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Research article

Null controllability of a coupled nonlinear parabolic system

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Abstract: This paper deals with the null controllability of a coupled nonlinear parabolic system. The coefficients of the system operators depend on the states, and the control acts through the first equation. To solve the control problem of the linearized system, we use maximum regularity results in the spaces $L^q(0,T,L^p(\Omega))$, and we use Liusternick's inverse function theorem for the nonlinear controllability problem. In addition, as an appendix, we prove the well-posedness of the system.

Keywords: Quasi-linear parabolic problem; controllability; strong solution to PDE's
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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \leq 3$, be an open bounded with a boundary $\partial\Omega$ in the class C^2, $T > 0$, $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. Consider the following thermodiffusion system:

$$\left| \begin{array}{lll} y_t - \nabla(a(y,z)\nabla y) + f(y,z) = v\chi_\omega & \text{in} & Q, \\ z_t - \nabla(b(y,z)\nabla z) + g(y,z) = 0 & \text{in} & Q, \\ y = z = 0 & \text{on} & \Sigma, \\ y(0) = y_0,\ z(0) = z_0, & \text{on} & \Omega, \end{array} \right. \tag{1.1}$$

where y represents the temperature of a body, z is its chemical potential, v is the control function that acts on the first equation, and $a(y,z)$ and $b(y,z)$ are the diffusion coefficients (in this case see [1]). Additionally, the reaction terms f, g depend on the state, which turns (1.1) into a semilinear parabolic system. Consider $y_0, z_0 \in W_0^{1,p}(\Omega)$ with $p > 3$, $a, b \in C_b^2(\mathbb{R}^2)$ with $0 < a_0 < a(r,s)$, $0 < b_0 < b(r,s)$, $f, g \in C_b^1(\mathbb{R}^2)$ with $g_r(0,0) \neq 0$.

In this work, we will prove the existence of a local null control for system (1.1) when the initial data $y_0, z_0 \in W_0^{1,p}(\Omega)$ with $3 < p \leq 6$, thereby applying Liusternick's right inverse function theorem.

This work is a natural extension for a equation such as the following:

$$\begin{vmatrix} y_t - \nabla(a(y)\nabla y) + f(y) = v\chi_\omega & \text{in} & Q, \\ y = 0 & \text{on} & \Sigma, \\ y(0) = y_0, & \text{on} & \Omega, \end{vmatrix} \tag{1.2}$$

with $y_0 \in W_0^{1,p}(\Omega)$, $3 < p \leq 6$ (recently proved by Manghi et al. in [2]). Then, this work improves the results for equation (1.2) proved by Fernández-Cara et al. in [3], where the authors considered $H_0^1(\Omega) \cap H^3(\Omega)$, since $H_0^1(\Omega) \cap H^3(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$. Such an improvement for the initial data is due to differences in the approach.

In [3], Fernández-Cara et al. studied the regularity of the associated linear system through additional estimates, where the authors used derivatives for this system, which increased the regularity of the initial data. In our work and in [2], the regularity of the linear system is obtained using results that invoke maximum regularity in spaces $L^q(0, T; L^p(\Omega))$, without using derivatives on this system. Moreover, this kind of system has applications in Physics, Engineering and Biology.

For the controllability of system (1.2) with only one equation in the one-dimensional case and data in $H_0^1(\Omega) \cap H^2(\Omega)$, we can refer to [4]. In the case of controllability in dimension 2 and 3 and initial data in $H_0^1(\Omega) \cap H^3(\Omega)$, we can refer to [3]. For the controllability of system (1.1) with non-local terms, that is, when $a = a\Big(\int_\Omega y, \int_\Omega z\Big)$, $b = b\Big(\int_\Omega y, \int_\Omega z\Big)$, we can refer to [5]. Lastly, for the case of (1.1) with only one equation and
$a = a(y, \nabla y)$, we can refer to [6, 7]. Both system (1.1) with these non-local terms and system (1.2) can be studied following ideas from our work, thereby considering initial data in spaces with less regularity.

In general, there is a large bibliography for the controllability of quasilinear and semilinear parabolic systems. Among them, we can refer to [8–12]. The following bibliographies were useful for the well-posedness of (1.1) (proved in the appendix): [6, 13–17].

2. Controllability of the linearized system

In this section, we will study the controllability of the following linearized system:

$$\begin{vmatrix} y_t - a(0,0)\Delta y + f_r(0,0)y + f_s(0,0)z = v\chi_\omega + h & \text{in} & Q, \\ z_t - b(0,0)\Delta z + g_r(0,0)y + g_s(0,0)z = k & \text{in} & Q, \\ y = z = 0, & \text{on} & \Sigma, \\ y(0) = y_0, z(0) = z_0 & \text{on} & \Omega, \end{vmatrix} \tag{2.1}$$

which can be rewritten in the following equivalent form:

$$\begin{vmatrix} y_t - \alpha\Delta y + A_1 y + A_2 z = v\chi_\omega + h & \text{in} & Q, \\ z_t - \beta\Delta z + B_1 y + B_2 z = k & \text{in} & Q, \\ y = z = 0 & \text{on} & \Sigma, \\ y(0) = y_0, z(0) = z_0 & \text{on} & \Omega. \end{vmatrix} \tag{2.2}$$

The adjoint system of (2.2) is given by the following:

$$\left|\begin{array}{lll} -\varphi_t - \alpha\Delta\varphi + A_1\varphi + B_1\psi = F_1 & \text{in} & Q, \\ -\psi_t - \beta\Delta\psi + A_2\varphi + \beta_2\psi = F_2 & \text{in} & Q, \\ \varphi = \psi = 0 & \text{on} & \Sigma, \\ \varphi(T) = \varphi^T, \psi(T) = \psi^T & \text{on} & \Omega \end{array}\right. \tag{2.3}$$

We will prove a Carleman inequality for System (2.3). For this, we need to establish the function α_0 given by the following lemma.

Lemma 2.1. *There is a function $\alpha_0 \in C^2(\bar{\Omega})$ which satisfies $\alpha_0 > 0$ in Ω, $\alpha_0 = 0$ in $\partial\Omega$ and $|\nabla\alpha_0| > 0$ in $\overline{\Omega} - \omega_0$.*

The proof of this lemma can be found in [18].

Let $m \in C^\infty([0,T])$ with $m(t) \geq T^2/8$ in $[0, T/2]$ and $m(t) = t(T-t)$ in $[T/2, T]$. We define the following:

$$\phi(x,t) = \frac{e^{\lambda\alpha_0(x)}}{m(t)} \;, \alpha(x,t) = \frac{e^{R\lambda} - e^{\lambda\alpha_0}}{m(t)} = \frac{\bar{\alpha}(x)}{m(t)}$$

with $R > \|\alpha_0\|_\infty + \ln(2)$, $\lambda > 0$.

Next, we will use the following notation to the operator $I(s,\lambda,z)$:

$$I(s,\lambda,z) = \int_Q e^{-2s\alpha}[(s\phi)^{-1}(|z_t|^2 + |\Delta z|^2) + \lambda^2 s\phi|\nabla z|^2 + \lambda^4(s\phi)^3|z|^2]$$

Now, let us consider the following Carleman Inequality, which was proved in [18].

Proposition 2.2. *Consider $F_1, F_2 \in L^2(Q)$, and $\varphi^T, \psi^T \in L^2(\Omega)$. If $B_1 \neq 0$, then there exist constants $\lambda_0 = \lambda_0(\Omega,\omega)$, $C_0 = C_0(\Omega,\omega)$ and $S_0 = S_0(\Omega,\omega,\alpha,\beta,C_1,1/B_1)$ such that the solution of (2.3) satisfies the following:*

$$I(s,\lambda,\varphi) + I(s,\lambda,\psi) \leq C_0\Big(\int_Q e^{-2s\alpha}[\lambda^4(s\phi)^3|F_1|^2 + |F_2|^2] + \int_0^T\int_\omega e^{-2s\alpha}\lambda^8(s\phi)^7|\varphi|^2\Big)$$

for all $\lambda \geq \lambda_0$, $s \geq s_0(T + T^2)$.

Define, $\alpha_1 = \min\limits_{x\in\bar{\Omega}} \bar{\alpha}(x)$ and $\alpha_2 = \max\limits_{x\in\bar{\Omega}} \bar{\alpha}(x)$.

From the definition of R, we deduce that $2\alpha_1 \geq \alpha_2$; thus,

$$e^{\frac{s\alpha_1}{m}} < e^{\frac{s\alpha}{m}} \leq e^{\frac{s\alpha_2}{m}} \leq e^{\frac{2s\alpha_1}{m}}$$

We will denote $\rho_k = e^{s\alpha}m^{\frac{k}{2}}$.

Note that, from the hypotheses, we have $B_1 = g_r(0,0) \neq 0$.

Theorem 2.3. ***(Controllability of the linear system)*** *Consider $3 < p \leq 6$ and $p < q < \infty$. Let us assume that the functions h, k satisfy $\rho_3 h, \rho_3 k \in L^q(0,T,L^p(\Omega))$. Then, (2.3) is null controllable at time $T > 0$, that is, for each $y_0, z_0 \in W_0^{\cdot p}(\Omega)$, there exist null control $v \in L^2(\omega \times (0,T))$ and associated states that satisfy the following:*

$$
\begin{aligned}
|\hat{\rho}_{19}y|_{L^q(0,T,W^{2,p}(\Omega)} + |\hat{\rho}_{19}z|_{L^q(0,T,W^{2,p}(\Omega))} + |\hat{\rho}_{19}y|_{C^0(0,T,W^{1,p}(\Omega))} \\
+ |\hat{\rho}_{19}z|_{C^0([0,T];W^{1,p}(\Omega))} + |(\hat{\rho}_{19}y)_t|_{L^q(0,T,L^p(\Omega))} + |(\hat{\rho}_{19}z)_t|_{L^q(0,T,L^p(\Omega))} \\
\leq |\rho_3 h|_{L^q(0,T,L^p(\Omega))} + |\rho_0 k|_{L^q(0,T,L^p(\Omega))} + |y_0|_{W_0^{1,p}(\Omega)} + |z_0|_{W_0^{1,p}(\Omega)}.
\end{aligned}
\tag{2.4}
$$

Next, we will prove three lemmas that will be used to prove Theorem 2.3.

Lemma 2.4. *Under the conditions of the Theorem 2.3, there exist $v \in L^2(\omega \times (0,T))$, (y,z), and an associated solution of* (2.2) *that satisfies the following:*

$$
\int_Q \rho_3^2|y|^2 + \rho_0^2|z|^2 dxdt + \int_0^T \int_\omega \rho_7^2|v|^2 \leq c\Big(|y_0|^2 + |z_0|^2 + \int_Q \rho_3^2|h|^2 + \rho_3^2|k|^2\Big).
$$

Proof. Let us define $P_0 = \{(\varphi_1,\varphi_2) \in [C^2(Q)]^2 : \varphi_1 = \varphi_2 = 0 \text{ in } (0,T) \times \partial\Omega\}$. Let $\omega \subset \omega_1$ and $\chi_{\omega_1} \in C_0^\infty(\omega_1)$ such that $0 \leq \chi_{\omega_1} \leq 1, \chi_{\omega_1} = 1$ in ω. Consider the bilinear form as follows:

$$
\begin{aligned}
\theta((\varphi_1,\varphi_2),(\tilde{\varphi}_1,\tilde{\varphi}_2)) &= \int_Q \rho_3^{-2} L_1^*(\varphi_1,\varphi_2) L_1^*(\tilde{\varphi}_1,\tilde{\varphi}_2) dxdt \\
&\quad + \int_Q \rho_0^{-2} L_2^*(\varphi_1,\varphi_2) L_2^*(\tilde{\varphi}_1,\tilde{\varphi}_2) dxdt + \int_0^T \int_\omega \chi_{\omega_1} \rho_7^{-2} \varphi_1 \tilde{\varphi}_1 dxdt
\end{aligned}
$$

where

$$
\begin{aligned}
L_1^*(\varphi_1,\varphi_2) &= -\varphi_{1t} - \alpha\Delta\varphi_1 + A_1\varphi_1 + B_1\varphi_2, \\
L_2^*(\varphi_1,\varphi_2) &= -\varphi_{2t} - \beta\Delta\varphi_2 + A_2\varphi_1 + B_2\varphi_2.
\end{aligned}
$$

From Carleman's inequality, we conclude that $\theta(\cdot,\cdot)$ is an inner product in P_0. Thus, P is considered as the completion of P_0 with the inner product $\theta(\cdot,\cdot)$, that is,

$$
P = \bar{P}_0^{\theta(\cdot,\cdot)}.
$$

Additionally, define $S : P \to \mathbb{R}$ by the foillowing:

$$
S(\varphi_1,\varphi_2) = (y_0, \varphi_1(0)) + (z_0, \varphi_2(0)) + \int_Q h\varphi_1 + k\varphi_2 \, dxdt
$$

Using Carleman's inequality, it is proven that S is bounded, and since θ is coercive in P with the inner product generated by θ, then from Lax-Milgram theorem, it can be verified that there exists a unique $\varphi = (\varphi_1,\varphi_2) \in P$ solution of

$$
\theta(\varphi,\omega) = S(\omega), \quad \forall \omega \in P. \tag{2.5}
$$

Define $y = -\rho_3^{-2} L_1^*\varphi$, $z = -\rho_0^{-2} L_2^*\varphi$, $\bar{v} = -\rho_7^{-2}\varphi_1$ and $v = \bar{v}\,|_{(0,T)\times\omega}$; from (2.5) with $\omega = \varphi$, we obtain the following:

$$
\int_Q \rho_3^2|y|^2 + \rho_0^2|z|^2 dxdt + \int_0^T \int_\omega \rho_7^2|v|^2 \leq c\Big(|y_0|^2 + |z_0|^2 + \int_Q \rho_3^2|h|^2 + \rho_3^2|k|^2\Big)
$$

□

Lemma 2.5. *The control function* v *and the associated state* (y,z) *have the following regularity:* $\rho_{15}v \in L^2(0,T,H^2(\omega))$*,* $(\rho_{15}v)_t \in L^2(Q)$*,* $\rho_{15}v \in C^0(0,T,H^1(\omega))$*.*

Proof. It is verified that

$$\rho_{15}\bar{v} = -\rho_{15}\rho_7^{-2}\varphi_1 = -e^{-s\alpha}m^{\frac{1}{2}} = -(\rho_{-1})^{-1}\varphi_1 = -\rho_{-1}^{-1}\varphi_1.$$

Consequently,

$$\begin{aligned}(\rho_{15}\bar{v})_t + \alpha\Delta(\rho_{15}\bar{v}) &= [(\rho_{-1}^{-1})_t + \alpha\Delta(\rho_{-1}^{-1})]\varphi_1 - \rho_{-1}^{-1}(\varphi_{1t} + \alpha\Delta\varphi_1) - 2\alpha\nabla(\rho_1^{-1})\nabla\varphi_1 \\ &= \rho_{-1}^{-1}L_1^*\varphi - A_1\rho_{-1}^{-1}\varphi_1 - B_1\rho_{-1}^{-1}\varphi_2 - (\rho_{-1}^{-1})_t\varphi_1 - \alpha\Delta(\rho_{-1}^{-1})\varphi_1 - 2\alpha\nabla(\rho_{-1}^{-1})\nabla\varphi_1 \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.\end{aligned}$$

Additionally, we have the following inequality:

$$|I_1| = |\rho_{-1}^{-1}\rho_3^2 y| = \rho_7|y| \le c\rho_3|y|.$$

Thus, $|I_1| \in L^2(Q)$.

Applying Calerman's inequality to systems, one has the following:

$$L_1^*\varphi = -\rho_3^2 y \text{ and } L_2^*\varphi = -\rho_0^2 z.$$

Additionally, we have the following:

$$\int_Q \rho_1^{-2}(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2) + \rho_3^{-2}(|\varphi_1|^2 + |\varphi_2|^2) \le c\int_Q \rho_3^2|y|^2 + \rho_0^2|z|^2 dtdx + \int_{\omega\times(0,T)} \rho_7^2|v|^2$$

From the last inequality, we obtain the following:

$$\begin{aligned}|I_2| &\le c\rho_{-1}^{-1}|\varphi_1| \le c\rho_3^{-1}|\varphi_1| \\ |I_3| &\le c\rho_{-1}^{-1}|\varphi_2| \le c\rho_3^{-1}|\varphi_2| \\ |I_4| &\le c|(\rho_{-1}^{-1})_t\varphi_1| \le c\rho_3^{-1}|\varphi_1| \\ |I_5| &\le c|\Delta(\rho_{-1}^{-1})\varphi_1| \le c\rho_3^{-1}|\varphi_1| \\ |I_6| &\le c|\nabla(\rho_{-1}^{-1})\nabla\varphi_1| \le c\rho_1^{-1}|\nabla\varphi_1|.\end{aligned}$$

Then, I_2, I_3, I_4, I_5 and I_6 belongs to $L^2(Q)$.

Furthermore, $\rho_{15}\bar{v}(T) = -\rho_{-1}^{-1}\varphi_1(T) = 0$.

From a parabolic regularity, we have $\rho_{15}\bar{v} \in L^2(0,T,H_0^1(\Omega) \cap H^2(\Omega))$, $(\rho_{15}\bar{v})_t \in L^2(Q)$, $\rho_{15}\bar{v} \in c^0(0,T,H_0^1(\Omega))$ and then $\rho_{15}v \in L^2(0,T,H^2(\omega))$, $(\rho_{15}v)_t \in L^2(Q)$, $\rho_{15}v \in c^0(0,T,H^1(\omega))$.

Following the same ideas as Propositions 2.5 and 2.6 in [3], we have the following Lemma.

Lemma 2.6. *Under the hypotheses of Theorem 2.3, the associated state* (y,z) *with the control function* v *of* (2.2) *satisfies the following:*

$$\sup_{[0,T]}\int_\Omega \rho_5^2(|y|^2 + |z|^2) + \rho_7^2(|\nabla y|^2 + |\nabla z|^2)dx + \int_Q \rho_7^2(|y_t^2 + |z_t|^2 + |\Delta y|^2 + |\Delta z|^2)dxdt$$
$$\le c(\|y_0\|^2 + \|z_0\|^2 + \int_Q \rho_3^2(|h|^2 + |k|^2)dxdt).$$

Proof of Theorem 2.3

Consider $\hat{\rho}_{19} = e^{\frac{s\alpha_1}{m(t)}} m(t)^{\frac{19}{2}}$. From (2.2), we have the following:

$$\begin{aligned} &(\hat{\rho}_{19}y)_t - \alpha\Delta(\hat{\rho}_{19}y) + A_1(\hat{\rho}_{19}y) = \hat{\rho}_{19}v\chi_\omega + \hat{\rho}_{19}h + \hat{\rho}_{19t}y - A_2\rho_{19}z = g_1 \\ &(\hat{\rho}_{19}z)_t - \beta\Delta(\hat{\rho}_{19}z) + B_2(\hat{\rho}_{19}z) = \hat{\rho}_{19}k + \hat{\rho}_{19t}z - B_1\hat{\rho}_{19}y = g_2 \end{aligned} \tag{2.6}$$

Let's analyze g_1 and g_2.

Indeed, from Lemma 2.5, we have that $\rho_{15}v \in C^0(0, T, H^1(\omega))$, where

$$\hat{\rho}_{19}v \in C^0(0, T, H^1(\omega)) \hookrightarrow L^q(0, T, L^p(\Omega)) \tag{2.7}$$

for $1 \leq q \leq \infty$ and $1 \leq p \leq 6$.

Additionally, from Lemma 2.6, we have $\rho_7 y, \rho_7 z \in L^\infty(0, T, H_0^1(\Omega))$.

As $|\hat{\rho}_{19t}| \leq c|\rho_{15}| \leq c_1|\rho_7|$ and $|\hat{\rho}_{19}| \leq c|\rho_7|$, then we obtain the following:

$$\hat{\rho}^{19t}y, A_2\hat{\rho}_{19}z, \hat{\rho}_{19t}z, B_1\hat{\rho}_{19}y \in L^\infty(0, T, H_0^1(\Omega)) \hookrightarrow L^q(0, T, L^p(\Omega)),$$

for $1 \leq q \leq \infty$, $1 \leq p \leq 6$. Thus, $g_1, g_2 \in L^q(0, T, L^p(\Omega))$ for $1 \leq q \leq \infty$, $1 \leq p \leq 6$.

Therefore, from the maximum regularity results for parabolic equations, one has:

$$\hat{\rho}_{19}y, \hat{\rho}_{19}z \in L^q(0, T, W^{2,p}(\Omega))$$

and

$$(\hat{\rho}_{19}y)_t, (\hat{\rho}_{19}z)_t \in L^q(0, T, L^p(\Omega)),$$

for q, p determined by the hypothesis $\rho_3 h, \rho_3 k \in L^q(0, T, L^p(\Omega))$.

Furthermore,

$$\begin{aligned} &|\hat{\rho}_{19}y|_{L^q(0,T,W^{2,p}(\Omega))} + |\hat{\rho}_{19}z|_{L^q(0,T,W^{2,p}(\Omega)} + |(\hat{\rho}_{19}y)_t|_{L^q(0,T,L^p(\Omega))} + |(\hat{\rho}_{19}z)_t|_{L^q(0,T,L^p(\Omega))} \\ &\qquad \leq c(|g_1|_{L^q(0,T,L^p(\Omega))} + |g_2|_{L^q(0,T,L^p(\Omega))} + |y_0|_{W_0^{1,p}(\Omega)} + |z_0|_{W_0^{1,p}(\Omega)}). \end{aligned}$$

If we consider the hypothesis $3 < p < q < \infty$, from Proposition 4.1 in the appendix, we have that $\hat{\rho}_{19}y, \hat{\rho}_{19}z \in C^0(0, T, W^{1,p}(\Omega))$. However, to study the nonlinear problem, we will need the following immersion:

$$W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega),$$

which is valid for $p > 3$. Then, we complete the proof of the theorem. □

3. Nonlinear Problem

We can reduce the control problem associated with the nonlinear system (1.1) by solving the following abstract equation:

$$H(y, z, v) = (0, 0, y_0, z_0), \tag{3.1}$$

where $H : Y \to Z$ is a map between two Banach spaces, which are appropriate spaces with conveniently chosen weights. To solve problem (3.1), we will use the right inverse theorem , namely Liusternik's Theorem.

Theorem 3.1. *Let Y and Z be Banach spaces, B_r be an open ball centered at the origin in Y, and consider a C^1 mapping $H : B_r(0) \subset Y \to Z$. Assume that the derivative of H at the origin, $H'(0) : Y \to Z$ is onto and set $\xi_0 := H(0)$. Then, there exists constant $\epsilon > 0$, $\kappa > 0$ and a mapping $W : B_\epsilon(\xi_0) \subset Z \to Y$, with the subsequent properties holding for each $\xi \in B_\epsilon(\xi_0)$:*
a) *$W(\epsilon) \in B_r$;*
b) *$H(W(\epsilon)) = \xi$; and*
c) *$\|W(\epsilon)\|_y \leq \kappa\|\xi - H(0)\|_z$.*

Considering $3 < p \leq 6$ and $p < q < \infty$, we define the spaces of functions with weights by Y, F, Z as follows:

$Y = \{(y, z, v) : v \in L^2(\omega \times (0, T)), \rho_7 v \in L^2(\omega \times (0, T)), \rho_3\bar{h}, \rho_3\bar{k} \in L^q(0, T, L^p(\Omega))$ with $\bar{h} = y_t - a(0, 0)\Delta y + f_r(0, 0)y + f_s(0, 0)z - \chi_\omega v$, $\bar{k} = z_t - b(0, 0)\Delta z + g_r(0, 0)y + g_s(0, 0)z$, $\hat{\rho}_{19}y, \hat{\rho}_{19}z \in L^q(0, T, W^{2,p}(\Omega))$, $(\hat{\rho}_{19}y)_t, (\hat{\rho}_{19}z)_t \in L^q(0, T, L^p(\Omega), y(0), z(0) \in W_0^{1,p}(\Omega), y_{|\Sigma} = 0, z_{|\Sigma} = 0\}$, with the norm

$$\begin{aligned}\|(y, z, v)\|_Y = {} & \|\rho_7 v\|_{L^2(\omega\times(0,T))} + \|\rho_3\bar{h}\|_{L^q(0,T,L^p(\Omega))} + \|\rho_3\bar{k}\|_{L^q(0,T,L^p(\Omega))} + |\hat{\rho}_{19}y|_{L^q(0,T,W^{2,p}(\Omega))} \\ & + |\hat{\rho}_{19}z|_{L^q(0,T,W^{2,p}(\Omega))} + \|(\hat{\rho}_{19}y)_t\|_{L^q(0,T,L^p(\Omega))} + \|(\hat{\rho}_{19}z)_t\|_{L^q(0,T,L^p(\Omega))} \\ & + |y(0)|_{W_0^{1,p}(\Omega)} + |z(0)|_{W_0^{1,p}(\Omega)} ,\end{aligned}$$

$F = \{g : \rho_3 g \in L^q(0, T, L^p(\Omega))\}$ with the norm $\|g\|_F = \|\rho_3 g\|_{L^q(0,T,L^p(\Omega))}$, and we consider the product space $Z = F \times F \times W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ with the norm

$$\|(h, k, y_0, z_0)\|_Z = \|h\|_F + \|k\|_F + \|y_0\|_{W_0^{1,p}(\Omega)} + \|z_0\|_{W_0^{1,p}(\Omega)} .$$

We define the application $H : Y \to Z$ by the following:

$$\begin{aligned}H(y, z, v) &= (y_t - \nabla \cdot (a(y, z)\nabla y) + f(y, z) - v\chi_\omega, z_t - \nabla \cdot (b(y, z)\nabla z) + g(y, z), y(0), z(0)) \\ &= (H_1(y, z, v), H_2(y, z, v), H_3(y, z, v), H_4(y, z, v)).\end{aligned}$$

Remark 1: In the definition of the space Y, the conditions in $\rho_3\bar{h}$, $\rho_3\bar{k}$, $y(0)$, $z(0)$, $y_{|\Sigma}$ and $z_{|\Sigma}$ make the elements (y, z, v) of the space Y have the same regularity as the pair state-control (y, z, v) of the linear system from Theorem 2.3 , which verify the estimate (2.4).

Theorem 3.2. *If a, b, f, g satisfy the hypotheses given in the introduction, then there exists $\epsilon > 0$ such that if*

$$\|(y_0, z_0)\|_{W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)} < \epsilon,$$

then there exists a solution (y, z) of the nonlinear system (1.1) *which satisfies $y(T) = 0$, $z(T) = 0$.*

Proof. To prove Theorem 3.2, we will use Theorem 3.1 to prove that the abstract equation (3.1) has a solution. However, to do this, we need to prove the following three lemmas.

Lemma 3.3. *Let $H : Y \to Z$ be the application defined above. Then, H is well defined and continuous.*

Proof. We will prove that $H_1(y,z,v) \in F$. First, we have the following:

$$\begin{aligned}|H_1(y,z,v)|_F^q &= \int_0^T |\rho_3(y_t - \nabla\cdot(a(y,z)) + f(y,z) - v\chi_\omega|^q_{L^p(\omega)} \\ &\le c\Big(\int_0^T |\rho_3(y_t - a(0,0)\Delta y + f_r(0,0)y + f_s(0,0)z - v\chi_\omega|^q_{L^p(\Omega)}\Big) \\ &\quad + \int_0^T |\rho_3(\nabla\cdot(a(y,z)-a(0,0))\nabla y)|^q_{L^p(\Omega)} + \int_0^T |\rho_3 f(y,z)|^q_{L^p(\Omega)} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Using the definition of the space Y, one has the following:

$$I_1 \le \|(y,z,v)\|_Y.$$

Additionally, we have the following:

$$\begin{aligned}I_2 &= \int_0^T |\rho_3((a_r(y,z)|\nabla y|^2 + a_s(y,z)\nabla z\nabla y) + (a(y,z)-a(0,0))\Delta y)|^q_{L^p} \\ &\le c_1\Big(\int_0^T |\rho_3|\nabla y|^2|^q_{L^p(\Omega)} + \int_0^T \big|\rho_3|\nabla z||\nabla y|\big|^q_{L^p(\Omega)} + \int_0^T |\rho_3(|y|+|z|)\Delta y|^q_{L^p(\Omega)}\Big) \\ &= K_1 + K_2 + K_3\end{aligned}$$

Since $2\alpha_1 > \alpha_2$, we have the following:

$$\begin{aligned}|\rho_3\hat{\rho}^{-2}| &= |e^{s\alpha}e^{\frac{-2s\alpha_1}{m}}m^{\frac{-35}{2}}| \\ &\le |e^{\frac{s(\alpha_2-2\alpha_1)}{m}}m^{\frac{-35}{2}}| \le c.\end{aligned} \tag{3.2}$$

Thus, from (2.4), the immersion $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is valid for $n = 2$ and 3, $p > 3$ and from (3.2), we have the following:

$$\begin{aligned}|K_1| &\le c_1\int^T \Big(\int_\Omega \rho_3^p|\nabla y|^{2p}_{R^n}\Big)^{\frac{q}{p}} \\ &= c_1\int_0^T \Big(\int_\Omega (\rho_3\hat{\rho}_{19}^{-2})^p|\hat{\rho}_{19}\nabla y|^p_{R^n}|\hat{\rho}_{19}\nabla y|^p_{R^n}\Big)^{\frac{q}{p}} \\ &\le c_2\int_0^T |\hat{\rho}_{19}\nabla y|^q_{L^\infty}|\hat{\rho}_{19}\nabla y|^q_{L^p(\Omega)} \\ &\le c_2\int_0^T |\hat{\rho}_{19}\nabla y|^q_{W^{1,p}(\Omega)}|\hat{\rho}_{19}y|^q_{W^{1,p}(\Omega)} \\ &\le c_3\int_0^T |\hat{\rho}_{19}y|^q_{W^{2,p}(\Omega)}|\hat{\rho}_{19}y|^q_{W^{1,p}(\Omega)} \\ &\le c_3|\hat{\rho}_{19}y|^q_{C^0(0,T,W^{1,p}(\Omega))}|\hat{\rho}_{19}y|^q_{L^q(0,T,W^{2,p}(\Omega))} \\ &\le c_4\Big(|\rho_3\bar{h}|^q_{L^q(0,T,L^p(\Omega))} + |\rho_0\bar{k}|^q_{L^q(0,T,L^p(\Omega))} + |y_0|_{W_0^{1,p}(\Omega)} + |z_0|^q_{W_0^{1,p}(\Omega)}\Big)^2.\end{aligned}$$

The analysis of K_2 is similar. Let's analyze K_3 as follows:

$$
\begin{aligned}
|K_3| &\le c\int_0^T \Big(\int_\Omega \rho_3^p(|y|^p+|z|^p)|\Delta y|^p\Big)^{\frac{q}{p}} \\
&\le c\int_0^T \rho_3^q\Big(\int_\Omega |y|^p|\Delta y|^p+\int_\Omega |z|^p|\Delta y|^p\Big)^{\frac{q}{p}} \\
&\le c\int_0^T \rho_3^q(|y|_\infty^q+|z|_\infty^q)|\Delta y|^q_{L^p(\Omega)} \\
&\le c\int_0^T (\rho_3\hat{\rho}_{19}^{-2})^q(|\hat{\rho}_{19}y|_\infty^q+|\hat{\rho}_{19}z|_\infty^q)|\hat{\rho}_{19}\Delta y|^q_{L^p(\Omega)} \\
&\le c\int_0^T (|\hat{\rho}_{19}y|^q_{W^{1,p}(\Omega)}+|\hat{\rho}_{19}z|^q_{W^{1,p}(\Omega)})(|\hat{\rho}_{19}y|^q_{W^{2,p}(\Omega)}) \\
&\le c\Big(|\hat{\rho}_{19}y|^q_{C^0(0,T,W^{1,p}(\Omega))}+|\hat{\rho}_{19}z|^q_{C^0(0,T,W^{1,p}(\Omega))}\Big)(|\hat{\rho}_{19}y|_{L^q(0,T,W^{2,p}(\Omega))}\Big).
\end{aligned}
$$

□

From and Remark 1, it follows that

$$|k_3| \le c\Big(|\rho_3\bar{h}|^q_{L^q(0,T,L^p(\Omega))}+|\rho_0\bar{k}|^q_{L^q(0,T,L^p(\Omega)}+|y_0|^q_{W_0^{1,p}(\Omega)}+|z_0|^q_{W_0^{1,p}(\Omega)}\Big)^2.$$

Lemma 3.4. *The application $H : Y \to Z$ is continuously differentiable.*

Proof. We can write H in the following form:

$$H(y,z,v) = \mathcal{H}_1(y,z,v)+\mathcal{H}_2(y,z,v),$$

where

$$\mathcal{H}_1(y,z,v) = (y_t - v\chi_\omega, z_t, y(0), z(0)),$$

and

$$\mathcal{H}_2(y,z,v) = (-\nabla\cdot(a(y,z)\nabla y)+f(y,z), -\nabla\cdot(b(y,z)\nabla z)+g(y,z),0,0).$$

As $\mathcal{H}_1(y,z,v)$ is linear, then H_1 is continuously differentiable. Then, it will be enough to analyze $\mathcal{H}_2(y,z,v)$. For $(y,z,v),(\bar{y},\bar{z},\bar{v})$ belonging to Y and $\epsilon>0$, one has the following:

$$
\begin{aligned}
\frac{1}{\epsilon}[\mathcal{H}_2((y,z,v)+\epsilon(\bar{y},\bar{z},\bar{v}))-\mathcal{H}_2(y,z,v)] &= \frac{1}{\epsilon}(-\nabla(a(y+\epsilon\bar{y},z+\epsilon\bar{z})\nabla(y+\epsilon\bar{y})-a(y,z)\nabla y)) \\
&\quad+\frac{1}{\epsilon}(f(y+\epsilon\bar{y},z+\epsilon\bar{z})-f(y,z)),\frac{1}{\epsilon}(-\nabla(b(y+\epsilon\bar{y},z+\epsilon\bar{z})\nabla cz+\epsilon\bar{z}) \\
&\quad-b(y,z)\nabla z)+\frac{1}{\epsilon}(g(y+\epsilon\bar{y},z+\epsilon\bar{z})-g(y,z)),0,0).
\end{aligned}
$$

The linear application $D_G\mathcal{H}_2(y,z,v)\in L(Y,Z)$ is defined by the following:

$$
\begin{aligned}
D_G\mathcal{H}_2(y,z,v)(\bar{y},\bar{z},\bar{v}) &= (-\nabla(a_y(y,z)\bar{y}\nabla y+a_z(y,z)\bar{z}\nabla y+a(y,z)\nabla\bar{y})+f_y\bar{y}+f_z\bar{z}, \\
&\quad-\nabla\cdot(b_y(y,z)\bar{y}\nabla z+b_z(y,z)\bar{z}\nabla z+b(y,z)\nabla\bar{z})+g_y\bar{y}+g_z\bar{z},0,0).
\end{aligned}
$$

Using similar arguments as in Lemma 3.3, we can prove that

$$\frac{1}{\epsilon}[\mathcal{H}_2((y,z,v)+\epsilon(\bar{y},\bar{z},\bar{v}))-\mathcal{H}_2(y,z,v)]$$

strongly converges in Z for the operator $D_G\mathcal{H}_2$. Thus, $\mathcal{H}_2$ is G-differentiable and

$$\mathcal{H}_2' = D_G\mathcal{H}_2.$$

Therefore, H is G-differentiable and

$$H'(y,z,v) = \mathcal{H}_1 + D_G\mathcal{H}_2.$$

Additionally, using Lemma 3.3 and Lebesgue's dominated convergence theorem, it can be proven that

$$\begin{array}{rcll} DH & : & Y & \rightarrow \ L(y,z) \\ & & (y,z,v) & \mapsto \ DH(y,z,v) = H'(y,z,v) \end{array} \tag{3.3}$$

is continuous with the topologies of Y and $L(Y,Z)$. Thus, H is F-differentiable and its derivative is H'. □

Lemma 3.5. *The application $H'(0,0,0): Y \to Z$ is surjective.*

Proof. The proof is a consequence of Theorem 2.3 since the surjectivity of $H'(0,0,0)$ is equivalent to solving the control problem of the linearized system (2.1). □

4. Appendix

Proposition 4.1. *Consider $3 < p < q < \infty$. If $u \in L^q(0,T,W^{2,p}(\Omega))$ and $u' \in L^q(0,T,L^p(\Omega))$, then $u \in C^0(0,T,W^{1,p}(\Omega))$.*

Proof. Consider u a regular function with a compact support contained in Ω. Thus, we have the following:

$$\begin{aligned} \frac{d}{dt}\int_\Omega |\nabla u|^p &= \int_\Omega \frac{d}{dt}(|\nabla u|^2)^{\frac{p}{2}} \\ &= \int_\Omega p((|\nabla u|^2)^{\frac{p}{2}-1})(\nabla u \nabla u') \\ &= p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u' \\ &\quad - p\int_\Omega \nabla(|\nabla u|^{p-2}\nabla u)u' + \int_{\partial\Omega} p|\nabla u|^{p-2}\nabla u \cdot u' \cdot \vec{\eta} \\ &= \int_\Omega (p(p-2)(|\nabla u|^{p-4}\nabla u \nabla(u_{x_i})u_{x_i} + p|\nabla u|^{p-2}\Delta u)u' \\ &\leq c\int_\Omega (p(p-2)|\nabla u|^{p-2}|D^2u| + p|\nabla u|^{p-2}|\Delta u|)|u'|. \end{aligned} \tag{4.1}$$

Since $\frac{1}{p}+\frac{1}{p}+\frac{1}{\frac{p}{p-2}}=1$, then

$$\frac{d}{dt}\int_\Omega |\nabla u|^p \le c_1\Big(\int_\Omega (\nabla u|^{p-2})^{\frac{p}{p-2}}\Big)^{\frac{p-2}{p}}\Big(\int_\Omega |D^2u|^p\Big)^{\frac{1}{p}}\Big(\int_\Omega |u'|^p\Big)^{\frac{1}{p}}$$

$$\le c_1\Big(\int_\Omega |\nabla u|^p\Big)^{\frac{p-2}{p}}|u|_{W^{2,p}(\Omega)}|u'|_{L^p(\Omega)}.$$

Integrating in $[0,\ T]$, as $\frac{1}{q}+\frac{1}{q}+\frac{1}{\frac{q}{q-2}}=1$ and using that $\frac{p-2}{q-2}<1$, one has the following:

$$\int_\Omega |\nabla u(t)|^p \le \int_\Omega |\nabla u(0)|^p + c_1\Big(\int_0^T\Big(\int_\Omega |\nabla u|^p\Big)^{\frac{p-2}{p}\frac{q}{q-2}}\Big)^{\frac{q-2}{q}}\cdot |u|_{L^q(0,T,W^{2,p}(\Omega))}\cdot |u'|_{L^q(0,T,L^p(\Omega))}$$

$$\le |u_0|^p_{W_0^{1,p}(\Omega)} + c_1|u|^{q-2}_{L^q(0,T,W^{1,p}(\Omega))}|u|_{L^q(0,T,W^{2,p}(\Omega))}\cdot |u'|_{L^q(0,T,L^p(\Omega))}.$$

Then, the result follows using density arguments. □

□

4.1. Well-posedness

Theorem 4.2. *Consider $y_0, z_0 \in H_0^1(\Omega)\cap H^3(\Omega)$; there exists $\epsilon > 0$ sufficiently small such that if*

$$|y_0|_{H^3(\Omega)} + |z_0|_{H^3(\Omega)} \le \epsilon,$$

then (1.1) *has a unique strong solution of the following system:*

$$\begin{cases} y_t - \nabla(a(y,z)\nabla y) + f(y,z) = 0 & in \quad \Omega\times(0,T),\\ z_t - \nabla(b(y,z)\nabla z) + g(y,z) = 0 & in \quad \Omega\times(0,T),\\ y = z = 0 & on \quad \Sigma,\\ y(0) = y_0, z(0) = z_0. \end{cases} \tag{4.2}$$

Proof of theorem

Let $(\omega_i)_i$ be a special basis of $H_0^1(\Omega)$, where $-\Delta\omega_i = \lambda\omega_i$ and $V_m = [\omega_1, ..., \omega_m]$ is the space generated by the first m functions ω_i. Consider the approximate problem:

Then $y_m(t) = \sum_{i=1}^m g_{im}(t)w_i$, and $z_m(t) = \sum_{i=1}^m h_{im}(t)w_i$ solves the following system:

$$\left|\begin{array}{lll} (y_m', \omega_i) - (\nabla(a(y_m,z_m)\nabla y_m)\omega_i) + (f(y_m,z_m),\omega_i) = 0, & \text{in} & \omega \in V_m\\ (z_m', \tilde\omega_i) - (\nabla(b(y_m,z_m)\nabla z_m)\tilde\omega_i) + (g(y_m,z_m),\tilde\omega_i) = 0, & \text{in} & \tilde\omega \in V_m\\ y_m(0) = y_{0m} \to y_0 & \text{in} & H^3(\Omega)\cap H_0^1(\Omega),\\ z_m(0) = z_{0m} \to z_0 & \text{in} & H^3(\Omega)\cap H_0^1(\Omega), \end{array}\right. \tag{4.3}$$

Consider $\omega = -\Delta y_m$ and $\tilde{\omega} = -\Delta z_m$ in (4.11). Then, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|\nabla y_m|^2 + |\nabla z_m|^2) + \min\{a_0, b_0\}(|\Delta y_m|^2 + |\Delta z_m|^2) + \beta(|\nabla y_m|^2 + |\nabla z_m|^2) \\ &\qquad \leq k_1(|\Delta y_m|^3 + |\Delta z_m|^2) + k_2(|\Delta z_m||\Delta y_m|^2 + |\Delta y_m||\Delta z_m|^2) + 2\gamma(|\nabla z_m||\nabla y_m|) \end{aligned} \tag{4.4}$$

Now, consider $\omega = -\Delta y_m'$ and $\tilde{\omega} = -\Delta z_m'$ in (4.11). Then, one has the following:

$$\begin{aligned} &\frac{1}{2}(|\nabla y_m'|^2 + |\nabla z_m'|^2) + \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega a(y_m, z_m)|\Delta y_m|^2 + \int_\Omega b(y_m, z_m)|\Delta z_m|^2\Big) \\ &\qquad \leq \delta(|\Delta y_m'|^2 + |\Delta z_m'|^2) + c_\delta(|\Delta z_m|^4 + |\Delta y_m|^4) + c(|\nabla y_m|^2 + |\nabla z_m|^2). \end{aligned} \tag{4.5}$$

Applying derivatives to $(4.2)_1$ and $(4.2)_2$ with respect to t and considering $\omega = -\Delta y_m'$ and $\tilde{\omega} = -\Delta z_m'$ in these new equations, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|\nabla y_m'|^2 + |\nabla z_m'|^2) + \min\{a_0, b_0\}(|\Delta y_m'|^2 + |\Delta z_m'|^2) \\ &\qquad \leq c(|\Delta y_m'|^2 + |\Delta z_m'|^2)(|\Delta y_m|^2 + |\Delta z_m|^2 + |\Delta y_m| + |\Delta z_m|) \end{aligned} \tag{4.6}$$

If we sum (4.4) - (4.6) and denote $c_0 = \min\{a_0, b_0\}$, then we have the following:

$$\begin{aligned} &\frac{d}{dt}\Big(|\nabla y_m|^2 + |\nabla z_m|^2 + \int_\Omega a(y_m, z_m)|\Delta y_m|^2 + b(y_m, z_m)|\Delta z_m|^2 + |\nabla y_m'|^2 + |\nabla z_m'|^2\Big) \\ &\qquad + c(|\Delta y_m|^2 + |\Delta z_m|^2 + |\Delta y_m'|^2 + |\Delta z_m'|^2 + |\nabla y_m|^2 + |\nabla z_m|^2) \\ &\qquad + (|\Delta y_m|^2 + |\Delta z_m|^2)\Big\{\frac{c_0}{2} - c_2(|\Delta y_m| + |\Delta z_m| + |\Delta y_m|^2 + |\Delta z_m|^2)\Big\} \\ &\qquad + (|\Delta y_m'|^2 + |\Delta z_m'|^2)\Big\{\frac{c_0}{2} - c_3(|\Delta y_m| + |\Delta z_m| + |\Delta y_m|^2 + |\Delta z_m|^2)\Big\} \leq 0. \end{aligned} \tag{4.7}$$

Proceeding in a standard way and using

$$|\nabla y_m'(0)| + |\nabla z_m'(0)| \leq c(|\Delta y_0|^3 + |\Delta z_0|^3 + |y_0|_{H^3}|\Delta y_0| + |z_0|_{H^3}|\Delta z_0| + |y_0|_{H^3} + |z_0|_{H^3}),$$

from (4.7), for $\epsilon > 0$ sufficiently small with

$$|y_0|_{H^3(\Omega)} + |z_0|_{H^3(\Omega)} \leq \epsilon,$$

we obtain the following estimate:

$$\begin{aligned} &|y_m|_{L^\infty(0,T,H_0^1)} + |z_m|_{L^\infty(0,T,H_0^1)} + |y_m'|_{L^\infty(0,T,H_0^1)} + |z_m'|_{L^\infty(0,T,H_0^1)} \\ &\qquad |y_m|_{L^\infty(0,T,H^2)} + |z_m|_{L^\infty(0,T,H^2)} + |y_m'|_{L^2(0,T,H^2)} + |z_m'|_{L^2(0,T,H^2)} \leq c, \end{aligned} \tag{4.8}$$

where, we obtain a solution to Theorem 4.2 taking this to the limit as $m \to +\infty$.

Author contributions

Jéssica Manghi, Mauro A. Rincon and Juan Límaco: Writing-original draft, supervision, formal analysis, methodology, supervision.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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