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Research article

Non linear equivalence of some scaling invariant norms for solutions of incompressible Navier-Stokes equations

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Abstract: Motivated by the study of the possible breakdown of regularity for solutions of the incompressible Navier-Stokes system in the whole three-dimensional space $\mathbb{R}^3$, we prove that the norm $\|u\|_{L^\infty_T(B^{\frac{1}{2}}_{2\infty})}$ and the quantity $\displaystyle \sup_{I\subset[0,T[} \frac{1}{\sqrt{|I|}} \int_I \|\nabla u(t)\|^2_{L^2} dt$ are non linearly equivalent. The proofs rely on the Duhamel formula, dyadic localization in the Fourier space, and for one of them, on the energy inequality.

Keywords: incompressible Navier-Stokes; Besov spaces
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1. Introduction

In this paper, we consider the three-dimensional incompressible Navier-Stokes system in $\mathbb{R}^3$

$$(NS) \quad \begin{cases} \partial_t u + u\cdot\nabla u - \Delta u = -\nabla p \\ \operatorname{div} u = 0 \quad \text{and} \quad u_{|t=0} = u_0 \,. \end{cases}$$

This system has two main properties:

1. The energy estimate: for any smooth enough solution on the interval $[t_0, t_1]$, we have

$$(E) \quad \frac{1}{2}\|u(t_1)\|^2_{L^2} + \int_{t_0}^{t_1} \|\nabla u(t)\|^2_{L^2} dt \leq \frac{1}{2}\|u(t_0)\|^2_{L^2}\,;$$

2. The scaling invariance: if u is a solution on an interval $[0, T)$, then for any positive λ, the vector field u_λ defined by

$$u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$$

is a solution of (NS) on the interval $[0, \lambda^{-2}T)$.

A basic and seminal result about Navier-Stokes equations is the following one, due to J. Leray (see [1]).

Theorem 1.1. *Let u_0 be a divergence-free vector field, the components of which belong to the Sobolev space $H^1(\mathbb{R}^3)$. Then there exists a unique maximal solution u on $[0, T^\star[$ continuous with value in $H^1(\mathbb{R}^3)$ that satisfies the energy inequalities (E). Moreover, a constant c exists such that*

$$T^\star \geq c\|\nabla u_0\|_{L^2}^{-4} \tag{1.1}$$

And we have

$$\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} \leq c \Longrightarrow T^\star = +\infty\,.$$

Let us point out that scaling invariant quantities play a key role in the study of non linear partial differential equations in general and in particular for (NS). Let us exhibit some scaling invariant quantities:

1. $\sup\limits_{t\leq T^\star} \sqrt{T^\star - t}\,\|\nabla u(t)\|_{L^2}^2\,,$
2. the norm $L^p([0,T]; H^{\frac{1}{2}+\frac{2}{p}})$ and $L^p([0,T]; L^q_x)$ with $\dfrac{2}{p} + \dfrac{3}{q} = 1$.

In [1], J. Leray conjectured that in some case, $T^\star$ is finite because of the existence of a solution of the form

$$u(t,x) = \frac{1}{\sqrt{T^\star - t}} U\Big(\frac{x}{\sqrt{T^\star - t}}\Big) \tag{1.2}$$

called a backward self-similar solution. In 1996, in [2], J. Nečas, M. Růžička and V. Šverák proved that such a solution must be identically 0. In 2003, L. Eskauriaza, G. Seregin and V. Šverák proved in [3] the following very important generalization:

$$T^\star < \infty \Longrightarrow \limsup_{t\to T^\star} \|u(t)\|_{L^3} = \infty\,. \tag{1.3}$$

In 2020, T. Tao proved in [4] an effective blow-up rate, namely

$$T^\star < \infty \Longrightarrow \limsup_{t\to T^\star} \frac{\|u(t)\|_{L^3}}{\Big(\log\log\log\Big(1 - \frac{t}{T^\star}\Big)\Big)^\theta} = \infty$$

for some positive constant θ. Then, in [5], T. Barker and C. Prange used this result in order to improve the assertion (1.3) by

$$T^\star < \infty \Longrightarrow \limsup_{t\to T^\star} \int_{\mathbb{R}^3} \frac{|u(t,x)|}{\Big(\log\log\log\log\Big(e^{e^{3e^e}} + |u(t,x)|\Big)\Big)^\theta} = \infty \tag{1.4}$$

for some positive constant θ.

Inequality (1.1) translated in time implies in particular that

$$\|\nabla u(t)\|_{L^2}^2 \geq \frac{c}{\sqrt{T^\star - t}}\,\cdot \tag{1.5}$$

One natural question we can address is the following: do we have

$$T^\star < \infty \Longrightarrow \limsup_{t\to T^\star} \sqrt{T^\star - t}\,\|\nabla u(t)\|_{L^2}^2 = \infty\,? \tag{1.6}$$

The purpose of this work is to realize that this question is strongly related to the following one: do we have

$$T^\star < \infty \Longrightarrow \limsup_{t\to T^\star} \|u(t)\|_{B^{\frac12}} = \infty \tag{1.7}$$

where $B^{\frac12}$ denotes the homogeneous Besov space $\dot B^{\frac12}_{2,\infty}$? For the reader's convenience, let us recall the definition of homogeneous Besov spaces and define the norm we are going to use (see for instance, [6] for more details about these spaces).

Definition 1.2. *For a real number, let us define the norm $B^s_{2,r}$ as the space of distributions such that*

$$\|u\|_{\dot B^s_{2,r}} \stackrel{\text{def}}{=} \big\|(2^{js}\Delta_j u)_{j\in\mathbb{Z}}\big\|_{\ell^r(\mathbb{Z})} \quad \textit{with} \quad \Delta_j u \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{2^j\le|\xi|\le 2^{j+1}}u).$$

It is obvious that, in the case when r equals to 2, the norm $\|\cdot\|_{\dot B^s_{2,2}}$ is equivalent to the homogeneous Sobolev norm $\displaystyle\int_{\mathbb{R}^d}|\xi|^{2s}|\widehat{u}(\xi)|^2d\xi$. Let us also notice that the function

$$r \longmapsto \|u\|_{\dot B^s_{2,r}}$$

is a non increasing function. Because of the Sobolev embedding of $H^{\frac12}$ into L^3, Assertion (1.3) implies in particular that

$$T^\star < \infty \Longrightarrow \limsup_{t\to T^\star} \|u(t)\|_{H^{\frac12}} = \infty\,.$$

For s between 0 and 1, the norm $\|\cdot\|_{B^s_{2,\infty}}$ can be expressed in terms of physical space. A constant C exists such that

$$C^{-1}\|u\|_{\dot B^s_{2,\infty}} \le \sup_{y\in\mathbb{R}^d\setminus\{0\}} \frac{\|u(\cdot - y) - u\|_{L^2}}{|y|^s} \le C\|u\|_{\dot B^s_{2,\infty}}.$$

Let us notice that the two questions (1.6) and (1.7) are quite different in the following sense: the space $B^{\frac12}$ contains homogeneous functions of degree -1, which is not the case neither for L^3 involded in (1.3) nor for the space used in (1.4). Let us notice that the space $B^{\frac12}$ is continuously included in the weak L^3 space we denote by L^3_w. In [7], T. Barker, G. Seregin, and V. Šverák prove some weak stability results for bounded in-time solutions with value in L^3_w and also uniqueness for solutions which are continuous in time with value in L^3_w.

Now let us first define

$$(\star) \qquad \mathcal{N}_T(u) \stackrel{\text{def}}{=} \sup_{I\subset[0,T[} \frac{1}{\sqrt{|I|}}\int_I \|\nabla u(t)\|_{L^2}^2 dt \le C\,.$$

Let us make some comments about this quantity. First of all, for any interval $I = [t_0, t_1]$ included in $[0, T]$,

$$\int_{t_0}^{t_1} \|\nabla u(t)\|_{L^2}^2 dt \quad\le\quad \Big(\sup_{t\le T}\sqrt{T^\star - t}\,\|\nabla u(t)\|_{L^2}^2\Big)\int_{t_0}^{t_1}\frac{1}{\sqrt{T^\star - t}}dt$$

$$\leq \frac{1}{2}\Big(\sup_{t\leq T}\sqrt{T^\star - t}\,\|\nabla u(t)\|_{L^2}^2\Big)\big(\sqrt{T^\star - t_0} - \sqrt{T^\star - t_1}\big)$$
$$\leq \frac{1}{2}\Big(\sup_{t\leq T}\sqrt{T^\star - t}\,\|\nabla u(t)\|_{L^2}^2\Big)\sqrt{t_1 - t_0}\,.$$

Thus we have

$$\mathcal{N}_{T^\star}(u) \leq \sup_{t\leq T^\star}\sqrt{T^\star - t}\,\|\nabla u(t)\|_{L^2}^2. \tag{1.8}$$

Moreover, thanks to the Sobolev embeddings, we have

$$\sup_{I\subset[0,T[}\frac{1}{\sqrt{|I|}}\int_I\Big(\int_{B(x_0,\sqrt{|I|})}|u(t,x)|^6dx\Big)^{\frac{1}{3}}dt \leq C\mathcal{N}_T(u).$$

The quantity on the left is a particular case of a family of quantities that ensures local regularity when they are small enough (see Theorem 6 of [8]).

The purpose of this article is to prove that the two quantities $\|u\|_{L^\infty_T(B^{\frac{1}{2}})}$ and $\mathcal{N}_T(u)$ are equivalent but non linearly, in the sense of the following theorem.

Theorem 1.3. *Let us consider a regular solution of* (NS) *on a time interval* $[0,T[$. *Then a constant* C *exists such that*

$$\|u\|_{L^\infty([0,T[;B^{\frac{1}{2}})} \leq (2\|u_0\|_{B^{\frac{1}{2}}} + C\mathcal{N}_T(u))\sqrt{\log(\|u_0\|_{B^{\frac{1}{2}}} + CT\mathcal{N}_t(u) + e)} \tag{1.9}$$

Conversely, a constant C *exists such that*

$$\mathcal{N}_T(u) \leq C\Big(\|u\|^2_{L^\infty(I;B^{\frac{1}{2}})} + \|u\|^3_{L^\infty(I;B^{\frac{1}{2}})}\sqrt{\log(2C\|u\|_{L^\infty(I;B^{\frac{1}{2}})})}\Big). \tag{1.10}$$

The proof of this theorem will be the purpose of the second section. Before doing it, let us make some remarks and state some corollaries.

First of all, let us notice that Inequalities (1.8) and (1.9) imply that

$$\sup_{t\leq T^\star}\sqrt{T^\star - t}\,\|\nabla u(t)\|_{L^2}^2 < \infty \Longrightarrow u\in L^\infty([0,T^\star[;B^{\frac{1}{2}}).$$

Let us notice that E. Poulon proved in [9] an analog of this result using profile theory. Let us also mention that G. Seregin and D. Zhou proved in [10] an analog of inequality (1.10) where the norm $\|\cdot\|_{L^\infty(I;B^{\frac{1}{2}})}$ is replaced by the $\|\cdot\|_{L^\infty(I;B^{-1}_{\infty,\infty})}$ norm (which is smaller) but with term on the left-handside being local in space norms instead of global in space here.

Corollary 1.4. *Let us consider a semi-regular solution of* (NS) *on the maximal time interval* $[0,T^\star[$ *which in addition belongs to the space* $L^\infty([0,T^\star[;B^{\frac{1}{2}})$. *If* $T^\star$ *is finite, a constant* C *exists such that*

$$\liminf_{t\to T^\star}\sqrt{T^\star - t}\,\|\nabla u(t)\|_{L^2}^2 \leq C\|u\|^2_{L^\infty(I;B^{\frac{1}{2}})}\Big(1 + \|u\|_{L^\infty([0,T^\star[;B^{\frac{1}{2}})}\sqrt{\log(2C\|u\|_{L^\infty([0,T^\star[;B^{\frac{1}{2}})})}\Big).$$

Proof. Let A be a positive real number such that

$$\exists t_A\ /\ \forall t\geq t_A\,,\ \|\nabla u(t)\|_{L^2}^2 \geq \frac{A^2}{\sqrt{T^\star - t}}\,\cdot \tag{1.11}$$

Then, we have

$$\begin{aligned}\mathcal{N}_{T^\star}(u) &\geq \frac{1}{\sqrt{T^\star - t_A}}\int_{t_A}^{T^\star} \|\nabla u(t)\|_{L^2}^2 dt\\ &\geq \frac{1}{\sqrt{T^\star - t_A}}\int_{t_A}^{T^\star} \frac{A^2}{\sqrt{T^\star - t'}}\, dt'\\ &\geq \frac{A^2}{2}\cdot\end{aligned}$$

As the inequality is true for any real number A satisfying (1.11), this implies that

$$\mathcal{N}_{T^\star}(u) \geq \frac{1}{2}\liminf_{t\to T^\star} \|\nabla u(t)\|_{L^2}^2\,.$$

This proves the corollary. □

Let us notice that, if the solution u remains bounded in $B^{\frac{1}{2}}_{2,\infty}$, and if $\sqrt{T^\star - t}\,\|\nabla u(t)\|_{L^2}^2$ does not remain bounded; this implies that this quantity must oscillate between a finite quantity and larger and larger values.

Corollary 1.5. *Let us consider a semi-regular solution of* (NS) *on the maximal time interval* $[0, T^\star[$ *which in addition belongs to the space* $L^\infty([0, T^\star[; B^{\frac{1}{2}})$*. Then for any subinterval* $I = [t_1, t_0]$ *of* $[0, T^\star[$,

$$\frac{1}{\sqrt{|I|}}\int_I \|u(t)\|_{\dot B^{\frac{3}{2}}_{2,1}}\, dt \leq C\Big(\|u\|^2_{L^\infty(I;B^{\frac{1}{2}})} + \|u\|^3_{L^\infty(I;B^{\frac{1}{2}})}\sqrt{\log(2C\|u\|_{L^\infty(I;B^{\frac{1}{2}})})}\Big).$$

We shall prove this corollary at the beginning of the next section as a warm-up. Let us simply remark that, thanks to Fourier inversion theorem, we have the L^∞ norm is less than or equal to a constant times the $B^{\frac{3}{2}}_{2,1}$ norm. Thus, the Corollary implies in particular that

$$\frac{1}{\sqrt{|I|}}\int_I \|u(t)\|_{L^\infty} dt \lesssim \|u\|^2_{L^\infty(I;B^{\frac{1}{2}})} + \|u\|^3_{L^\infty(I;B^{\frac{1}{2}})}\sqrt{\log(2C\|u\|_{L^\infty(I;B^{\frac{1}{2}})})}.$$

2. Proof of Theorem 1.3

The proofs presented here have in common:

- the use of frequency localization operator introduced in Definition 1.2 applied to the Duhamel formula,
- the use of a cutoff in the non linear term.

More precisely the non linear term writes after a use of the Duhamel formula and in terms of Fourier transform

$$\int_{t_0}^{t_1} e^{-(t_1-t')|\xi|^2}\mathcal{F}(\mathbb{P}\,\mathrm{div}(u(t')\otimes u(t')))(\xi)dt' \tag{2.1}$$

In terms of frequency localization, it can be translated as

$$\forall t\in[t_0,t_1]\,,\quad \begin{aligned}&\|\Delta_j u(t)\|_{L^2} \leq e^{-(t-t_0)2^{2j}}\|\Delta_j u(t_0)\|_{L^2} + I_j(t) \quad\text{with}\\ &\mathcal{I}_j(t) \overset{\text{def}}{=} \int_{t_0}^{t} e^{-(t_1-t')2^{2j}} 2^j \big\|\Delta_j(u(t')\otimes u(t'))\big\|_{L^2} dt'.\end{aligned} \tag{2.2}$$

The idea is to consider a part of the integral where the quantity $(t-t')2^{2j}$ is large and another one where the quantity $(t-t')2^{2j}$ is bounded.

Proof of Corollary 1.5. It is close to the proof of Lemma 3.2 of [11]. We use the product law as stated, for instance, in Theorem 2.52 on page 92 of [6]):

$$\|ab\|_{B^{\frac12}_{2,1}} \leq C\|\nabla a\|_{L^2}\|\nabla b\|_{L^2}.$$

Plugging this into (2.2) and multiplying by $2^{\frac{3j}{2}}$, we infer that, for any t in the interval $[t_0, t_1]$,

$$\begin{aligned} 2^{\frac{3j}{2}}\|\Delta_j u(t)\|_{L^2} &\leq e^{-t2^{2j}}2^{\frac{3j}{2}}\|\Delta_j u(t_0)\|_{L^2} + \int_{t_0}^t e^{-(t-t')2^{2j}}2^{\frac{5j}{2}}\big\|\Delta_j(u(t')\otimes u(t'))\big\|_{L^2}dt' \\ &\leq e^{-t2^{2j}}2^{\frac{3j}{2}}\|\Delta_j u(t_0)\|_{L^2} + \int_{t_0}^t e^{-(t-t')2^{2j}}2^{2j}d_j(t')\|\nabla u(t')\|_{L^2}^2dt' \end{aligned}$$

where, for all time t, $(d_j(t))_{j\in\mathbb{Z}}$ is a function of time with value in the sphere of $\ell^1(\mathbb{Z})$. Then, by integration on the time interval $I \stackrel{\text{def}}{=} [t_0, t_1]$, we infer that

$$\begin{aligned} \int_I 2^{\frac{3j}{2}}\|\Delta_j u(t)\|_{L^2}dt \leq \Big(\int_I e^{-t2^{2j}}dt\Big)2^{\frac{3j}{2}}\|\Delta_j u(t_0)\|_{L^2} \\ + \int_{\mathbb{R}^2} e^{-(t-t')2^{2j}}2^{2j}d_j(t')\mathbf{1}_{t'\leq t}(t',t)\mathbf{1}_I(t')\|\nabla u(t')\|_{L^2}^2dt'dt. \end{aligned}$$

Integrating first with respect to the variable t in the non linear term gives

$$\int_I 2^{\frac{3j}{2}}\|\Delta_j u(t)\|_{L^2}dt \leq \Big(\int_I e^{-t2^{2j}}dt\Big)2^{\frac{3j}{2}}\|\Delta_j u(t_0)\|_{L^2} + \int_I\sum_j d_j(t')\|\nabla u(t')\|_{L^2}^2dt'.$$

Taking the sum over j, we infer, thanks to permutation of sum and integral and by definition of the $\|\cdot\|_{B^{\frac32}_{21}}$ and $\|\cdot\|_{B^{\frac12}}$, that

$$\begin{aligned} \int_I \|u(t)\|_{B^{\frac32}_{2,1}}dt &\leq \int_I\sum_j e^{-t2^{2j}}2^j2^{\frac{j}{2}}\|\Delta_j u(t_0)\|_{L^2} + \int_I\|\nabla u(t')\|_{L^2}^2dt' \\ &\leq \|u_0\|_{B^{\frac12}}\int_I\sum_j e^{-t2^{2j}}2^jdt + \int_I\|\nabla u(t')\|_{L^2}^2dt'. \end{aligned}$$

Using that $\displaystyle\sum_j e^{-t2^{2j}}2^j \lesssim \frac{1}{\sqrt{t}}$, we infer that

$$\begin{aligned} \frac{1}{\sqrt{|I|}}\int_I 2^{\frac{3j}{2}}\|\Delta_j u(t)\|_{L^2}dt &\leq C\frac{\sqrt{t_1}-\sqrt{t_0}}{\sqrt{t_1-t_0}}\|u_0\|_{B^{\frac12}} + \int_I\|\nabla u(t')\|_{L^2}^2dt' \\ &\leq C\|u_0\|_{B^{\frac12}} + \frac{1}{\sqrt{|I|}}\int_I\|\nabla u(t')\|_{L^2}^2dt'. \end{aligned}$$

The corollary is proved using inequality (1.10). □

Proof of inequality (1.9). We decompose $\mathcal{I}_j$ of inequality (2.2) as $\mathcal{I}_j = \mathcal{J}_j(t) + \mathcal{K}_j(t)$ with

$$\begin{aligned} \mathcal{J}_j(t) &\stackrel{\text{def}}{=} \int_0^{t-2\Lambda^2 2^{-2j}} e^{-(t-t')2^{2j}} 2^{\frac{3j}{2}} \big\|\Delta_j(u(t')\otimes u(t'))\big\|_{L^2} dt' \quad \text{and} \\ \mathcal{K}_j(t) &\stackrel{\text{def}}{=} \int_{t-2\Lambda^2 2^{-2j}}^{t} e^{-(t-t')2^{2j}} 2^{\frac{3j}{2}} \big\|\Delta_j(u(t')\otimes u(t'))\big\|_{L^2} dt' \end{aligned} \tag{2.3}$$

where Λ is a positive real number that will be chosen later on. Product law in Besov spaces (see for instance Theorem 2.52, page 92 of [6]) implies that, for any t' in $[0,T[$,

$$2^{\frac{3j}{2}} \big\|\Delta_j(u(t')\otimes u(t'))\big\|_{L^2} \leq C 2^{2j} \|u(t')\|^2_{B^{\frac{1}{2}}}.$$

We infer that

$$\begin{aligned} \mathcal{J}_j(t) &\leq Ce^{-\Lambda^2} \|u\|^2_{L^\infty([0,t];B^{\frac{1}{2}})} 2^{2j} \int_0^t e^{-(t-t')2^{2j-1}} dt' \\ &\leq Ce^{-\Lambda^2} \|u\|^2_{L^\infty([0,t];B^{\frac{1}{2}})}. \end{aligned} \tag{2.4}$$

Using again product law in Besov spaces, we can write

$$\mathcal{K}_j(t) \leq C \int_{t-2\Lambda^2 2^{-2j}}^{t} e^{-(t-t')2^{2j}} 2^j \|u(t')\|^2_{B^1} dt'. \tag{2.5}$$

As the B^1-norm is less than the homogeneous H^1 norm, we infer that

$$\mathcal{K}_j(t) \leq C\Lambda \mathcal{N}_t(u).$$

Then, thanks to (2.2)–(2.4), we get, by definition of the $\|\cdot\|_{B^{\frac{1}{2}}}$ norm,

$$\|u\|_{L^\infty([0,t];B^{\frac{1}{2}})} \leq \|u_0\|_{B^{\frac{1}{2}}} + C_0\Lambda\mathcal{N}_t(u) + e^{-\Lambda^2} \|u\|^2_{L^\infty([0,t];B^{\frac{1}{2}})}. \tag{2.6}$$

Let us define

$$\begin{aligned} M(t) &\stackrel{\text{def}}{=} 2(\|u_0\|_{B^{\frac{1}{2}}} + C_0\mathcal{N}_t(u)) \log^{\frac{1}{2}}(\|u_0\|_{B^{\frac{1}{2}}} + C_0\mathcal{N}_t(u) + e) \quad \text{and} \\ \Lambda(t) &\stackrel{\text{def}}{=} \sqrt{\log(2M(t))}. \end{aligned}$$

Now let us introduce the time $\widetilde{T}$ such that

$$\widetilde{T} \stackrel{\text{def}}{=} \sup\{t \in [0,T[\,,\ \|u\|_{L^\infty([0,t];B^{\frac{1}{2}})} \leq M(t)\}.$$

By definition of $\Lambda(t)$ and $M(t)$, we get from (2.6), that, for any t less than $\widetilde{T}$,

$$\begin{aligned} \|u\|_{L^\infty([0,t];B^{\frac{1}{2}})} &\leq \|u_0\|_{B^{\frac{1}{2}}} + C_0\mathcal{N}_t(u) \log^{\frac{1}{2}}(2M(t)) + \frac{1}{2}M(t) \\ &\leq \|u_0\|_{B^{\frac{1}{2}}} + \frac{1}{2}M(t) + C_0\mathcal{N}_t(u)\Big(\log^{\frac{1}{2}}(4\|u_0\|_{B^{\frac{1}{2}}} + 2C_0\mathcal{N}_t(u) + e) \\ &\qquad + \log^{\frac{1}{2}}(\log^{\frac{1}{2}}(\|u_0\|_{B^{\frac{1}{2}}} + C_0\mathcal{N}_t(u) + e)\Big). \end{aligned}$$

□

Proof of inequality (1.10). This proof, in contrast with the two previous ones used the energy estimate in a crucial way to go from a purely supremum in time norm (the norm $\|\cdot\|_{L^\infty_T(B^{\frac12})}$) into a norm which includes local in-time energy. More precisely, the energy inequality between t_1 and t_0 (for $t_1 \geq t_0$) gives

$$\begin{aligned}\int_{t_0}^{t_1}\|\nabla u(t')\|_{L^2}^2 dt' &\leq \frac12\|u(t_0)\|_{L^2}^2 - \frac12\|u(t_1)\|_{L^2}^2\\ &\leq \frac12\big(u(t_0)-u(t_1)\big|u(t_0)+u(t_1)\big)_{L^2}.\end{aligned}$$

Now let us use the duality between the two spaces $B^{-\frac12}_{2,1}$ and $B^{\frac12}_{2,\infty}$; this consists simply in writing that

$$\begin{aligned}\frac12\big(u(t_0)-u(t_1)\big|u(t_0)+u(t_1)\big)_{L^2} &\leq \frac12\sum_j\Big(2^{-\frac j2}\Delta_j(u(t_0)-u(t_1))\Big|2^{\frac j2}(\Delta_j u(t_1)+\Delta_j u(t_0))\Big)_{L^2}\\ &\leq \|u(t_0)-u(t_1)\|_{\dot B^{-\frac12}_{2,1}}\|u\|_{L^\infty_T(B^{\frac12})}.\end{aligned} \tag{2.7}$$

Now let us estimate $\Delta(t_0,t_1) \stackrel{\text{def}}{=} \|u(t_0)-u(t_1)\|_{\dot B^{-\frac12}_{2,1}}$. In order to do it, let us write that

$$u(t_1)-u(t_0) = (e^{(t_1-t_0)\Delta}-\mathrm{Id})u(t_0) + \int_{t_0}^{t_1} e^{(t_1-t')\Delta}\,\mathbb{P}\,\mathrm{div}(u(t')\otimes u(t'))dt'. \tag{2.8}$$

Let us first estimate the quantity $\Big\|(e^{(t_1-t_0)\Delta}-\mathrm{Id})u(t_0)\Big\|_{\dot B^{-\frac12}_{2,1}}$.

Lemma 2.1. *We have the following estimate:*

$$\Big\|e^{\tau\Delta}v - v\|_{\dot B^{-\frac12}_{2,1}} \leq 2\sqrt{\tau}\|v\|_{\dot B^{\frac12}}.$$

Proof. Using Fourier-Plancherel formula, let us write that

$$\sum_j 2^{-\frac j2}\Big\|\Delta_j(e^{\tau\Delta}v-v)\Big\|_{L^2} = (2\pi)^{\frac32}\sum_j 2^{-\frac j2}\Big\|(1-e^{-\tau 2^{2j}})\widehat{v}\Big\|_{L^2(2^jC)}.$$

Splitting the sum into a "low frequency" part and a "high frequency" part, we can write

$$\begin{aligned}\sum_j 2^{-\frac j2}\Big\|\Delta_j(e^{\tau\Delta}v-v)\Big\|_{L^2} &\leq (2\pi)^{\frac32}\sum_{\sqrt{\tau}2^j\leq 1}\tau 2^{\frac{3j}{2}}\|\widehat{v}\|_{L^2(2^jC)} + (2\pi)^{\frac32}\sum_{\sqrt{\tau}2^j\geq 1}2^{-\frac j2}\|\widehat{v}\|_{L^2(2^jC)}\\ &\leq \sum_{\sqrt{\tau}2^j\leq 1}\tau 2^j 2^{\frac j2}\|\Delta_j v\|_{L^2} + \sum_{\sqrt{\tau}2^j\geq 1}2^{-j}2^{\frac j2}\|\Delta_j v\|_{L^2}\\ &\leq \Big(\sum_{\sqrt{\tau}2^j\leq 1}\tau 2^j + \sum_{\sqrt{\tau}2^j\geq 1}2^{-j}\Big)\|v\|_{B^{\frac12}}.\end{aligned}$$

The lemma is proved. □

Continuation of the proof of inequality (1.10) Plugging the result of the above lemma in Equality (2.8) gives,

$$\Delta(t_0,t_1) \leq 2\sqrt{t_1-t_0}\|u(t_0)\|_{B^{\frac12}} + \Delta_{\rm NL}(t_0,t_1) \quad \text{with}$$
$$\Delta_{\rm NL}(t_0,t_1) \stackrel{\rm def}{=} \Big\|\int_{t_0}^{t_1} e^{(t_1-t')\Delta}\,\mathbb{P}\,{\rm div}(u(t')\otimes u(t'))dt'\Big\|_{\dot B^{-\frac12}_{2,1}}. \tag{2.9}$$

Again, the estimate of $\Delta_{\rm NL}(t_0,t_1)$ relies on the splitting into a "low frequency" part and a "high frequency" part. Let Λ be a positive real parameter that will be chosen later on. Let us write that, by definition of the $\|\cdot\|_{B^{-\frac12}_{2,1}}$ norm,

$$\begin{aligned}
\Delta_{\rm NL}(t_0,t_1) &\leq \int_{t_0}^{t_1}\sum_j e^{-(t_1-t)2^{2j}}2^{-\frac j2}2^j\big\|\Delta_j(u(t)\otimes u(t))\big\|_{L^2}dt\\
&\leq \Delta^\flat_{\rm NL}(t_0,t_1)+\Delta^\sharp_{\rm NL}(t_0,t_1) \quad \text{with}\\
\Delta^\flat_{\rm NL}(t_0,t_1) &\stackrel{\rm def}{=} \int_{t_0}^{t_1}\sum_{2^j\sqrt{t_1-t}\leq\Lambda} 2^{\frac j2}\big\|\Delta_j(u(t)\otimes u(t))\big\|_{L^2}dt \quad \text{and}\\
\Delta^\sharp_{\rm NL}(t_0,t_1) &\stackrel{\rm def}{=} \int_{t_0}^{t_1}\sum_{2^j\sqrt{t_1-t}\geq\Lambda} e^{-(t_1-t)2^{2j}}2^{\frac j2}\big\|\Delta_j(u(t)\otimes u(t))\big\|_{L^2}dt\,.
\end{aligned} \tag{2.10}$$

Product law in Besov spaces as stated for instance in Theorem 2.52 page 92 of [6] implies that

$$\big\|\Delta_j(u(t)\otimes u(t))\|_{L^2} \leq C2^{\frac j2}\|u\|^2_{L^\infty(I;B^{\frac12})} \quad \text{with} \quad I \stackrel{\rm def}{=} [t_0,t_1].$$

By definition of $\Delta^\flat_{\rm NL}(t_0,t_1)$, we infer that

$$\begin{aligned}
\Delta^\flat_{\rm NL}(t_0,t_1) &\leq C\|u\|^2_{L^\infty(I;B^{\frac12})}\int_{t_0}^{t_1}\Big(\sum_{2^j\sqrt{t_1-t}\leq\Lambda}2^j\Big)dt\\
&\leq C\|u\|^2_{L^\infty(I;B^{\frac12})}\int_{t_0}^{t_1}\frac{\Lambda}{\sqrt{t_1-t}}dt\\
&\leq C\|u\|^2_{L^\infty(I;B^{\frac12})}\Lambda\sqrt{|I|}.
\end{aligned} \tag{2.11}$$

By definition of $\Delta^\sharp_{\rm NL}(t_0,t_1)$ we have

$$\Delta^\sharp_{\rm NL}(t_0,t_1) \leq e^{-\Lambda^2}\int_{t_0}^{t_1}\sum_j 2^{\frac j2}\big\|\Delta_j(u(t)\otimes u(t))\|_{L^2}dt.$$

Using the product law $\|ab\|_{B^{\frac12}_{2,1}} \leq C\|\nabla a\|_{L^2}\|\nabla b\|_{L^2}$ (see for instance, Theorem 2.52 page 92 of [6]), we infer that

$$\Delta^\sharp_{\rm NL}(t_0,t_1) \leq Ce^{-\Lambda^2}\int_{t_0}^{t_1}\|\nabla u(t)\|^2_{L^2}dt.$$

Plugging this inequality and inequality (2.11) into inequality (2.10), we get that, for any positive real number Λ,

$$\Delta_{\rm NL}(t_0,t_1) \leq C\|u\|^2_{L^\infty(I;B^{\frac12})}\Lambda\sqrt{|I|} + Ce^{-\Lambda^2}\int_{t_0}^{t_1}\|\nabla u(t')\|^2_{L^2}dt'.$$

Plugging this inequality in inequality (2.9) gives, for any positive real number Λ,

$$\Delta(t_0, t_1) \leq C\sqrt{t_1 - t_0}(\|u(t_0)\|_{B^{\frac{1}{2}}} + \|u\|^2_{L^\infty(I;B^{\frac{1}{2}})}\Lambda) + Ce^{-\Lambda^2}\int_{t_0}^{t_1} \|\nabla u(t)\|^2_{L^2} dt.$$

Then, using inequality (2.7), we infer that, for any positive real number Λ,

$$\int_{t_0}^{t_1} \|\nabla u(t)\|^2_{L^2} dt \leq C\sqrt{t_1 - t_0}(\|u\|^2_{L^\infty(I;B^{\frac{1}{2}})} + \|u\|^3_{L^\infty(I;B^{\frac{1}{2}})}\Lambda) + C\|u\|_{L^\infty(I;B^{\frac{1}{2}})} e^{-\Lambda^2}\int_{t_0}^{t_1} \|\nabla u(t)\|^2_{L^2} dt.$$

Then, choosing $\Lambda = \sqrt{\log(2C\|u\|_{L^\infty(I;B^{\frac{1}{2}})})}$ ensures the result. □

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declare no conflict of interest.

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