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Research article

Stability of Navier-Stokes-Oseen flows

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Abstract: This paper studies the stability of a weak mild solution of the Navier–Stokes–Oseen equations in the solenoidal Lorentz space $L^3_{\sigma,\mathrm{w}}$. Our approach relies on dual space pair and suitable estimates in our setting for the Oseen semigroup. Therefore, we get a new result for the stability of a weak mild solution following the initial datum and external force.

Keywords: stability; Navier–Stokes–Oseen equations; Oseen operator; rotating and translating obstacle; solenoidal Lorentz spaces
Mathematics Subject Classification: 35B35, 35Q30, 35Q35, 76D07

1. Introduction

Let Ω be an exterior domain with a smooth boundary complemented by an obstacle in $\mathbb{R}^3$. We are concerned with the Navier–Stokes–Oseen equations

$$\begin{cases} D_t u + (u\cdot\nabla)u - \Delta u + kD_3 u & & \\ -((\omega\times x)\cdot\nabla)u + \omega\times u + \nabla p = \operatorname{div} F & & \text{in } \Omega\times(0,\infty), \\ \operatorname{div} u = 0 & & \text{in } \Omega\times(0,\infty), \\ u(x,t) = \omega\times x - u_\infty & & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & & \text{in } \Omega, \\ \lim_{|x|\to\infty} u(x,t) = 0 & & \text{for all } t\in(0,\infty). \end{cases} \tag{1.1}$$

This system describes the dynamics of incompressible viscous fluid flows passing a translating and rotating obstacle, in which $\omega = a\mathbf{e}_3, \mathbf{e}_3 = (0,0,1)^T$ and $u_\infty = k\mathbf{e}_3$ are, respectively, the angular velocity and the translational velocity of an obstacle; $u = u(x,t) = (u_1,u_2,u_3)$ is the velocity field of the fluid;

$p = p(x,t)$ is the pressure of the fluid; and $F = F(x,t) = (F_{js})_{j,s=1,2,3}$ is the external force. Here $D_t = \partial/\partial_t$ and $\nabla = (D_1, D_2, D_3)^T$ with $D_i = \partial/\partial_{x_i}$, $i = 1,2,3$. Note that $\text{div}F = (\sum_{s=1}^{3} D_s F_{js})_{j=1,2,3}$. Considering the case of fixed obstacles, i.e., $a = k = 0$, then this system becomes the Navier–Stokes equations.

To study the system (1.1), a common approach is to use the Helmholtz projection to eliminate the pressure function. Applying the Helmholtz projection $\mathbb{P}$ into the system (1.1), we have

$$\begin{cases} D_t u + \mathbb{P}((u\cdot\nabla)u) + \mathcal{L}_{a,k}u = \mathbb{P}\text{div}F & \text{in } \Omega\times(0,\infty), \\ \text{div}\, u = 0 & \text{in } \Omega\times(0,\infty), \\ u(x,t) = \omega\times x - u_\infty & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = \mathbb{P}u_0(x) & x\in\Omega, \end{cases} \tag{1.2}$$

where $\mathcal{L}_{a,k}u = \mathbb{P}\left[-\Delta u + kD_3 u - ((\omega\times x)\cdot\nabla)u + \omega\times u\right]$. We call the operator $\mathcal{L}_{a,k}$ *the Oseen operator*. See Section 2 for the Helmholtz projection and the Oseen operators. By $\text{div}\, u = 0$, the system (1.2) is rewritten as follows:

$$\begin{cases} D_t u + \mathcal{L}_{a,k}u = \mathbb{P}\text{div}(F - u\otimes u) & \text{in } \Omega\times(0,\infty), \\ u(x,t) = \omega\times x - u_\infty & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = \mathbb{P}u_0(x) & x\in\Omega. \end{cases} \tag{1.3}$$

Many authors have studied the Navier–Stokes–Oseen equations, so hard to give a complete list of research results on this topic. Therefore, we review only some previous works related to our purposes. In the case $a = k = 0$, Kozono and Shimizu [1] have proved the unique existence of global mild solutions with small initial data in the solenoidal Lorentz spaces $L^p_{\sigma,\text{w}}(\mathbb{R}^n)$, and the unique existence of time-global weak mild solutions with small initial data in the solenoidal Lorentz spaces $L^n_{\sigma,\text{w}}(\Omega), n\geq 3$ was shown by Yamazaki [2]. In the case $k = 0$, the unique existence of time-local mild solutions in the spaces $L^p_\sigma(\Omega)$ have been proved by Geissert, Heck and Hieber [3]. Duoc [4] showed the unique existence of time-local mild solutions to the system in (1.3) in the solenoidal Lorentz spaces $L^{3,q}_\sigma(\Omega), q<\infty$ and the unique existence of time-global weak mild solutions to the system (1.3) in the solenoidal Lorentz space $L^3_{\sigma,\text{w}}(\Omega)$. In addition, the unique existence of time-global mild solutions of (1.3) in the solenoidal Lorentz spaces $L^3_{\sigma,\text{w}}(\Omega)$ was proved by [5].

Let $\tilde{u} \in C_b((0,\infty), L^3_{\sigma,\text{w}}(\Omega))$ be the weak mild solution of the system (1.3) corresponding to the external force $\tilde{F}$, where $C_b((0,\infty), L^3_{\sigma,\text{w}}(\Omega)) = \{u : (0,\infty) \to L^3_{\sigma,\text{w}}(\Omega)$ is a continuous function such that $\sup\limits_{t>0}\|u(t)\|_{3,\text{w}} < \infty\}$. We note that the unique existence of the solution $\tilde{u}\in C_b((0,\infty), L^3_{\sigma,\text{w}}(\Omega))$ is guaranteed by [4]. Our goal in this paper is to show the stability of the solution $\tilde{u}$ in the solenoidal Lorentz space $L^3_{\sigma,\text{w}}(\Omega)$ following initial datum and external force. To study the stability of the solution $\tilde{u}$ in the solenoidal Lorentz space $L^3_{\sigma,\text{w}}(\Omega)$, we set $z(x,t) = u(x,t) - \tilde{u}(x,t)$ and $G = F - \tilde{F}$. It is easy to check that z satisfies the following system:

$$\begin{cases} D_t z + \mathcal{L}_{a,k}z = \mathbb{P}\text{div}(G - z\otimes z - \tilde{u}\otimes z - z\otimes\tilde{u}) & \text{in } \Omega\times(0,\infty), \\ z(x,t) = 0 & \text{on } \partial\Omega\times(0,\infty), \\ z(x,0) = z_0(x) & x\in\Omega,\ z_0\in L^3_{\sigma,\text{w}}(\Omega). \end{cases} \tag{1.4}$$

We now study the system in (1.4). Using the dual space pairs and observing the estimates of the Oseen semigroup, we establish the unique existence and properties of the solution z. From that, we get the results for the stability of $\tilde{u}$. Thus, this paper is organized as follows. Section 2 is designed to provide some preliminaries about the Oseen operators and solenoidal Lorentz spaces. In Section 3, we recall the definition of a weak mild solution of the system (1.4) and then prove our main results in this paper.

2. Preliminaries

In this section, we recall the definition of solenoidal Lorentz spaces and provide some properties of strongly continuous semigroups generated by Oseen operators.

2.1. Solenoidal Lorentz spaces and Helmholtz projection

For $1 \le r \le \infty$ and $1 \le q \le \infty$, let $L^{r,q}(\Omega)$ denote the Lorentz space on Ω defined by

$$L^{r,q}(\Omega) = \{f \in L^1(\Omega) + L^\infty(\Omega) : \|f\|_{r,q} < \infty\},$$

with the norm

$$\|f\|_{r,q} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{r}} f^{**}(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 1 \le q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{r}} f^{**}(t) & \text{if } q = \infty. \end{cases}$$

Here $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$, $f^*(t) = \inf\{s > 0 : m(\{x \in \Omega : |f(x)| > s\}) \le t\}$, for $t \ge 0$, and m denotes the 3-dimensional Lebesgue measure.

Note that $L^{r,r}(\Omega) = L^r(\Omega)$ for $r \in (1,\infty]$ and $L^{1,\infty}(\Omega) = L^1(\Omega)$. Moreover, $L^{r,\infty}(\Omega), r \in (1,\infty)$, is called the weak-$L^r$ space and is denoted by $L^r_{\mathrm{w}}(\Omega) := L^{r,\infty}(\Omega), \|\cdot\|_{r,\mathrm{w}} := \|\cdot\|_{r,\infty}$. In addition, the Lorentz space is also defined for $r \in (0,1), q \in (0,\infty]$ and $r \in [1,\infty], q \in (0,1)$ (see Komatsu [6]).

On the other hand, for $1 \le q \le \infty$, the Lorentz spaces can be described by using interpolation pairs as follows:

$$L^{r,q}(\Omega) = (L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,q} \quad \text{for} \quad \frac{1}{r} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ with } 1 < r < \infty \text{ and } 0 < \theta < 1.$$

Readers can refer to [6–9] for the definition of Lorentz spaces and the properties of these spaces. From [4, Lemma 1.1], we obtain

Lemma 2.1. *Let $1 \le p, p_1, p_2 \le \infty$, and $1 \le q, q_1, q_2 \le \infty$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. If $f \in L^{p_1,q_1}(\Omega)$, $g \in L^{p_2,q_2}(\Omega)$ then $fg \in L^{p,q}(\Omega)$ and*

$$\|fg\|_{p,q} \le 2^{\frac{1}{p}} \|f\|_{p_1,q_1} \|g\|_{p_2,q_2}.$$

Let us assume

$$C^\infty_{0,\sigma}(\Omega) := \{v \in C^\infty_0 : \operatorname{div} v = 0 \text{ in } \Omega\},$$
$$L^r_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^r}}, \ r \in (1,\infty).$$

Let $\mathbb{P} = \mathbb{P}_r$ be the Helmholtz projection on $L^r(\Omega)$, which means the projection onto $L^r_\sigma(\Omega)$ corresponding to the following Helmholtz decomposition of L^r-vector fields (see [3, 10]):

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus \{\nabla p \in L^r(\Omega) : p \in L^r_{\mathrm{loc}}(\overline{\Omega})\}.$$

We now give notation of the solenoidal Lorentz spaces which are defined by

$$L^{r,q}_\sigma(\Omega) := (L^{r_0}_\sigma(\Omega), L^{r_1}_\sigma(\Omega))_{\theta,q}$$

with $1 < r_0 < r < r_1 < \infty,\ 1 \le q \le \infty$ and $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. If $q = \infty$, then $L^r_{\sigma,\mathrm{w}}(\Omega) := L^{r,\infty}_\sigma(\Omega)$. By interpolation theory, the Helmholtz projection above defines a bounded projection $\mathbb{P} = \mathbb{P}_{r,q}$ on Lorentz space $L^{r,q}(\Omega)$ and

$$L^{r,q}_\sigma(\Omega) = \mathrm{Im}\mathbb{P}_{r,q}.$$

We also have (see [10, Theorem 5.2])

$$L^{r,q}(\Omega) = L^{r,q}_\sigma(\Omega) \oplus \{\nabla p \in L^{r,q}(\Omega) : p \in L^{r,q}_{\mathrm{loc}}(\overline{\Omega})\}.$$

Furthermore, if $1 \le q < \infty$ then

$$(L^{r,q}_\sigma(\Omega))' = L^{r',q'}_\sigma(\Omega) \quad \text{here } r' = \frac{r}{r-1},\ q' = \frac{q}{q-1} \text{ and } q' = \infty \text{ if } q = 1.$$

2.2. Oseen operators

Let us now recall the Oseen operator in the space $L^r_\sigma(\Omega)$ with $1 < r < \infty$. We define the linear operators $\mathcal{L}_{a,k}$ and $\mathcal{L}'_{a,k}$ in $L^r_\sigma(\Omega)$ by

$$D(\mathcal{L}_{a,k}) := \left\{u \in L^r_\sigma(\Omega) \cap W^{2,r}(\Omega) : u|_{\partial\Omega} = 0 \text{ and } ((\omega \times x)\cdot\nabla)u \in L^r(\Omega)\right\},$$
$$\mathcal{L}_{a,k}u := \mathbb{P}\left[-\Delta u + kD_3u - ((\omega \times x)\cdot\nabla)u + \omega \times u\right] \quad \text{for } u \in D(\mathcal{L}_{a,k}),$$

and

$$\mathcal{L}'_{a,k}u = \mathbb{P}\left[-\Delta u - kD_3u + ((\omega \times x)\cdot\nabla)u + \omega \times u\right] \quad \text{for } D(\mathcal{L}'_{a,k}) = D(\mathcal{L}_{a,k}).$$

We call $\mathcal{L}_{a,k}$ the Oseen operator in $L^r_\sigma(\Omega)$. Moreover, the Oseen operator $-\mathcal{L}_{a,k}$ is a generator of the bounded C_0-semigroup $(e^{-t\mathcal{L}_{a,k}})_{t\ge0}$ on $L^r_\sigma(\Omega)$, and if $\mathcal{L}^*_{a,k}$ is an adjoint operator of $\mathcal{L}_{a,k}$ then $\mathcal{L}^*_{a,k} = \mathcal{L}'_{a,k}$, see [11, 12].

By interpolation theory, $(e^{-t\mathcal{L}_{a,k}})_{t\ge0}$ is also the bounded C_0-semigroup in the solenoidal Lorentz space $L^{r,q}_\sigma(\Omega)$ with $1 \le q < \infty$ and is strongly continuous on $(0,\infty)$ in $L^r_{\sigma,\mathrm{w}}(\Omega)$. Moreover, we can transfer the $L^p - L^q$ decay estimates obtained by Shibata in [12, Theorem 3] for $(e^{-t\mathcal{L}_{a,k}})_{t\ge0}$ on $L^r_\sigma(\Omega)$ to the $L^{r,q} - L^{p,q}$ decay estimates for that semigroup on the space $L^{r,q}_\sigma(\Omega)$. We now list some important properties of the semigroup $(e^{-t\mathcal{L}_{a,k}})_{t\ge0}$ on the solenoidal Lorentz spaces in the paper [5, Proposition 2.2].

Lemma 2.2. *Let $1 < r < \infty,\ 1 \le q \le \infty$ and denote by $\|f\|_{r,q}$ the norm in the space $L^{r,q}_\sigma(\Omega)$. Then, the following inequalities hold.*

(i) For $1 < p \le r < \infty$

$$\|e^{-t\mathcal{L}_{a,k}}f\|_{r,q},\ \|e^{-t\mathcal{L}'_{a,k}}f\|_{r,q} \le Mt^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|f\|_{p,q}. \tag{2.1}$$

(ii) Furthermore, when $1 < p \le r \le 3$ and $1 \le q < \infty$, we have

$$\|\nabla e^{-t\mathcal{L}_{a,k}}f\|_{r,q},\ \|\nabla e^{-t\mathcal{L}'_{a,k}}f\|_{r,q} \le Mt^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|f\|_{p,q}. \tag{2.2}$$

(iii) For $1 < p < r < \infty,\ 1 \le q < \infty$ *then*

$$\|e^{-t\mathcal{L}_{a,k}} f\|_{r,q},\ \|e^{-t\mathcal{L}'_{a,k}} f\|_{r,q} \le Mt^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|f\|_{p,\frac{q}{q-1}}. \tag{2.3}$$

(iv) Moreover, when $1 < p < r \le 3$ *and* $1 \le q < \infty$*, we have*

$$\|\nabla e^{-t\mathcal{L}_{a,k}} f\|_{r,q},\ \|\nabla e^{-t\mathcal{L}'_{a,k}} f\|_{r,q} \le Mt^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|f\|_{p,\frac{q}{q-1}}. \tag{2.4}$$

(v) For $r \ge 3$ *and* $f \in L_\sigma^{\frac{r-1}{r},1}(\Omega)$*, we have*

$$\int_0^\infty \|\nabla e^{-t\mathcal{L}'_{a,k}} f\|_{\frac{3r}{2r-3},1}\, dt \le M\|f\|_{\frac{r}{r-1},1}. \tag{2.5}$$

3. Stability of weak mild solutions

To prove the stability of the weak mild solution $\tilde{u}$ in $L^3_{\sigma,\mathrm{w}}(\Omega)$, we will rewrite the system (1.4) in an abstract form and then study the unique existence and properties of the solution z.

$$\begin{cases} D_t z + \mathcal{L}_{a,k} z = \mathbb{P}\,\mathrm{div}\,(G - z\otimes z - \tilde{u}\otimes z - z\otimes\tilde{u}), \quad t > 0, \\ z|_{t=0} = z_0 \in L^3_{\sigma,\mathrm{w}}(\Omega). \end{cases} \tag{3.1}$$

Now, we restate the concept of weak mild solutions.

Definition 3.1. A continuous function $z : (0,\infty) \to L^3_{\sigma,\mathrm{w}}(\Omega)$ is a *weak mild solution* of the system (3.1) if it is a solution of the equation

$$\langle\, z(t), \varphi \,\rangle = \langle\, e^{-t\mathcal{L}_{a,k}} z_0, \varphi \,\rangle - \int_0^t \langle\, (G - z\otimes z - \tilde{u}\otimes z - z\otimes\tilde{u})(\tau), \nabla e^{-(t-\tau)\mathcal{L}'_{a,k}} \varphi \,\rangle\, d\tau$$

for all $\varphi \in L_\sigma^{\frac{3}{2},1}(\Omega)$ and $t > 0$.

For $u \in C_b((0,\infty), L^r_{\sigma,\mathrm{w}}(\Omega))$, denote the norm $\|u\|_{\infty,r,\mathrm{w}} = \sup_{t>0}\|u(t)\|_{r,\mathrm{w}}$. The results on the stability of $\tilde{u}$ are as follows.

Theorem 3.2. *Assume that* $\tilde{u} \in C_b((0,\infty), L^3_{\sigma,\mathrm{w}}(\Omega))$ *and* $G \in C_b((0,\infty), L^{\frac{3}{2}}_{\sigma,\mathrm{w}}(\Omega)^{3\times 3})$*. Let* $r \in (3,\infty)$*. Then the constants* $\delta > 0$ *and* $K > 0$ *exist such that if*

$$\|z_0\|_{3,\mathrm{w}} + \|G\|_{\infty,\frac{3}{2},\mathrm{w}} + \|\tilde{u}\|_{\infty,3,\mathrm{w}} < \delta,$$

then the following assertions hold true.

(i) *The system* (3.1) *has a unique weak mild solution* z *in* $C_b((0,\infty), L^3_{\sigma,\mathrm{w}}(\Omega))$ *satisfying*

$$\|z\|_{\infty,3,\mathrm{w}} \le K(\|z_0\|_{3,\mathrm{w}} + \|G\|_{\infty,\frac{3}{2},\mathrm{w}}).$$

(ii) *If G satisfies* $\sup_{t>0} t^{\frac{1}{2}-\frac{3}{2r}}\|G(t)\|_{\frac{3r}{r+3},\mathrm{w}} < \infty$, *then there are constants $\delta_1 \in (0,\delta)$ and $K_1 > 0$ such that if*

$$\|z_0\|_{3,\mathrm{w}} + \max\{\|G\|_{\infty,\frac{3}{2},\mathrm{w}}, \sup_{t>0} t^{\frac{1}{2}-\frac{3}{2r}}\|G(t)\|_{\frac{3r}{r+3},\mathrm{w}}\} + \|\tilde{u}\|_{\infty,3,\mathrm{w}} < \delta_1, \tag{3.2}$$

then the solution z satisfies

$$\|z(t)\|_{r,\mathrm{w}} \le K_1(\|z_0\|_{3,\mathrm{w}} + \max\{\|G\|_{\infty,\frac{3}{2},\mathrm{w}}, \sup_{t>0} t^{\frac{1}{2}-\frac{3}{2r}}\|G(t)\|_{\frac{3r}{r+3},\mathrm{w}}\})t^{-\frac{1}{2}+\frac{3}{2r}} \quad \textit{for all} \;\; t > 0.$$

(iii) *Let $p \in (\frac{3r}{r+3}, 3)$ and assume that the condition* (3.2) *holds. If* $\sup_{t>0} t^{\frac{3}{2p}-\frac{3}{2r}}\|G(t)\|_{\frac{3r}{r+3},\mathrm{w}} < \infty$ *and* $z_0 \in L^p_{\sigma,\mathrm{w}}(\Omega)$, *then*

$$\sup_{t>0} t^{-\frac{1}{2}+\frac{3}{2p}}\|z(t)\|_{3,\mathrm{w}} + \sup_{t>0}\|z(t)\|_{p,\mathrm{w}} < \infty.$$

Remark 3.3. By (i), the solution $\tilde{u}$ is stable in $L^3_{\sigma,\mathrm{w}}(\Omega)$ following initial datum and external force. Furthermore, from (iii), if $G = 0$, this solution is $L^{3,\infty}$-asymptotically stable, as the initial datum is better.

Proof. For $z \in C_b((0,\infty), L^3_{\sigma,\mathrm{w}}(\Omega))$, we define the map T by $z \mapsto Tz$ such that for each $t > 0$, one has

$$\langle\, (Tz)(t), \varphi \,\rangle = \langle\, e^{-t\mathcal{L}_{a,k}} z_0, \varphi \,\rangle - \int_0^t \langle\, H(z)(\tau), \nabla e^{-(t-\tau)\mathcal{L}'_{a,k}} \varphi \,\rangle \, d\tau$$

for all $\varphi \in L^{\frac{3}{2},1}_\sigma(\Omega)$, where $H(z) = G - z \otimes z - \tilde{u} \otimes z - z \otimes \tilde{u}$.

Fixed $t > 0$, by (2.1) and dual inequality, we have

$$\begin{aligned} |\langle\, (Tz)(t), \varphi \,\rangle| \le &\quad |\langle\, e^{-t\mathcal{L}_{a,k}} z_0, \varphi \,\rangle| + \int_0^t \left|\langle\, -H(z)(\tau), \nabla e^{-(t-\tau)\mathcal{L}'_{a,k}} \varphi \,\rangle\right| d\tau \\ \le &\quad M\|z_0\|_{3,\mathrm{w}}\|\varphi\|_{\frac{3}{2},1} + \int_0^t \|H(z)(\tau)\|_{\frac{3}{2},\mathrm{w}}\|\nabla e^{-(t-\tau)\mathcal{L}'_{a,k}} \varphi\|_{3,1}\, d\tau. \end{aligned}$$

By Lemma 2.1, we have

$$z(t) \otimes z(t) + \tilde{u}(t) \otimes z(t) + z(t) \otimes \tilde{u}(t) \in L^{\frac{3}{2}}_{\sigma,\mathrm{w}}(\Omega)^{3\times 3}$$

and

$$\|z(t) \otimes z(t) + \tilde{u}(t) \otimes z(t) + z(t) \otimes \tilde{u}(t)\|_{\frac{3}{2},\mathrm{w}} \le 2^{\frac{2}{3}}(\|z(t)\|^2_{3,\mathrm{w}} + 2\|z(t)\|_{3,\mathrm{w}}\|\tilde{u}(t)\|_{3,\mathrm{w}}).$$

Therefore,

$$\|H(z)\|_{\infty,\frac{3}{2},\mathrm{w}} \le \|G\|_{\infty,\frac{3}{2},\mathrm{w}} + 2^{\frac{2}{3}}(\|z\|^2_{\infty,3,\mathrm{w}} + 2\|z\|_{\infty,3,\mathrm{w}}\|\tilde{u}\|_{\infty,3,\mathrm{w}}). \tag{3.3}$$

Thus,

$$|\langle\, (Tz)(t), \varphi \,\rangle| \le M\|z_0\|_{3,\mathrm{w}}\|\varphi\|_{\frac{3}{2},1} + \|H(z)\|_{\infty,\frac{3}{2},\mathrm{w}} \int_0^t \|\nabla e^{-(t-\tau)\mathcal{L}'_{a,k}} \varphi\|_{3,1}\, d\tau$$

$$\leq M\|z_0\|_{3,\mathrm{w}}\|\varphi\|_{\frac{3}{2},1} + \|H(z)\|_{\infty,\frac{3}{2},\mathrm{w}} \int_0^\infty \|\nabla e^{-\tau\mathcal{L}'_{a,k}}\varphi\|_{3,1}\,d\tau.$$

By (2.5)

$$|\langle\,(Tz)(t),\varphi\,\rangle| \leq M\|z_0\|_{3,\mathrm{w}}\|\varphi\|_{\frac{3}{2},1} + M\|H(z)\|_{\infty,\frac{3}{2},\mathrm{w}}\|\varphi\|_{\frac{3}{2},1}.$$

Hence, $(Tz)(t) \in L^3_{\sigma,\mathrm{w}}(\Omega)$ and by (3.3)

$$\begin{aligned}\|(Tz)(t)\|_{3,\mathrm{w}} &\leq M\|z_0\|_{3,\mathrm{w}} + M\|H(z)\|_{\infty,\frac{3}{2},\mathrm{w}}\\ &\leq M[\|z_0\|_{3,\mathrm{w}} + \|G\|_{\infty,\frac{3}{2},\mathrm{w}} + 2^{\frac{2}{3}}(\|z\|^2_{\infty,3,\mathrm{w}} + 2\|z\|_{\infty,3,\mathrm{w}}\|\tilde{u}\|_{\infty,3,\mathrm{w}})]\end{aligned} \tag{3.4}$$

for all $t > 0$.

For $t_2 > t_1 > 0$, we have

$$\begin{aligned}\langle\,(Tz)(t_2) - (Tz)(t_1),\varphi\,\rangle &= \langle\, e^{-t_2\mathcal{L}_{a,k}}z_0 - e^{-t_1\mathcal{L}_{a,k}}z_0,\varphi\,\rangle\\ &\quad - \int_0^{t_1}\langle\, H(z)(t_2-\tau) - H(z)(t_1-\tau), \nabla e^{-\tau\mathcal{L}'_{a,k}}\varphi\,\rangle\,d\tau\\ &\quad - \int_{t_1}^{t_2}\langle\, H(z)(t_2-\tau), \nabla e^{-\tau\mathcal{L}'_{a,k}}\varphi\,\rangle\,d\tau.\end{aligned}$$

Similar to the above, we obtain

$$\begin{aligned}\|(Tz)(t_2) - (Tz)(t_1)\|_{3,\mathrm{w}} &\leq \|e^{-t_2\mathcal{L}_{a,k}}z_0 - e^{-t_1\mathcal{L}_{a,k}}z_0\|_{3,\mathrm{w}}\\ &\quad + t_1 M \sup_{\tau\in(0,t_1]}\|H(z)(t_2-\tau) - H(z)(t_1-\tau)\|_{\frac{3}{2},\mathrm{w}}\\ &\quad + M\|H(z)\|_{\infty,\frac{3}{2},\mathrm{w}}\,|t_2 - t_1|.\end{aligned}$$

Since the functions $e^{-t\mathcal{L}_{a,k}}z_0$ and $H(z)$ are continuous on $(0,\infty)$, it follows that the function Tz is also continuous. Thus, $Tz \in C_b((0,\infty), L^3_{\sigma,\mathrm{w}}(\Omega))$.

Let B_ρ be a closed ball in $C_b((0,\infty), L^3_{\sigma,\mathrm{w}}(\Omega))$ centered at 0 with a radius ρ. We will choose ρ such that $T : B_\rho \to B_\rho$ and is a contractive mapping. The discussion is similar to the estimate of $\|(Tz)(t)\|_{3,\mathrm{w}}$, and we have

$$\|(Tz_1)(t) - (Tz_2)(t)\|_{3,\mathrm{w}} \leq 2^{\frac{2}{3}}(\|z_1\|_{\infty,3,\mathrm{w}} + \|z_2\|_{\infty,3,\mathrm{w}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}})\|z_1 - z_2\|_{\infty,3,\mathrm{w}}$$

for all $z_1, z_2 \in C_b((0,\infty), L^3_{\sigma,\mathrm{w}}(\Omega))$ and $t > 0$. Therefore, for $z, z_1, z_2 \in B_\rho$, we get a system of inequalities

$$\begin{cases} M[\|z_0\|_{3,\mathrm{w}} + \|G\|_{\infty,\frac{3}{2},\mathrm{w}} + 2^{\frac{2}{3}}(\rho^2 + 2\rho\|\tilde{u}\|_{\infty,3,\mathrm{w}})] & \leq \rho,\\ 2^{\frac{2}{3}}(2\rho + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}) & \leq \frac{1}{2}.\end{cases}$$

Therefore, $\delta > 0$ exists such that if

$$\|z_0\|_{3,\mathrm{w}} + \|G\|_{\infty,\frac{3}{2},\mathrm{w}} + \|\tilde{u}\|_{\infty,3,\mathrm{w}} < \delta,$$

then the system above has solution $\rho > 0$. So, $T : B_\rho \to B_\rho$ is a contractive mapping. This leads to the system (3.1) having a unique solution in $C_b((0,\infty), L^3_{\sigma,\mathrm{w}}(\Omega))$. By (3.4), a constant $K > 0$ exists such that

$$\|z\|_{\infty,3,\mathrm{w}} \leq K(\|z_0\|_{3,\mathrm{w}} + \|G\|_{\infty,\frac{3}{2},\mathrm{w}}).$$

To prove (ii), we set Banach space

$$\mathbb{M} = \left\{ v \in C_b((0,\infty), L^3_{\sigma,\mathrm{w}}(\Omega)) : \sup_{t>0} t^{\frac{1}{2}-\frac{3}{2r}} \|v(t)\|_{r,\mathrm{w}} < \infty \right\}$$

endowed with the norm $\|v\|_{\mathbb{M}} := \max\{\|v\|_{\infty,3,\mathrm{w}}, \sup_{t>0} t^{\frac{1}{2}-\frac{3}{2r}} \|v(t)\|_{r,\mathrm{w}}\}$. Put

$$\|G\|_{\mathbf{M}} = \max\{\|G\|_{\infty,\frac{3}{2},\mathrm{w}}, \sup_{t>0} t^{\frac{1}{2}-\frac{3}{2r}} \|G(t)\|_{\frac{3r}{r+3},\mathrm{w}}\}.$$

For $z \in \mathbb{M}$, we have

$$\begin{aligned}
\left| \int_0^t \langle -H(z)(t-\tau), \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \rangle \, d\tau \right| &\le \int_0^t \left| \langle -H(z)(t-\tau), \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \rangle \right| d\tau \\
&\le \int_0^{\frac{t}{2}} \left| \langle -H(z)(t-\tau), \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \rangle \right| d\tau \\
&\quad + \int_{\frac{t}{2}}^t \left| \langle -H(z)(t-\tau), \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \rangle \right| d\tau. \qquad (3.5)
\end{aligned}$$

By dual inequality, Lemma 2.1, and (2.5), we get

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \left| \langle -H(z)(t-\tau), \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \rangle \right| d\tau \\
&\le \int_0^{\frac{t}{2}} \|H(z)(t-\tau)\|_{\frac{3r}{r+3},\mathrm{w}} \left\| \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \right\|_{\frac{3r}{2r-3},1} d\tau \\
&\le \int_0^{\frac{t}{2}} [\|G(t-\tau)\|_{\frac{3r}{r+3},\mathrm{w}} + 2^{\frac{r+3}{3r}} (\|z(t-\tau)\|_{3,\mathrm{w}} + 2\|\tilde{u}(t-\tau)\|_{3,\mathrm{w}}) \|z(t-\tau)\|_{r,\mathrm{w}}] \left\| \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \right\|_{\frac{3r}{2r-3},1} d\tau \\
&\le \left(\frac{t}{2}\right)^{-\frac{1}{2}+\frac{3}{2r}} [\|G\|_{\mathbf{M}} + 2^{\frac{r+3}{3r}} (\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}) \|z\|_{\mathbb{M}}] \int_0^{\frac{t}{2}} \left\| \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \right\|_{\frac{3r}{2r-3},1} d\tau \\
&\le M_1 \left(\frac{t}{2}\right)^{-\frac{1}{2}+\frac{3}{2r}} (\|G\|_{\mathbf{M}} + \|z\|^2_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}} \|z\|_{\mathbb{M}}) \|\varphi\|_{\frac{r}{r-1},1}. \qquad (3.6)
\end{aligned}$$

On the other hand, by (2.2)

$$\begin{aligned}
&\int_{\frac{t}{2}}^t \left| \langle -H(z)(t-\tau), \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \rangle \right| d\tau \le \int_{\frac{t}{2}}^t \|H(z)(t-\tau)\|_{\frac{3}{2},\mathrm{w}} \left\| \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \right\|_{3,1} d\tau \\
&\le \int_{\frac{t}{2}}^t [\|G(t-\tau)\|_{\frac{3}{2},\mathrm{w}} + (\|z(t-\tau)\|_{3,\mathrm{w}} + 2\|\tilde{u}(t-\tau)\|_{3,\mathrm{w}}) \|z(t-\tau)\|_{3,\mathrm{w}}] \left\| \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \right\|_{3,1} d\tau \\
&\le M(\|G\|_{\mathbf{M}} + \|z\|^2_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}} \|z\|_{\mathbb{M}}) \int_{\frac{t}{2}}^\infty \tau^{-\frac{3}{2}+\frac{3}{2r}} \|\varphi\|_{\frac{r}{r-1},1} d\tau \\
&\le M_2 t^{-\frac{1}{2}+\frac{3}{2r}} (\|G\|_{\mathbf{M}} + \|z\|^2_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}} \|z\|_{\mathbb{M}}) \|\varphi\|_{\frac{r}{r-1},1}. \qquad (3.7)
\end{aligned}$$

By (3.5), (3.6), and (3.7),

$$\left| \int_0^t \langle -H(z)(t-\tau), \nabla e^{-\tau \mathcal{L}'_{a,k}} \varphi \rangle \, d\tau \right| \le \widetilde{M} t^{-\frac{1}{2}+\frac{3}{2r}} (\|G\|_{\mathbf{M}} + \|z\|^2_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}} \|z\|_{\mathbb{M}}) \|\varphi\|_{\frac{r}{r-1},1}$$

for all $\varphi \in C_{0,\sigma}^{\infty}(\Omega)$. Thus,

$$\sup_{t>0} t^{\frac{1}{2}-\frac{3}{2r}} \|(Tz)(t)\|_{r,\mathrm{w}} \le M\|z_0\|_{3,\mathrm{w}} + \widetilde{M}\left(\|G\|_{\mathbf{M}} + \|z\|_{\mathbb{M}}^2 + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}\|z\|_{\mathbb{M}}\right). \tag{3.8}$$

Combining (3.4) and (3.8), we get

$$\|Tz\|_{\mathbb{M}} \le C(\|z_0\|_{3,\mathrm{w}} + \|G\|_{\mathbf{M}} + \|z\|_{\mathbb{M}}^2 + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}\|z\|_{\mathbb{M}}).$$

Similarly, for $z_1, z_2 \in \mathbb{M}$, we have

$$\|Tz_1 - Tz_2\|_{\mathbb{M}} \le C(\|z_1\|_{\mathbb{M}} + \|z_2\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}})\|z_1 - z_2\|_{\mathbb{M}}.$$

Let $\rho \in (0,1)$ and consider $\|z\|_{\mathbb{M}} \le \rho$. Then, $\delta_1 \in (0,\delta)$ exists such that if

$$\|z_0\|_{3,\mathrm{w}} + \|G\|_{\mathbf{M}} + \|\tilde{u}\|_{\infty,3,\mathrm{w}} < \delta_1,$$

then there is a $\rho \in (0,1)$ satisfying

$$\begin{cases} C(\|z_0\|_{3,\mathrm{w}} + \|G\|_{\mathbf{M}} + \rho^2 + 2\rho\|\tilde{u}\|_{\infty,3,\mathrm{w}}) & \le \rho, \\ C(2\rho + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}) & \le \frac{1}{2}. \end{cases}$$

Thus, the system (3.1) has a unique solution z in $\mathbb{M}$ and $K_1 > 0$ exists such that

$$\|z\|_{\mathbb{M}} \le K_1(\|z_0\|_{3,\mathrm{w}} + \|G\|_{\mathbf{M}}).$$

Hence,

$$\|z(t)\|_{r,\mathrm{w}} \le K_1(\|z_0\|_{3,\mathrm{w}} + \|G\|_{\mathbf{M}})t^{-\frac{1}{2}+\frac{3}{2r}} \quad \text{for all } \; t>0.$$

We now prove (iii). For $t > 0$ and $\varphi \in C_{0,\sigma}^{\infty}(\Omega)$, we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \left|\langle -H(z)(t-\tau), \nabla e^{-\tau\mathcal{L}_{a,k}'}\varphi\rangle\right| d\tau \\
&\le \int_0^{\frac{t}{2}} \left[\|G(t-\tau)\|_{\frac{3r}{r+3},\mathrm{w}} + 2^{\frac{r+3}{3r}}(\|z(t-\tau)\|_{3,\mathrm{w}} + 2\|\tilde{u}(t-\tau)\|_{3,\mathrm{w}})\|z(t-\tau)\|_{r,\mathrm{w}}\right] \left\|\nabla e^{-\tau\mathcal{L}_{a,k}'}\varphi\right\|_{\frac{3r}{2r-3},1} d\tau \\
&\le \left(\frac{t}{2}\right)^{-\frac{3}{2p}+\frac{3}{2r}} \sup_{s>0} s^{\frac{3}{2p}-\frac{3}{2r}}\|G(s)\|_{\frac{3r}{r+3},\mathrm{w}} \int_0^{\frac{t}{2}} \left\|\nabla e^{-\tau\mathcal{L}_{a,k}'}\varphi\right\|_{\frac{3r}{2r-3},1} d\tau \\
&+ 2^{\frac{r+3}{3r}}\left(\frac{t}{2}\right)^{-\frac{3}{2p}+\frac{3}{2r}} (\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}) \int_0^{\frac{t}{2}} (t-\tau)^{\frac{3}{2p}-\frac{3}{2r}}\|z(t-\tau)\|_{r,\mathrm{w}} \left\|\nabla e^{-\tau\mathcal{L}_{a,k}'}\varphi\right\|_{\frac{3r}{2r-3},1} d\tau \\
&\le M_1\left(\frac{t}{2}\right)^{-\frac{3}{2p}+\frac{3}{2r}} \left(\sup_{s>0} s^{\frac{3}{2p}-\frac{3}{2r}}\|G(s)\|_{\frac{3r}{r+3},\mathrm{w}} + (\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}) \sup_{s\in(0,t]} s^{\frac{3}{2p}-\frac{3}{2r}}\|z(s)\|_{r,\mathrm{w}}\right)\|\varphi\|_{\frac{r}{r-1},1},
\end{aligned}$$

and

$$\int_{\frac{t}{2}}^{t} \left|\langle -H(z)(t-\tau), \nabla e^{-\tau\mathcal{L}_{a,k}'}\varphi\rangle\right| d\tau$$

$$\le \int_{\frac{t}{2}}^{t} [\|G(t-\tau)\|_{\frac{3r}{r+3},\mathrm{w}} + 2^{\frac{r+3}{3r}}(\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}})\|z(t-\tau)\|_{r,\mathrm{w}}] \left\|\nabla e^{-\tau\mathcal{L}'_{a,k}}\varphi\right\|_{\frac{3r}{2r-3},1} d\tau$$
$$\le \Big(\sup_{s>0} s^{\frac{3}{2p}-\frac{3}{2r}}\|G(s)\|_{\frac{3r}{r+3},\mathrm{w}} \int_{\frac{t}{2}}^{t} \frac{(t-\tau)^{-\frac{3}{2p}+\frac{3}{2r}}}{\tau} d\tau$$
$$+ 2^{\frac{r+3}{3r}}(\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}) \sup_{s\in(0,t]} s^{\frac{3}{2p}-\frac{3}{2r}}\|z(s)\|_{r,\mathrm{w}} \int_{\frac{t}{2}}^{t} \frac{(t-\tau)^{-\frac{3}{2p}+\frac{3}{2r}}}{\tau} d\tau\Big)\|\varphi\|_{\frac{r}{r-1},1}$$
$$\le M_2 t^{-\frac{3}{2p}+\frac{3}{2r}}(\sup_{s>0} s^{\frac{3}{2p}-\frac{3}{2r}}\|G(s)\|_{\frac{3r}{r+3},\mathrm{w}} + (\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}) \sup_{s\in(0,t]} s^{\frac{3}{2p}-\frac{3}{2r}}\|z(s)\|_{r,\mathrm{w}})\|\varphi\|_{\frac{r}{r-1},1}.$$

Thus,

$$t^{\frac{3}{2p}-\frac{3}{2r}}\|z(t)\|_{r,\mathrm{w}} \le M\|z_0\|_{p,\mathrm{w}} + \widetilde{M}m + \widetilde{M}(\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}) \sup_{s\in(0,t]} s^{\frac{3}{2p}-\frac{3}{2r}}\|z(s)\|_{r,\mathrm{w}}, \tag{3.9}$$

where $m = \sup_{s>0} s^{\frac{3}{2p}-\frac{3}{2r}}\|G(s)\|_{\frac{3r}{r+3},\mathrm{w}}$. On the other hand

$$s^{\frac{3}{2p}-\frac{3}{2r}}\|z(s)\|_{r,\mathrm{w}} = s^{\frac{3}{2p}-\frac{1}{2}} s^{\frac{1}{2}-\frac{3}{2r}}\|z(s)\|_{r,\mathrm{w}} \le t^{\frac{3}{2p}-\frac{1}{2}}\|z\|_{\mathbb{M}} < \infty$$

for all $s \in (0,t]$. Therefore, by (3.9), $C > 0$ exists such that

$$\sup_{s\in(0,t]} s^{\frac{3}{2p}-\frac{3}{2r}}\|z(s)\|_{r,\mathrm{w}} \le C(\|z_0\|_{p,\mathrm{w}} + m)$$

for all $t > 0$. So,

$$\alpha := \sup_{t>0} t^{\frac{3}{2p}-\frac{3}{2r}}\|z(t)\|_{r,\mathrm{w}} < \infty. \tag{3.10}$$

Because of $p > \frac{3r}{r+3} > \frac{3}{2}$, we have $\frac{p}{p-1} < \frac{3r}{2r-3} < 3$. Therefore

$$\int_0^t \left|\langle -H(z)(t-\tau), \nabla e^{-\tau\mathcal{L}'_{a,k}}\varphi\rangle\right| d\tau$$
$$\le \int_0^t [\|G(t-\tau)\|_{\frac{3r}{r+3},\mathrm{w}} + 2^{\frac{r+3}{3r}}(\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}})\|z(t-\tau)\|_{r,\mathrm{w}}] \left\|\nabla e^{-\tau\mathcal{L}'_{a,k}}\varphi\right\|_{\frac{3r}{2r-3},1} d\tau$$
$$\le C(m + \alpha(\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}))\|\varphi\|_{\frac{p}{p-1},1} \int_0^t (t-\tau)^{-\frac{3}{2p}+\frac{3}{2r}} \tau^{-1+\frac{3}{2p}-\frac{3}{2r}} d\tau$$
$$= C(m + \alpha(\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}))\|\varphi\|_{\frac{p}{p-1},1} \int_0^1 (1-\tau)^{-\frac{3}{2p}+\frac{3}{2r}} \tau^{-1+\frac{3}{2p}-\frac{3}{2r}} d\tau$$

for all $\varphi \in C_{0,\sigma}^{\infty}(\Omega)$. Hence

$$\|z(t)\|_{p,\mathrm{w}} \le M\|z_0\|_{p,\mathrm{w}} + \widetilde{C}(m + \alpha(\|z\|_{\mathbb{M}} + 2\|\tilde{u}\|_{\infty,3,\mathrm{w}}))$$

for all $t > 0$. So,

$$\beta := \sup_{t>0} \|z(t)\|_{p,\mathrm{w}} < \infty. \tag{3.11}$$

By interpolation theory for Lorentz spaces (see [8, 9]), we have

$$L^3_{\mathrm{w}}(\Omega) = (L^p_{\mathrm{w}}(\Omega), L^r_{\mathrm{w}}(\Omega))_{\theta,\infty} \quad \text{with } \theta = \frac{r(3-p)}{3(r-p)}.$$

Therefore, from (3.10) and (3.11), we obtain

$$\|z(t)\|_{3,\mathrm{w}} \le \|z(t)\|^{1-\theta}_{p,\mathrm{w}} \|z(t)\|^{\theta}_{r,\mathrm{w}} \le \beta^{1-\theta}\alpha^{\theta} t^{\theta\left(\frac{3}{2r}-\frac{3}{2p}\right)} = Ct^{\frac{1}{2}-\frac{3}{2p}}$$

for all $t > 0$. □

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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