Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 17(4): 955–970.
DOI: 10.3934/cam.2025039
Received: 13 November 2024
Revised: 11 July 2025
Accepted: 02 September 2025
Published: 03 December 2025

Research article

On some Liouville theorems for *p*-Laplace type operators

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Abstract: The aim of this note is to examine Liouville-type theorems for p-Laplacian-type operators. Guided by the Laplacian case, analogous results are established for the p-Laplacian and sums of operators of this type.

Keywords: p-Laplace operator; Liouville theorem; Schrödinger equation; nonlinear operators; anisotropic Laplace operator; double phase problem
Mathematics Subject Classification: 35A01, 35B53, 35D30, 35F25

1. Introduction and notation

It is well known, and it goes back to Liouville, that if u is harmonic, bounded function in $\mathbb{R}^n$ then u has to be a constant, i.e., if

$$-\Delta u = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n)$$

and u is bounded, then u is constant (see for instance [1,2]). The problem is much more subtle when the equation above has a lower-order term, i.e., if u is a solution to the Schrödinger equation

$$-\Delta u + bu = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n) \tag{1.1}$$

for some function $b \geq 0$. If $n = 2$ and $b \neq 0$, then every bounded solution to (1.1) is equal to 0. The situation is radically different when $n > 2$. To sketch the situation, if b is not decaying too quickly at infinity, then bounded solutions to (1.1) are vanishing. On the contrary, for functions b with fast decay, equation (1.1) can have bounded nontrivial solutions (see, for instance, [3–6]).

The goal of this note is to investigate the situation when the Laplacian is replaced by the p-Laplacian. The expectation in this case is as follows. For $p \geq n$, every bounded solution u to

$$-\Delta_p u = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n)$$

has to be constant. But when $1 < p < n$ and b decays fast enough, then one can exhibit nontrivial bounded solutions u to

$$-\Delta_p u + b|u|^{p-2}u = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n).$$

This is what we would like to investigate in a slightly more general framework. Recall that the p-Laplacian is defined as

$$\Delta_p u := \partial_{x_i}\{|\nabla u|^{p-2}\partial_{x_i}u\} = \nabla \cdot \{|\nabla u|^{p-2}\nabla u\}$$

with the summation convention in i, i.e., in the above formula one sums in i for $i = 1, \dots, n$. We will address these issues for p-Laplacian type operators, the archetype of which is

$$-\nabla \cdot \{a(x, u)|\nabla u|^{p-2}\nabla u\}.$$

We also discuss cases for sums of p-Laplace type operators

$$\partial_{x_k}\Big(\sum_{i=1}^{N} a_i(x, u)|\nabla u|^{p_i-2}\partial_{x_k}u\Big),$$

which are involved in double phase problems (see, for example, [7–9], and references therein), and, in particular, model the anisotropic $\vec{p}$-Laplace operator (see, for instance, [10]).

We note that our method to establish non-existence of non-constant bounded solutions u to

$$\partial_{x_k}\Big(\sum_{i=1}^{N} a_i(x, u)|\nabla u|^{p_i-2}\partial_{x_k}u\Big) + b(x, u) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n)$$

is not new; it relies on the classic truncation techniques obtained by multiplying the equation with suitable test-functions, and using the coercivity properties of the principal differential operator (see, e.g., [3, 11, 12]). The novelty in this paper is that we could adapt this technique to the case of sums of p-Laplace type operators and the anisotropic $\vec{p}$-Laplace operator.

The paper is divided as follows. The two next sections provide Liouville-type results in different situations, getting in particular inspiration from the case of the Laplacian where b is chosen with a relatively slow decay at infinity. In Section 4, we give an example of a nontrivial bounded solution when the lower-order term of the operator vanishes at infinity. Finally, in the last section, we briefly explain how the arguments developed in Theorem 3.1 can be extended in the case of several operators.

For interesting related topics, we refer to [11, 13–20].

2. p-Laplacian type operators for "$p \geq n$"

Let us denote by $a_i(x, u)$, $i = 1, \cdots, N$ Carathéodory functions such that for some positive constants λ, Λ one has for $i = 1, \cdots, N$

$$\lambda \leq a_i(x, u) \leq \Lambda \qquad \text{a.e. } x \in \mathbb{R}^n, \ \forall u \in \mathbb{R}.$$

Let $p_1, \cdots, p_N$ be real numbers such that

$$1 < p_1 \le p_2 \le \cdots \le p_N.$$

Denote also by $b : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $(x, u) \mapsto b(x, u)$ a Carathéodory function satisfying

$$b(\cdot, u)v \in L^1_{loc}(\mathbb{R}^n) \qquad \text{for every } u, v \in W^{1,p_N}_{loc}(\mathbb{R}^n),$$

and

$$b(x, u)u \ge 0 \qquad \text{for a.e. } x \in \mathbb{R}^n,\ \forall u \in \mathbb{R}. \tag{2.1}$$

Suppose now that u is a *solution* to

$$-\partial_{x_k}\Big(\sum_{i=1}^{N} a_i(x, u)|\nabla u|^{p_i-2}\partial_{x_k}u\Big) + b(x, u) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{2.2}$$

i.e., $u \in W^{1,p_N}_{\ell oc}(\mathbb{R}^n)$ and for every bounded open subset Ω of $\mathbb{R}^n$

$$\int_\Omega \sum_{i=1}^{N} a_i(x, u)|\nabla u|^{p_i-2}\nabla u \cdot \nabla v + b(x, u)v = 0 \qquad \forall v \in W^{1,p_N}_0(\Omega). \tag{2.3}$$

Then, one can show :

Theorem 1. *Suppose that $p_i \ge n$, for all $i = 1, \cdots, N$. Then, the only bounded solutions to* (2.2) *are the constants.*

Proof. Set

$$A(x, u(x), \xi) = \sum_{i=1}^{N} a_i(x, u(x))|\xi|^{p_i-2}\xi$$

for a.e. $x \in \Omega$, and every $\xi \in \mathbb{R}^n$. One has, if we denote by a dot the scalar product

$$A(x, u(x), \xi) \cdot \xi \ge \lambda \sum_{i=1}^{N} |\xi|^{p_i}, \tag{2.4}$$

and

$$|A(x, u(x), \xi)| \le \Lambda \sum_{i=1}^{N} |\xi|^{p_i-1} \tag{2.5}$$

for a.e. $x \in \Omega$, and every $\xi \in \mathbb{R}^n$. Let us denote by ρ a smooth, nonnegative function on $\mathbb{R}^n$ such that

$$\rho = 1 \text{ on } B_{\frac{1}{2}},\ \ \rho = 0 \text{ outside } B_1,\ \ |\nabla\rho| \le K \tag{2.6}$$

for some constant K (B_r denotes the ball of center 0 and radius r). If u is a weak solution to (2.2) and if $p \ge p_N$, then one has that

$$v := u\rho^p\left(\frac{\cdot}{r}\right) \in W^{1,p_N}_0(B_r).$$

Thus, from (2.3) one derives, omitting the measures of integration

$$\int_{B_r} A(x,u(x),\nabla u(x))\cdot\nabla\{u\,\rho^p(\tfrac{x}{r})\} + b(x,u(x))\,u(x)\rho^p(\tfrac{x}{r}) = 0,$$

which is equivalent to

$$\int_{B_r} A(x,u(x),\nabla u(x))\cdot\nabla u\,\rho^p(\tfrac{x}{r}) + b(x,u(x))\,u(x)\rho^p(\tfrac{x}{r}) = -p\int_{B_r\setminus B_{\frac{r}{2}}} A(x,u(x),\nabla u(x))\cdot\nabla\{\rho(\tfrac{x}{r})\}\,\rho^{p-1}(\tfrac{x}{r})u.$$

Using (2.4)–(2.6), recalling that $\nabla\{\rho(\frac{x}{r})\} = \frac{1}{r}\nabla\rho(\frac{x}{r})$, we get by (2.1) that

$$\begin{aligned}\lambda\int_{B_r}\sum_{i=1}^N |\nabla u|^{p_i}\,\rho^p(\tfrac{x}{r}) &\le \frac{pK\Lambda}{r}\sum_{i=1}^N\int_{B_r\setminus B_{\frac{r}{2}}} |\nabla u|^{p_i-1}\rho^{p-1}(\tfrac{x}{r})|u|\\ &\le \frac{pK\Lambda}{r}\sum_{i=1}^N\int_{B_r\setminus B_{\frac{r}{2}}} |\nabla u|^{p_i-1}\rho^{\frac{p(p_i-1)}{p_i}}\rho^{p-\frac{p(p_i-1)}{p_i}-1}|u|\\ &= \frac{pK\Lambda}{r}\sum_{i=1}^N\int_{B_r\setminus B_{\frac{r}{2}}} |\nabla u|^{p_i-1}\rho^{\frac{p}{p_i'}}\rho^{\frac{p-p_i}{p_i}}|u|\end{aligned}$$

with $p_i' = \frac{p_i}{p_i-1}$. Using Hölder's inequality in this last integral, one sees that

$$\lambda\int_{B_r}\sum_{i=1}^N |\nabla u|^{p_i}\,\rho^p(\tfrac{x}{r}) \le \sum_{i=1}^N\left[\int_{B_r\setminus B_{\frac{r}{2}}}|\nabla u|^{p_i}\rho^p(\tfrac{x}{r})\right]^{\frac{1}{p_i'}}\left[\int_{B_r\setminus B_{\frac{r}{2}}}\rho^{p-p_i}(\tfrac{x}{r})|u|^{p_i}\right]^{\frac{1}{p_i}}\frac{pK\Lambda}{r}. \tag{2.7}$$

Then, by the Young inequality

$$\sum_i a_i b_i \le \varepsilon\sum_i a_i^{p_i'} + C_\varepsilon\sum_i b_i^{p_i} \tag{2.8}$$

holding for all $\varepsilon > 0$, $a_i, b_i \ge 0$ with some constant $C_\varepsilon > 0$, we get

$$\begin{aligned}\lambda\int_{B_r}\sum_{i=1}^N |\nabla u|^{p_i}\,\rho^p(\tfrac{x}{r}) &\le \varepsilon\int_{B_r\setminus B_{\frac{r}{2}}}\sum_{i=1}^N|\nabla u|^{p_i}\rho^p(\tfrac{x}{r}) + C_\varepsilon\sum_{i=1}^N\int_{B_r\setminus B_{\frac{r}{2}}}\frac{1}{r^{p_i}}\rho^{p-p_i}(\tfrac{x}{r})|u|^{p_i}\\ &\le \varepsilon\int_{B_r\setminus B_{\frac{r}{2}}}\sum_{i=1}^N|\nabla u|^{p_i}\rho^p(\tfrac{x}{r}) + C_\varepsilon\sum_{i=1}^N\int_{B_r\setminus B_{\frac{r}{2}}}\frac{|u|^{p_i}}{r^{p_i}}.\end{aligned}$$

Recall that $p \ge p_i\ \forall i$. Let us assume that

$$\sum_{i=1}^N\frac{1}{r^{p_i}}\int_{B_r\setminus B_{\frac{r}{2}}}|u|^{p_i} \quad \text{is bounded independently of r.} \tag{2.9}$$

Then, choosing $\varepsilon = \frac{\lambda}{2}$, one derives that

$$\int_{B_{\frac{r}{2}}} \sum_{i=1}^{N} |\nabla u|^{p_i} \quad \text{is bounded independently of r}$$

and thus, since this integral is nondecreasing in r for every i, we can conclude that

$$\lim_{r\to\infty} \int_{B_r} |\nabla u|^{p_i} \ \text{ exists.}$$

Going back to (2.7), applying (2.9), one easily derives that for some constants C,

$$\begin{aligned}
\lambda \int_{B_{\frac{r}{2}}} \sum_{i=1}^{N} |\nabla u|^{p_i} &\leq C \sum_{i=1}^{N} \left[\int_{B_r \setminus B_{\frac{r}{2}}} |\nabla u|^{p_i} \right]^{\frac{1}{p_i'}} \left[\int_{B_r \setminus B_{\frac{r}{2}}} \frac{1}{r^{p_i}} |u|^{p_i} \right]^{\frac{1}{p_i}} \\
&\leq C \sum_{i=1}^{N} \left[\int_{B_r} |\nabla u|^{p_i} - \int_{B_{\frac{r}{2}}} |\nabla u|^{p_i} \right]^{\frac{1}{p_i'}} \to 0 \quad \text{when } r \to \infty.
\end{aligned}$$

Thus, in case (2.9) holds, $\nabla u = 0$, and so u is constant. Note, if $p_i \geq n$ for every i and if u is bounded, then it is easy to see that (2.9) holds. This completes the proof of the theorem. □

Remark 1. *The condition* (2.9) *is weaker than assuming boundedness of u. Of course, if* $b(x, u)$ *is not identically equal to* 0*, the constant in Theorem 1 vanishes. Also, using the structure assumptions* (2.4) *and* (2.5)*, one sees that the theorem above can be extended to more general operators. For instance, with a summation in k for*

$$-\sum_{i=1}^{N} (\partial_{x_k} a_i^k(x, u) |\nabla u|^{p_i - 2} \partial_{x_k} u).$$

In this case, the k-component of $A(x, u, \xi)$ *is given by*

$$-\sum_{i=1}^{N} a_i^k(x, u) |\xi|^{p_i - 2} \xi_k$$

and provided $a_i^k \geq \lambda$ *one has*

$$A(x, u, \xi) \cdot \xi \geq \lambda \sum_{i=1}^{N} |\xi|^{p_i}$$

(2.5) *being easy to establish if the* a_i^k *are bounded.*

Similarly, for instance, for the so-called anisotropic pseudo $\vec{p}$*-Laplace operator*

$$-\partial_{x_k} \{ a^k(x, u) |\partial_{x_k} u|^{p_k - 2} \partial_{x_k} u \}$$

(see, for example, [10, 21, 22]), the k-component of $A(x, u, \xi)$ *is given by*

$$a^k(x, u) |\xi_k|^{p_k - 2} \xi_k$$

and provided $a^k \geq \lambda$ it holds

$$A(x,u,\xi)\cdot\xi = \sum_{k=1}^{n} a^k(x,u)|\xi_k|^{p_k} \geq \lambda \sum_{k=1}^{n} |\xi_k|^{p_k}.$$

The proof of Theorem 1 follows the same pattern in this case, (2.7) *being replaced by*

$$\lambda \sum_{k=1}^{n} \int_{B_r} |\partial_{x_k} u|^{p_k}\, \rho^p(\tfrac{x}{r}) \leq \frac{C}{r} \sum_{k=1}^{n} \left[\int_{B_r\setminus B_{\frac{r}{2}}} |\partial_{x_k}|^{p_k} \rho^p(\tfrac{x}{r})\right]^{\frac{1}{p_k'}} \left[\int_{B_r\setminus B_{\frac{r}{2}}} \rho^{p-p_k}(\tfrac{x}{r})|u|^{p_k}\right]^{\frac{1}{p_k}}$$

and the result holds for $p_k \geq n$, $\forall k$.

3. p-Laplacian type operators for "N = 1" but "p" arbitrary

In this section, we would like to show that, in case the lower-order term $b(x,u)$ in equation (2.2) is stronger, one can extend Theorem 1 to every $1 < p < \infty$. To avoid technicalities, we will restrict ourselves to the case of one single operator of p-Laplacian type (that is, we take $N = 1$), postponing to the last section (Section 5) the possible extensions. Thus, for some $p > 1$, we suppose that u is a *solution* to

$$-\partial_{x_k}(a(x,u)|\nabla u|^{p-2}\partial_{x_k} u) + b(x,u) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{3.1}$$

i.e., $u \in W^{1,p}_{\ell oc}(\mathbb{R}^n)$ and for every bounded open subset Ω of $\mathbb{R}^n$,

$$\int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u\cdot\nabla v + b(x,u)v = 0 \qquad \forall v \in W^{1,p}_0(\Omega). \tag{3.2}$$

We suppose, of course, that $a(x,u)$ is a Carathéodory function satisfying

$$\lambda \leq a(x,u) \leq \Lambda \quad \text{a.e. } x \in \mathbb{R}^n,\ \forall u \in \mathbb{R}. \tag{3.3}$$

Theorem 2. *Suppose, in addition to* (2.1)*, that there are $0 < \ell < p$ and constants c, $R > 0$ such that for every $r \geq R$,*

$$b(x,u)u \geq \frac{c}{r^\ell}|u|^p \qquad \forall\, |x| \geq r. \tag{3.4}$$

Then every bounded solution to (3.1) *vanishes.*

Proof. Let ρ be a function satisfying (2.6). Taking as test function in (3.2)

$$v = u\,\rho^p(\frac{\cdot}{r}),$$

we get

$$\int_{B_r} a(x,u)|\nabla u|^{p-2}\nabla u\cdot\nabla\{u\,\rho^p(\tfrac{x}{r})\} + b(x,u)\,u\,\rho^p(\frac{x}{r}) = 0.$$

This implies easily

$$\int_\Omega a(x,u)|\nabla u|^p\rho^p(\frac{x}{r}) + b(x,u)u\rho^p(\tfrac{x}{r}) = -p\int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u\cdot\nabla\{\rho(\tfrac{x}{r})\}\rho^{p-1}u. \tag{3.5}$$

Arguing as in the previous section, one derives (see (3.3), (3.4))

$$\int_{B_r} \lambda|\nabla u|^p \rho^p + b(x,u)u\rho^p \le \frac{pK\Lambda}{r} \int_{B_r \setminus B_{\frac{r}{2}}} |\nabla u|^{p-1} \rho^{p-1} |u|. \tag{3.6}$$

Applying Hölder's inequality, the estimate (3.6) becomes

$$\begin{aligned} \int_{B_r} \lambda|\nabla u|^p \rho^p + b(x,u)u\rho^p &\le \frac{pK\Lambda}{r} \left[\int_{B_r \setminus B_{\frac{r}{2}}} |\nabla u|^p \rho^p \right]^{\frac{1}{p'}} \left[\int_{B_r \setminus B_{\frac{r}{2}}} |u|^p \right]^{\frac{1}{p}} \\ &\le \frac{pK\Lambda}{r} \left[\int_{B_r \setminus B_{\frac{r}{2}}} |\nabla u|^p \rho^p \right]^{\frac{1}{p'}} \left[\int_{B_r \setminus B_{\frac{r}{2}}} \frac{r^\ell}{c} b(x,u)u \right]^{\frac{1}{p}} \\ &\le \frac{pK\Lambda}{c^{\frac{1}{p}} r^{1-\frac{\ell}{p}}} \left[\int_{B_r \setminus B_{\frac{r}{2}}} |\nabla u|^p \rho^p \right]^{\frac{1}{p'}} \left[\int_{B_r \setminus B_{\frac{r}{2}}} b(x,u)u \right]^{\frac{1}{p}}. \end{aligned} \tag{3.7}$$

Using the Young inequality

$$ab \le \frac{1}{p'} a^{p'} + \frac{1}{p} a^p, \quad \forall\, a, b \ge 0,$$

we get

$$\int_{B_{\frac{r}{2}}} \lambda|\nabla u|^p + b(x,u)u \le \frac{pK\Lambda}{\lambda p' c^{\frac{1}{p}} r^{1-\frac{\ell}{p}}} \int_{B_r \setminus B_{\frac{r}{2}}} \lambda|\nabla u|^p \rho^p + \frac{pK\Lambda}{p c^{\frac{1}{p}} r^{1-\frac{\ell}{p}}} \int_{B_r \setminus B_{\frac{r}{2}}} b(x,u)u.$$

Thus, for some constant $C > 0$,

$$\int_{B_{\frac{r}{2}}} \lambda|\nabla u|^p + b(x,u)u \;\le \frac{C}{r^{1-\frac{\ell}{p}}} \int_{B_r} \lambda|\nabla u|^p + b(x,u)u.$$

Iterating this formula, one derives

$$\begin{aligned} \int_{B_{\frac{r}{2^{k+1}}}} \lambda|\nabla u|^p + b(x,u)u &\le \frac{C}{(r/2^k)^{1-\frac{\ell}{p}}} \int_{B_{\frac{r}{2^k}}} \lambda|\nabla u|^p + b(x,u)u \\ &= \frac{C\, 2^{k(1-\frac{\ell}{p})}}{r^{1-\frac{\ell}{p}}} \int_{B_{\frac{r}{2^k}}} \lambda|\nabla u|^p + b(x,u)u \\ &\le \frac{C^k\, 2^{k\frac{k+1}{2}(1-\frac{\ell}{p})}}{r^{k(1-\frac{\ell}{p})}} \int_{B_{\frac{r}{2}}} \lambda|\nabla u|^p + b(x,u)u \end{aligned}$$

and so,

$$\int_{B_{\frac{r}{2^{k+1}}}} \lambda|\nabla u|^p + b(x,u)u \le \frac{C(k,\ell,p)}{r^{k(1-\frac{\ell}{p})}} \int_{B_{\frac{r}{2}}} \lambda|\nabla u|^p + b(x,u)u \tag{3.8}$$

for some constant $C(k,\ell,p) > 0$ depending on C, k, ℓ, and p. Going back to (3.7), we have

$$\int_{B_r} \lambda|\nabla u|^p \rho^p + b(x,u)u\rho^p \le \frac{pK\Lambda}{r} \left[\int_{B_r \setminus B_{\frac{r}{2}}} |\nabla u|^p \rho^p \right]^{\frac{1}{p'}} \left[\int_{B_r \setminus B_{\frac{r}{2}}} |u|^p \right]^{\frac{1}{p}}$$

$$\leq \frac{pK\Lambda}{r}\frac{1}{\lambda^{\frac{1}{p'}}}\left[\int_{B_r\setminus B_{\frac{r}{2}}}\lambda|\nabla u|^p\rho^p\right]^{\frac{1}{p'}}\left[\int_{B_r\setminus B_{\frac{r}{2}}}|u|^p\right]^{\frac{1}{p}}$$

$$\leq \frac{pK\Lambda}{r}\frac{1}{\lambda^{\frac{1}{p'}}}\left[\int_{B_r\setminus B_{\frac{r}{2}}}\lambda|\nabla u|^p\rho^p + b(x,u)u\rho^p\right]^{\frac{1}{p'}}\left[\int_{B_r\setminus B_{\frac{r}{2}}}|u|^p\right]^{\frac{1}{p}}$$

and thus, for some constant $C > 0$,

$$\left[\int_{B_r}\lambda|\nabla u|^p\rho^p + b(x,u)u\rho^p\right]^{\frac{1}{p}} \leq \frac{C}{r}\left[\int_{B_r\setminus B_{\frac{r}{2}}}|u|^p\right]^{\frac{1}{p}}$$

which leads to

$$\int_{B_{\frac{r}{2}}}\lambda|\nabla u|^p + b(x,u)u \leq \left(\frac{C}{r}\right)^p\int_{B_r\setminus B_{\frac{r}{2}}}|u|^p.$$

If u is uniformly bounded by assumption, then one gets

$$\int_{B_{\frac{r}{2}}}\lambda|\nabla u|^p + b(x,u)u \leq Cr^{n-p}. \tag{3.9}$$

for some other constant C. From (3.8), we derive then

$$\int_{B_{\frac{r}{2^{k+1}}}}\lambda|\nabla u|^p + b(x,u)u \leq \frac{C(k,\ell,p)}{r^{k(1-\frac{\ell}{p})}}\,C\,r^{n-p} \to 0 \qquad \text{as } r\to\infty$$

when $k(1-\frac{\ell}{p}) > n-p$. This completes the proof of Theorem 2. □

Remark 2. *From* (3.9) *one can get the result for $p > n$. Note also that* (3.4) *holds with $\ell = 0$ when one has*

$$b(x,u)u \geq c|u|^p \qquad \textit{for a.e. } x\in\mathbb{R}^d \textit{ and every } u\in\mathbb{R}. \tag{3.10}$$

4. Existence of a nontrivial solution for "$N = 1$" and "$p < n$"

In this section, we would like to construct a nontrivial bounded solution to the equation

$$-\Delta_p u + b\,|u|^{p-2}u = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{4.1}$$

when $b = b(x)$ is nonnegative. Here, a function u is called a *solution* to (4.1) if $u\in W^{1,p}_{\ell oc}(\mathbb{R}^n)$ and for every open bounded subset $\Omega\subseteq\mathbb{R}^n$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v + b(x)|u|^{p-2}uv = 0 \;\; \forall v\in W^{1,p}_0(\Omega). \tag{4.2}$$

Recall that B_k denotes the ball of center 0 and radius k. Then, for every $k\in\mathbb{N}$, there exists a unique solution u_k to the variational inequality

$$\begin{cases} u_k\in K = \{v\in W^{1,p}(B_k) : v = 1 \text{ on } \partial B_k\}, \\ \displaystyle\int_{B_k}|\nabla u_k|^{p-2}\nabla u_k\cdot\nabla(v-u_k) + b(x)|u_k|^{p-2}u_k(v-u_k) \geq 0 \;\; \forall v\in K. \end{cases} \tag{4.3}$$

We refer, for instance, to [23, 24], or the Remark 3 below.

1. Claim: $0 \le u_k \le 1$ on B_k

Recall that $w^+(x) := \max\{0, w(x)\}$ denotes the positive part of a function w and $w^- := (-w)^+$ the negative part. Then, taking $v = u_k^+$ as a test function in (4.3) and by using that $u_k^+ - u_k = u_k^-$, it comes

$$\begin{aligned} 0 &\le \int_{B_k} |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla u_k^- + b\,|u_k|^{p-2}u_k u_k^- \\ &= -\int_{B_k} |\nabla u_k^-|^{p-2}\nabla u_k^- \cdot \nabla u_k^- + b\,|u_k^-|^{p-2}u_k^- u_k^- \le 0, \end{aligned}$$

from which we can conclude that

$$\int_{B_k} |\nabla u_k^-|^p + b|u_k^-|^p = 0.$$

Thus, $u_k^- = 0$ on B_k, which implies that $u_k \ge 0$ on B_k.

It should be noted that $u_k \pm (u_k - 1)^+ \in K$. Thus, taking $v = u_k \pm (u_k - 1)^+$ in (4.3), one gets

$$\int_{B_k} |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla(u_k - 1)^+ + b|u_k|^{p-2}u_k(u_k - 1)^+ = 0$$

and hence,

$$\int_{B_k} |\nabla u_k|^{p-2}\nabla(u_k - 1) \cdot \nabla(u_k - 1)^+ = -\int_{B_k} b|u_k|^{p-2}u_k(u_k - 1)^+ \le 0.$$

Thus, $(u_k - 1)^+ = 0$, i.e., $u_k \le 1$.

2. Claim: $u_{k+1} \le u_k$ on B_k

Clearly $(u_{k+1} - u_k)^+ \in W_0^{1,p}(B_k)$. We now suppose that this function is extended by 0 on B_{k+1}. Taking $v = u_k \pm (u_{k+1} - u_k)^+$ in (4.3), we get that

$$\int_{B_k} |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla(u_{k+1} - u_k)^+ + b|u_k|^{p-2}u_k(u_{k+1} - u_k)^+ = 0.$$

Similarly, taking $v = u_{k+1} \pm (u_{k+1} - u_k)^+$ in (4.3) gives

$$\int_{B_k} |\nabla u_{k+1}|^{p-2}\nabla u_{k+1} \cdot \nabla(u_{k+1} - u_k)^+ + b|u_{k+1}|^{p-2}u_{k+1}(u_{k+1} - u_k)^+ = 0.$$

By subtraction, we obtain that

$$\begin{aligned} \int_{B_k} \{|\nabla u_{k+1}|^{p-2}\nabla u_{k+1} - |\nabla u_k|^{p-2}\nabla u_k\} \cdot \nabla(u_{k+1} - u_k)^+ \\ + b\{|u_{k+1}|^{p-2}u_{k+1} - |u_k|^{p-2}u_k\}(u_{k+1} - u_k)^+ = 0. \end{aligned}$$

Thus, for some constant $c_p > 0$, we get that (see, for example, [23, Proposition 17.3])

$$c_p \int_{B_k} (|\nabla u_{k+1}| + |\nabla u_k|)^{p-2} |\nabla (u_{k+1} - u_k)^+|^2 \le 0,$$

implying that $(u_{k+1} - u_k)^+ = 0$ on B_k, which is $u_{k+1} \le u_k$ on B_k.

From Claim 1 and Claim 2, we derive that

$$u_k(x) \to u(x) \qquad \text{pointwise for a.e. } x \in \mathbb{R}^n, \tag{4.4}$$

where $u : \mathbb{R}^n \to \mathbb{R}$ is a function satisfying

$$0 \le u \le 1 \qquad \text{on } \mathbb{R}^n.$$

3. Claim: If b is radially symmetric, so are u_k and u.

If $R = (R_{j,k})$ is an orthogonal transformation, then one has with the summation convention

$$\begin{aligned} \nabla\{v(Rx)\} &= (\partial_{y_j} v(Rx) \partial_{x_i} R_{j,k} x_k) \\ &= (R_{j,i} \partial_{y_j} v(Rx)) = R^T \{\nabla v\}(Rx). \end{aligned}$$

Thus, a change of variable yields that

$$\int_{B_k} |\nabla\{u_k(Rx)\}|^{p-2} \nabla\{u_k(Rx)\} \cdot \nabla\{(v(Rx) - u_k(Rx))\} \\ + b|u_k(Rx)|^{p-2} u_k(Rx)(v(Rx) - u_k(Rx)) \ge 0$$

for any $v \in W_0^{1,p}(B_k),\ v = 1$ on ∂B_k. Choosing $v(R^T x)$, we see, by uniqueness of u_k that

$$u_k(Rx) = u_k(x)$$

for any orthogonal transformation R.

Remark 3. *Taking $v = u_k \pm \varphi$ for $\varphi \in W_0^{1,p}(B_k)$ in* (4.3)*, one sees that u_k satisfies*

$$u_k \in K \quad \text{and} \quad \int_{B_k} |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \varphi + b|u_k|^{p-2} u_k \varphi = 0 \ \ \forall \varphi \in W_0^{1,p}(B_k), \tag{4.5}$$

that is, u_k is a weak solution *of the nonlinear Dirichlet problem*

$$\begin{aligned} -\Delta_p u_k + b\,|u_k|^{p-2} u_k &= 0 \quad \text{in } B_k, \\ u_k &= 1 \quad \text{on } \partial B_k. \end{aligned}$$

Note that u_k is also the unique minimiser on K to

$$J(v) = \int_{B_k} |\nabla v|^p + b|v|^p.$$

From now on, we suppose that

$$\begin{aligned} &b \in L^1(\mathbb{R}^n) \text{ and } b \text{ is radially symmetric with compact support, i.e.,} \\ &b(x) = b(|x|) = 0 \qquad \text{for all } |x| = r \geq r_0. \end{aligned} \tag{4.6}$$

Since the function $1 \in K$, one has then

$$\int_{B_k} |\nabla u_k|^p + b|u_k|^p = J(u_k) \leq J(1) = \int_{\mathbb{R}^n} b < +\infty.$$

Thus, up to a subsequence,

$$\nabla u_k \rightharpoonup \nabla u \text{ in } L^p(\Omega) \tag{4.7}$$

for every bounded subdomain Ω of $\mathbb{R}^n$.

4. Differential equation satisfied by u_k and u.

If $u_k = u_k(r)$, then

$$\nabla u_k = u_k'(r)\nabla r = u_k'(r)\frac{x}{r} \quad \text{ and } \quad |\nabla u_k| = |u_k'(r)|.$$

From this, it follows that

$$\begin{aligned} \nabla \cdot (|\nabla u_k|^{p-2}\nabla u_k) &= \partial_{x_i}(|u_k'|^{p-2}u_k'\tfrac{x_i}{r}) \\ &= |u_k'|^{p-2}u_k'\partial_{x_i}\{\tfrac{x_i}{r}\} + (|u_k'|^{p-2}u_k')'\tfrac{x_i}{r}\tfrac{x_i}{r} \\ &= |u_k'|^{p-2}u_k'(\tfrac{n}{r}) + |u_k'|^{p-2}u_k'x_i(-\tfrac{1}{r^2})\frac{x_i}{r} + (|u_k'|^{p-2}u_k')' \\ &= |u_k'|^{p-2}u_k'(\tfrac{n-1}{r}) + (|u_k'|^{p-2}u_k')' \\ &= \frac{1}{r^{n-1}}\Big(|u_k'|^{p-2}u_k'(n-1)r^{n-2} + r^{n-1}(|u_k'|^{p-2}u_k')'\Big) \\ &= \frac{1}{r^{n-1}}(|u_k'|^{p-2}u_k'r^{n-1})'. \end{aligned}$$

Thus from (4.5), one derives that u_k satisfies

$$\frac{1}{r^{n-1}}(|u_k'|^{p-2}u_k'r^{n-1})' = b|u_k|^{p-2}u_k \qquad \text{for } 0 < r < k,$$

which is equivalent to

$$(|u_k'|^{p-2}u_k'r^{n-1})' = r^{n-1}b|u_k|^{p-2}u_k \qquad \text{for } 0 < r < k,$$

and again, equivalent to

$$|u_k'(r)|^{p-2}u_k'(r) = \frac{1}{r^{n-1}}\int_0^r s^{n-1}b|u_k|^{p-2}u_k ds \qquad \text{for } 0 < r < k.$$

Setting $\Psi(x) = |x|^{p-2}x$ for $x \in \mathbb{R}$, then ψ is bijective on $\mathbb{R}$ and its inverse is $\Psi^{-1}(x) = |x|^{\frac{1}{p-1}} \operatorname{sign} x$, where $\operatorname{sign} x$ denotes the sign of x. One gets

$$u_k' = \Psi^{-1}\Big(\frac{1}{r^{n-1}}\int_0^r s^{n-1}b|u_k|^{p-2}u_k ds\Big) \qquad \text{for } 0 < r < k. \tag{4.8}$$

From (4.7), one has up to a subsequence still labelled by k

$$\nabla u_k = u_k'\frac{x}{r} \rightharpoonup u'\frac{x}{r} \text{ in } L^p(\Omega) \tag{4.9}$$

for every open and bounded subset $\Omega \subseteq \mathbb{R}^n$. Thus, by using (4.4), multiplying (4.8) with x/r for $r > 0$ and subsequently passing to the limit, we arrive at

$$u'(r)\frac{x}{r} = \frac{x}{r}\Psi^{-1}(\frac{1}{r^{n-1}}\int_0^r s^{n-1}b|u|^{p-2}uds) \qquad \text{for } r > 0,$$

which is equivalent to

$$u'(r) = \Psi^{-1}(\frac{1}{r^{n-1}}\int_0^r s^{n-1}b|u|^{p-2}uds) \qquad \text{for } r > 0,$$

and

$$\Psi(u') = |u'|^{p-2}u' = \frac{1}{r^{n-1}}\int_0^r s^{n-1}b|u|^{p-2}uds \qquad \text{for } r > 0.$$

Multiplying the last equation by r^{n-1} and subsequently differentiating it; shows that u satisfies

$$-\frac{1}{r^{n-1}}(|u'|^{p-2}u'r^{n-1})' + b|u|^{p-2}u = 0 \qquad \text{in } (0,\infty),$$

that is, u satisfies the same equation as u_k in all $\mathbb{R}^n$.

We would like to show now that u is nontrivial.

5. The limit of u_k cannot be identically 0, that is, u is nontrivial.

Due to the definition of b, one has that

$$(|u_k'|^{p-2}u_k'r^{n-1})' = 0 \text{ for } r \geq r_0.$$

Thus,

$$|u_k'|^{p-2}u_k'r^{n-1} = C_k \text{ for } r \geq r_0.$$

where C_k is some constant. Thus, for $r \geq r_0$ one has

$$u_k' = \Psi^{-1}(\frac{C_k}{r^{n-1}}) = |C_k|^{\frac{1}{p-1}} \operatorname{sign} C_k \frac{1}{r^{\frac{n-1}{p-1}}}.$$

Integrating between r_0 and r, we get

$$u_k(r) - u_k(r_0) = |C_k|^{\frac{1}{p-1}} \operatorname{sign} C_k \int_{r_0}^r \frac{1}{r^{\frac{n-1}{p-1}}}. \tag{4.10}$$

Now, if $u_k(r) \to 0$ pointwise, (4.10) implies that $C_k \to 0$. On the other hand, choosing $r = k$ in (4.10) gives that

$$1 - u_k(r_0) = |C_k|^{\frac{1}{p-1}} \operatorname{sign} C_k \int_{r_0}^k \frac{1}{r^{\frac{n-1}{p-1}}} \tag{4.11}$$

for every $k \geq r_0$. If $n > p$, then the integral above converges, and so, we arrive at a contradiction when we send $k \to \infty$ in (4.11). Thus, we have proved

Theorem 3. *In the case $p < n$ ($n > 2$ in the case of the Laplacian), one can find functions b satisfying* (4.6) *such that equation* (4.1) *admits a nontrivial bounded solution.*

5. Concluding remarks

We would like to show briefly here how Theorem 2 can be extended in the case of several p-Laplacian type operators. Suppose that u is a solution to (2.2). Arguing as in (3.5) and (3.6), one gets that

$$\int_{B_r} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} \rho^p(\tfrac{x}{r}) + b(x,u)u\rho^p(\tfrac{x}{r}) \leq \frac{pK\Lambda}{r} \int_{B_r\setminus B_{\frac{r}{2}}} \sum_{i=1}^{N} |\nabla u|^{p_i-1} \rho^{\frac{p}{p_i'}} \rho^{\frac{p-p_i}{p_i}} |u|. \tag{5.1}$$

Using the Hölder inequality, we derive

$$\int_{B_r} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} \rho^p(\tfrac{x}{r}) + b(x,u)u\rho^p(\tfrac{x}{r})$$
$$\leq \frac{pK\Lambda}{r} \sum_{i=1}^{N} \Big(\int_{B_r\setminus B_{\frac{r}{2}}} |\nabla u|^{p_i}\rho^p \Big)^{\frac{1}{p_i'}} \Big(\int_{B_r\setminus B_{\frac{r}{2}}} \rho^{p-p_i}|u|^{p_i} \Big)^{\frac{1}{p_i}}.$$

Assuming then for $|x|$ large enough and for all i

$$b(x,u)u \geq \frac{c}{r^\ell}|u|^{p_i}, \quad c > 0, \ell < p_1 \leq p_i$$

we get

$$\int_{B_r} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} \rho^p(\tfrac{x}{r}) + b(x,u)u\rho^p(\tfrac{x}{r})$$
$$\leq \frac{pK\Lambda}{r^{1-\frac{\ell}{p_1}}} \sum_{i=1}^{N} \Big(\int_{B_r\setminus B_{\frac{r}{2}}} |\nabla u|^{p_i}\rho^p \Big)^{\frac{1}{p_i'}} \Big(\int_{B_r\setminus B_{\frac{r}{2}}} \rho^{p-p_i}\frac{1}{c}b(x,u)u \Big)^{\frac{1}{p_i}}.$$

Then, applying the Young inequality

$$ab \leq \frac{1}{p_i'}a^{p_i'} + \frac{1}{p_i}b^{p_i}, \quad a, b \geq 0$$

to the latter estimate, one easily sees that for some constant $C > 0$,

$$\int_{B_r} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} \rho^p(\tfrac{x}{r}) + b(x,u)u\rho^p(\frac{x}{r}) \leq \frac{C}{r^{1-\frac{\ell}{p_1}}} \int_{B_r\setminus B_{\frac{r}{2}}} \sum_{i=1}^{N} \lambda|\nabla u|^{p_i}\rho^p + \rho^{p-p_i}b(x,u)u.$$

Thus, if $p \geq p_i$, for some constant $C > 0$, we get

$$\int_{B_{\frac{r}{2}}} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} + b(x,u)u \leq \frac{C}{r^{1-\frac{\ell}{p_1}}} \int_{B_r} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} + b(x,u)u.$$

Iterating this formula, one gets

$$\int_{B_{\frac{r}{2^{k+1}}}} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} + b(x,u)u \leq \frac{C(k,\ell,p_1)}{r^{k(1-\ell/p_1)}} \int_{B_{\frac{r}{2}}} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} + b(x,u)u. \tag{5.2}$$

Going back to (5.1) and using (2.8) (taking $\varepsilon = \frac{1}{2}$), we obtain that

$$\int_{B_r} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} \rho^p + b(x,u)u\rho^p \leq \varepsilon \int_{B_r} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} \rho^p + C_\varepsilon \int_{B_r} \sum_{i=1}^{N} \frac{|u|^{p_i}}{r^{p_i}}$$

and

$$\int_{B_{\frac{r}{2}}} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} + b(x,u)u \leq 2\, C_\varepsilon \int_{B_r} \sum_{i=1}^{N} \frac{|u|^{p_i}}{r^{p_i}}.$$

If u is bounded, this leads to

$$\int_{B_{\frac{r}{2}}} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} + b(x,u)u \leq C \sum_{i=1}^{N} r^{n-p_i}.$$

By (5.2), it follows that

$$\int_{B_{\frac{r}{2^{k+1}}}} \lambda \sum_{i=1}^{N} |\nabla u|^{p_i} + b(x,u)u \leq \frac{C(k,\ell,p_1)}{r^{k(1-\frac{\ell}{p_1})}}\, C \sum_{i=1}^{N} \frac{1}{r^{p_i-n}} \to 0$$

as $r \to \infty$, provided $k(1 - \frac{\ell}{p_1}) > n - p_i$. This completes the proof in this case.

Author contributions

Both authors have contributed equally to the development of the research results presented in this article in terms of *conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, supervision, validation, visualization, writing – original draft writing – review & editing*.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Dedication: To Tom Sideris, an elegant scholar.

The second author's research was partially supported by the two Australian Research Council grants DP200101065 and DP220100067.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. L. C. Evans, *Partial Differential Equations*, Volume 19 of *Graduate Studies in Mathematics*, American Math. Society, Providence, 1998.
2. M. H. Protter, H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.
3. H. Brezis, M. Chipot, Y. Xie, Some remarks on Liouville type theorems, In World Scientific Edt., editor, *Proceedings of the international conference in nonlinear analysis*, 43–65, Hsinchu, Taiwan 2006, 2008.
4. A. Grigor'yan, Bounded solutions of the Schrödinger equation on non-compact Riemannian manifolds, *J. Sov. Math.*, **51** (1990), 2340–2349. https://doi.org/10.1007/BF01094993
5. A. Grigor'yan, W. Hansen, A Liouville property for Schrödinger operators, *Math. Ann.*, **312** (1998), 659–716. https://doi.org/10.1007/s002080050241
6. R. G. Pinsky, A probabilistic approach to a Liouville-type problem for Schrödinger operators, Preprint, 2006.
7. P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, *Calc. Var. Partial Differential Equations*, **57** (2018), 48. https://doi.org/10.1007/s00526-018-1332-z
8. P. Marcellini, Regularity and existence of solutions of elliptic equations with p,q-growth conditions, *J. Differential Equations*, **90** (1991), 1–30.
9. V. Zhikov, On variational problems and nonlinear elliptic equations with nonstandard growth conditions, *J. Math. Sci.*, **173** (2011), 463–570. https://doi.org/10.1007/s10958-011-0260-7
10. B. Brandolini, F. C. Cîrstea, Anisotropic elliptic equations with gradient-dependent lower order terms and L^1 data, *Math. Eng.*, **5** (2023), 1–33. https://doi.org/10.3934/mine.2023073
11. E. N. Dancer, D. Daners, D. Hauer, A Liouville theorem for p-harmonic functions on exterior domains, *Positivity*, **19** (2015), 577–586. https://doi.org/10.1007/s11117-014-0316-2
12. E. Mitidieri, S. I. Pohozaev, Towards a unified approach to nonexistence of solutions for a class of differential inequalities, *Milan J. Math.*, **72** (2004), 129–162. https://doi.org/10.1007/s00032-004-0032-7
13. I. Birindelli, F. Demengel, Some Liouville theorems for the p-Laplacian, *Proceedings of the 2001 Luminy Conference on Quasilinear Elliptic and Parabolic Equations and System, Electron. J. Differ. Equ. Conf.*, **8** (2002), 35–46.
14. M. Chipot, *Elliptic Equations: An Introductory Course, Second edition*, Birkhäuser, 2024. https://doi.org/10.1007/978-3-031-54123-0
15. Y. Du, *Order Structure and Topological Methods in Nonlinear Partial Differential Equations*, World Scientific, Taipei, 2006.
16. A. Farina, C. Mercuri, M. Willem, A Liouville theorem for the p-Laplacian and related questions, *Calc. Var. Partial Differential Equations*, **58** (2019), 13. https://doi.org/10.1007/s00526-019-1596-y
17. M. Meier, Liouville theorem for nonlinear elliptic equations and systems, *Manuscripta Math.*, **29** (1979), 207–228. https://doi.org/10.1007/BF01303628

18. P. Pucci, J. Serrin, *The Maximum Principle*, volume 73 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, 2007.
19. P. Quittner, P. Souplet, *Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States*, Birkhäuser, 2007.
20. Y. Pinchover, A. Tertikas, K. Tintarev, A Liouville-type theorem for the p-laplacian with potential terms, *Ann. I. H. Poincaré*, **25** (2008), 357–368. https://doi.org/10.1016/j.anihpc.2006.12.004
21. J. Vétois, A priori estimates for solutions of anisotropic elliptic equations, *Nonlinear Anal.*, **71** (2009), 3881–3905. https://doi.org/10.1016/j.na.2009.02.076
22. J. Vétois, Strong maximum principles for anisotropic elliptic and parabolic equations, *Adv. Nonlinear Stud.*, **12** (2012), 101–114.
23. M. Chipot, *Elliptic Equations: An Introductory Course*, Birkhäuser, 2009. https://doi.org/10.1007/978-3-7643-9982-5
24. D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, volume 31 of *Classic Appl. Math.* SIAM, Philadelphia, 2000.