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Research article

The constant in the Sobolev inequality and the boundedness of subelliptic operators on compact Lie groups

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Abstract: Given a compact connected Lie group G, we estimated the constant for the Sobolev inequality for an arbitrary Hörmander sub-Laplacian on G. The constant was estimated in terms of quantities depending on the intrinsic geometry of the group and on the Lebesgue parameters. We also provided applications of this result to the estimation of the operator norm for L^p-L^q-bounded subelliptic pseudo-differential operators in subelliptic Hörmander classes.

Keywords: compact Lie groups; Sobolev inequality; subelliptic pseudo-differential operators
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1. Introduction

This paper aims to estimate the constant for the Sobolev inequality for the sub-Laplacian on compact Lie groups. More precisely, if $\mathbf{X} = \{X_1, \cdots, X_k\}$ is a Hörmander system of vector fields, that is, the X_i's vector fields and their commutators generate the Lie algebra, then we want to estimate the constant $C_{p,q}$ in the very well-known inequality

$$\|u\|_{L^q(G)} \leq C_{p,q}\|(1 + \mathcal{L})^{\frac{\alpha}{2}} u\|_{L^p(G)}, \tag{1.1}$$

where $\mathcal{L} = -\sum_{i=1}^{k} X_i^2$ is the positive sub-Laplacian associated with the system $\mathbf{X} = \{X_1, \ldots, X_k\}$. We follow the convention used in [1] and define the sub-Laplacian with a minus sign to ensure that $\mathcal{L}$ is a positive (self-adjoint) operator. This choice aligns with standard practices in spectral analysis and facilitates the definition of fractional powers $(I + \mathcal{L})^{\alpha/2}$ via the spectral theorem. The Lebesgue parameters p, q are related by the admissibility condition

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q},\ 1 < p < \infty,\ 0 \leq \alpha < \frac{Q}{p},\ p \leq q < \infty. \tag{1.2}$$

Here, we denote by Q the homogeneous dimension associated with the Carnot–Carathéodory distance defined by the Hörmander system of vector fields $\mathbf{X} = \{X_1, \dots, X_k\}$. Then, the Sobolev inequality is a natural tool in the analysis of the problem

$$(I + \mathcal{L})^{\frac{\alpha}{2}} u = f, \tag{1.3}$$

and in this paper, we analyze even the case when, instead of the operator $(I + \mathcal{L})^{\frac{\alpha}{2}}$, one considers an arbitrary subelliptic operator T, for instance, in any subelliptic Hörmander class on a compact Lie group.

To illustrate our approach, before presenting the main result of this work, we are going to present some historical aspects of the Sobolev inequality and its relevance in the geometric analysis. The history starts with a 1936 published paper from S. Sobolev in which he proved several *a priori inequalities*. Among them, he showed the following inequality, which is now called the *Sobolev inequality:*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)}, \tag{1.4}$$

where $1 < p < n$, $p^* = \frac{np}{n-p}$, and $C = C(n, p) > 0$ is a positive constant. This paper does not study the case when $p = 1$. Nevertheless, on a Riemannian manifold M, the case when $p = 1$ is given by

$$\|f\|_q \leq C\|\nabla f\|_1, \quad \forall f \in C_0^\infty(M), \tag{1.5}$$

which is interesting by itself due to its equivalence with the following *isoperimetric inequality*:

$$(\mathrm{Vol}_n(\Omega))^{1/q} \leq C\,\mathrm{Vol}_{n-1}(\partial\Omega), \tag{1.6}$$

on a smooth bounded set Ω where we have denoted by $\partial\Omega$ its corresponding boundary. For instance, if we consider $\Omega = B(t)$ to be the geodesic ball, the above isoperimetric inequality can be seen as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(B(t)) \geq C^{-1} V(t)^{1/q},$$

since $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(B(t)) = \mathrm{Vol}_{n-1}(\partial B(t))$. This shows that the inequality (1.5) with $q = \frac{D}{D-1}$ implies $\mathrm{Vol}(B(t)) \geq ct^D$, where $D > 1$, see [1]. This explains the relation between the Sobolev inequality and the growth of the function $\mathrm{Vol}(B(t))$. Later in 1958, Gagliardo and Nirenberg proved independently the Sobolev inequality on $\mathbb{R}^2$. The same results can be proven for L^p spaces, in a similar way as it was proven in the case of $p = 1$.

Nevertheless, in more recent times, the best constant of the Sobolev inequality was obtained by Talenti in [2] and Aubin in [3]. In recent decades, the Sobolev inequality has been one of the most important tools in the research of several subelliptic partial differential equations (PDEs) and in the analysis of variational problems. Due to this fact, several authors have considered its extension from the Euclidean setting to other general frameworks. Indeed, the Sobolev inequality was proved on general stratified groups in [4], on general unimodular Lie groups in [1], on general locally compact unimodular groups in [5], on general noncompact Lie groups in [6, 7], on graded Lie groups in [8], and on general Lie groups in [9]. We also note that the best constant of the Sobolev inequality on graded groups was proved in [10].

Among more recent works, we also mention recent results related to Sobolev spaces on Lie groups including the proof of some embedding theorems and algebra properties [6] and the constant for the Sobolev embedding on Lie groups in [7], where the authors studied the Sobolev inequality and the best constant in the case where G is a non-compact Lie group of polynomial growth. Therefore, this paper and the ones mentioned above comprehensively analyze the constant for the Sobolev inequality for the sub-Laplacian on compact Lie groups G.

Our main result is an estimate for the constant of the embedding $L^p_\alpha(G) \hookrightarrow L^q(G)$, when $1 < p < \infty, 0 \leqslant \alpha < d/p$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, of the form $CS(p,q)$, where

$$S(p,q) := \min\left(\frac{q^{1/p'}}{p-1}, \frac{p'^{1/q}}{q'-1}\right) \tag{1.7}$$

and C depends only on the group and its chosen sub-Riemannian structure. Here and throughout the paper, given any $p \in (1,\infty)$, we denote by p' its conjugate exponent, that is, $p' = p/(p-1)$. The following is the main result of this paper.

Theorem 1.1. *Let $p \in (1,\infty), \alpha \in [0, Q/p)$, and $q \in [p,\infty)$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then there exists $A_1 = A_1(G, \mathbf{X}) > 0$ such that for all $f \in L^p_\alpha(G)$,*

$$\|f\|_{L^q(G)} \leqslant A_1 S(p,q) \|f\|_{L^p_\alpha(G)}.$$

The Sobolev space $L^p_\alpha(G)$, which appears in the above estimate, is defined in Section 3 through the operator $(I+\mathcal{L})^{\alpha/2}$; see Equation (3.6). We observe that as an application to the previous result, we can estimate the constant of boundedness for subelliptic pseudo-differential operators on a compact Lie group. Although this is not the point of discussion in this paper, we note that if $\Psi^{m,\mathcal{L}}_{\rho,\delta}(G)$ denotes the corresponding class of subelliptic pseudo-differential operators associated with a Hörmander system of vector fields $\mathbf{X} = \{X_1, \cdots, X_k\}$ (see the work [11] of the second author with Ruzhansky for this notation and other details), we have the following interesting consequence of Theorem 1.1.

Theorem 1.2. *Let G be a compact Lie group and Q the Hausdorff dimension associated with the Laplacian $\mathcal{L} = \mathcal{L}_X$, where $\mathbf{X} = \{X_1, \cdots, X_k\}$ is a system of vector fields satisfying the Hörmander's condition of order κ. For $0 \leqslant \delta < \rho \leqslant 1$, let us consider the pseudo-differential operator $T \in \Psi^{m,\mathcal{L}}_{\rho,\delta}(G)$ of order*

$$m \leqslant -Q\left(\frac{1}{p} - \frac{1}{q}\right) - Q(1-\rho)\left|\frac{1}{q} - \frac{1}{2}\right|.$$

Then $T : L^p(G) \to L^q(G)$ is bounded and

$$\|Tf\|_{L^p(G)} \leqslant \|T(I+\mathcal{L})^{\alpha/2}\|_{L^q(G)\to L^q(G)} A_1(G,X) S(p,q) \|f\|_{L^q(G)}, \tag{1.8}$$

where $p \in (1,\infty)$, $q \in [p,\infty)$, and α is defined by the admissibility condition $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$.

Proof. Note that the operator $T(I+\mathcal{L})^{\alpha/2}$ is bounded from $L^q(G)$ to itself in view of the Fefferman theorem proved in [11, Chapter 6, Theorem 6.9]. The estimate in (1.8) is a consequence of the factorization $T = T(I+\mathcal{L})^{\alpha/2}(I+\mathcal{L})^{-\alpha/2}$ and of the Sobolev inequality in Theorem 1.1. □

We observe that in Theorem 1.2 the boundedness of the operator $T : L^p(G) \to L^q(G)$ has been obtained by the second author, Kumar, Delgado, and Ruzhansky in [12]. However, the contribution in this paper is the size of the operator norm estimated in (1.8), namely, the appearance of the constant $S(p, q)$ in such an inequality, which by itself is interesting.

This work is organized as follows. In Section 2, we give the preliminaries about the sub-Riemannian geometry on a compact Lie group, and in Section 3, we prove our main result. Finally, we illustrate our main theorem by giving the analysis of the Sobolev constant in the case of SU(2) in Section 4.

2. Preliminaries

Throughout the paper, we use the following notation: $d\lambda$ stands for the Haar measure on a compact Lie group G, with Lie algebra $\mathfrak{g} \simeq T_{e_G}G$, where e_G is the neutral element of G, and let

$$\mathbf{X} = \{X_1, \cdots, X_k\} \subset \mathfrak{g}$$

be a system of C^∞-vector fields. For all $I = (i_1, \cdots, i_\omega) \in \{1, 2, \cdots, k\}^\omega$ of length $\omega \geqslant 1$, we denote by

$$X_I := [X_{i_1}, [X_{i_2}, \cdots [X_{i_{\omega-1}}, X_{i_\omega}] \cdots]]$$

a commutator of length ω, where $X_I := X_i$ when $\omega = 1$ and $I = (i)$. The system $\mathbf{X}$ is said to satisfy Hörmander's condition of step (or order) κ if $\mathfrak{g} = \text{span}\{X_I : |I| \leq \kappa\}$, that is, the vector fields X_j, $j = 1, \ldots, k$, together with their commutator up to length κ, generate the whole Lie algebra $\mathfrak{g}$. Here, we assume that there is no subset

$$\mathbf{Y} = \{Y_1, \cdots, Y_\ell\} \subset \mathbf{X},$$

with $\ell < k$, of smooth vector fields such that $\mathfrak{g} = \text{span}\{Y_I : |I| \leq \kappa\}$.

Given a system $\mathbf{X} = \{X_1, \cdots, X_k\}$ of Hörmander's vector fields, then the operator defined as

$$\mathcal{L} \equiv \mathcal{L}_{\mathbf{X}} := -(X_1^2 + \cdots + X_k^2)$$

is a hypoelliptic operator in view of Hörmander's theorem on sums of the squares (see Hörmander [13]). In particular the operator $\mathcal{L}$ is also subelliptic, and it is called the subelliptic Laplacian associated with the system $\mathbf{X}$, or for short, simply a sub-Laplacian. It is clear from the definition that one can define different sub-Laplacians by using different systems of Hörmander's vector fields (that satisfy the Hörmander condition of different steps).

Let us now introduce the Hausdorff dimension associated with the sub-Laplacian $\mathcal{L}$. For all $x \in G$, let $H_x^\omega G$ be the linear subspace of the tangent space T_xG generated by the X_i's and by all the Lie brackets

$$[X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \cdots, [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots, X_{j_\omega}]]],$$

with $\omega \leqslant \kappa$. Then, clearly, the Hörmander condition can be stated as $H_x^\kappa G = T_xG$ for all $x \in G$, where the following inclusions hold:

$$H_x^1 G \subset H_x^2 G \subset H_x^3 G \subset \cdots \subset H_x^{\kappa-1} G \subset H_x^\kappa G = T_xG, \ \ x \in G.$$

Note that the dimension of every $H_x^\omega G$ is constant in $x \in G$, so we set $\dim H^\omega G := \dim H_x^\omega G$, for all $x \in G$, and have that the Hausdorff dimension can be defined as

$$Q := \dim(H^1 G) + \sum_{i=1}^{\kappa-1}(i+1)(\dim H^{i+1}G - \dim H^i G). \tag{2.1}$$

Here, $d_C(\cdot,\cdot)$ will be the associated left-invariant Carnot-Carathéodory distance, $|x| = d_C(x,e)$ the distance between the neutral element on the group e to the element element x, B_r the ball centered at e of radius r, and $\lambda(B_r)$ the measure of the ball B_r with respect to the left-Haar measure λ.

We recall that in compact Lie groups there exists a relation between $\lambda(B_r)$ and the Hausdorff dimension Q as follows: $C^{-1}r^Q \leqslant \lambda(B_r) \leqslant Cr^Q$ for all $r \in (0,1]$ where $C > 0$ does not depend on r.

3. The constant in the Sobolev inequality on a compact Lie group

Our main result is an estimate for the constant of the embedding $L^p_\alpha(G) \hookrightarrow L^q(G)$, when $1 < p < \infty$, $0 \leqslant \alpha < Q/p$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, of the form $CS(p,q)$, where

$$S(p,q) := \min\left(\frac{q^{1/p'}}{p-1}, \frac{p'^{1/q}}{q'-1}\right) \tag{3.1}$$

and C depends only on the group and its chosen sub-Riemannian structure. Here and throughout the paper, given any $p \in (1,\infty)$, we denote by p' its conjugate exponent, that is, $p' = p/(p-1)$. Remember that the convolution between two functions f and g is defined as follows:

$$(f * g)(x) = \int_G f(xy)g(y^{-1})\mathrm{d}\lambda(y),\ x \in G. \tag{3.2}$$

Since G is a compact Lie group, it is unimodular; thus, the left and right Haar measures coincide, and we may use $\breve{g}(x) := g\left(x^{-1}\right)$ in convolution estimates.

Proposition 3.1 (Young's inequality)**.** *If $1 \le p \le q \le \infty$ and $r \ge 1$ are such that $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$, then*

$$\begin{aligned} \|f * g\|_{L^q(G)} &\le \|f\|_{L^p(G)} \|\breve{g}\|_{L^r(G)}^{r/p'} \|g\|_{L^r(G)}^{r/q},\ q < \infty, \\ \|f * g\|_{L^\infty(G)} &\le \|f\|_{L^p(G)} \|g\|_{L^r(G)}, \end{aligned} \tag{3.3}$$

where $\breve{g}(x) = g(x^{-1})$.

Let $\mathcal{L}$ denote the Laplacian on a compact Lie group G associated with $\mathbf{X} = \{X_1, \cdots, X_k\}$, the system of Hörmander's vector fields, as

$$\mathcal{L} = -\sum_{j=1}^{k} X_j^2. \tag{3.4}$$

Proposition 3.2. *There exist constants $b, c > 0$, such that the convolution kernel p_t of $e^{-t\mathcal{L}}$ has the following estimate:*

$$p_t(x) \le c(1 \wedge t)^{-Q/2} e^{-b\frac{|x|^2}{t}}, \tag{3.5}$$

where $1 \wedge t := \min(1,t)$.

Remark 3.3. This estimate follows from [1, Theorem IX.1.3], where Gaussian upper bounds for the heat kernel on compact Lie groups are established. See also [7, Equation (2.3)]; in our case, since the group is compact, we have $\mathfrak{c}(\delta) = 0$.

When $p \in (1, \infty)$, and $\alpha > 0$ we define the Sobolev spaces $L^p_\alpha(G)$ as the set of functions $f \in L^p(G)$ such that $(I + \mathcal{L})^{\alpha/2} f \in L^p(G)$, endowed with the norm:

$$\|f\|_{L^p_\alpha(G)} = \left\|(I + \mathcal{L})^{\alpha/2} f\right\|_{L^p(G)}. \tag{3.6}$$

Here, the fractional powers of $(I + \mathcal{L})$ are defined via the spectral theorem applied to the non-negative self-adjoint operator $\mathcal{L}$ on $L^2(G)$; see also [14] for a general theory of fractional powers of operators.

Moreover, on compact Lie groups, the operator $(I + \mathcal{L})^{\alpha/2}$ admits a convolution representation. That is, there exists a convolution kernel G^α such that

$$(I + \mathcal{L})^{\alpha/2} f = f * G^\alpha.$$

We decompose this kernel as

$$G^\alpha = G^\alpha \mathbb{1}_{B_1} + G^\alpha \mathbb{1}_{B_1^c},$$

and we define

$$G^{\alpha,\mathrm{loc}} := G^\alpha \mathbb{1}_{B_1}, \quad G^{\alpha,\mathrm{glob}} := G^\alpha \mathbb{1}_{B_1^c}.$$

The following result is adapted from [7], where the corresponding estimate is proved in the setting of non-compact Lie groups.

Remark 3.4. (Global vs. local behavior of the kernel G^α on compact Lie Groups.)

In the context of compact Lie groups, the global estimate for the convolution kernel G^α naturally follows from the spectral properties of the sub-Laplacian $\mathcal{L}$ and the finite volume of the group. This compact structure guarantees that outside the unit ball B_1, the kernel G^α decays exponentially, a property that simplifies many integral estimates and functional inequalities. Consequently, while the local estimate requires careful analysis near the origin, the global behavior is inherently controlled by the compact geometry of G; see [1] and [7, Section 2]. This observation allows us to focus on the local behavior in the subsequent Lemma (3.5), as the global decay is already structurally ensured.

We now estimate the global part of the kernel G^α, which appears in the proof of Theorem 1.1 through the operator $\tilde{K}_\alpha$. This is the content of the following lemma.

Lemma 3.5. *There exists $C = C(G, \mathbf{X}) > 0$ such that, for $\alpha \in (0, Q)$ and $x \in G$,*

$$\left|G^{\alpha,\mathrm{loc}}(x)\right| \leqslant C \frac{\alpha}{Q - \alpha} |x|^{\alpha - Q} \mathbb{1}_{B_1}(x). \tag{3.7}$$

Proof. The convolution kernel G^α of the operator $(I + \mathcal{L})^{\alpha/2}$ can be written as

$$G^\alpha = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} \mathrm{e}^{-t} p_t \mathrm{d}t. \tag{3.8}$$

Remark 3.6. Formula (3.8) follows from the Bochner integral representation of the fractional power of the positive operator $(I + \mathcal{L})^{\alpha/2}$ via spectral calculus. For sub-Laplacians on Lie groups, this representation is well known (see, e.g., [6, Section 4]).

By Proposition 3.2, and using the estimate for the convolution kernel, we obtain the representation:

$$G^\alpha \le \frac{c}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1}(1\wedge t)^{-Q/2} \mathrm{e}^{-\left(t+b\frac{|x|^2}{t}\right)} \mathrm{d}t. \tag{3.9}$$

We study the integral given in (3.9) in two cases, where our analysis emphasizes the case where $|x| \le 1$.

Case 1: When $|x| \geqslant 1$,

$$\begin{aligned}
G^\alpha &\le \frac{c}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1}(1\wedge t)^{-Q/2} \mathrm{e}^{-\left(t+b\frac{|x|^2}{t}\right)} \mathrm{d}t \\
&\le \frac{c}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1}(1\wedge t)^{-Q/2} \mathrm{e}^{-\frac{1}{2}\left(t+b\frac{|x|^2}{t}+\sqrt{2b}|x|\right)} \mathrm{d}t \\
&\overset{|x|\ge 1}{\le} \frac{c}{\Gamma(\alpha/2)} \mathrm{e}^{-\frac{1}{2}\sqrt{2b}|x|} \int_0^\infty t^{\alpha/2-1}(1\wedge t)^{-Q/2} \mathrm{e}^{-\frac{t}{2}-\frac{b|x|^2}{2t}}\ \mathrm{d}t \\
&\leqslant C_1 \mathrm{e}^{-b_0|x|},
\end{aligned} \tag{3.10}$$

since $t + b|x|^2/t \geqslant \frac{1}{2}\left(t + b|x|^2/t + \sqrt{2b}|x|\right)$ (provided that $(z+y)^2 \leqslant 2(z^2+y^2)$, $z = \sqrt{at}$, $y = \sqrt{\frac{b}{t}}|x|$, and that $2 > \sqrt{2}$) and the exponential function is increasing. We observe that this estimate is trivial since G is compact.

Case 2: When $|x| \leqslant 1$, splitting the integral we have:

$$\begin{aligned}
G^\alpha(x) &\leqslant \frac{c}{\Gamma(\alpha/2)} \mathrm{e}^{-\frac{1}{2}\sqrt{2b}|x|} \int_0^\infty t^{\alpha/2-1}(1\wedge t)^{-Q/2} \mathrm{e}^{-\frac{t}{2}-\frac{b|x|^2}{2t}}\ \mathrm{d}t \\
&\leqslant C\alpha\left(\int_0^1 t^{(\alpha-Q)/2-1}\mathrm{e}^{-b|x|^2/t}\ \mathrm{d}t + \int_1^\infty t^{\alpha/2-1}\mathrm{e}^{-t}\mathrm{e}^{-\frac{b|x|^2}{2t}}\ \mathrm{d}t\right) \\
&=: C\alpha\left(G_1(x) + G_2(x)\right).
\end{aligned} \tag{3.11}$$

Note that the function $G_2(x) \leqslant C$. Since $\alpha \in (0, Q)$, let $A := \frac{\alpha}{2} - 1$, and we study two cases; first when $A \in (-1, 0)$, and second, when $A \in (0, \frac{Q}{2} - 1)$, $Q \ge 2$. Note that in the first case, $t^{\alpha/2-1} \leqslant 1$ given that $t^A \leqslant 1$ since $t \ge 1$. In the second case, we split G_2 as follows:

$$\begin{aligned}
G_2(x) &= \int_1^M t^{\alpha/2-1}\mathrm{e}^{-t}\ \mathrm{d}t + \int_M^\infty t^{\alpha/2-1}\mathrm{e}^{-t}\ \mathrm{d}t \\
&\leqslant C(M-1) + \int_M^\infty \mathrm{e}^{-\frac{t}{2}}\ \mathrm{d}t = C_2,
\end{aligned} \tag{3.12}$$

since $h(t) = t^{\alpha/2-1}\mathrm{e}^{-t}$ is a continuous function on a compact set $[1, M]$ and $t^{\alpha/2-1} \leqslant e^{\frac{t}{2}}$.

Now, let us analyze the function $G_1(x)$. Taking $u = \frac{|x|^2}{t}$ in $G_1(x)$, we obtain the following:

$$\begin{aligned}
G_1(x) &= |x|^{\alpha-Q}\left(\int_{|x|^2}^1 u^{(Q-\alpha)/2-1}\mathrm{e}^{-bu}\mathrm{d}u + \int_1^\infty u^{(Q-\alpha)/2-1}\mathrm{e}^{-bu}\mathrm{d}u\right) \\
&\leqslant C|x|^{\alpha-Q}\left(\frac{1}{Q-\alpha}\left(1-|x|^{Q-\alpha}\right)+1\right).
\end{aligned} \tag{3.13}$$

We are given that in the first integral $e^{-bu} \leq 1$ and in the second integral $u^{(Q-\alpha)/2-1} \leqslant e^{\frac{b}{2}u}$. Finally, we observe that the estimate for $G_1(x)$ is multiplied by α (as in equation 3.11), yielding the constant $\frac{\alpha}{Q-\alpha}$ in the final bound. □

3.1. Proof of Theorem 1.1

The aim of this subsection is to prove our main theorem.

When $\alpha = 0$ and hence $q = p$, the statement is the trivial embedding $L^p(G) \hookrightarrow L^p(G)$. Define:

$$K_\alpha(x) = |x|^{\alpha-Q}\mathbb{1}_{B_1}(x), \quad \tilde{K}_\alpha(x) = \mathrm{e}^{-b_0|x|}\mathbb{1}_{B_1^c}(x). \tag{3.14}$$

To estimate the operator $(I+\mathcal{L})^{\alpha/2}$, we decompose its convolution kernel into local and global parts. This leads us to introduce the kernels K_α and $\tilde{K}_\alpha$, which allow separate control of singular and decaying contributions, and play a central role in establishing the convolution estimates in (3.15). Let us consider the case where $\alpha > 0$. We claim that:

$$\begin{aligned} \|f * K_\alpha\|_{L^q(G)} &\leqslant C(G,\mathbf{X})\frac{Q-\alpha}{\alpha}\frac{q^{1/p'}}{p-1}\|f\|_{L^p(G)}, \\ \left\|f * \tilde{K}_\alpha\right\|_{L^q(G)} &\leqslant C(G,\mathbf{X})\|f\|_{L^p(G)}. \end{aligned} \tag{3.15}$$

By combining the above bounds (3.15) and Lemma (3.5), we obtain that:

$$\begin{aligned} \left\|(I+\mathcal{L})^{-\alpha/2} f\right\|_{L^q(G)} &\leqslant C(G,\mathbf{X})\frac{Q-\alpha}{\alpha}\frac{q^{1/p'}}{p-1}\|f\|_{L^p(G)} + C(G,\mathbf{X})\|f\|_{L^p(G)} \\ &= A_1(G,\mathbf{X})Q(p,q)\|f\|_{L^p(G)}. \end{aligned} \tag{3.16}$$

Observe that $q^{1/p'}/(p-1)$ is bounded away from zero when $q \geqslant p > 1$. Assuming the claims for a moment, we complete the proof. Observe that the condition $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ is invariant under the involution $(p,q) \mapsto (q',p')$. Set $Q(p,q) = \frac{q^{1/p'}}{p-1}$. By duality, and from (3.16), we have:

$$\left\|(I+\mathcal{L})^{-\alpha/2} f\right\|_{L^{p'}(G)} \leqslant A_1 Q(p,q)\|f\|_{L^{q'}(G)}, \tag{3.17}$$

that is, switching the roles of the pairs (p,q) and (q',p'),

$$\left\|(I+\mathcal{L})^{-\alpha/2} f\right\|_{L^q(G)} \leqslant A_1 Q\left(q',p'\right)\|f\|_{L^p(G)}. \tag{3.18}$$

From the inequalities (3.16) and (3.18), we conclude that

$$\left\|(I+\mathcal{L})^{-\alpha/2} f\right\|_{L^q(G)} \leqslant A_1 \min\left(Q(p,q), Q\left(q',p'\right)\right)\|f\|_{L^p(G)}. \tag{3.19}$$

Now, making a change of variables in the above estimate (3.19) ($f := (I+\mathcal{L})^{-\alpha/2}\tilde{f}$), we conclude

$$\|f\|_{L^q(G)} \leqslant A_1 S(p,q)\|f\|_{L^p_\alpha(G)}.$$

Since $S(p,q) = \min\left(Q(p,q), Q\left(q',p'\right)\right)$, it remains to prove the bound (3.16) follows by:

Remark 3.7. The Carnot-Carathéodory distance on a compact Lie group G is also symmetric in view of its G-invariance [1], and note that:

$$\check{\tilde{K}}_\alpha(x) = \tilde{K}_\alpha(x^{-1}) = e^{-b_0|x^{-1}|}\mathbb{1}(x^{-1}) = e^{-b_0|x|}\mathbb{1}(x) = \tilde{K}_\alpha(x). \tag{3.20}$$

By applying Young's inequality (3.3):

$$\begin{aligned} \left\|f * \tilde{K}_\alpha\right\|_{L^q(G)} &\leqslant \|f\|_{L^p(G)} \underbrace{\left\|\check{\tilde{K}}_\alpha\right\|_{L^r(G)}^{r/p'}}_{\check{\tilde{K}}_\alpha = \tilde{K}_\alpha} \left\|\check{\tilde{K}}_\alpha\right\|_{L^r(G)}^{r/q} \\ &\leqslant \|f\|_{L^p(G)} \left\|\tilde{K}_\alpha\right\|_{L^r(G)}^{r(1/p'+1/q)}, \end{aligned} \tag{3.21}$$

where $r \in (1,\infty)$ is such that $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$. Since G is a compact Lie group, we have for some $M_0 > 0$ the following inequalities:

$$\begin{aligned} \left\|\tilde{K}_\alpha\right\|_{L^r(G)}^r &= \int_G |\tilde{K}_\alpha(x)|^r \mathrm{d}\lambda(x) \overset{(3.14)}{=} \int_G \left(\mathrm{e}^{-b_0|x|}\mathbb{1}_{B_1^c}(x)\right)^r \mathrm{d}\lambda(x) \\ &= \int_{B_1^c} \mathrm{e}^{-rb_0|x|}\mathrm{d}\lambda(x) = \int_{\bigcup_{k=0}^{M_0}\{x\in G: 2^k<|x|<2^{k+1}\}} \mathrm{e}^{-rb_0|x|}\mathrm{d}\lambda(x) \\ &\leqslant \sum_{k=0}^{M_0} \int_{\{x\in G:2^k<|x|<2^{k+1}\}} \mathrm{e}^{-rb_0|x|}\mathrm{d}\lambda(x) \leqslant C\int_{B_1^c} \mathrm{e}^{-rb_0|x|}\mathrm{d}\lambda(x) \\ &\leqslant C\sum_{k=0}^{M_0} \int_{\{x\in G:2^k<|x|<2^{k+1}\}} \mathrm{e}^{-rb_0|x|}\mathrm{d}\lambda(x) \overset{(3.23)}{\leqslant} C\sum_{k=0}^{M_0} \mathrm{e}^{-rb_0 2^k} \leqslant C, \end{aligned} \tag{3.22}$$

which combined with (3.21) implies (3.15). Note that the sum given in (3.22) is finite since the group is compact.

Observation 3.8. The following estimate arises as an intermediate step in the computation of $\left\|\tilde{K}_\alpha\right\|_{L^r(G)}^r$ above:

$$\begin{aligned} &\int_{\{x\in G:2^k<|x|<2^{k+1}\}} \mathrm{e}^{-rb_0|x|}\mathrm{d}\lambda(x) = \frac{\mathrm{e}^{-rb_0\cdot 2^k}}{rb_0} - \frac{\mathrm{e}^{-rb_0\cdot 2^{k+1}}}{rb_0} \\ &= \frac{1}{rb_0}\left(\mathrm{e}^{-rb_0\cdot 2^k} - \mathrm{e}^{-rb_0\cdot 2^{k+1}}\right) \leqslant \left(\mathrm{e}^{-rb_0\cdot 2^k} - \mathrm{e}^{-rb_0\cdot 2^{k+1}}\right) \leqslant \mathrm{e}^{-rb_0\cdot 2^k}. \end{aligned} \tag{3.23}$$

The rest of the proof will be devoted to show (3.15).

Indeed, for $0 < s \leqslant 1$, define $K_{\alpha,s}^{(1)} = K_\alpha \cdot \mathbb{1}_{B_s}$ and $K_{\alpha,s}^{(2)} = K_\alpha \cdot \mathbb{1}_{B_s^c}$. Notice that $K_{\alpha,s}^{(1)} = \check{K}_{\alpha,s}^{(1)}$ and that the same holds for $K_{\alpha,s}^{(2)}$. Let now $\tilde{p} \in (1,\infty)$ and $\tilde{q} \in (\tilde{p},\infty)$ be such that $\frac{1}{\tilde{q}} = \frac{1}{\tilde{p}} - \frac{\alpha}{Q}$, and observe that

$$(\alpha - Q)\tilde{p}' + Q = -\frac{Q\tilde{p}'}{\tilde{q}}, \quad \frac{\tilde{p}}{\tilde{q}} = 1 - \tilde{p}\frac{\alpha}{Q}, \quad \frac{1}{\tilde{p}'}\left(1 - \frac{\tilde{p}}{\tilde{q}}\right) = (\tilde{p}-1)\frac{\alpha}{Q}. \tag{3.24}$$

By Young's inequality (3.3), there exists $C > 0$ depending only on G and $\mathbf{X}$ such that

$$\begin{aligned}\left\|f * K_{\alpha,s}^{(1)}\right\|_{L^{\tilde{p}}(G)} &\leqslant \|f\|_{L^{\tilde{p}}(G)} \left\|K_{\alpha,s}^{(1)}\right\|_{L^{r}(G)}^{1/\tilde{p}} \left\|\check{K}_{\alpha,s}^{(1)}\right\|_{L^{r}(G)}^{1/\tilde{p}'}, \quad \left(\frac{1}{\tilde{p}} + \frac{1}{r} = 1 + \frac{1}{\tilde{p}'}\right) \\ &= \|f\|_{L^{\tilde{p}}(G)} \left\|K_{\alpha,s}^{(1)}\right\|_{L^{1}(G)}^{1/\tilde{p}} \left\|\check{K}_{\alpha,s}^{(1)}\right\|_{L^{1}(G)}^{1/\tilde{p}'}, \quad (r = 1) \\ &\overset{(3.26)}{\leqslant} C\frac{1}{\alpha} s^{\alpha} \|f\|_{L^{\tilde{p}}(G)}.\end{aligned} \tag{3.25}$$

Remark 3.9. Using *polar coordinates* on G, see, e.g., [15, Theorem 1], we have:

$$\begin{aligned}\|K_{\alpha,s}^{(1)}\|_{L^1(G)} &= \int_G |K_{\alpha,s}^{(1)}(x)| \mathrm{d}\lambda(x) = \int_G |x|^{\alpha-Q} \mathbb{1}_{B_1}(x) \mathbb{1}_{B_s}(x) \mathrm{d}\lambda(x) \\ &= \int_0^s \int_{|x|\leq 1} l^{Q-1} l^{(\alpha-Q)} \mathrm{d}\sigma \mathrm{d}l \lesssim \int_0^s l^{\alpha-1} \mathrm{d}l = \frac{s^{\alpha}}{\alpha},\end{aligned} \tag{3.26}$$

and the same works for $\left\|\check{K}_{\alpha,s}^{(1)}\right\|_{L^1(G)}$, therefore:

$$\left\|K_{\alpha,s}^{(1)}\right\|_{L^1(G)}^{1/\tilde{p}} \left\|\check{K}_{\alpha,s}^{(1)}\right\|_{L^1(G)}^{1/\tilde{p}'} = \left(\frac{s^{\alpha}}{\alpha}\right)^{1/\tilde{p}} \left(\frac{s^{\alpha}}{\alpha}\right)^{1/\tilde{p}'} = \left(\frac{s^{\alpha}}{\alpha}\right)^{\frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'}} \leqslant \frac{s^{\alpha}}{\alpha}, \tag{3.27}$$

and

$$\left\|f * K_{\alpha,s}^{(2)}\right\|_{L^{\infty}(G)} \leqslant \|f\|_{L^{\tilde{p}}(G)} \left\|\check{K}_{\alpha,s}^{(2)}\right\|_{L^{\tilde{p}'}(G)} \leqslant C\left(\frac{\bar{q}}{Q\bar{p}'}\right)^{1/\bar{p}'} \left(s^{-Q\bar{p}'/\tilde{q}} - 1\right)^{1/\tilde{p}'} \|f\|_{L^{\tilde{p}}(G)}. \tag{3.28}$$

This estimate follows from standard heat kernel bounds for subelliptic operators on Lie groups. A similar argument can be found in [7, Equation 3.7], where analogous decay behavior is established.

For $t > 0$, we now set:

$$s(t) = \left[1 + \frac{Q\tilde{p}'}{\tilde{q}}\left(\frac{t}{2}\right)^{\tilde{p}'}\right]^{-\frac{\tilde{q}}{Q\tilde{p}'}}, \tag{3.29}$$

and observe that $s(t) \leqslant 1$ for every $t > 0$. By (3.30),

$$\left\|f * K_{\alpha,s(t)}^{(2)}\right\|_{L^{\infty}(G)} \leqslant C\frac{t}{2}\|f\|_{L^{\tilde{p}}(G)} \quad \forall t > 0. \tag{3.30}$$

Thus, with C, the same constant as in (3.28) and (3.30),

$$\begin{aligned}&\sup_{t>0} t\lambda\left(\{x : |f * K_{\alpha}(x)| > t\}\right)^{1/\tilde{q}} = \sup_{t>0} t\lambda\left(\left\{x : |f * K_{\alpha}(x)| > t\left(\frac{\|f\|_{L^{\tilde{p}}(G)}}{\|f\|_{L^{\tilde{p}}(G)}}\right)\right\}\right)^{1/\tilde{q}} \\ &\leqslant C\|f\|_{L^{\tilde{p}}(G)} \sup_{t>0} t\lambda\left(\left\{x : \left|f * K_{\alpha,s(t)}^{(1)}(x)\right| > C\frac{t}{2}\|f\|_{L^{\tilde{p}}(G)}\right\}\right)^{1/\tilde{q}}, \quad \text{in view of (3.30).}\end{aligned} \tag{3.31}$$

By (3.28) and Chebychev's inequality, we get:

$$
\begin{aligned}
&\sup_{t>0} t\lambda\left(\left\{x : \left|f * K^{(1)}_{\alpha,s(t)}(x)\right| > C\frac{t}{2}\|f\|_{L^{\tilde p}(G)}\right\}\right)^{1/\tilde q} \\
&\leqslant \sup_{t>0} t\left[\left(\frac{2}{Ct\|f\|_{L^{\tilde p}(G)}}\right)^{\tilde p}\left\|f * K^{(1)}_{\alpha,s(t)}\right\|^{\tilde p}_{L^{\tilde p}(G)}\right]^{1/\tilde q}, \\
&\leqslant \sup_{t>0} t\left(\frac{Ct\|f\|_{L^{\tilde p}(G)}}{2}\right)^{-\tilde p/\tilde q}\left(\frac{s(t)^{\alpha}}{\alpha}\right)^{\tilde p/\tilde q} C^{\tilde p/\tilde q}\|f\|^{\tilde q}_{L^{\tilde p}(G)} \\
&\overset{(3.29)}{=} \left(\frac{2}{\alpha}\right)^{\tilde p/\tilde q}\sup_{t>0} t^{1-\tilde p/\tilde q}\left[1+\frac{Q\tilde p'}{\tilde q}\left(\frac{t}{2}\right)^{\tilde p'}\right]^{-\frac{1}{p}\left(1-\frac{\tilde p}{\tilde q}\right)} \\
&\overset{(3.33)}{=} \left(\frac{2}{\alpha}\right)^{\tilde p/\tilde q}\sup_{t>0}\left(2u\left(\frac{Q\tilde p'}{\tilde q}\right)^{-\frac{1}{\tilde p'}}\right)^{1-\tilde p/\tilde q}\left[1+u^{\tilde p'}\right]^{-\frac{1}{\tilde p'}\left(1-\frac{\tilde p}{\tilde q}\right)} \\
&= \left(\frac{2^{\tilde p/\tilde q}2^{1-\frac{\tilde p}{\tilde q}}}{\alpha^{\tilde p/\tilde q}}\right)\left(\frac{Q\tilde p'}{\tilde q}\right)^{-\frac{1}{\tilde p'}(1-\tilde p/\tilde q)}\sup_{u>0} u^{1-\tilde p/\tilde q}\left[1+u^{\tilde p'}\right]^{-\frac{1}{\tilde p'}\left(1-\frac{\tilde p}{\tilde q}\right)} \\
&= \left(\frac{2}{\alpha^{\tilde p/\tilde q}}\right)\left(\frac{\tilde q}{Q\tilde p'}\right)^{\frac{1}{\tilde p'}\left(1-\frac{\tilde p}{\tilde q}\right)}\sup_{u>0} u^{1-\tilde p/\tilde q}\left[1+u^{\tilde p'}\right]^{-\frac{1}{\tilde p'}\left(1-\frac{\tilde p}{\tilde q}\right)},
\end{aligned}
\tag{3.32}
$$

and since we make the following change of variables in (3.32),

$$
u = \left(\frac{Q\tilde p}{\tilde q}\right)^{\frac{1}{\tilde p}}\frac{t}{2} \Longleftrightarrow t = 2u\left(\frac{Q\tilde p'}{\tilde q}\right)^{-\frac{1}{\tilde p'}}. \tag{3.33}
$$

It is now easy to see that, for every $\tilde p$ and $\tilde q$,

$$
\sup_{u>0} u^{1-\tilde p/\tilde q}\left(1+u^{\tilde p'}\right)^{-\frac{1}{\tilde p}\left(1-\frac{\vec{\tilde p}}{q}\right)} \overset{(v=u^{\tilde p'})}{=} \sup_{v>0}[v/(1+v)]^{\frac{1}{\tilde p'}\left(1-\frac{\tilde p}{\tilde q}\right)} = 1. \tag{3.34}
$$

Moreover, by (3.25), we end up with the inequality

$$
\begin{aligned}
\|f * K_\alpha\|_{L^{\tilde q,\infty}(G)} &= \sup_{t>0} t\lambda\left(\{x : |f * K_\alpha(x)| > t\}\right)^{\frac{1}{\tilde q}} \\
&\overset{(3.31)}{\leqslant} C\|f\|_{L^{\tilde p}(G)}\underbrace{\sup_{t>0} t\lambda\left(\left\{x : \left|f * K^{(1)}_{\alpha,s(t)}(x)\right| > C\frac{t}{2}\|f\|_{L^{\tilde p}(G)}\right\}\right)^{1/\tilde q}}_{(3.32)} \\
&\overset{(3.32)}{\leqslant} \left(\frac{2}{\alpha^{\tilde p/\tilde q}}\right)\left(\frac{\tilde q}{Q\tilde p'}\right)^{\frac{1}{\tilde p'}\left(1-\frac{\tilde p}{\tilde q}\right)}\|f\|_{L^{\tilde p}(G)} \\
&\overset{(3.24)}{\leqslant} C\alpha^{\tilde p\alpha/Q-1}\left(\frac{\tilde q}{Q\tilde p}\right)^{(\tilde p-1)\alpha/Q}\|f\|_{L^{\tilde p}(G)}, \quad \left(\frac{\tilde p}{\tilde q} = 1-\frac{\alpha}{Q}\tilde p\right).
\end{aligned}
\tag{3.35}
$$

In other words, the operator defined by $\mathcal{K}_\alpha f = f * K_\alpha$ is of weak type $(\tilde p, \tilde q)$ for every $\tilde p, \tilde q$ such that $\frac{1}{\tilde q} = \frac{1}{\tilde p} - \frac{\alpha}{Q}, 1 < \tilde p < \tilde q < \infty, 0 < \alpha < Q$.

In a similar way, we can also prove that $\mathcal{K}_\alpha$ is of weak type $(1, \tilde q)$ for $\frac{1}{\tilde q} = 1 - \frac{\alpha}{Q}$ and $0 < \alpha < Q$. Indeed, the estimate (3.28) holds also for $\tilde p = 1$ and

$$\left\|f * K_{\alpha,s}^{(2)}\right\|_{L^\infty(G)} \leqslant C\|f\|_{L^1(G)} \times \begin{cases} s^{\alpha-Q} & \text{if } s < 1 \\ 0 & \text{if } s \geqslant 1. \end{cases} \tag{3.36}$$

We now set:

$$s(t) = \begin{cases} \left(1 + \frac{t}{2}\right)^{1/(\alpha-Q)} & t \geqslant 2 \\ 1 & 0 < t < 2, \end{cases} \tag{3.37}$$

which is $\leqslant 1$. Then (3.35) holds also in this case and we obtain as above that:

$$\begin{aligned} &\sup_{t>0} t\lambda\left(\{x : |f * K_\alpha(x)| > t\}\right)^{1/\tilde{q}} \\ &\leqslant C\|f\|_{L^1(G)} \sup_{t>0} t\lambda\left(\left\{x : \left|f * K_{\alpha,s(t)}^{(1)}(x)\right| > C\frac{t}{2}\|f\|_{L^{\tilde{p}}(G)}\right\}\right)^{1/\tilde{q}} \\ &\leqslant C\|f\|_{L^1(G)} \sup_{t>0} t\left(\frac{2}{Ct\|f\|_{L^1(G)}} \mid f * K_{\alpha,s(t)}^{(1)}\|_{L^1(G)}\right)^{1/\tilde{q}}. \end{aligned} \tag{3.38}$$

Here we have to distinguish the supremum in two cases:

Case 1: $0 < t < 2$:

$$\begin{aligned} \sup_{0<t<2} t\left(\frac{2}{Ct\|f\|_{L^1(G)}} \left\|f * K_{\alpha,s(t)}^{(1)}\right\|_{L^1(G)}\right)^{1/\tilde{q}} &\leqslant \sup_{0<t<2} t\left(\frac{t\|f\|_{L^1(G)}}{2}\right)^{-1/\tilde{q}} \left(\frac{s(t)^\alpha}{\alpha}\right)^{1/\tilde{q}} \|f\|_{L^1(G)}^{1/\tilde{q}} \\ &= \sup_{0<t<2} t\left(\frac{t\|f\|_{L^1(G)}}{2}\right)^{-1/\tilde{q}} \left(\frac{1}{\alpha}\right)^{1/\tilde{q}} \|f\|_{L^1(G)}^{1/\tilde{q}} \\ &= 2\alpha^{-1/\tilde{q}}. \end{aligned} \tag{3.39}$$

Case 2: $t \geq 2$:

$$\begin{aligned} \sup_{t\geqslant 2} t\left(\frac{2}{Ct\|f\|_{L^1(G)}} \left\|f * K_{\alpha,s(t)}^{(1)}\right\|_{L^1(G)}\right)^{1/\tilde{q}} &\leqslant \sup_{t\geqslant 2} t\left(\frac{t\|f\|_{L^1(G)}}{2}\right)^{-1/\tilde{q}} \left(\frac{s(t)^\alpha}{\alpha}\right)^{1/\tilde{q}} \|f\|_{L^1(G)}^{1/\tilde{q}} \\ &\leqslant C \sup_{t\geqslant 2} t^{1-\frac{1}{q}} \left(\frac{2}{\alpha}\right)^{1/\tilde{q}} \left(\frac{t}{2}\right)^{-1/Q} \\ &= C2^{1/\tilde{q}}\alpha^{-1/\tilde{q}} \sup_{t\geqslant 2} t^{1-\frac{1}{q}} \left(\frac{t}{2}\right)^{-1/Q} \\ &= C\alpha^{-1/\tilde{q}}. \end{aligned} \tag{3.40}$$

This proves that

$$\|f * K_\alpha\|_{L^{\tilde{q},\infty}(G)} \leqslant C\alpha^{-1/\tilde{q}}\|f\|_{L^1(G)}.$$

We shall now use the Marcinkiewicz interpolation theorem for two specific choices of the couple $(\tilde{p}, \tilde{q})$. Being $p \in (1, \infty), q \in (p, \infty)$, and $\alpha/Q = 1/p - 1/q$ as in the statement, we define

$$\left(\frac{1}{p_1}, \frac{1}{q_1}\right) = \left(1, 1 - \frac{\alpha}{Q}\right), \quad \left(\frac{1}{p_2}, \frac{1}{q_2}\right) = \left(\frac{\alpha}{Q} + \frac{1}{q+1}, \frac{1}{q+1}\right). \tag{3.41}$$

Remark 3.10. For computational reasons, we observe that:

$$\begin{cases} \frac{1}{p_1} = 1 \\ \frac{1}{q_1} = 1 - \frac{\alpha}{Q} = 1 - \frac{1}{p} + \frac{1}{q} = \frac{qp+p-q}{qp} \\ \frac{1}{p_2} = \frac{\alpha}{Q} + \frac{1}{q+1} = \frac{(q-p)}{pq} + \frac{1}{(q+1)} = \frac{(q-p)(q+1)+pq}{pq(q+1)} = \frac{(q^2+q-pq-p+qp)}{pq(q+1)} = \frac{(q^2+q-p)}{pq(q+1)} \\ \frac{1}{q_2} = \frac{1}{q+1}, \end{cases} \tag{3.42}$$

where

$$\begin{cases} p_1 = 1 \\ q_1 = \frac{Q}{(Q-\alpha)} \equiv \frac{qp}{qp+p-q} \\ p_2 = \frac{Q(q+1)}{(q+1)\alpha+Q} = \frac{pq(q+1)}{(q^2+q-p)} \\ q_2 = (q+1). \end{cases} \tag{3.43}$$

Note that above, $\mathcal{K}_\alpha$ is both of weak type $(1, q_1)$ and of weak type (p_2, q_2), with norms $M(1, q_1)$ and $M(p_2, q_2)$, respectively, given by

$$\begin{aligned} M(1, q_1) &= \alpha^{-(1-\alpha/Q)} \\ M(p_2, q_2) &= \left(\frac{Q^{\alpha/Q}}{\alpha}\right)\left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{\alpha/Q+1/(q+1)}} \left[\left(1 - \frac{\alpha}{Q} - \frac{1}{q+1}\right)(q+1)\right]^{\frac{1}{1+Q/(\alpha(q+1))} - \frac{\alpha}{Q}}. \end{aligned} \tag{3.44}$$

We select

$$\theta = \frac{1 - \frac{1}{p}}{1 - \frac{\alpha}{Q} - \frac{1}{q+1}}, \tag{3.45}$$

and therefore $1/p = (1-\theta)/p_1 + \theta/p_2$ and $1/q = (1-\theta)/q_1 + \theta/q_2$. Thus, $\mathcal{K}_\alpha$ is of strong type (p, q), i.e., bounded from $L^p(G)$ to $L^q(G)$, with the norm bounded by

$$CM_0(1, q_1, p_2, q_2)^{1/q} M(1, q_1)^{1-\theta} M(p_2, q_2)^{\theta}, \tag{3.46}$$

from where we get precisely (3.15). If we observe that

$$M_0(1, q_1, p_2, q_2)^{1/q} M(1, q_1)^{1-\theta} M(p_2, q_2)^{\theta} \leqslant C \frac{Q-\alpha}{\alpha} \frac{q^{1/p'}}{p-1}, \tag{3.47}$$

then we get (3.15), which concludes the proof of the theorem. We now prove (3.47).

Estimate for $M_1 := M(1, q_1)$. Observe that:

$$M_1 = \alpha^{-(1-\alpha/Q)} = \alpha^{-1}\alpha^{\alpha/Q}\frac{Q^{\alpha/Q}}{Q^{\alpha/Q}} = \alpha^{-1}Q^{\alpha/Q}\underbrace{(\alpha/Q)^{\alpha/Q}}_{\leq 1} \leq \alpha^{-1}Q^{\alpha/Q} \leq \alpha^{-1}Q,$$

since $\alpha/Q \leq 1$.

Estimate for M_0**:** Note that

$$M_0 := M_0(1, q_1, p_2, q_2) = \frac{q(p_2/p)^{q_2/p_2}}{q_2 - q} + \frac{q/p^{q_1}}{q - q_1}. \tag{3.48}$$

Let us consider y to be defined as below, and let us also consider the identities for q_2/p_2 and for p_2/p presented below. Indeed, observe that

$$y := \frac{\alpha}{Q}(q+1) = \left(\frac{1}{p} - \frac{1}{q}\right)(q+1) = \frac{(q-p)(q+1)}{pq} = \frac{q^2+q-pq-p}{pq},$$

and we have,

$$\begin{aligned} 1+y &= \frac{Q+\alpha(q+1)}{Q} \equiv \frac{qp+(q-p)(q+1)}{qp} = \frac{qp+q^2+q-pq-p}{qp} \\ &= \frac{q^2+q-p}{pq}, \\ \text{and that, } 1+y+\frac{1}{q} &= \frac{q^2+q-p}{pq} + \frac{1}{q} = \frac{(q^2+q-p)q+pq}{pq^2} = \frac{q^3+q^2-pq+pq}{pq^2} \\ &= \frac{q^3+q^2}{pq^2} = \frac{(q+1)q^2}{pq^2} = \frac{(q+1)}{p}. \end{aligned} \tag{3.49}$$

Also let us remark that

$$\begin{aligned} \frac{q_2}{p_2} &= \frac{(q+1)}{\left(\frac{Q(q+1)}{(q+1)\alpha+Q}\right)} = \frac{(q+1)((q+1)\alpha+Q)}{Q(q+1)} = \frac{(q+1)\alpha+Q}{Q} = 1+y, \\ \frac{p_2}{p} &= \frac{Q(q+1)}{((q+1)\alpha+Q)p} = \frac{\left(\frac{pq(q+1)}{(q^2+q-p)}\right)}{p} = \frac{pq}{(q^2+q-p)}\frac{(q+1)}{p} \\ &= \frac{1}{(1+y)}\frac{(q+1)}{p} = \frac{1}{(1+y)}\left(1+y+\frac{1}{q}\right). \end{aligned} \tag{3.50}$$

The analysis above implies that

$$\begin{aligned} \left(\frac{p_2}{p}\right)^{q_2/p_2} &= \left(\frac{p_2}{p}\right)^{(1+y)} = \left(\frac{1}{(1+y)}\left(1+y+\frac{1}{q}\right)\right)^{(1+y)} \\ &= \left(1+y+\frac{1}{q}\right)^{(1+y)}(1+y)^{-(1+y)}. \end{aligned} \tag{3.51}$$

Therefore:

$$\begin{aligned} M_0 &= \frac{q\,(p_2/p)^{q_2/p_2}}{q_2-q} + \frac{q/p^{q_1}}{q-q_1} = q\,(p_2/p)^{1+y} + \frac{q/p^{q_1}}{q-q_1} \\ &= q\left(1+y+\frac{1}{q}\right)^{(1+y)}(1+y)^{-(1+y)} + \underbrace{\frac{q/p^{q_1}}{q-q_1}}_{:=A}. \end{aligned} \tag{3.52}$$

We analyze the term A:

$$A := \frac{q/p^{q_1}}{q-q_1} = \frac{\left(\frac{q}{p^{q_1}}\right)}{q-q_1} = \frac{q}{(q-q_1)}\frac{1}{p^{q_1}} = \left(1+\frac{p'}{q}\right)\frac{1}{p^{q_1}} = \left(1+\frac{p'}{q}\right)p^{-q_1}, \tag{3.53}$$

where

$$\begin{aligned} q-q_1 &= q-\frac{qp}{(qp+p-q)} = \frac{q(qp+p-q)-qp}{(qp+p-q)} = \frac{q^2p+qp-q^2-pq}{(qp+p-q)} \\ &= \frac{q^2p-q^2}{(qp+p-q)} = \frac{(p-1)q^2}{(qp+p-q)} \\ \frac{1}{q-q_1} &= \frac{(qp+p-q)}{(p-1)q^2} = \frac{(qp+p-q)}{(p-1)q}\frac{1}{q}. \end{aligned} \tag{3.54}$$

Then, for $q-q_1$, we have the identities

$$\begin{aligned} \frac{q}{q-q_1} &= \frac{(qp+p-q)}{(p-1)q} = \left(1+\frac{p'}{q}\right), \\ q_1 &:= \frac{qp}{qp+p-q}, \end{aligned} \tag{3.55}$$

and

$$\text{and } -\frac{p'q}{(q+p')} = -\frac{(p-1)}{(qp-q+p)}\frac{pq}{(p-1)} = -\frac{pq}{(qp-q+p)} = -q_1. \tag{3.56}$$

We analyze the term $C(p,q)$:

$$\begin{aligned} C(p,q) &= p^{-p'q/(q+p')}\left(1+\frac{p'}{q}\right) = p^{-q_1}\left(1+\frac{\frac{p}{(p-1)}}{q}\right) \\ &= p^{-q_1}\left(1+\frac{p}{q(p-1)}\right) = p^{-q_1}\left(\frac{q(p-1)+p}{q(p-1)}\right) \\ &= p^{-q_1}\underbrace{\left(\frac{(qp-q+p)}{q(p-1)}\right)}_{\frac{q}{(q-q_1)}} = A. \end{aligned} \tag{3.57}$$

We get

$$M_0 = q\left(1+y+\frac{1}{q}\right)^{(1+y)}(1+y)^{-(1+y)} + \underbrace{C(p,q)}_{:=A}. \tag{3.58}$$

Moreover,

$$\left(y+1+\frac{1}{q}\right)^{1+y}(1+y)^{-(1+y)} = \left[\left(1+\frac{1}{q(1+y)}\right)^{q(1+y)}\right]^{1/q} \leqslant \mathrm{e}, \tag{3.59}$$

since $q(1+y)\geqslant 1$ and, in view of the estimate $\left(1+\frac{1}{x}\right)^x \leqslant e$, for $x\geqslant 1$. Thus, $M_0 \leqslant \mathrm{e}\, q + C(p,q)$.

Estimate for $M_2 := M(p_2,q_2)$:

$$M_2 = \left(\frac{Q^{\alpha/Q}}{\alpha}\right)\left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{\alpha/Q+1/(q+1)}}\left[\left(1-\frac{\alpha}{Q}-\frac{1}{q+1}\right)(q+1)\right]^{\frac{1}{1+Q/(\alpha(q+1))}-\frac{\alpha}{Q}}. \tag{3.60}$$

We are going to estimate M_2^θ:

$$\begin{aligned} M_2^\theta &= \left(\frac{Q^{\alpha/Q}}{\alpha}\right)^\theta \left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{\alpha/Q+1/(q+1)}\theta} \left[\left(1-\frac{\alpha}{Q}-\frac{1}{q+1}\right)(q+1)\right]^{\theta\frac{1}{1+Q/(\alpha(q+1))}-\frac{\alpha}{Q}\theta} \\ &= Q^{(\alpha/Q)\theta}\alpha^{-\theta}\left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{\alpha/Q+1/(q+1)}\theta} \left[\left(1-\frac{\alpha}{Q}-\frac{1}{q+1}\right)(q+1)\right]^{\theta\frac{1}{1+Q/(\alpha(q+1))}-\frac{\alpha}{Q}\theta} \\ &\overset{\left(\frac{\alpha}{Q}\theta\in(0,\theta)\right)}{\leqslant} Q^\theta\alpha^{-\theta}\left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{\alpha/Q+1/(q+1)}\theta} \left[\left(1-\frac{\alpha}{Q}-\frac{1}{q+1}\right)(q+1)\right]^{\theta\frac{1}{1+Q/(\alpha(q+1))}-\frac{\alpha}{Q}\theta}. \end{aligned} \tag{3.61}$$

Observe that with $z := \frac{\alpha}{Q} + \frac{1}{(q+1)}$, we have that $\theta = \frac{1-\frac{1}{p}}{1-\frac{\alpha}{Q}-\frac{1}{q+1}} = \left(\frac{1-\frac{1}{p}}{1-z}\right)$, and

$$\begin{aligned} &\left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{\alpha/Q+1/(q+1)}\theta} \left[\left(1-\frac{\alpha}{Q}-\frac{1}{q+1}\right)(q+1)\right]^{\theta\frac{1}{1+Q/(\alpha(q+1))}-\frac{\alpha}{Q}\theta} \\ &= \left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{z}\theta} \left[(1-z)(q+1)\right]^{\theta\frac{\alpha/Q}{\alpha/Q+1/(q+1)}-\frac{\alpha}{Q}\theta} = \left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{z}\theta} \left[(1-z)(q+1)\right]^{\theta\frac{\alpha/Q}{z}-\frac{\alpha}{Q}\theta} \\ &= \left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{z}\theta} \left[(1-z)(q+1)\right]^{(\frac{1}{z}-1)\frac{\alpha}{Q}\theta} = \left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{z}\theta} \left[(1-z)(q+1)\right]^{(\frac{1-z}{z})\frac{\alpha}{Q}\theta} \\ &= \left(\frac{\alpha}{Q}\right)^{\frac{\alpha/Q}{z}\left(\frac{1-\frac{1}{p}}{1-z}\right)} \left[(1-z)(q+1)\right]^{(\frac{1-z}{z})\frac{\alpha}{Q}\left(\frac{1-\frac{1}{p}}{1-z}\right)} \\ &= \left[\left(\frac{\alpha}{Q}\right)^{\frac{1}{1-z}}(q+1)\right]^{(1-1/p)\frac{\alpha/Q}{z}} (1-z)^{(1-1/p)\frac{\alpha/Q}{z}}. \end{aligned} \tag{3.62}$$

Note also that

$$\begin{aligned} \frac{\alpha/Q}{\alpha/Q+1/(q+1)} &= \left(\frac{\alpha}{Q}\right)\frac{1}{\alpha/Q+1/(q+1)} = \frac{\alpha}{Qz} = \left(\frac{q-p}{pq}\right)\frac{1}{z} \\ &= \left(\frac{q-p}{pq}\right)\frac{1}{\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{(q+1)}\right)} = \left(\frac{q-p}{pq}\right)\left(\frac{1}{\left(\frac{(q-p)(q+1)+pq}{(q+1)}\right)}\right) \\ &= \frac{(q-p)(q+1)}{(q-p)(q+1)+pq} = \frac{(q-p)(q+1)}{q(q+1)-pq-p+pq} = \frac{(q-p)(q+1)}{q(q+1)-p} \leqslant 1. \end{aligned} \tag{3.63}$$

As a consequence we have that

$$\begin{cases} \left(\frac{\alpha}{Q}\right)^{\frac{1}{1-z}} \leqslant \frac{\alpha}{Q}, \\ (1-z)^{(1-1/p)\frac{\alpha/Q}{z}} \leqslant 1. \end{cases} \tag{3.64}$$

Observe now that by (3.63) and since:

$$2\frac{q}{p} \geqslant \frac{q(q+1)-p}{pq} \geqslant \frac{q}{p} \geqslant 1, \tag{3.65}$$

one gets

$$\left[\left(\frac{\alpha}{Q}\right)(q+1)\right]^{(1-1/p)\frac{\alpha/Q}{z}} \leqslant 2\left(\frac{q}{p}\right)^{1/p'}. \tag{3.66}$$

This proves that $M_2^\theta \leqslant 2Q^\theta (q/p)^{1-1/p}\alpha^{-\theta}$.

$$\begin{aligned} M_2^\theta &\overset{(3.62)}{\leqslant} Q^\theta\alpha^{-\theta}\left[\left(\frac{\alpha}{Q}\right)^{\frac{1}{1-z}}(q+1)\right]^{(1-1/p)\frac{\alpha/Q}{z}} \underbrace{(1-z)^{(1-1/p)\frac{\alpha/Q}{z}}}_{\leqslant 1} \\ &\overset{(3.66)}{\leqslant} 2Q^\theta (q/p)^{1-1/p}\alpha^{-\theta}. \end{aligned} \tag{3.67}$$

Putting the remarks above together, we have proved that:

$$\begin{aligned} M_0^{1/q} M_1^{1-\theta} M_2^\theta &\leqslant (\mathrm{e}q + C(p,q))^{1/q}\left(\alpha^{-1}Q\right)^{1-\theta}\left(2Q^\theta (q/p)^{1-1/p}\alpha^{-\theta}\right) \\ &\leqslant 2Q\alpha^{-1}(\mathrm{e}q + C(p,q))^{1/q}(q/p)^{1-1/p}. \end{aligned} \tag{3.68}$$

It remains to estimate the term in the parentheses in the right-hand side. Observe first that:

$$(\mathrm{e}q + C(p,q))^{1/q} \leqslant (\mathrm{e}q)^{1/q} + C(p,q)^{1/q} \leqslant 2\mathrm{e} + C(p,q)^{1/q},$$

and then that:

$$C(p,q)^{1/q} \leqslant \left(1+\frac{p'}{q}\right)^{1/q} = \frac{Q-\alpha}{Q}p'\left(1+\frac{p'}{q}\right)^{1/q-1} \leqslant \frac{Q-\alpha}{Q}p'.$$

After observing that $(Q-\alpha)p'/Q \geqslant 1$, the proof of (3.13) is complete. The analysis above implies (3.1) and completes the proof.

4. The case of SU(2)

In this section, we shall estimate the Sobolev constant for the Lie group $G = \mathrm{SU}(2)$, which consists of 2×2 matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}.$$

Its associated Lie algebra consists of 2×2 Hermitian, anti-symmetric, and traceless matrices, which iare generated by three elements:

$$\mathfrak{su}(2) = \operatorname{span}\left\{X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right\}. \tag{4.1}$$

These elements satisfy the following relations:

$$[X,Y] = Z, [Y,Z] = X, [Z,X] = Y. \tag{4.2}$$

Let $\mathbf{X} = \{X, Y\} \subset \mathfrak{su}(2)$ be a system of C^∞-vector fields, and such a system satisfies the Hörmander's condition of step (or order) $\kappa = 2$ since

$$\mathfrak{g} = \operatorname{span}\{X, Y, [X,Y] = Z\}.$$

We call $\mathbf{X}$ a system of Hörmander's vector fields of the Lie algebra $\mathfrak{su}(2)$, and then the sub-Laplacian associated to this system is defined as

$$\mathcal{L} \equiv \mathcal{L}_{\mathbf{X}} := -X^2 - Y^2. \tag{4.3}$$

In this case the Hausdorff dimension is $Q = 4$, since the Hörmander condition can be stated as $H_x^2\mathrm{SU}(2) = T_x\mathrm{SU}(2)$ for all $x \in \mathrm{SU}(2)$. Indeed, note that

$$H_x^1\mathrm{SU}(2) = \mathrm{span}\{X, Y\},$$

and that

$$H_x^2\mathrm{SU}(2) = \mathrm{span}\{X, Y, [X, Y] = Z\}.$$

These remarks imply that

$$Q := \dim(H^1\mathrm{SU}(2)) + 2(\dim H^2\mathrm{SU}(2) - \dim H^1\mathrm{SU}(2)) = 2 + 2(3 - 2) = 4. \tag{4.4}$$

The Sobolev embedding constant of the Lie group $G = \mathrm{SU}(2)$ is given in terms of the Hausdorff dimension $Q = 4$ as follows:

Corollary 4.1. *Let $1 < p < \infty, 0 \leqslant \alpha < 4/p$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{4}$. The Sobolev embedding constant is given in terms of the Hausdorff dimension $Q = 4$ by*

$$S(p, q) := \min\left(\frac{q^{1/p'}}{p-1}, \frac{p'^{1/q}}{q'-1}\right). \tag{4.5}$$

Author contributions

Authors of this paper have equally contributed to it.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest. Data sharing does not apply to this article, as no data sets were generated or analyzed during the current study.

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