb{N}$.

At this stage, in order to end this parameter-based stability analysis, we need to have a closer look at the partition of the delay axis and to define the delay intervals for which the roots are located in $\mathbb{C}_-$. In fact, for each critical frequency $\omega_\pm$, we associate a set of critical delays. Since the roots are simple, as discussed in [8], an *invariant crossing property* holds and we can compute a partition of the critical delays in two ordered sets $\mathcal{T}_+$ (critical root crossing $i\mathbb{R}$ toward instability) and $\mathcal{T}_-$ (critical root crossing $i\mathbb{R}$ toward stability). Since $\mathcal{T}_\pm$ are associated to the crossing frequencies $\omega_\pm$, we will simply write $\mathcal{T}_\pm(\omega_\pm)$:

$$\begin{cases} \mathcal{T}_+(\omega_+) := \left\{ \dfrac{1}{\omega_+} \left[i\, Log \dfrac{i\beta\omega_+}{\omega_+^2 - \frac{n^2\pi^2}{\ell^2} - i\alpha\omega_+} + 2k\pi \right] \geq 0, \quad k \in \mathbb{Z} \right\}, \\ \mathcal{T}_-(\omega_-) := \left\{ \dfrac{1}{\omega_-} \left[i\, Log \dfrac{i\beta\omega_-}{\omega_-^2 - \frac{n^2\pi^2}{\ell^2} - i\alpha\omega_-} + 2k\pi \right] \geq 0, \quad k \in \mathbb{Z} \right\}, \end{cases} \tag{73}$$

and let $\tau_{m,\pm}$ be the smallest positive delay value of the corresponding set $\mathcal{T}_\pm$, independent of the parameter values, $\tau_{m,+} \neq \tau_{m,-}$.

Thus, by taking into account the classification (i.1)–(i.2), one gets:

(i.1) If $\alpha + \beta > 0$, we will always have $\tau_{m,+} > \tau_{m,-}$ corresponding to the fact that the first delay interval is stable, followed by a second delay interval which is unstable showing the so-called *destabilizing effect* induced by the delay parameter.

(i.2) If $\alpha + \beta < 0$, two situations may occur: first, $\tau_{m,+} > \tau_{m,-}$, which simply says that the second delay interval still corresponds to an unstable delay interval, and if $\tau_{m,+} < \tau_{m,-}$, since there are 2 unstable roots at $\tau = 0$, these roots will cross the imaginary axis towards stability leading to a first stable delay interval showing that the delay may have a *stabilizing effect* on the dynamics.

Consider now the particular case when $\alpha = 0$. Then, it is easy to observe that the corresponding quasipolynomials correspond to the closed-loop system of an oscillator controlled by a delayed derivative operator.

9. Application to the transverse vibration control of a string

In this section, the transverse vibration of a string is considered for the illustration of the distributed control scheme described in this work. For a perfectly elastic string of length $l > 0$ with both ends fixed, having constant density and elasticity parameters, the motion equation is described by the following partial derivative equation:

$$\begin{cases} U_{\mathbf{tt}}(\mathbf{x}, \mathbf{t}) - c^2\, U_{\mathbf{xx}}(\mathbf{x}, \mathbf{t}) = -V(\mathbf{x}, \mathbf{t}), \quad (\mathbf{x}, \mathbf{t}) \in (0, l) \times (0, +\infty), \\ U(0, \mathbf{t}) = 0, \quad U(l, \mathbf{t}) = 0, \quad \mathbf{t} > 0, \\ U(\mathbf{x}, 0) = f(\mathbf{x}), \quad U_{\mathbf{t}}(\mathbf{x}, 0) = g(\mathbf{x}), \quad \mathbf{x} \in (0, l), \end{cases} \tag{$\tilde{\mathcal{P}}$}$$

where $U(\mathbf{x}, \mathbf{t})$ denotes the transversal displacement of a point belonging to the string, located at position $\mathbf{x}$ and given at time $\mathbf{t}$; $V(\mathbf{x}, \mathbf{t})$ denotes the control which is an applied body force in the whole domain $(0, l)$; and f and g are smooth univariate functions that are specified further below.

Remark 9.1. *The wave equation problem modeled by equation* ($\mathcal{P}$) *has used some normalized (*i.e.*, dimensionless) time t and position x variables in such a way that their corresponding relations to the* *considered true variables are* $\mathbf{x} = l\,\dfrac{x}{\ell}$ *and* $\mathbf{t} = d\,t$ *with* $d := \dfrac{l}{c\,\ell}$. *In this case, the normalized transversal displacement is related to the original one by the relation*

$$\begin{aligned} u(x,t) &:= U(\mathbf{x},\mathbf{t}),\\ &= U(l\,x, d\,t). \end{aligned} \tag{74}$$

The delay-based output feedback control law, applied in a distributed manner on the domain defined in $(\tilde{\mathcal{P}})$, is

$$\begin{cases} V(\mathbf{x},\mathbf{t}) := \dfrac{\alpha}{d}\,U_{\mathbf{t}}(\mathbf{x},\mathbf{t}) + \dfrac{\beta}{d}\,U_{\mathbf{t}}(\mathbf{x},\mathbf{t}-\boldsymbol{\tau}), \quad (\mathbf{x},\mathbf{t}) \in (0,l)\times(0,+\infty),\\ U_{\mathbf{t}}(\mathbf{x},\mathbf{t}-\boldsymbol{\tau}) = 0, \quad \mathbf{t} \in (0,\boldsymbol{\tau}), \end{cases} \tag{$\tilde{C}$}$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^*$ are the scalar gains computed as in Theorems 6.1 or 6.4; and $\boldsymbol{\tau} := d\,\tau$ is the delay used here as a control parameter.

9.1. Finite difference scheme

In order to check numerically the efficiency of the proposed control scheme, we propose to perform a finite difference discretization of the above problem by using the approximations described in the sequel.

Let us define the constant space step $\Delta x > 0$, the constant time step $\Delta t > 0$, and the numerical sequences

$$\mathbf{x}_i := i\,\Delta x\ (i = 0,1,2,\ldots,N), \quad \mathbf{t}_j := j\,\Delta t\ (j = 0,1,2,\ldots,n), \quad \text{and} \quad U_{i,j} := U(\mathbf{x}_i,\mathbf{t}_j), \tag{75}$$

where $N, n \in \mathbb{N}$ are such that

$$N\,\Delta x = l \quad \text{and} \quad n\,\Delta t = T_f, \tag{76}$$

with T_f standing for the final time of the simulation. Let us denote $U_{\mathbf{t}^{i,j}} := \left.\dfrac{\partial U(\mathbf{x},\mathbf{t})}{\partial \mathbf{t}}\right|_{\substack{\mathbf{x}=\mathbf{x}_i,\\ \mathbf{t}=\mathbf{t}_j}}$, $U_{\mathbf{x}^{i,j}} := \left.\dfrac{\partial U(\mathbf{x},\mathbf{t})}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_i,\\ \mathbf{t}=\mathbf{t}_j}}$, $U_{\mathbf{tt}^{i,j}} := \left.\dfrac{\partial^2 U(\mathbf{x},\mathbf{t})}{\partial \mathbf{t}^2}\right|_{\substack{\mathbf{x}=\mathbf{x}_i,\\ \mathbf{t}=\mathbf{t}_j}}$, and $U_{\mathbf{xx}^{i,j}} := \left.\dfrac{\partial^2 U(\mathbf{x},\mathbf{t})}{\partial \mathbf{x}^2}\right|_{\substack{\mathbf{x}=\mathbf{x}_i,\\ \mathbf{t}=\mathbf{t}_j}}$.

For all $\mathbf{t} > 0$, the boundary conditions in $(\tilde{\mathcal{P}})$ are transcribed as follows with $j \in \mathbb{N}^*$:

$$\begin{aligned} U(0,\mathbf{t}) = U(l,\mathbf{t}) \equiv 0 &\longrightarrow & U_{0,j} = U_{N,j} = 0,\\ U_{\mathbf{t}}(0,\mathbf{t}) = U_{\mathbf{t}}(l,\mathbf{t}) \equiv 0 &\longrightarrow & U_{\mathbf{t}^{0,j}} = U_{\mathbf{t}^{N,j}} = 0,\\ U_{\mathbf{tt}}(0,\mathbf{t}) = U_{\mathbf{tt}}(l,\mathbf{t}) \equiv 0 &\longrightarrow & U_{\mathbf{tt}^{0,j}} = U_{\mathbf{tt}^{N,j}} = 0,\\ \Rightarrow U_{\mathbf{xx}}(0,\mathbf{t}) = U_{\mathbf{xx}}(l,\mathbf{t}) \equiv 0 &\longrightarrow & U_{\mathbf{xx}^{0,j}} = U_{\mathbf{xx}^{N,j}} = 0. \end{aligned} \tag{77}$$

For all $\mathbf{x} \in (0,l)$, the initial conditions read, for $i = 0,1,2,\ldots,N$,

$$\begin{aligned} U(\mathbf{x},0) \equiv f(\mathbf{x}) &\longrightarrow & U_{i,0} = f(i\,\Delta x),\\ U_{\mathbf{t}}(\mathbf{x},0) \equiv g(\mathbf{x}) &\longrightarrow & U_{\mathbf{t}^{i,0}} = g(i\,\Delta x),\\ U_{\mathbf{x}}(\mathbf{x},0) \equiv f'(\mathbf{x}) &\longrightarrow & U_{\mathbf{x}^{i,0}} = f'(i\,\Delta x),\\ U_{\mathbf{xx}}(\mathbf{x},0) \equiv f''(\mathbf{x}) &\longrightarrow & U_{\mathbf{xx}^{i,0}} = f''(i\,\Delta x). \end{aligned} \tag{78}$$

By the continuity of $U(\mathbf{x}, \mathbf{t})$ in the variable $\mathbf{t}$, the initial conditions are then

$$\begin{aligned} U(0,0) = U(l,0) = 0 \Rightarrow f(0) = f(l) = 0 \quad &\longrightarrow \quad U_{0,0} = U_{N,0} = 0, \\ U_{\mathbf{t}}(0,0) = U_{\mathbf{t}}(l,0) = 0 \Rightarrow g(0) = g(l) = 0 \quad &\longrightarrow \quad U_{\mathbf{t}^{0,0}} = U_{\mathbf{t}^{N,0}} = 0. \end{aligned} \tag{79}$$

To summarize, at this stage, the functions f and g should satisfy

$$\begin{aligned} &f(0) = 0, \quad f(l) = 0, \quad g(0) = 0, \quad g(l) = 0, \\ &f''(0) = 0, \quad f''(l) = 0. \end{aligned} \tag{80}$$

We consider the following approximations of the first-order partial derivatives:

$$U_{\mathbf{t}^{i,j}} = \frac{1}{2\,\Delta t}\Big(3\,U_{i,j} - 4\,U_{i,j-1} + U_{i,j-2}\Big), \qquad i = 0, 1, \ldots, N, \quad j \in \mathbb{N} \backslash \{0, 1\}, \tag{81}$$

$$U_{\mathbf{x}^{i,j}} = \frac{1}{2\Delta x}\Big(U_{i+1,j} - U_{i-1,j}\Big), \qquad i = 1, 2, \ldots, N-1, \quad j \in \mathbb{N}, \tag{82}$$

and the ones for the second-order partial derivatives:

$$U_{\mathbf{tt}^{i,j}} = \frac{1}{\Delta t^2}\Big(U_{i,j} - 2\,U_{i,j-1} + U_{i,j-2}\Big), \qquad i = 0, 1, \ldots, N, \quad j \in \mathbb{N} \backslash \{0, 1\}, \tag{83}$$

$$U_{\mathbf{xx}^{i,j}} = \frac{1}{\Delta x^2}\Big(U_{i+1,j} - 2\,U_{i,j} + U_{i-1,j}\Big), \qquad i = 1, 2, \ldots, N-1, \quad j \in \mathbb{N}. \tag{84}$$

For $j = 1$, $i = 0, 1, \ldots, N$, using a first-order Taylor expansion, we set

$$\begin{aligned} U_{\mathbf{t}^{i,1}} &= U_{\mathbf{t}^{i,0}} + \Delta t\, U_{\mathbf{tt}^{i,0}}, \\ &= g(i\,\Delta x) + \Delta t\, c^2 f''(i\,\Delta x). \end{aligned} \tag{85}$$

For $i = 0$ or $i = N$, and $j \in \mathbb{N}$,

$$\begin{aligned} U_{\mathbf{x}^{0,j}} &= \frac{1}{\Delta x}\Big(U_{1,j} - U_{0,j}\Big) = \frac{1}{\Delta x} U_{1,j}, \qquad U_{\mathbf{xx}^{0,j}} = 0, \\ U_{\mathbf{x}^{N,j}} &= \frac{1}{\Delta x}\Big(U_{N,j} - U_{N-1,j}\Big) = -\frac{1}{\Delta x} U_{N-1,j}, \quad U_{\mathbf{xx}^{N,j}} = 0. \end{aligned} \tag{86}$$

By denoting $V_{i,j} := V(\mathbf{x}_i, \mathbf{t}_j)$, the delay-based output feedback control law $(\tilde{C})$, applied on the domain, is discretized into the numerical sequence given by

$$\begin{cases} V_{i,j} = \dfrac{\alpha}{d}\, U_{\mathbf{t}^{i,j}}, \qquad j = 0, 1, \ldots, \delta - 1, \\ V_{i,j} = \dfrac{\alpha}{d}\, U_{\mathbf{t}^{i,j}} + \dfrac{\beta}{d}\, U_{\mathbf{t}^{i,j-\delta}}, \qquad j = \delta, \delta + 1, \ldots, n, \end{cases} \tag{87}$$

where $\delta := \lfloor \tau/\Delta t \rfloor$ denotes the integer part of $\tau/\Delta t$, and α, β are real scalar parameters that set the control law as indicated by Theorems 6.1 or 6.4.

The numerical finite difference schemes for both the uncontrolled wave equation in $(\tilde{\mathcal{P}})$, and the closed-loop system, obtained by combining $(\tilde{\mathcal{P}})$ and $(\tilde{C})$, are now ready to be laid.

The constant parameters $\mu := \dfrac{c\,\Delta t}{\Delta x}$ and $\theta := \dfrac{\Delta t}{d}$ are introduced for the sequel.

9.2. Uncontrolled case

Concerning $(\tilde{\mathcal{P}})$ when $V(\mathbf{x},\mathbf{t}) \equiv 0$ (no applied control, i.e., $\alpha = \beta = 0$), the finite difference scheme reads as

$$\begin{cases} U_{\mathbf{tt}^{i,j}} = c^2\, U_{\mathbf{xx}^{i,j}}, \quad i = 0,1,2,\ldots,N, \quad \forall j \in \mathbb{N}, \\ U_{0,j} = U_{N,j} = 0, \quad \forall j \in \mathbb{N}, \\ U_{i,0} = f(i\,\Delta x), \quad i = 0,1,2,\ldots,N, \quad f(0) = f(l) = 0, \quad f''(0) = f''(l) = 0, \\ U_{t^{i,0}} = g(i\,\Delta x), \quad i = 0,1,2,\ldots,N, \quad g(0) = g(l) = 0, \end{cases} \tag{88}$$

leading to the following implicit scheme:

$$\begin{cases} -\mu^2\, U_{i+1,j} + (1 + 2\,\mu^2)\, U_{i,j} - \mu^2\, U_{i-1,j} = 2\, U_{i,j-1} - U_{i,j-2}, \\ \qquad\qquad i = 1,2,\ldots,N-1, \quad j \in \mathbb{N}\setminus\{0,1\}, \\ U_{i,0} = f(i\,\Delta x), \quad i = 0,1,\ldots,N, \\ U_{i,1} = U_{i,0} + \Delta t\, U_{\mathbf{t}^{i,0}}, \\ \quad = f(i\,\Delta x) + \Delta t\, g(i\,\Delta x), \quad i = 0,1,\ldots,N, \\ U_{0,j} = U_{N,j} = 0, \quad j \in \mathbb{N}. \end{cases} \tag{89}$$

9.3. Closed-loop case

When α, β, and τ in $(\tilde{C})$ are set following Theorems 6.1 or 6.4, the finite difference scheme of the closed-loop system reads as

$$\begin{cases} U_{\mathbf{tt}^{i,j}} - c^2\, U_{\mathbf{xx}^{i,j}} = -V_{i,j}, \\ V_{i,j} = \dfrac{\alpha}{d}\, U_{\mathbf{t}^{i,j}} + \dfrac{\beta}{d}\, U_{\mathbf{t}^{i,j-\delta}}, \quad i = 0,1,\ldots,N, \quad j \in \mathbb{N}, \\ U_{0,j} = U_{N,j} = 0, \quad \forall j \in \mathbb{N}, \\ U_{i,0} = f(i\,\Delta x), \quad i = 0,1,2,\ldots,N, \quad f(0) = f(l) = 0, \quad f''(0) = f''(l) = 0, \\ U_{t^{i,0}} = g(i\,\Delta x), \quad i = 0,1,2,\ldots,N, \quad g(0) = g(l) = 0, \end{cases}$$

where, for all $i = 0,1,\ldots,N$, $j \in \mathbb{N}$, $U_{\mathbf{t}^{i,j-\delta}} = 0$ for any $\delta \in \mathbb{N}$ such that $j - \delta < 0$. It becomes

$$\begin{cases} -\mu^2\, U_{i+1,j} + \left(1 + 2\,\mu^2 + \dfrac{3}{2}\theta\alpha\right) U_{i,j} - \mu^2\, U_{i-1,j} = 2\,(1 + \theta\alpha)\; U_{i,j-1} \\ \qquad - \left(1 + \dfrac{1}{2}\theta\alpha\right) U_{i,j-2} - \dfrac{3}{2}\theta\beta\, U_{i,j-\delta} + 2\theta\beta\, U_{i,j-\delta-1} - \dfrac{1}{2}\theta\beta\, U_{i,j-\delta-2}, \\ \qquad\qquad i = 1,2,\ldots,N-1, \quad j \in \mathbb{N}\setminus\{0,1\}, \\ U_{i,0} = f(i\,\Delta x), \quad i = 0,1,\ldots,N, \\ U_{i,1} = U_{i,0} + \Delta t\, U_{\mathbf{t}^{i,0}}, \\ \quad = f(i\,\Delta x) + \Delta t\, g(i\,\Delta x), \quad i = 0,1,\ldots,N, \\ U_{0,j} = U_{N,j} = 0, \quad j \in \mathbb{N}, \\ \quad \text{where, for all } i, j \in \mathbb{N},\ U_{i,j-\delta} = 0 \text{ for any } \delta \in \mathbb{N} \text{ such that } j - \delta < 0. \end{cases} \tag{90}$$

This numerical scheme is implicit. However, since it is linear, it can easily be transformed into an explicit numerical scheme and written in a matricial form. The study of stability, consistency, and convergence of both numerical schemes is not presented for brevity.

9.4. Energy function

For illustration purposes, let us consider the energy function representing the total energy of the string. It is given by

$$\begin{aligned} E(t) &:= \frac{1}{2}\int_0^{\ell} \left(|u_t(x,t)|^2 + |u_x(x,t)|^2\right) dx, \\ &= \frac{1}{2}\frac{d}{c}\int_0^{l} \left(|U_{\mathbf{t}}(\mathbf{x},\mathbf{t})|^2 + c^2|U_{\mathbf{x}}(\mathbf{x},\mathbf{t})|^2\right) d\mathbf{x}. \end{aligned} \tag{91}$$

Using a forward Euler approximation, this function is approximated by

$$E(j\,\Delta t) \simeq \frac{\Delta x}{2}\frac{d}{c}\sum_{i=0}^{N-1} {U_{\mathbf{t}^{i,j}}}^2 + c^2\,{U_{\mathbf{x}^{i,j}}}^2, \qquad j \in \mathbb{N}. \tag{92}$$

The energy function in (91) is computed using the previous numerical schemes, for both the open-loop and the closed-loop cases. For $j \in \mathbb{N}\setminus\{0,1\}$, given the boundary conditions in $(\tilde{\mathcal{P}})$, this function is computed as

$$E(j\,\Delta t) \simeq \frac{1}{8\,\mu\,\theta}\sum_{i=0}^{N-1}\left(3\,U_{i,j} - 4\,U_{i,j-1} + U_{i,j-2}\right)^2 + \frac{\mu}{2\,\theta}\,{U_{1,j}}^2 + \frac{\mu}{8\,\theta}\sum_{i=1}^{N-1}\left(U_{i+1,j} - U_{i-1,j}\right)^2. \tag{93}$$

Moreover, for $j = 0$ and $j = 1$, and considering (85), we get

$$E(0) \simeq \frac{d\,\Delta x}{2\,c}\sum_{i=0}^{N-1} g(i\,\Delta x)^2 + c^2\,f'(i\,\Delta x)^2, \tag{94}$$

$$\begin{aligned} E(1\,\Delta t) \simeq{}& \frac{d\,\Delta x}{2\,c}\sum_{i=0}^{N-1}\left(g(i\,\Delta x) + \Delta t\,c^2\,f''(i\,\Delta x)\right)^2 + \frac{\mu}{2\,\theta}\,{U_{1,1}}^2 \\ &+ \frac{\mu}{8\,\theta}\sum_{i=1}^{N-1}\left(f\left((i+1)\,\Delta x\right) - f\left((i-1)\,\Delta x\right) + \Delta t\,g\left((i+1)\,\Delta x\right) - g\left((i-1)\,\Delta x\right)\right)^2. \end{aligned} \tag{95}$$

9.5. Numerical simulations

Let us now consider the numerical values of Table 1 for the considered string of problem $(\tilde{\mathcal{P}})$. To ensure stability and convergence of both numerical schemes (89) and (90), we set $\Delta x = 0.05$, $\Delta t = 0.005$, and $T_f = 100$. To cope with all the requirements described in 9.1, the initial conditions are chosen as follows:

$$\begin{aligned} f(x) &:= A\,\sin\left(\frac{n\,\pi}{l}x\right), \\ g(x) &\equiv 0. \end{aligned}$$

Table 1. Features of the string.

Length (m) (with $\ell = 1$)	l	10
Wave propagation speed (m/s)	c	1.118
Time scaling factor (s)	d	8.9443
Wave magnitude	A	1

Two values of n have been considered to emphasize the results of both Theorems 6.1 and 6.4: $n = 1$ and $n = 3$, where the corresponding simulations are reported in Figure. 4 and Figure. 5, respectively. For those values, the time responses of the autonomous system to the initial conditions above are shown in Figure. 4(a) & 5(a), respectively. For the closed-loop behavior, the time responses are simulated for both main results. The computed parameters of the control law ($\tilde{C}$) are summarized as follows for each case:

Case 1 (Theorem 6.1) $n = 1, \alpha = 0, \beta \simeq 4.3079, \tau \simeq 0.095583$ giving $s_1 \simeq -6.4659$.

Case 2 (Theorem 6.4) $n = 1, \alpha = 6.0793, \beta \simeq 0.0071129, \tau = 1.0$ giving $s_1 \simeq -2.21$.

Case 3 (Theorem 6.1) $n = 3, \alpha = 0, \beta \simeq 12.924, \tau \simeq 0.031861$ giving $s_3 \simeq -19.398$.

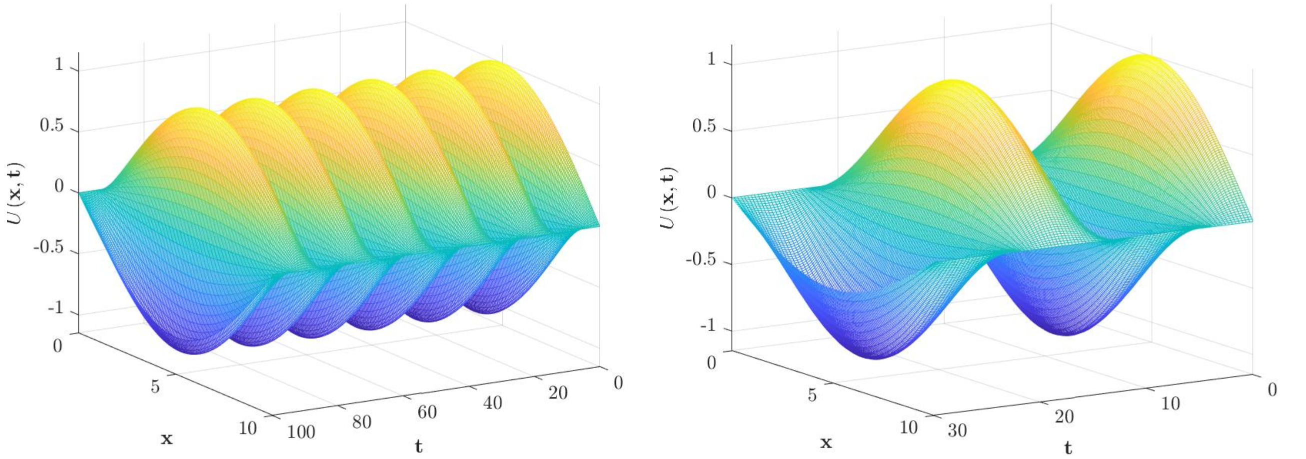
Case 4 (Theorem 6.4) $n = 3, \alpha = 18.761, \beta \simeq 5.473 \cdot 10^{-6}, \tau = 1.0$ giving $s_3 \simeq -8.4619$.

The numerical simulations illustrate the exponential decay of the closed-loop system solutions for both types of settings, that is, with the control parameters α, β, and τ set according to Theorem 6.1 and Theorem 6.4.

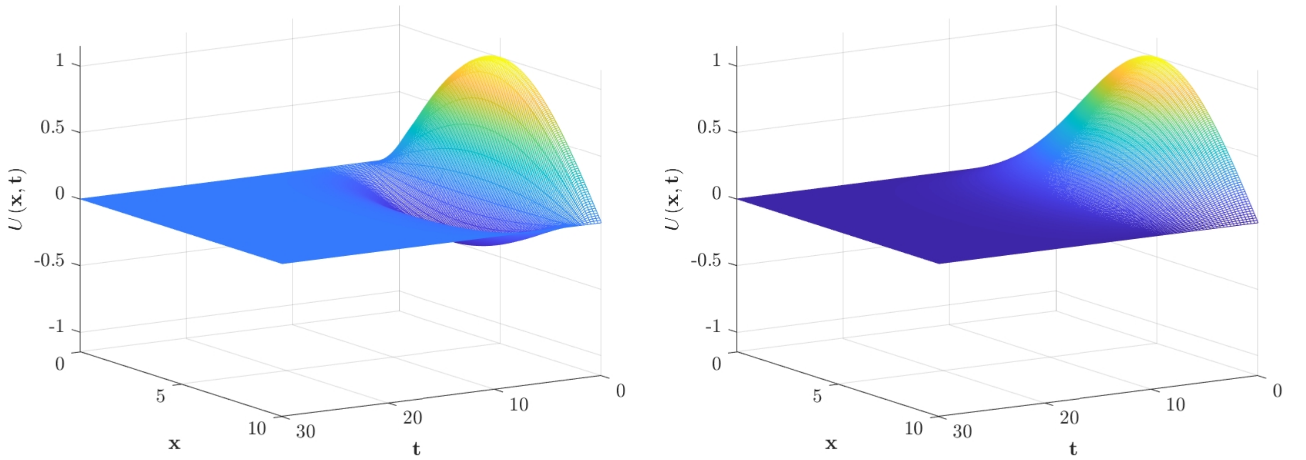
Recall that Theorem 6.4 gives a delay-independent stability property while Theorem 6.1 leads to a delay-dependent one with $\alpha = 0$. The latter leads to closed-loop dynamics that are more sensitive to small variations affecting the system's parameters, for example, because of parametric uncertainties. It is also sensitive to small variations impacting the controller parameters, for example, due to round-off errors during the numerical implementation. In such a situation, this small variation can cause the closed-loop spectral roots, not only the real dominant one but also the complex-conjugate pairs, to migrate toward the imaginary axis, resulting in the apparition of damped oscillating modes one can observe on the time response numerical simulations. This is what Case 1 (Figures 4(b) & 4(d)) and Case 3 (Figures 5(b) & 5(d)) illustrate.

On the other hand, the settings due to Theorem 6.4, with $\alpha > \beta$, give a slower closed-loop dynamics but are however more robust to round-off errors affecting the control parameters, especially the delay τ that is quite larger than in the previous case, or round-off propagating errors due to the inner properties of the used numerical scheme. This is what Case 2 (Figures 4(c) & 4(e)) and Case 4 (Figures 5(c) & 5(e)) aim to illustrate.

These observations are confirmed by looking at the numerical values of the control parameters for the four proposed cases. A large delay in the control law results in a low value of β weighting the delayed measurement, leaving the instantaneous measurement $u_t(x, t)$ having a predominant effect on the closed-loop system. Contrarily, when α is enforced to zero, the resulting delay τ is very small compared to the previous one.

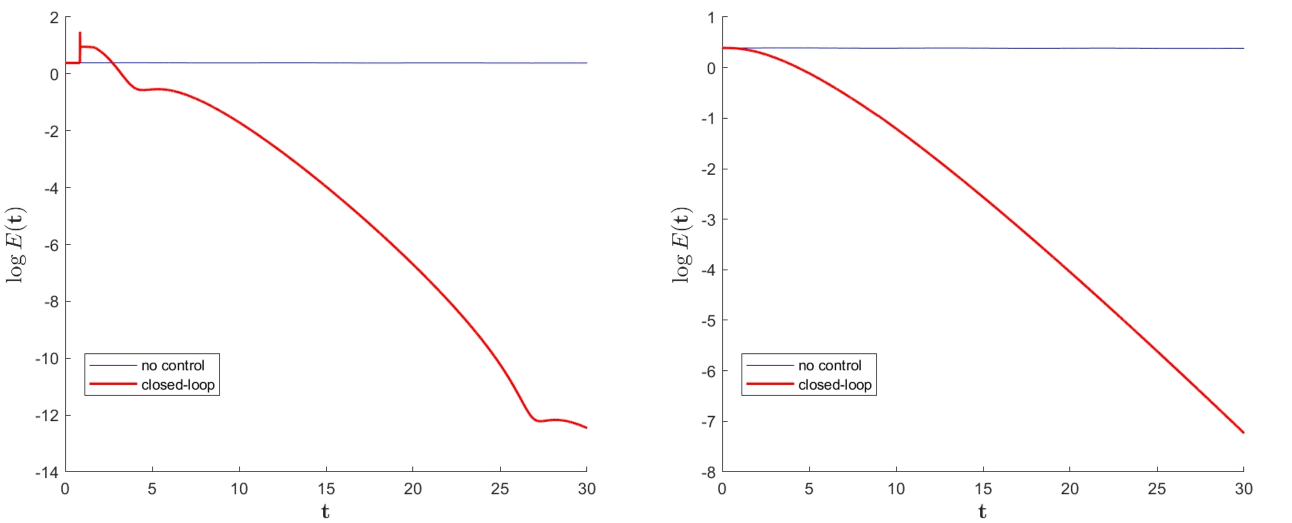


(a) Without control with a focus on the right.



(b) Closed-loop control – case 1.

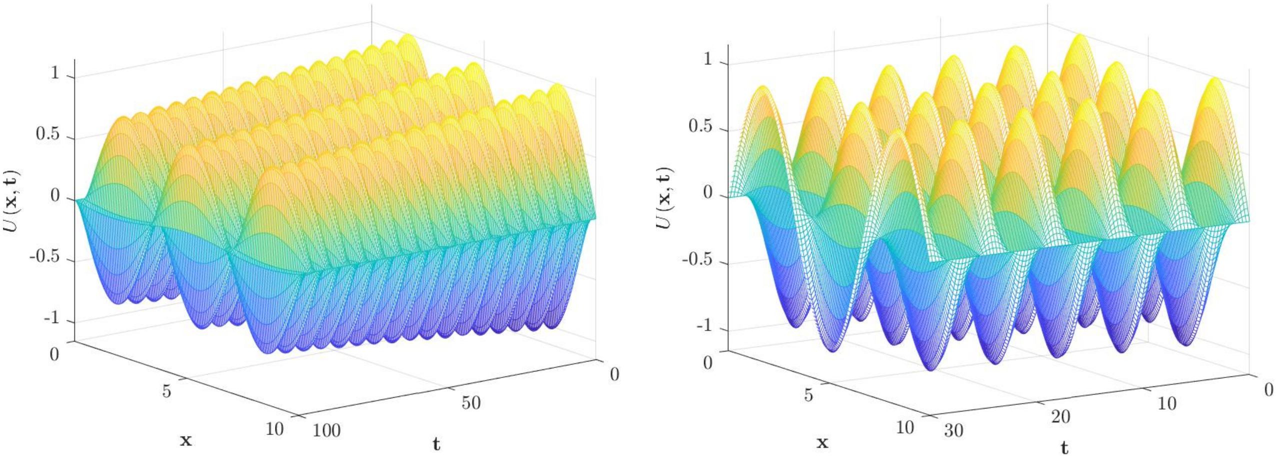
(c) Closed-loop control – case 2.



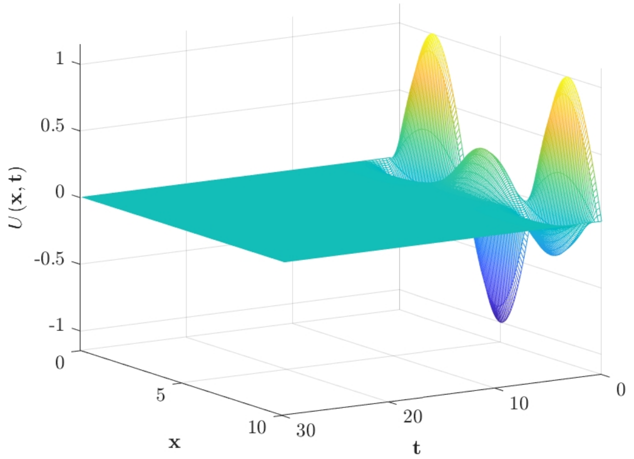
(d) Energy function – case 1.

(e) Energy function – case 2.

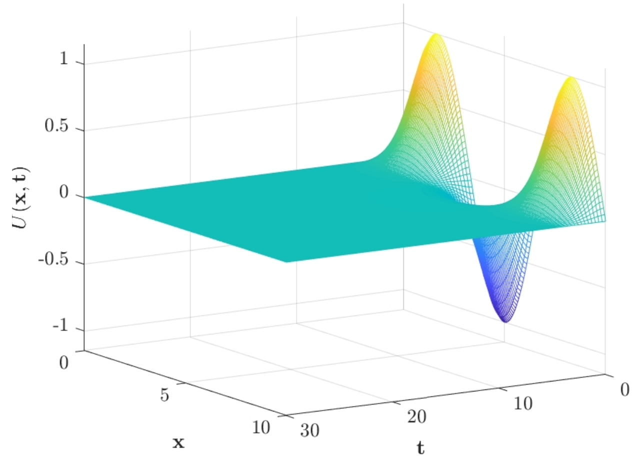
Figure 4. Time simulations for $n = 1$.



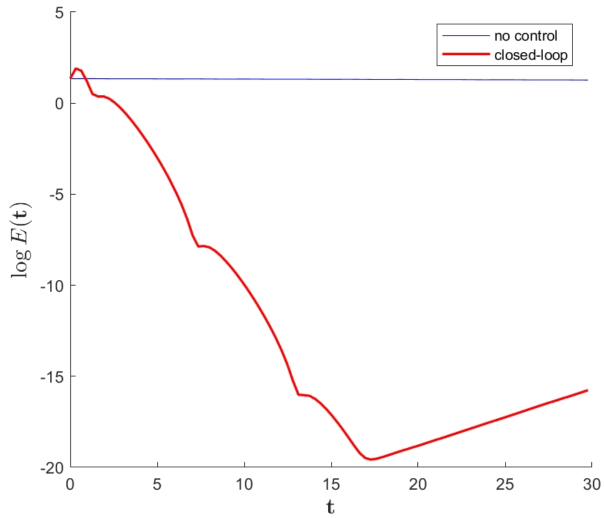
(a) Without control with a focus on the right.



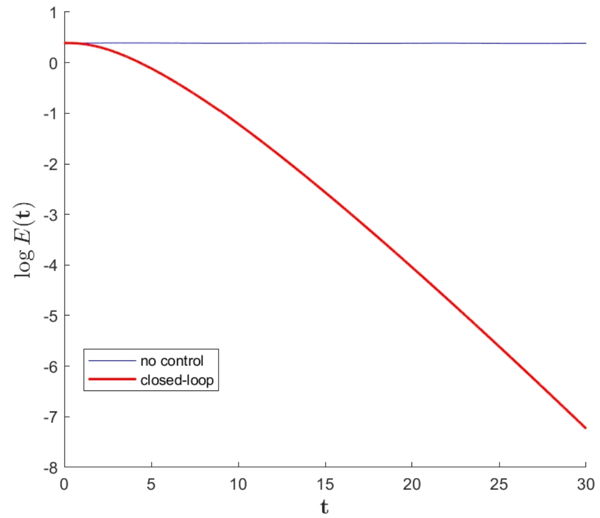
(b) Closed-loop control – case 3.



(c) Closed-loop control – case 4.



(d) Energy function – case 3.



(e) Energy function – case 4.

Figure 5. Time simulations for $n = 3$.

Author contributions

Kaïs Ammari was involved in conceptualization, methodology, and writing the original draft. Islam Boussaada was involved in conceptualization, methodology, software, and writing the original draft. Silviu-Iulian Niculescu was involved in methodology and writing the original draft. Sami Tliba was involved in methodology, software, and writing the original draft.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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