Communications in Analysis and Mechanics

Research article

Well-posedness of classical solutions to the 3D isentropic compressible Navier-Stokes-Poisson equations with degenerate viscosities and vacuum

Peng Lu[1,*] **and Shaojun Yu**[2]

[1] Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou, Zhejiang 310018, P.R. China

[2] School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, P.R. China

* **Correspondence:** Email: plu25@zstu.edu.cn.

Abstract: We consider the isentropic compressible Navier-Stokes-Poisson equations with degenerate viscosities and vacuum in a three-dimensional torus. The local well-posedness of classical solution is established by introducing a "quasi-symmetric hyperbolic"–"degenerate elliptic" coupled structure to control the behavior of the velocity of the fluid near the vacuum and give some uniform estimates. In particular, the initial data allows vacuum in an open set and we do not need any initial compatibility conditions.

Keywords: density-dependent viscosities; compressible Navier-Stokes-Poisson equations; vacuum; local well-posedness; classical solutions

Mathematics Subject Classification: 35Q35, 35A09, 76N10

1. Introduction

In this paper, we consider the compressible Navier-Stokes-Poisson (NSP) equations for the dynamics of charged particles of electrons (see [1]). The equations can be written as

$$\begin{cases} \rho_t + \text{div}(\rho u) = 0, \\ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) = \text{div}\mathbb{S} + \rho\nabla\Phi, \\ \Delta\Phi = \rho - b(x), \end{cases} \tag{1.1}$$

for $t > 0$ and $x \in \mathbb{T}^3$, with initial data

$$t = 0 : (\rho, u, \Phi) = (\rho_0(x), u_0(x), \Phi_0(x)) \quad \text{for} \quad x \in \mathbb{T}^3, \tag{1.2}$$

where $\mathbb{T}^3$ is a three-dimensional torus, $x = (x_1, x_2, x_3)^\top \in \mathbb{T}^3$ and $t \geq 0$ denote the spatial coordinate and time coordinate, respectively. $\rho \geq 0$ is the mass density, and $u = (u^1, u^2, u^3)^\top$ is the fluid velocity. The

pressure P of the polytropic fluid satisfies

$$P(\rho) = A\rho^\gamma, \tag{1.3}$$

where $A > 0$ and $\gamma > 1$ are the gas constants. Φ is the electrostatic potential satisfying

$$m_\Phi(t) \triangleq \int_{\mathbb{T}^3} \Phi(t, x)\mathrm{d}x = 0, \quad \forall\, t \geq 0. \tag{1.4}$$

Moreover, $\mathbb{S}$ is the viscosity stress tensor given by

$$\mathbb{S} = 2\mu(\rho)\mathfrak{D}(u) + \lambda(\rho)\mathrm{div}u\mathbb{I}_3, \tag{1.5}$$

where

$$\mathfrak{D}(u) = \frac{\nabla u + (\nabla u)^\top}{2}$$

is the deformation tensor, $\mathbb{I}_3$ is the 3×3 identity matrix, and

$$\mu(\rho) = \alpha\rho^\delta, \quad \lambda(\rho) = \beta\rho^\delta, \tag{1.6}$$

where μ is the shear viscosity coefficient, $\lambda + \frac{2}{3}\mu$ is the bulk viscosity coefficient, (α, β, δ) are constants satisfying

$$\alpha > 0, \quad 2\alpha + 3\beta \geq 0, \quad \delta > 1, \tag{1.7}$$

and the function $b(x)$ is the doping profile satisfying the following compatibility condition

$$\int_{\mathbb{T}^3} b(x)\mathrm{d}x = \int_{\mathbb{T}^3} \rho_0(x)\mathrm{d}x,$$

which describes the density of fixed, positively charged background ions.

From the mathematical point of view, the NSP equations are the compressible Navier-Stokes equations coupled with the Poisson equation. Indeed, if the electrostatic effects are absent, then system (1.1)–(1.3) will be reduced to the isentropic compressible Navier-Stokes equations. For the constant viscous fluid, there is rich literature on the well-posedness of classical solutions to compressible Navier-Stokes equations. When $\inf_x \rho_0(x) > 0$, the local well-posedness of classical solutions follows from the standard symmetric hyperbolic-parabolic structure satisfying the well-known Kawashima's condition (cf. [2]), which has been extended to a global one by Matsumura-Nishida [3] near the nonvacuum equilibrium. When $\inf_x \rho_0(x) = 0$, the first main issue is the degeneracy of the time evolution operator, which makes it difficult to describe the behavior of the velocity field near the vacuum. For this case, the local-in-time well-posedness of strong solutions with vacuum was first solved by Cho-Choe-Kim [4] and Cho-Kim [5] in $\mathbb{R}^3$, where they introduced an initial compatibility condition to compensate the lack of a positive lower bound of density. Later, Huang-Li-Xin [6] extended the local solution to a global one under some initial smallness assumptions in $\mathbb{R}^3$. Jiu-Li-Ye [7] proved the global existence of classical solution with arbitrarily large data and vacuum in $\mathbb{R}$. For local well-posedness results without initial compatibility condition, we refer to Gong-Li-Liu-Zhang [8], Huang [9] and the references therein.

When viscosity coefficients are density-dependent, the Navier-Stokes system have received extensive attentions in recent years, especially for the case with vacuum, where the well-posedness of solutions

becomes more challenging due to the degenerate viscosity. In fact, the high-order regularity estimates of the velocity in [4, 6, 7] ($\delta = 0$) strongly rely on the uniform ellipticity of the Lamé operator. While for $\delta > 0$, $\mu(\rho)$ vanishes as the density function connects to vacuum continuously, thus it is difficult to adapt the approach of the constant viscosity case. A remarkable discovery of a new mathematical entropy function was made by Bresch-Desjardins [10] for the viscosity satisfying some mathematical relation, which provides additional regularity on some derivative of the density. This observation was applied widely in proving the global existence of weak solutions with vacuum for Navier-Stokes equations and some related models; see Bresch-Desjardins [10], Bresch-Vasseur-Yu [11], Jiu-Xin [12], Li-Xin [13], Mellet-Vasseur [14], Vasseur-Yu [15], and so on. Then, we turn our attention to the study of classical solutions. When $\delta = 1$, Li-Pan-Zhu [16] obtained the local existence of 2-D classical solution with far field vacuum, which also applies to the 2-D shallow water equations. When $1 < \delta \le \min\left\{3, \frac{\gamma+1}{2}\right\}$, by making full use of the symmetrical structure of the hyperbolic operator and the weak smoothing effect of the elliptic operator, Li-Pan-Zhu [17] established the local existence of 3-D classical solutions with arbitrarily large data and vacuum, see also Geng-Li-Zhu [18] for more related results, and Xin-Zhu [19] for the global existence of classical solution under some initial smallness assumptions in homogeneous Sobolev space. When $0 < \delta < 1$, Xin-Zhu [20] obtained the local existence of 3-D local classical solution with far field vacuum, Cao-Li-Zhu [21] proved the global existence of 1-D classical solution with large initial data. Some other interesting results and discussions can also be found in Germain-Lefloch [22], Guo-Li-Xin [23], Lions [24], Yang-Zhao [25], and the references therein.

Concerning the NSP system, there are also extensive studies about the local and global well-posedness of solutions. For constant viscous fluid (i.e., $\delta = 0$ in (1.6)), Donatelli [26] obtained the local and global existence of weak solutions to 3-D isentropic Navier-Stokes-Poisson equations with vacuum in a bounded domain. Li-Matsumura-Zhang [27] established the global existence and large time behavior to classical solution for Cauchy problem with small data. Zheng [28] established the global existence for Cauchy problem in Besov space. Tan-Wang-Wang [29] established the global existence of classical solutions and obtained the time decay rates of the solution. Liu-Xu-Zhang [30] established the global well-posedness of strong solutions to the Cauchy problem with large oscillations and vacuum. Liu-Luo-Zhong [31] considered the radially symmetric case, and established the global well-posedness of classical solutions with large initial data exterior to a ball. Chen-Huang-Shi [32] considered the initial-boundary-value problem in a 3-D bounded domain with slip boundary condition. The global well-posedness of classical solution was established with small initial energy but possibly large oscillations and vacuum.

For the case of density-dependent viscosity coefficients (i.e., $\delta > 0$ in (1.6)), the problem is much more challenging due to the degeneration of viscosities near the vacuum and hence the obtained results are limited. Ducomet et al. [33] studied the global stability of the weak solutions to the Cauchy problem of the NSP equations with non-monotone pressure as $\gamma > \frac{4}{3}$. In [34] they also considered the Cauchy problem for the NSP equations of spherically symmetric motions, including both constant viscosities and density-dependent viscosities, and proved the global stability of the weak solutions provided that $\gamma > 1$. Zlotinik [35] studied the long-time behavior of the spherically symmetric weak solutions near a hard core by giving global-in-time bounds for the solutions. Ye-Dou [36] studied the global existence of weak solutions to the compressible NSP equations with density-dependent viscosities for $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ in (1.6) in a three-dimensional torus. Recently, Yu [37] considered the three-dimensional Cauchy problem with $\delta > \frac{3}{2}$ in (1.6), and established the global existence and H^2-decay rate of the strong solution.

It should be pointed out that, in spite of the above significant achievements, a lot of questions remain open, including the local well-posedness of classical solutions in multi-dimensions with vacuum. In this paper, we consider the 3-D isentropic compressible Navier-Stokes-Poisson equations with degenerate viscosities and vacuum in a torus, where the viscosities depend on the density in a super-linear power law (i.e., $\delta > 1$ in (1.6)), and obtain the local existence of classical solutions. One of the main difficulties is that the appearance of the vacuum leads to degeneration in the elliptic part of the momentum equation, which makes it difficult to estimate the velocity u. To this end, we reformulate the original system and make use of the "quasi-symmetric hyperbolic"–"degenerate elliptic" coupled structure of the new system to control the behavior of the velocity of the fluid near the vacuum and give uniform estimates. The second main difficulty is that, although the equation of electrostatic potential Φ is linear, it turns to be a nonlinear one in the higher-order estimates after reformulation. By employing the Poincaré inequality on a three-dimensional torus and taking advantage of the precise structure of the system, we can establish the H^4-estimates of Φ.

Here and throughout this paper, we adopt the following simplified notations: for any $p, r \in [1, \infty]$ and integer $k, s \geq 0$, we denote

$$|f|_p = \|f\|_{L^p(\mathbb{T}^3)}, \quad \|f\|_s = \|f\|_{H^s(\mathbb{T}^3)}, \quad D^{k,r} = \left\{ f \in L^1_{loc}(\mathbb{T}^3) : \left|\nabla^k f\right|_r < +\infty \right\},$$

$$D^k = D^{k,2}, \quad |f|_{D^{k,r}} = \|f\|_{D^{k,r}(\mathbb{T}^3)}, \quad |f|_{D^k} = \|f\|_{D^k(\mathbb{T}^3)}, \quad \int f = \int_{\mathbb{T}^3} f \, dx,$$

$$\|(f, g)\|_X = \|f\|_X + \|g\|_X, \quad \|f\|_{X \cap Y} = \|f\|_X + \|f\|_Y.$$

Definition 1.1. *Let $T > 0$ be a finite constant. A solution (ρ, u, Φ) to the problem (1.1)–(1.2) is called a regular solution in $[0, T] \times \mathbb{T}^3$ if (ρ, u, Φ) satisfies this problem in the sense of distribution and:*

(1) $\rho \geq 0$, $\rho^{\frac{\delta-1}{2}} \in C\left([0, T]; H^3\right)$, $\rho^{\frac{\gamma-1}{2}} \in C\left([0, T]; H^3\right)$;
(2) $u \in C([0, T]; H^{s'}) \cap L^\infty\left(0, T; H^3\right)$, $\rho^{\frac{\delta-1}{2}} \nabla^4 u \in L^2\left(0, T; L^2\right)$;
(3) $u_t + u \cdot \nabla u = 0$ as $\rho(t, x) = 0$;
(4) $\Phi \in C\left([0, T]; H^3\right) \cap L^\infty\left(0, T; H^4\right)$,

where $s' \in [2, 3)$ is a constant.

Our main theorem can be stated as follows.

Theorem 1.1. *Assume that*

$$\delta \in (1, 2] \cup \{3\}, \quad \gamma > 1.$$

If the initial data (ρ_0, u_0, Φ_0) satisfies the following regularity conditions:

$$\rho_0 \geq 0, \quad \left(\rho_0^{\frac{\gamma-1}{2}}, \rho_0^{\frac{\delta-1}{2}}, u_0\right) \in H^3, \tag{1.8}$$

and $b(x) \in H^2$, then there exists a positive time T_* and a unique regular solution $(\rho, u, \Phi)(t, x)$ in $[0, T_*] \times \mathbb{T}^3$ to the problem (1.1)–(1.2) satisfying:

$$\sup_{0 \leq t \leq T_*} \left(\|\rho^{\frac{\gamma-1}{2}}\|_3^2 + \|\rho^{\frac{\delta-1}{2}}\|_3^2 + \|u\|_3^2 + \|\Phi\|_4^2 \right) + \int_0^t |\rho^{\frac{\delta-1}{2}} \nabla^4 u|_2^2 ds \leq C^0,$$

for a positive constant $C^0 = C^0(A, \gamma, \delta, \alpha, \beta, \rho_0, u_0, b(x))$. Actually, (ρ, u, Φ) satisfies the problem (1.1)– (1.2) classically in positive time $(0, T_*]$.

Remark 1.1. *The initial data allows vacuum in an open set and we do not need any initial compatibility conditions. In addition, there are no smallness conditions imposed on the initial data.*

The rest of this paper is organized as follows. §2 is dedicated to the preliminary lemmas to be used later. §3 is devoted to proving the well-posedness of local classical solutions, i.e., Theorem 1.1.

2. Preliminaries

In this section, we list some basic lemmas to be used later. The following well-known Gagliardo-Nirenberg inequality will be used (see [38]).

Lemma 2.1. *For $p \in [2, 6]$, $q \in (1, \infty)$, and $r \in (3, \infty)$, there exists a constant $C > 0$ which may depend on q, r such that for $f \in H^1$ with $\int_{\mathbb{T}^3} f \mathrm{d}x = 0$ and $g \in L^q \cap D^{1,r}$, it holds*

$$|f|_p \le C|f|_2^{\frac{6-p}{2p}} |\nabla f|_2^{\frac{3p-6}{2p}}, \tag{2.1}$$

$$|g|_\infty \le C|g|_q^{\frac{q(r-3)}{3r+q(r-3)}} |\nabla g|_r^{\frac{3r}{3r+q(r-3)}}. \tag{2.2}$$

The next several lemmas contain some Sobolev inequalities on the product estimates, the interpolation estimates, the composite function estimates, etc., which can be found in many works, see Majda [39].

Lemma 2.2. [39] *For constants $s \in \mathbb{N}_+$, $r, p, q \in [1, \infty]$ satisfying*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

and functions $f, g \in W^{s,p} \cap W^{s,q}$, there exists a constant C only depending on s, such that

$$|\nabla^s(fg) - f\nabla^s g|_r \le C \left(|\nabla f|_p |\nabla^{s-1} g|_q + |\nabla^s f|_p |g|_q \right), \tag{2.3}$$

$$|\nabla^s(fg) - f\nabla^s g|_r \le C \left(|\nabla f|_p |\nabla^{s-1} g|_q + |\nabla^s f|_q |g|_p \right), \tag{2.4}$$

where $\nabla^s f$ ($s > 1$) stands for the set of all partial derivatives $\partial_x^\xi f$ with $|\xi| = s$.

Lemma 2.3. [39] *If functions $f, g \in H^s$ and $s > \frac{3}{2}$, then $fg \in H^s$, and there exists a constant C only depending on s such that*

$$\|fg\|_s \le C\|f\|_s\|g\|_s. \tag{2.5}$$

Lemma 2.4. [39] *If $f \in H^s$, then for any $r \in [0, s]$, there exists a constant C only depending on s such that*

$$\|f\|_r \le C|f|_2^{1-\frac{r}{s}} \|f\|_s^{\frac{r}{s}}. \tag{2.6}$$

Lemma 2.5. [39] (1) *If $f, g \in H^s \cap L^\infty$ and $|\zeta| \le s$, then there exists a constant C only depending on s such that*

$$\left\|\partial_x^\zeta(fg)\right\|_s \le C \left(|f|_\infty |\nabla^s g|_2 + |g|_\infty |\nabla^s f|_2\right). \tag{2.7}$$

(2) *Let $u(x)$ be a continuous function taking its values in some open set G such that $u \in H^s \cap L^\infty$, and $g(u)$ be a smooth vector-valued function on G. Then for any $s \ge 1$, there exists a constant C only depending on s such that*

$$|\nabla^s g(u)|_2 \le C \left\|\frac{\partial g}{\partial u}\right\|_{s-1} |u|_\infty^{s-1} |\nabla^s u|_2. \tag{2.8}$$

Lemma 2.6. [39] *If function sequence $\{w_n\}_{n=1}^{\infty}$ converges weakly in a Hilbert space X to w, then w_n converges strongly to w in X if and only if*

$$\|w\|_X \geq \limsup_{n \to \infty} \|w_n\|_X.$$

The next one will show some compactness results from the Aubin-Lions lemma.

Lemma 2.7. [40] *Let X_0, X, and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Then:*

- *Let G be bounded in $L^p(0, T; X_0)$ for $1 \leq p < \infty$, and $\frac{\partial G}{\partial t}$ be bounded in $L^1(0, T; X_1)$. Then G is relatively compact in $L^p(0, T; X)$.*
- *Let F be bounded in $L^{\infty}(0, T; X_0)$ and $\frac{\partial F}{\partial t}$ be bounded in $L^p(0, T; X_1)$ with $p > 1$. Then F is relatively compact in $C(0, T; X)$.*

The last lemma will be used to show the time continuity for the higher-order terms of our solution:

Lemma 2.8. [41] *If $f(t, x) \in L^2\left(0, T; L^2\right)$, then there exists a sequence s_k such that*

$$s_k \to 0, \quad and \quad s_k |f(s_k, x)|_2^2 \to 0, \quad as \quad k \to +\infty.$$

3. Local-in-time well-posedness of solutions

This section is devoted to proving Theorem 1.1. To this end, we first reformulate the original Cauchy problem (1.1)–(1.2) into a more convenient form in terms of some new variables, and then establish the local well-posedness of the smooth solution to the reformulated system.

3.1. Reformulation

Inspired by [17, 18], we introduce two new quantities,

$$\phi = \rho^{\frac{\gamma-1}{2}}, \quad \varphi = \rho^{\frac{\delta-1}{2}},$$

to rewrite (1.1)–(1.2) as

$$\begin{cases} \varphi_t + u \cdot \nabla \varphi + \frac{\delta-1}{2} \varphi \operatorname{div} u = 0, \\ \phi_t + u \cdot \nabla \phi + \frac{\gamma-1}{2} \phi \operatorname{div} u = 0, \\ u_t + u \cdot \nabla u + \frac{A\gamma}{\gamma-1} \nabla \phi^2 + \varphi^2 L(u) = \frac{\delta}{\delta-1} Q(u) \cdot \nabla \varphi^2 + \nabla \Phi, \\ \Delta \Phi = \varphi^{\frac{2}{\delta-1}} - b, \\ (\varphi, W, \Phi)|_{t=0} = (\varphi_0(x), W_0(x), \Phi_0(x)), \quad x \in \mathbb{T}^3, \end{cases}$$

which can be rewritten into a new system that consists of a transport equation for φ, a linear elliptic equation for Φ, and a "quasi-symmetric hyperbolic"–"degenerate elliptic" coupled system with some

special lower-order source terms for (ϕ, u):

$$
\begin{cases}
\varphi_t + u \cdot \nabla \varphi + \frac{\delta-1}{2} \varphi \mathrm{div} u = 0, \\
\underbrace{A_0 W_t + \sum_{j=1}^3 A_j(W) \partial_j W}_{\text{symmetric hyperbolic}} + \underbrace{\varphi^2 \mathbb{L}(W)}_{\text{degenerate elliptic}} = \underbrace{\mathbb{Q}(W) H(\varphi) + \begin{pmatrix} 0 \\ a_1 \nabla \Phi \end{pmatrix}}_{\text{lower-order source}}, \\
\Delta \Phi = \varphi^{\frac{2}{\delta-1}} - b, \\
(\varphi, W, \Phi)|_{t=0} = (\varphi_0(x), W_0(x), \Phi_0(x)), \quad x \in \mathbb{T}^3,
\end{cases}
\tag{3.1}
$$

where $W(x) = (\phi, u)^\top(x)$, $W_0(x) = (\phi_0, u_0)^\top(x)$ and

$$
A_0 = \begin{pmatrix} 1 & 0 \\ 0 & a_1 \mathbb{I}_3 \end{pmatrix}, \quad A_j(W) = \begin{pmatrix} u_j & \frac{\gamma-1}{2} \phi e_j^\top \\ \frac{\gamma-1}{2} \phi e_j & a_1 u_j \mathbb{I}_3 \end{pmatrix}, \quad j = 1, 2, 3,
$$

$$
\mathbb{L}(W) = \begin{pmatrix} 0 \\ a_1 L(u) \end{pmatrix}, \quad L(u) = -\alpha(\Delta u + \nabla \mathrm{div} u) - \beta \nabla \mathrm{div} u,
$$

$$
H(\varphi) = \begin{pmatrix} 0 \\ \nabla \varphi^2 \end{pmatrix}, \quad \mathbb{Q}(W) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{a_1 \delta}{\delta-1} Q(u) \end{pmatrix},
$$

$$
Q(u) = \alpha(\nabla u + (\nabla u)^\top) + \beta \mathrm{div} u \mathbb{I}_3, \quad a_1 = \frac{(\gamma-1)^2}{4A\gamma}.
$$

Theorem 3.1. *If initial data* (φ_0, W_0, Φ_0) *satisfies*

$$
\varphi_0 \geq 0, \quad (\varphi_0, W_0) \in H^3,
\tag{3.2}
$$

and $b(x) \in H^2$, *then there exists a time* $T_* > 0$ *and a unique regular solution* (φ, ϕ, u, Φ) *in* $[0, T_*] \times \mathbb{T}^3$ *to the problem* (3.1) *satisfying:*

$$
\begin{aligned}
&\varphi \in C\left([0, T^*]; H^3\right), \quad \phi \in C\left([0, T^*]; H^3\right), \\
&\Phi \in C\left([0, T^*]; H^3\right) \cap L^\infty\left(0, T^*; H^4\right), \\
&u \in C\left([0, T^*]; H^{s'}\right) \cap L^\infty\left(0, T^*; H^3\right), \quad s' \in [2, 3), \\
&\varphi \nabla^4 u \in L^2\left(0, T^*; L^2\right), \quad u_t \in C\left([0, T^*]; H^1\right) \cap L^2\left(0, T^*; D^2\right).
\end{aligned}
\tag{3.3}
$$

Moreover, we have

$$
\sup_{0 \leq t \leq T_*} \left(\|\phi\|_3^2 + \|\varphi\|_3^2 + \|u\|_3^2 + \|\Phi\|_4^2 \right)(t) + \int_0^t |\varphi \nabla^4 u|_2^2 \mathrm{d}s \leq C^0,
$$

for a positive constant $C^0 = C^0(\gamma, \delta, \varphi_0, W_0, \Phi_0, b(x))$.

3.2. Linearization

Let T be any positive time. In order to solve the nonlinear problem (3.1), we should consider the following linearized approximation problem for (φ, ϕ, u, Φ):

$$
\begin{cases}
\varphi_t + v \cdot \nabla \varphi + \frac{\delta-1}{2} \tilde{\varphi} \operatorname{div} v = 0, \\
A_0 W_t + \sum\limits_{j=1}^{3} A_j(V) \partial_j W + (\varphi^2 + \eta^2) \mathbb{L}(W) = \mathbb{Q}(V) H(\varphi) + \begin{pmatrix} 0 \\ a_1 \nabla \Phi \end{pmatrix}, \\
\Delta \Phi = \varphi^{\frac{2}{\delta-1}} - b, \\
(\varphi, W, \Phi)|_{t=0} = (\varphi_0(x), W_0(x), \Phi_0(x)), \quad x \in \mathbb{T}^3,
\end{cases}
\tag{3.4}
$$

where $\eta \in (0, 1]$ is a constant, $W = (\phi, u)^\top$, $V = (\tilde{\phi}, v)^\top$, and $W_0 = (\phi_0, u_0)^\top$. $(\tilde{\varphi}, \tilde{\phi})$ are all known functions and $v = (v_1, v_2, v_3)^\top \in \mathbb{R}^3$ is a known vector satisfying the initial assumption $(\tilde{\varphi}, \tilde{\phi}, v)(t = 0, x) = (\varphi_0, \phi_0, u_0)(x)$ and

$$
\begin{aligned}
&\tilde{\varphi} \in C([0, T]; H^3), \quad \tilde{\varphi}_t \in C([0, T]; H^2), \quad \tilde{\phi} \in C([0, T]; H^3), \\
&\tilde{\phi}_t \in C([0, T]; H^2), \quad v \in C([0, T]; H^{s'}) \cap L^\infty(0, T; H^3), \\
&\tilde{\varphi} \nabla^4 v \in L^2(0, T; L^2), \quad v_t \in C([0, T]; L^2) \cap L^2(0, T; D^1).
\end{aligned}
\tag{3.5}
$$

Moreover, we assume that

$$
\varphi_0 \geq 0, \quad \phi_0 \geq 0, \quad (\varphi_0, W_0) \in H^3.
\tag{3.6}
$$

Now we have the following existence result of a strong solution (φ, ϕ, u, Φ) to (3.4) by the standard methods at least when $\eta > 0$:

Lemma 3.1. *Assume that the initial data (φ_0, W_0) satisfy (3.6). Then there exists a constant $T^* > 0$, such that for any $T \in [0, T^*]$, there exists a unique strong solution (φ, ϕ, u, Φ) in $[0, T] \times \mathbb{T}^3$ to (3.4) when $\eta > 0$ such that*

$$
\begin{aligned}
&\varphi \in C([0, T]; H^3), \quad \phi \in C([0, T]; H^3), \quad \Phi \in L^\infty(0, T; H^4) \cap C([0, T]; H^3), \\
&u \in C([0, T]; H^3) \cap L^2(0, T; D^4), \quad u_t \in C([0, T]; H^1) \cap L^2(0, T; D^2).
\end{aligned}
\tag{3.7}
$$

Proof. First, the local-in-time existence and regularities of a unique solution φ in $(0, T) \times \mathbb{T}^3$ to the equations $(3.4)_1$ can be obtained by the standard theory of characteristics, see [42, Section 3.2].

Next, since $\varphi \in C([0, T]; H^3)$ and $\frac{2}{\delta-1} \in [2, \infty) \cup \{1\}$, we have $\varphi^{\frac{2}{\delta-1}} \in C([0, T]; H^2)$. Consequently, the existence and $L^\infty(0, T; H^4)$ regularity of a unique solution Φ in $(0, T) \times \mathbb{T}^3$ to the equations $(3.4)_3$ can be obtained by the standard theory of elliptical equations, see [42, Section 6.2–6.3] for details. To obtain the $C([0, T]; H^3)$ regularity of Φ, we note that $(3.4)_1$ implies

$$
\left(\varphi^{\frac{2}{\delta-1}} \right)_t = \varphi^{\frac{3-\delta}{\delta-1}} \left(-v \cdot \nabla \varphi - \frac{\delta-1}{2} \tilde{\varphi} \operatorname{div} v \right) \in L^\infty(0, T; H^1).
$$

Therefore, for any $t, s \in [0, T]$, we have

$$
\left\| \varphi^{\frac{2}{\delta-1}}(t) - \varphi^{\frac{2}{\delta-1}}(s) \right\|_1 \leq C \int_s^t \left\| \left(\varphi^{\frac{2}{\delta-1}} \right)_t (\tau) \right\|_1 \, d\tau \to 0, \quad \text{as } s \to t.
$$

Then standard elliptical estimates and Poincaré inequality imply

$$\|\Phi(t) - \Phi(s)\|_3 \leq C \left\| \varphi^{\frac{2}{\delta-1}}(t) - \varphi^{\frac{2}{\delta-1}}(s) \right\|_1 \to 0, \qquad \text{as } s \to t.$$

This gives the $C([0, T]; H^3)$ regularity of Φ.

Finally, when $\eta > 0$, based on the regularities of φ and Φ, it is not difficult to solve W from the linear symmetric hyperbolic-parabolic coupled system $(3.4)_2$ to complete the proof, see [42, Chapter 7]. □

3.3. A priori estimates

In this subsection, we establish some local-in-time a priori estimates independent of η for the solution (φ, ϕ, u, Φ) to (3.4) obtained in Lemma 3.1. For this purpose, we fix a $T > 0$ and choose a positive constant c_0 large enough, such that

$$\|\varphi_0\|_3 + \|\phi_0\|_3 + \|u_0\|_3 + \|b(x)\|_2 \leq c_0. \tag{3.8}$$

Assume that there exist some time $T^* \in (0, T)$ and constants c_i $(i = 1, 2, 3)$ such that

$$\sup_{0 \leq t \leq T^*} \left(\|\tilde{\varphi}(t)\|_1^2 + \|\tilde{\phi}(t)\|_1^2 + \|v(t)\|_1^2 \right) + \int_0^{T^*} \left| \tilde{\varphi} \nabla^2 v \right|_2^2 dt \leq c_1^2,$$

$$\sup_{0 \leq t \leq T^*} \left(\|\tilde{\varphi}(t)\|_2^2 + \|\tilde{\phi}(t)\|_2^2 + \|v(t)\|_2^2 \right) + \int_0^{T^*} \left| \tilde{\varphi} \nabla^3 v \right|_2^2 dt \leq c_2^2, \tag{3.9}$$

$$\sup_{0 \leq t \leq T^*} \left(\|\tilde{\varphi}(t)\|_3^2 + \|\tilde{\phi}(t)\|_3^2 + \|v(t)\|_3^2 \right) + \int_0^{T^*} \left| \tilde{\varphi} \nabla^4 v \right|_2^2 dt \leq c_3^2,$$

where T^* and c_i $(i = 1, 2, 3)$ will be determined later (see (3.44)), and depend only on c_0 and the fixed constants A, α, β, γ, δ, and T.

Hereinafter, we use $C > 1$ to denote a generic positive constant depending only on fixed constants A, α, β, γ, δ, and T, but is independent of η.

Lemma 3.2. *Let* $T_1 = \min \left\{ T^*, (1 + Cc_3)^{-2} \right\}$. *Then for* $0 \leq t \leq T_1$,

$$\|\varphi(t)\|_3 \leq Cc_0, \quad \|\varphi_t(t)\|_2 \leq Cc_3^2, \quad \|\Phi(t)\|_4 \leq Cc_0^{\frac{2}{\delta-1}}.$$

Proof. Apply the operator ∂_x^ζ $(0 \leq |\zeta| \leq 3)$ to $(3.4)_1$, we have

$$(\partial_x^\zeta \varphi)_t + v \cdot \nabla \partial_x^\zeta \varphi = -(\partial_x^\zeta (v \cdot \nabla \varphi) - v \cdot \nabla \partial_x^\zeta \varphi) - \partial_x^\zeta (\tilde{\varphi} \operatorname{div} v). \tag{3.10}$$

Multiplying both sides of (3.10) by $\partial_x^\zeta \varphi$ and integrating over $\mathbb{T}^3$, we get

$$\frac{1}{2} \frac{d}{dt} |\partial_x^\zeta \varphi|_2^2 \leq C |\operatorname{div} v|_\infty |\partial_x^\zeta \varphi|_2^2 + C |\partial_x^\zeta (v \cdot \nabla \varphi - v \cdot \nabla \partial_x^\zeta \varphi)|_2 |\partial_x^\zeta \varphi|_2 + C |\partial_x^\zeta (\tilde{\varphi} \operatorname{div} v)|_2 |\partial_x^\zeta \varphi|_2. \tag{3.11}$$

First we consider the case where $|\zeta| \leq 2$,

$$|\partial_x^\zeta (v \cdot \nabla \varphi - v \cdot \nabla \partial_x^\zeta \varphi)|_2 \leq C(|\nabla v \cdot \nabla \varphi|_2 + |\nabla v \cdot \nabla^2 \varphi|_2 + |\nabla^2 v \cdot \nabla \varphi|_2) \leq C\|v\|_3 \|\varphi\|_2,$$
$$|\partial_x^\zeta (\tilde{\varphi} \operatorname{div} v)|_2 \leq C\|\tilde{\varphi}\|_2 \|v\|_3, \tag{3.12}$$

which yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_2 \le C\|v\|_3\|\varphi\|_2 + C\|\tilde{\varphi}\|_2\|v\|_3. \tag{3.13}$$

Then, according to Gronwall's inequality, one has

$$\|\varphi\|_2 \le (\|\varphi_0\|_2 + Cc_3^2 t)\exp(Cc_3 t) \le Cc_0, \tag{3.14}$$

for $0 \le T_1 = \min\{T^*, (1 + Cc_3)^{-2}\}$.

When $|\zeta| = 3$,

$$\begin{aligned}
|\partial_x^\zeta(v \cdot \nabla\varphi) - v \cdot \nabla\partial_x^\zeta\varphi|_2 &\le C(|\nabla^3 v \cdot \nabla\varphi|_2 + |\nabla^2 v \cdot \nabla^2\varphi|_2 + |\nabla v \cdot \nabla^3\varphi|_2) \\
&\le C\|v\|_3\|\varphi\|_3,
\end{aligned}$$

$$\begin{aligned}
|\partial_x^\zeta(\tilde{\varphi}\mathrm{div}v)|_2 &\le C(|\tilde{\varphi}\nabla^3\mathrm{div}v|_2 + |\nabla\tilde{\varphi} \cdot \nabla^2\mathrm{div}v|_2 + |\nabla^2\tilde{\varphi}\nabla\mathrm{div}v|_2 + |\nabla^3\tilde{\varphi}\mathrm{div}v|_2) \\
&\le C(|\tilde{\varphi}\nabla^4 v|_2 + \|\tilde{\varphi}\|_3\|v\|_3).
\end{aligned}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_3 \le C(\|v\|_3\|\varphi\|_3 + \|v\|_3\|\tilde{\varphi}\|_3 + |\tilde{\varphi}\nabla^4 v|_2). \tag{3.15}$$

According to Gronwall's inequality, we obtain that

$$\|\varphi\|_3 \le \left(\|\varphi_0\|_3 + c_3^2 t + \int_0^t |\tilde{\varphi}\nabla^4 v|_2 \mathrm{d}s\right)\exp(Cc_3 t). \tag{3.16}$$

Noting that

$$\int_0^t |\tilde{\varphi}\nabla^4 v|_2 \mathrm{d}s \le t^{\frac{1}{2}}\left(\int_0^t |\tilde{\varphi}\nabla^4 v|_2^2 \mathrm{d}s\right)^{\frac{1}{2}} \le c_3 t^{\frac{1}{2}}, \tag{3.17}$$

it follows from (3.16)–(3.17) that

$$\|\varphi\|_3 \le Cc_0, \tag{3.18}$$

for $0 \le t \le T_1$.

Observing that

$$\varphi_t = -v \cdot \nabla\varphi - \tilde{\varphi}\mathrm{div}v, \tag{3.19}$$

we easily have

$$\begin{aligned}
|\varphi_t|_2 &\le C(|v|_\infty|\nabla\varphi|_2 + |\tilde{\varphi}|_\infty|\mathrm{div}v|_2) \le Cc_2^2, \\
|\varphi_t|_{D^1} &\le C(|v|_\infty|\nabla^2\varphi|_2 + |\nabla v|_\infty|\nabla\varphi|_2 + |\nabla\tilde{\varphi}|_\infty|\nabla v|_2 + |\tilde{\varphi}|_\infty|\nabla^2 v|_2) \le Cc_3^2, \\
|\varphi_t|_{D^2} &\le C(|v|_\infty|\nabla^3\varphi|_2 + |\nabla v|_\infty|\nabla^2\varphi|_2 + |\nabla^2 v|_2|\nabla\varphi|_\infty \\
&\quad + |\nabla\tilde{\varphi}|_\infty|\nabla^2 v|_2 + |\tilde{\varphi}|_\infty|\nabla^3 v|_2) \le Cc_3^2.
\end{aligned}$$

It follows from $(3.4)_3$, (3.18), and standard elliptical estimates that

$$\|\Phi(t)\|_4 \le C\|\varphi^{\frac{2}{\delta-1}} - b(x)\|_2 \le C(\|\varphi^{\frac{2}{\delta-1}}\|_2 + \|b(x)\|_2) \le Cc_0^{\frac{2}{\delta-1}}.$$

Thus, we complete the proof of the lemma. □

Lemma 3.3. *Let* (φ, W, Φ) *be the solutions to (3.4). Then*

$$\|W(t)\|_1^2 + \int_0^t |\sqrt{\varphi^2 + \eta^2}\nabla^2 u|_2^2 \mathrm{d}s \le Cc_0^2,$$

for $0 \le t \le T_2 \triangleq \min\left\{T_1, (1 + Cc_3)^{-3}, (1 + Cc_0)^{-\frac{5-\delta}{\delta-1}}\right\}$.

Proof. Applying ∂_x^ζ to $(3.4)_3$, we have

$$
\begin{aligned}
&A_0\partial_x^\zeta W_t + \sum_{j=1}^3 A_j(V)\partial_j\partial_x^\zeta W + (\varphi^2 + \eta^2)\mathbb{L}(\partial_x^\zeta W) \\
&= \partial_x^\zeta \mathbb{Q}(V)H(\varphi) - \left(\partial_x^\zeta\Big(\sum_{j=1}^3 A_j(V)\partial_j W\Big) - \sum_{j=1}^3 A_j(V)\partial_j\partial_x^\zeta W\right) \\
&\quad - \left(\partial_x^\zeta((\varphi^2 + \eta^2)\mathbb{L}(W)) - (\varphi^2 + \eta^2)\mathbb{L}(\partial_x^\zeta W)\right) \\
&\quad + \left(\partial_x^\zeta(\mathbb{Q}(V)H(\varphi)) - \partial_x^\zeta \mathbb{Q}(V)H(\varphi)\right) + \begin{pmatrix} 0 \\ a_1\nabla\partial_x^\zeta\Phi \end{pmatrix}.
\end{aligned}
\tag{3.20}
$$

Multiplying (3.20) by $\partial_x^\zeta W$ on both sides and integrating over $\mathbb{T}^3$,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\partial_x^\zeta W)^\top A_0\partial_x^\zeta W + a_1\alpha|\sqrt{\varphi^2 + \eta^2}\nabla\partial_x^\zeta u|_2^2 + a_1(\alpha + \beta)|\sqrt{\varphi^2 + \eta^2}\mathrm{div}\partial_x^\zeta u|_2^2 \\
&= \frac{1}{2}\int (\partial_x^\zeta W)^\top \mathrm{div}A(V)\partial_x^\zeta W - a_1\int (\nabla(\varphi^2 + \eta^2)\cdot(\alpha\nabla\partial_x^\zeta u + (\alpha + \beta)\mathrm{div}\partial_x^\zeta u\mathbb{I}_3))\cdot\partial_x^\zeta u \\
&\quad + \frac{a_1\delta}{\delta - 1}\int (\nabla\varphi^2\partial_x^\zeta Q(v))\cdot\partial_x^\zeta u \\
&\quad - \int \left(\partial_x^\zeta\Big(\sum_{j=1}^3 A_j(V)\partial_j W\Big) - \sum_{j=1}^3 A_j(V)\partial_j\partial_x^\zeta W\right)\cdot\partial_x^\zeta W \\
&\quad - a_1\int \left(\partial_x^\zeta((\varphi^2 + \eta^2)L(u)) - (\varphi^2 + \eta^2)L(\partial_x^\zeta u)\right)\cdot\partial_x^\zeta u \\
&\quad + \frac{a_1\delta}{\delta - 1}\int \left(\partial_x^\zeta(\nabla\varphi^2\cdot Q(v)) - \nabla\varphi^2\cdot Q(\partial_x^\zeta v)\right)\cdot\partial_x^\zeta u \\
&\quad + a_1\int \nabla\partial_x^\zeta\Phi\cdot\partial_x^\zeta u \\
&\triangleq \sum_{i=1}^7 \mathrm{I}_i,
\end{aligned}
\tag{3.21}
$$

where $A(V) = (A_1(V), A_2(V), A_3(V))$ and $\mathrm{div}A(V) = \sum_{j=1}^3 \partial_j A_j(V)$.

When $|\zeta| \le 1$, we have

$$\mathrm{I}_1 = \frac{1}{2}\int (\partial_x^\zeta W)^\top \mathrm{div}A(V)\partial_x^\zeta W \le C|\nabla V|_\infty|\partial_x^\zeta W|_2^2$$

$$\leq C\|V\|_3|\partial_x^\zeta W|_2^2 \leq Cc_3|\partial_x^\zeta W|_2^2,$$

$$I_2 = -a_1 \int (\nabla(\varphi^2 + \eta^2) \cdot (\alpha\nabla\partial_x^\zeta u + (\alpha + \beta)\text{div}\partial_x^\zeta u\mathbb{I}_3)) \cdot \partial_x^\zeta u$$

$$\leq C|\nabla\varphi|_\infty|\varphi\nabla\partial_x^\zeta u|_2|\partial_x^\zeta u|_2$$

$$\leq \frac{a_1\alpha}{10}|\sqrt{\varphi^2 + \eta^2}\nabla\partial_x^\zeta u|_2^2 + Cc_0^2|\partial_x^\zeta u|_2^2,$$

$$I_3 = \frac{a_1\delta}{\delta - 1} \int (\nabla\varphi^2\partial_x^\zeta Q(v)) \cdot \partial_x^\zeta u$$

$$\leq C|\varphi|_\infty|\nabla\varphi|_\infty|\nabla\partial_x^\zeta v|_2|\partial_x^\zeta u|_2$$

$$\leq Cc_3^3|\partial_x^2 u|_2 \leq Cc_3^3|\partial_x^2 u|_2^2 + Cc_3^3,$$

$$I_4 = -\int \Big(\partial_x^\zeta(\sum_{j=1}^3 A_j(V)\partial_j W) - \sum_{j=1}^3 A_j(V)\partial_j\partial_x^\zeta W\Big) \cdot \partial_x^\zeta W$$

$$\leq C|\nabla V|_\infty|\nabla W|_2^2 \leq Cc_3|\nabla W|_2^2,$$

$$I_5 = -a_1\alpha \int \Big(\partial_x^\zeta((\varphi^2 + \eta^2)L(u)) - (\varphi^2 + \eta^2)L(\partial_x^\zeta u)\Big) \cdot \partial_x^\zeta u$$

$$\leq C|\nabla\varphi|_\infty|\varphi L(u)|_2|\partial_x^\zeta u|_2 \leq \frac{a_1\alpha}{10}|\sqrt{\varphi^2 + \eta^2}\nabla^2 u|_2^2 + Cc_0^2|\partial_x^\zeta u|_2^2,$$

$$I_6 = \frac{a_1\delta}{\delta - 1} \int \Big(\partial_x^\zeta(\nabla\varphi^2 \cdot Q(v)) - \nabla\varphi^2 \cdot Q(\partial_x^\zeta v)\Big) \cdot \partial_x^\zeta u$$

$$\leq C(|\varphi|_\infty|\nabla v|_\infty|\nabla^2\varphi|_2 + |\nabla v|_\infty|\nabla\varphi|_3|\nabla\varphi|_6)|\partial_x^\zeta u|_2$$

$$\leq Cc_3^3|\partial_x^\zeta u|_2 \leq Cc_3^3 + Cc_3^3|\partial_x^\zeta u|_2^2,$$

$$I_7 = a_1 \int \nabla\partial_x^\zeta\Phi \cdot \partial_x^\zeta u \leq C\|\Phi\|_2|\partial_x^\zeta u|_2 \leq Cc_0 + Cc_0^{\frac{4}{\delta-1}-1}|\partial_x^\zeta u|_2^2. \tag{3.22}$$

Then, it yields that

$$\frac{1}{2}\frac{\text{d}}{\text{d}t} \int (\partial_x^\zeta W)^\top A_0\partial_x^\zeta W + \frac{1}{2}a_1\alpha|\sqrt{\varphi^2 + \eta^2}\nabla^2 u|_2^2 \leq C\Big(c_3^3 + c_0^{\frac{4}{\delta-1}-1}\Big)\|W\|_1^2 + Cc_3^3. \tag{3.23}$$

By Gronwall's inequality, we have

$$\|W\|_1^2 + \int_0^t |\sqrt{\varphi^2 + \eta^2}\nabla^2 u|_2^2\text{d}s \leq C(\|W_0\|_1^2 + c_3^3 t)\exp\Big(Cc_3^3 t + Cc_0^{\frac{4}{\delta-1}-1}t\Big) \leq Cc_0^2, \tag{3.24}$$

for $0 \leq t \leq T_2 = \min\Big\{T_1, (1 + Cc_3)^{-3}, (1 + Cc_0)^{-\frac{5-\delta}{\delta-1}}\Big\}$. □

Lemma 3.4. *Let* (φ, W, Φ) *be the solutions to* (3.4). *Then*

$$|W(t)|_{D^2}^2 + \int_0^t |\sqrt{\varphi^2 + \eta^2}\nabla^3 u|_2^2\text{d}s \leq Cc_0^2,$$

$$\|\phi_t(t)\|_1 + |u_t(t)|_2 + \int_0^t |u_s|_{D^1}^2\text{d}s \leq Cc_3^3,$$

for $0 \leq t \leq T_2$.

Proof. Now we consider the terms on the righthand side of (3.21) when $|\zeta| = 2$. It follows from Lemma 2.1, (3.21), Hölder's inequality, Young's inequality, and integration by parts that

$$
\begin{aligned}
I_1 &= \frac{1}{2} \int (\partial_x^\zeta W)^\top \mathrm{div} A(V) \partial_x^\zeta W \leq C|\nabla V|_\infty |\partial_x^\zeta W|_2^2 \\
&\leq C\|V\|_3 |\partial_x^\zeta W|_2^2 \leq Cc_3 |\partial_x^\zeta W|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= -a_1 \int (\nabla(\varphi^2 + \eta^2) \cdot (\alpha \nabla \partial_x^\zeta u + (\alpha + \beta) \mathrm{div} \partial_x^\zeta u \mathbb{I}_3)) \cdot \partial_x^\zeta u \\
&\leq C|\nabla \varphi|_\infty |\varphi \nabla \partial_x^\zeta u|_2 |\partial_x^\zeta u|_2 \leq \frac{a_1 \alpha}{10} |\sqrt{\varphi^2 + \eta^2} \nabla \partial_x^\zeta u|_2^2 + Cc_0^2 |\partial_x^\zeta u|_2^2,
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
I_3 &= \frac{a_1 \delta}{\delta - 1} \int (\nabla \varphi^2 \partial_x^\zeta Q(v)) \cdot \partial_x^\zeta u \\
&\leq C|\varphi|_\infty |\nabla \varphi|_3 |\nabla \partial_x^\zeta v|_6 |\partial_x^\zeta u|_2 \\
&\leq Cc_3^3 |\partial_x^\zeta u|_2 \leq Cc_3^3 |\partial_x^\zeta u|_2^2 + Cc_3^3,
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= - \int \left(\partial_x^\zeta \left(\sum_{j=1}^3 A_j(V) \partial_j W \right) - \sum_{j=1}^3 A_j(V) \partial_j \partial_x^\zeta W \right) \cdot \partial_x^\zeta W \\
&\leq C|\nabla V|_\infty |\nabla^2 W|_2^2 + C|\nabla^2 V|_3 |\nabla W|_6 |\partial_x^\zeta W|_2 \leq Cc_3 |\nabla^2 W|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= -a_1 \int \left(\partial_x^\zeta ((\varphi^2 + \eta^2) L(u)) - (\varphi^2 + \eta^2) L(\partial_x^\zeta u) \right) \cdot \partial_x^\zeta u \\
&\leq C(|\nabla \varphi|_\infty^2 |\nabla^2 u|_2^2 + |\nabla \varphi|_\infty |\varphi \nabla^3 u|_2 |\nabla^2 u|_2) \\
&\leq \frac{a_1 \alpha}{10} |\sqrt{\varphi^2 + \eta^2} \nabla^3 u|_2^2 + Cc_0^2 |\nabla^2 u|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
I_6 &= \frac{a_1 \delta}{\delta - 1} \int \left(\partial_x^\zeta (\nabla \varphi^2 \cdot Q(v)) - \nabla \varphi^2 \cdot Q(\partial_x^\zeta v) \right) \cdot \partial_x^\zeta u \\
&\leq C(|\varphi|_\infty |\nabla^2 v|_6 |\nabla^2 \varphi|_3 + |\nabla \varphi|_\infty^2 |\nabla^2 v|_2 + |\nabla \varphi|_\infty |\nabla^2 \varphi|_3 |\nabla v|_6 \\
&\quad + |\varphi|_\infty |\nabla^3 \varphi|_2 |\nabla v|_\infty + |\nabla \varphi|_\infty |\nabla v|_\infty |\nabla^2 \varphi|_2) |\nabla^2 u|_2 \\
&\leq Cc_3^3 |\nabla^2 u|_2 \leq Cc_3^3 + Cc_3^3 |\nabla^2 u|_2^2,
\end{aligned}
$$

$$
I_7 = a_1 \int \nabla \partial_x^\zeta \Phi \cdot \partial_x^\zeta u \leq C\|\Phi\|_3 |\partial_x^\zeta u|_2 \leq Cc_0 + Cc_0^{\frac{4}{\delta-1}-1} |\partial_x^\zeta u|_2^2.
\tag{3.26}
$$

Then, it yields that

$$
\frac{1}{2} \frac{d}{dt} \int (\partial_x^\zeta W)^\top A_0 \partial_x^\zeta W + \frac{1}{2} a_1 \alpha |\sqrt{\varphi^2 + \eta^2} \nabla^3 u|_2^2 \leq C \left(c_3^3 + c_0^{\frac{4}{\delta-1}-1} \right) |W|_{D^2}^2 + Cc_3^3.
\tag{3.27}
$$

According to Gronwall's inequality, we have

$$
|W|_{D^2}^2 + \int_0^t |\sqrt{\varphi^2 + \eta^2} \nabla^3 u|_2^2 ds \leq C(|W_0|_{D^2}^2 + c_3^3 t) \exp \left(Cc_3^3 t + Cc_0^{\frac{4}{\delta-1}-1} t \right) \leq Cc_0^2,
\tag{3.28}
$$

for $0 \leq t \leq T_2$.

From the relation that

$$
\phi_t = -v \cdot \nabla \phi - \frac{\gamma - 1}{2} \tilde{\phi} \mathrm{div} u,
\tag{3.29}
$$

we have

$$
\begin{aligned}
|\phi_t|_2 &\le C|v \cdot \nabla\phi + \tilde{\phi}\mathrm{div}u|_2 \\
&\le C(|v|_\infty|\nabla\phi|_2 + |\tilde{\phi}|_\infty|\mathrm{div}u|_2) \\
&\le Cc_2^2, \\
|\phi_t|_{D^1} &\le C(|\nabla v|_\infty|\nabla\phi|_2 + |v|_\infty|\nabla^2\phi|_2 + |\nabla\tilde{\phi}|_\infty|\nabla u|_2 + |\tilde{\phi}|_\infty|\nabla^2 u|_2) \\
&\le Cc_3^2.
\end{aligned}
\tag{3.30}
$$

Noting that

$$
\begin{aligned}
&u_t + v \cdot \nabla u + \frac{2A\gamma}{\gamma - 1}\tilde{\phi}\nabla\phi - (\varphi^2 + \eta^2)(\alpha\Delta u + (\alpha+\beta)\nabla\mathrm{div}u) \\
&= \frac{\delta}{\delta - 1}(\alpha(\nabla v + (\nabla v)^\top) + \beta\mathrm{div}v\mathbb{I}_3) \cdot \nabla\varphi^2 + \nabla\Phi,
\end{aligned}
\tag{3.31}
$$

we get

$$
\begin{aligned}
|u_t|_2 &\le C(|v|_\infty|\nabla u|_2 + |\tilde{\phi}|_\infty|\nabla\phi|_2 + |\varphi^2 + \eta^2|_\infty|\nabla^2 u|_2 + |\varphi|_\infty|\nabla\varphi|_\infty|\nabla v|_2 + |\nabla\Phi|_2) \\
&\le Cc_2^3 + Cc_0^{\frac{2}{\delta-1}}.
\end{aligned}
\tag{3.32}
$$

For $|u_t|_{D^1}$,

$$
\begin{aligned}
|u_t|_{D^1} &\le C(|\nabla v|_6|\nabla u|_3 + |v|_\infty|\nabla^2 u|_2 + |\nabla\tilde{\phi}|_3|\nabla\phi|_6 + |\tilde{\phi}|_\infty|\nabla^2\phi|_2 \\
&\quad + |\sqrt{\varphi^2 + \eta^2}|_\infty|\sqrt{\varphi^2 + \eta^2}\nabla^3 u|_2 + |\varphi|_\infty|\nabla\varphi|_\infty|\nabla^2 u|_2 \\
&\quad + |\nabla\varphi|_\infty^2|\nabla v|_2 + |\varphi|_\infty|\nabla^2\varphi|_3|\nabla v|_6 + |\varphi|_\infty|\nabla\varphi|_\infty|\nabla^2 v|_2 + |\nabla^2\Phi|_2) \\
&\le Cc_2^3 + Cc_0^{\frac{2}{\delta-1}} + Cc_0|\sqrt{\varphi^2 + \eta^2}\nabla^3 u|_2,
\end{aligned}
\tag{3.33}
$$

which implies that

$$
\int_0^t |u_s|_{D^1}^2 \mathrm{d}s \le C\int_0^t (c_2^6 + Cc_0^{\frac{4}{\delta-1}} + c_0^2|\sqrt{\varphi^2 + \eta^2}\nabla^3 u|_2^2)\mathrm{d}s \le Cc_2^3,
\tag{3.34}
$$

for $0 \le t \le T_2$. $\qquad\square$

Lemma 3.5. *Let* (φ, W, Φ) *be the solutions to* (3.4). *Then*

$$
|W(t)|_{D^3}^2 + \int_0^t |\sqrt{\varphi^2 + \eta^2}\nabla^4 u|_2^2 \mathrm{d}s \le Cc_0^2,
$$

$$
|\phi_t(t)|_{D^2}^2 + |u_t(t)|_{D^1}^2 + \int_0^t |u_s|_{D^2}^2 \mathrm{d}s \le Cc_3^6,
$$

for $0 \le t \le T_3 \triangleq \min\{T_2, (1 + Cc_3)^{-4}\}$.

Proof. Now we consider the terms on the righthand side of (3.21) when $|\zeta| = 3$. It follows from Lemma 2.1, (3.21), Hölder's inequality, and Young's inequality that

$$
\mathrm{I}_1 = \frac{1}{2}\int (\partial_x^\zeta W)^\top \mathrm{div}A(V)\partial_x^\zeta W \le C|\nabla V|_\infty|\partial_x^\zeta W|_2^2
$$

$$\leq C\|V\|_3|\partial_x^\zeta W|_2^2 \leq Cc_3|\partial_x^\zeta W|_2^2,$$

$$I_2 = -a_1 \int (\nabla(\varphi^2 + \eta^2) \cdot (\alpha\nabla\partial_x^\zeta u + (\alpha+\beta)\mathrm{div}\partial_x^\zeta u\mathbb{I}_3)) \cdot \partial_x^\zeta u \tag{3.35}$$

$$\leq C|\nabla\varphi|_\infty|\varphi\nabla^4 u|_2|\partial_x^\zeta u|_2 \leq \frac{a_1\alpha}{10}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + Cc_0^2|\partial_x^\zeta u|_2^2,$$

$$I_3 = \frac{a_1\delta}{\delta-1} \int (\nabla\varphi^2\partial_x^\zeta Q(v)) \cdot \partial_x^\zeta u$$

$$\leq C(|\nabla\varphi|_\infty^2|\nabla^3 v|_2|\nabla^3 u|_2 + |\nabla^2\varphi|_3|\varphi\nabla^3 u|_6|\nabla^3 v|_2 + |\nabla\varphi|_\infty|\varphi\nabla^4 u|_2|\nabla^3 v|_2)$$

$$\leq Cc_3^3|\nabla^3 u|_2^2 + Cc_3^4 + \frac{a_1\alpha}{10}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2,$$

$$I_4 = -\int \left(\partial_x^\zeta(\sum_{j=1}^3 A_j(V)\partial_j W) - \sum_{j=1}^3 A_j(V)\partial_j\partial_x^\zeta W\right) \cdot \partial_x^\zeta W$$

$$\leq C(|\nabla V|_\infty|\nabla^3 W|_2^2 + |\nabla^2 V|_3|\nabla^2 W|_6|\nabla^3 W|_2 + |\nabla^3 V|_2|\nabla W|_\infty|\nabla W|_2)$$

$$\leq Cc_3|\nabla^3 W|_2^2,$$

$$I_5 = -a_1 \int \left(\partial_x^\zeta((\varphi^2+\eta^2)L(u)) - (\varphi^2+\eta^2)L(\partial_x^\zeta u)\right) \cdot \partial_x^\zeta u$$

$$\leq C(|\nabla\varphi|_\infty|\nabla^2\varphi|_3|\nabla^2 u|_6|\nabla^3 u|_2 + |\nabla^3\varphi|_2|\nabla^2 u|_3|\varphi\nabla^3 u|_6 + |\nabla\varphi|_\infty^2|\nabla^3 u|_2^2$$

$$+ |\nabla^2\varphi|_3|\varphi\nabla^3 u|_6|\nabla^3 u|_2 + |\nabla\varphi|_\infty|\varphi\nabla^4 u|_2|\nabla^3 u|_2)$$

$$\leq \frac{a_1\alpha}{10}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 + Cc_0^2|\nabla^3 u|_2^2,$$

$$I_6 \leq Cc_3^3|\nabla^3 u|_2^2 + Cc_3^4 + \frac{a_1\alpha}{10}|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2,$$

$$I_7 = a_1 \int \nabla\partial_x^\zeta\Phi \cdot \partial_x^\zeta u \leq C\|\Phi\|_4|\nabla^3 u|_2 \leq Cc_0 + Cc_0^{\frac{4}{\delta-1}-1}|\nabla^3 u|_2^2, \tag{3.36}$$

where I_6 was estimated through the argument in [18, Lemma 3.5]. Then, it yields that

$$\frac{1}{2}\frac{d}{dt}\int (\partial_x^\zeta W)^\top A_0\partial_x^\zeta W + \frac{1}{2}a_1\alpha|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 \leq C\left(c_3^3 + c_0^{\frac{4}{\delta-1}-1}\right)|W|_{D^3}^2 + Cc_3^4. \tag{3.37}$$

According to Gronwall's inequality, we have

$$|W|_{D^3}^2 + \int_0^t |\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2 ds \leq C(|W_0|_{D^2}^2 + c_3^4 t)\exp\left(Cc_3^3 t + Cc_0^{\frac{4}{\delta-1}-1}t\right) \leq Cc_0^2, \tag{3.38}$$

for $0 \leq t \leq T_3 = \min\{T_2, (1+Cc_3)^{-4}\}$.

For $|\phi_t|_{D^2}$, from (3.29), we have

$$\begin{aligned}|\phi_t|_{D^2} &\leq C(|\nabla^2 v|_3|\nabla\phi|_6 + |\nabla v|_3|\nabla^2\phi|_6 + |v|_\infty|\nabla^3\phi|_2 \\ &\quad + |\nabla^2\tilde\phi|_3|\nabla u|_6 + |\nabla\tilde\phi|_3|\nabla^2 u|_6 + |\tilde\phi|_\infty|\nabla^3 u|_2) \\ &\leq Cc_3^2.\end{aligned} \tag{3.39}$$

It follows from (3.33) that

$$|u_t|_{D^1} \leq Cc_2^3 + Cc_0^{\frac{2}{\delta-1}} + Cc_0|\sqrt{\varphi^2+\eta^2}\nabla^3 u|_2 \leq Cc_2^3 + Cc_0^{\frac{2}{\delta-1}}. \tag{3.40}$$

For $|u_t|_{D^2}$, from (3.31), we have

$$
\begin{aligned}
|u_t|_{D^2} &\le C(\|v\|_3\|u\|_3 + \|\tilde{\phi}\|_3\|\phi\|_3 + \|\varphi\|_3\|u\|_3 + \|\varphi\|_3^2\|u\|_3 + |\varphi|_\infty|\sqrt{\varphi^2 + \eta^2}\nabla^4 u|_2 + |\nabla^3\Phi|_2) \\
&\le Cc_3^3 + Cc_0^{\frac{2}{\delta-1}} + Cc_3|\sqrt{\varphi^2 + \eta^2}\nabla^4 u|_2,
\end{aligned}
\tag{3.41}
$$

which implies that

$$
\int_0^t |u_t|_{D^2}^2 \mathrm{d}s \le C\int_0^t \left(c_3^6 + Cc_0^{\frac{4}{\delta-1}} + c_3^2(|\sqrt{\varphi^2+\eta^2}\nabla^4 u|_2^2)\right)\mathrm{d}s \le Cc_3^4,
\tag{3.42}
$$

for $0 \le t \le T_3$. $\qquad\square$

Combining the estimates obtained in Lemmas 3.2–3.5, we have

$$
\begin{aligned}
&\|\varphi\|_3^2 \le Cc_0^2, \quad \|\varphi_t\|_2 \le Cc_3^2, \quad \|\Phi(t)\|_4 \le Cc_0^{\frac{2}{\delta-1}}, \\
&\|W\|_1^2 + \int_0^t |\varphi\nabla^2 u|_2^2 \mathrm{d}s \le Cc_0^2, \\
&|W|_{D^2}^2 + \int_0^t |\varphi\nabla^3 u|_2^2 \mathrm{d}s \le Cc_0^2, \\
&\|\phi_t\|_1^2 + |u_t|_2^2 + \int_0^t |u_t|_{D^1}^2 \mathrm{d}s \le Cc_3^6, \\
&|W|_{D^3}^2 + \int_0^t |\varphi\nabla^4 u|_2^2 \mathrm{d}s \le Cc_0^2, \\
&\|\phi_t\|_{D^2}^2 + |u_t|_{D^1}^2 + \int_0^t |u_t|_{D^2}^2 \mathrm{d}s \le Cc_3^6,
\end{aligned}
\tag{3.43}
$$

for $0 \le t \le T_3 = \min\{T_2, (1+Cc_3)^{-4}\}$.

Therefore, if we define the constants c_i ($i = 1, 2, 3$) and T^* by

$$
c_1 = c_2 = c_3 = C^{\frac{1}{2}}c_0, \quad T^* = \min\{T, (1+Cc_0)^{-4}, (1+Cc_0)^{-\frac{5-\delta}{\delta-1}}\},
\tag{3.44}
$$

then we deduce that

$$
\begin{aligned}
&\sup_{0\le t\le T^*}\left(\|\varphi(t)\|_1^2 + \|\phi(t)\|_1^2 + \|u(t)\|_1^2\right) + \int_0^{T^*}|\varphi\nabla^2 u|_2^2 \mathrm{d}t \le c_1^2, \\
&\sup_{0\le t\le T^*}\left(|\varphi(t)|_{D^2}^2 + |\phi(t)|_{D^2}^2 + \|u(t)\|_{D^2}^2\right) + \int_0^{T^*}|\varphi\nabla^3 u|_2^2 \mathrm{d}t \le c_2^2, \\
&\sup_{0\le t\le T^*}\left(|\varphi(t)|_{D^3}^2 + |\phi(t)|_{D^3}^2 + |u(t)|_{D^3}^2\right) + \int_0^{T^*}|\varphi\nabla^4 u|_2^2 \mathrm{d}t \le c_3^2, \\
&\sup_{0\le t\le T^*}\left(\|u_t\|_1^2 + \|\phi_t\|_2^2 + \|\varphi_t\|_2^2\right) + \int_0^{T^*}|u_t|_{D^2}^2 \mathrm{d}t \le Cc_3^6, \quad \sup_{0\le t\le T^*}\|\Phi\|_4^2 \le Cc_3^{\frac{4}{\delta-1}}.
\end{aligned}
\tag{3.45}
$$

3.4. Passing to the limit as $\eta \to 0$

Now we consider the system (3.4) when $\eta \to 0$ as follows:

$$
\begin{cases}
\varphi_t + v \cdot \nabla \varphi + \frac{\delta-1}{2} \tilde{\varphi} \operatorname{div} v = 0, \\
A_0 W_t + \sum_{j=1}^{3} A_j(V) \partial_j W + (\varphi^2 + \eta^2) \mathbb{L}(W) = \mathbb{Q}(V) H(\varphi) + \begin{pmatrix} 0 \\ a_1 \nabla \Phi \end{pmatrix}, \\
\Delta \Phi = \varphi^{\frac{2}{\delta-1}} - b, \\
(\varphi, W, \Phi)|_{t=0} = (\varphi_0(x), W_0(x), \Phi_0(x)), \quad x \in \mathbb{T}^3.
\end{cases}
\tag{3.46}
$$

Based on the local-in-time a priori estimates (3.45), we now establish the following local existence result.

Lemma 3.6. *Assume (φ_0, W_0) satisfies (3.6). Then there exists a time $T^* > 0$ and a unique strong solution (φ, W, Φ) in $[0, T^*] \times \mathbb{T}^3$ to (3.46) such that*

$$
\begin{aligned}
&\varphi \in C\left([0, T^*]; H^3\right), \quad \phi \in C\left([0, T^*]; H^3\right), \\
&u \in C\left([0, T^*]; H^{s'}\right) \cap L^\infty\left([0, T^*]; H^3\right), \quad s' \in [2, 3), \\
&\varphi \nabla^4 u \in L^2\left([0, T^*]; L^2\right), \quad u_t \in C\left([0, T^*]; H^1\right) \cap L^2\left([0, T^*]; D^2\right), \\
&\Phi \in C\left([0, T^*]; H^3\right) \cap L^\infty\left([0, T^*]; H^4\right).
\end{aligned}
\tag{3.47}
$$

Moreover, (φ, W, Φ) also satisfies the a priori estimates (3.45).

Proof. We prove the existence, uniqueness, and time-continuity in three steps.

Step 1. Existence. Due to Lemma 3.1 and the uniform estimates (3.45), for every $\eta > 0$, there exists a unique strong solution $(\varphi^\eta, W^\eta, \Phi^\eta)$ in $[0, T^*] \times \mathbb{T}^3$ to the linearized problem (3.4) satisfying estimates (3.45), where the time $T^* > 0$ is independent of η.

By virtue of the uniform estimates (3.45) independent of η and Lemma 2.7, we know that for any $R > 0$, there exists a subsequence of solutions (still denoted by) $(\rho^\eta, \varphi^\eta, W^\eta)$, which converges to a limit $(\varphi, W) = (\varphi, \phi, u)$ in the following strong sense:

$$
(\varphi^\eta, W^\eta) \to (\varphi, W) \quad \text{in } C\left([0, T^*]; H^2(B_R)\right), \quad \text{as } \eta \to 0.
\tag{3.48}
$$

Again by virtue of the uniform estimates (3.45) independent of η, we also know that there exists a subsequence of solutions (still denoted by) $(\varphi^\eta, W^\eta, \Phi^\eta)$, which converges to (φ, W, Φ) in the following weak or weak–$*$ sense:

$$
\begin{aligned}
(\varphi^\eta, W^\eta) &\rightharpoonup (\rho, \varphi, W) \text{ weakly–}* \text{ in } L^\infty\left([0, T^*]; H^3\left(\mathbb{T}^3\right)\right), \\
(\varphi_t^\eta, \phi_t^\eta) &\rightharpoonup (\rho_t, \varphi_t, \phi_t) \text{ weakly–}* \text{ in } L^\infty\left([0, T^*]; H^2\left(\mathbb{T}^3\right)\right), \\
\Phi^\eta &\rightharpoonup \Phi \text{ weakly–}* \text{ in } L^\infty\left([0, T^*]; H^4\left(\mathbb{T}^3\right)\right), \\
u_t^\eta &\rightharpoonup u_t \text{ weakly–}* \text{ in } L^\infty\left([0, T^*]; H^1\left(\mathbb{T}^3\right)\right), \\
u_t^\eta &\rightharpoonup u_t \text{ weakly in } L^2\left([0, T^*]; D^2\left(\mathbb{T}^3\right)\right),
\end{aligned}
\tag{3.49}
$$

which, along with the lower semi-continuity of weak convergence, implies that (φ, W, Φ) also satisfies the corresponding estimates (3.45) except those of $\varphi\nabla^4 u$.

Combining the strong convergence in (3.48) and the weak convergence in (3.49), we easily obtain that (ρ, φ, W) also satisfies the local estimates (3.45) and

$$\varphi^\eta \nabla^4 u^\eta \rightharpoonup \varphi\nabla^4 u \quad \text{weakly in} \quad L^2\left([0, T^*] \times \mathbb{T}^3\right). \tag{3.50}$$

Now we want to show that (ρ, φ, W) is a weak solution in the sense of distributions to the linearized problem (3.46). Multiplying (3.46)$_2$ by test function $f(t, x) = \left(f^1, f^2, f^3\right) \in C_c^\infty\left([0, T^*) \times \mathbb{T}^3\right)$ on both sides, and integrating over $[0, T^*] \times \mathbb{T}^3$, we have

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{T}^3} u^\eta \cdot f_t \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} (v \cdot \nabla) u^\eta \cdot f \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \frac{2A\gamma}{\gamma - 1} \tilde{\phi} \nabla \phi^\eta f \mathrm{d}x\mathrm{d}s \\
&= -\int u_0 \cdot f(0, x) + \int_0^t \int_{\mathbb{T}^3} \left((\varphi^\eta)^2 + \eta^2\right) L(u^\eta) f \mathrm{d}x\mathrm{d}s \\
&\quad - \frac{\delta}{\delta - 1} \int_0^t \int_{\mathbb{T}^3} Q(v) \cdot \nabla(\varphi^\eta)^2 f \mathrm{d}x\mathrm{d}s - a_1 \int_0^t \int_{\mathbb{T}^3} \nabla\Phi^\eta f \mathrm{d}x\mathrm{d}s.
\end{aligned}
\tag{3.51}
$$

Combining the strong convergence in (3.48) and the weak convergences in (3.49)–(3.50), and letting $\eta \to 0$ in (3.51), we have

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{T}^3} u \cdot f_t \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} (v \cdot \nabla) u \cdot f \mathrm{d}x\mathrm{d}s - \frac{2A\gamma}{\gamma - 1} \int_0^t \int_{\mathbb{T}^3} \tilde{\phi} \nabla \phi f \mathrm{d}x\mathrm{d}s \\
&= -\int u_0 \cdot f(0, x) + \int_0^t \int_{\mathbb{T}^3} \varphi^2 L(u) f \mathrm{d}x\mathrm{d}s \\
&\quad - \frac{\delta}{\delta - 1} \int_0^t \int_{\mathbb{T}^3} Q(v) \cdot \nabla\varphi^2 f \mathrm{d}x\mathrm{d}s - a_1 \int_0^t \int_{\mathbb{T}^3} \nabla\Phi f \mathrm{d}x\mathrm{d}s.
\end{aligned}
\tag{3.52}
$$

Thus it is obvious that (φ, W, Φ) is a weak solution in the sense of distributions to the linearized problem (3.46), satisfying the regularities

$$
\begin{aligned}
&(\varphi, \phi) \in C\left([0, T^*]; H^3\right), \quad \Phi \in C\left([0, T^*]; H^3\right) \cap L^\infty\left([0, T^*]; H^4\right), \\
&(\varphi_t, \phi_t) \in C\left([0, T^*]; H^2\right), \quad u \in C\left([0, T^*]; H^{s'}\right) \cap L^\infty\left([0, T^*]; H^3\right), \quad s' \in [2, 3), \\
&\varphi\nabla^4 u \in L^2\left([0, T^*]; L^2\right), \quad u_t \in C\left([0, T^*]; H^1\right) \cap L^2\left([0, T^*]; D^2\right).
\end{aligned}
\tag{3.53}
$$

Step 2. Uniqueness. Let (φ_1, W_1, Φ_1) and (φ_2, W_2, Φ_2) be two solutions obtained in the above step. We denote

$$\bar{\varphi} = \varphi_1 - \varphi_2, \quad \bar{W} = W_1 - W_2, \quad \bar{\Phi} = \Phi_1 - \Phi_2.$$

Then from (3.46)$_1$ and (3.46)$_3$, we have

$$\bar{\varphi}_t + v \cdot \nabla\bar{\varphi} = 0, \quad \Delta\bar{\Phi} = 0,$$

which implies that $\bar{\varphi} = 0, \bar{\Phi} = 0$. Let $\bar{W} = (\bar{\phi}, \bar{u})^\top$. From (3.46)$_3$ and $\varphi_1 = \varphi_2$, we have

$$A_0 \bar{W}_t + A_1(V)\bar{W}_x = -\varphi_1^2 \mathbb{L}(\bar{W}). \tag{3.54}$$

Then multiplying (3.54) by $\bar{W}$ on both sides, and integrating over $\mathbb{T}^3$, we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} &\int \bar{W}^\top A_0 \bar{W} + a_1\alpha\,|\varphi_1\nabla\bar{u}|_2^2 \\
&\leq C|\nabla V|_\infty |\bar{W}|_2^2 + |\nabla\varphi_1|_\infty\,|\bar{u}|_2\,|\varphi_1\nabla\bar{u}|_2 \\
&\leq \frac{a_1\alpha}{10}\,|\varphi_1\nabla\bar{u}|_2^2 + Cc_2^2|\bar{W}|_2^2,
\end{aligned}
\tag{3.55}
$$

which yields that

$$
\frac{\mathrm{d}}{\mathrm{d}t}|\bar{W}|_2^2 + |\varphi_1\nabla\bar{u}|_2^2 \leq Cc_2^2|\bar{W}|_2^2.
\tag{3.56}
$$

From Gronwall's inequality, we obtain that $\bar{W} = 0$, which gives the uniqueness.

Step 3. Time-continuity. First for φ, via the regularities shown in (3.53) and the classical Sobolev embedding theorem, we have

$$
\varphi \in C\left([0, T^*]; H^2\right) \cap C\left([0, T^*];\ \text{weak} - H^3\right).
\tag{3.57}
$$

Using the same arguments as in Lemma 3.2, we have

$$
\|\varphi(t)\|_3^2 \leq \left(\|\varphi_0\|_3^2 + C\int_0^t \left(\|\tilde{\varphi}\|_3^2\|v\|_3^2 + \left|\tilde{\varphi}\nabla^4 v\right|_2^2\right)\mathrm{d}s\right)\exp\left(C\int_0^t \|v\|_3\,\mathrm{d}s\right),
$$

which implies that

$$
\limsup_{t\to 0}\|\varphi(t)\|_3 \leq \|\varphi_0\|_3.
$$

Then according to Lemma 2.6 and (3.57), we know that φ is right continuous at $t = 0$ in H^3 space. From the reversibility on the time to equation $(3.46)_1$, we know

$$
\varphi \in C\left([0, T^*]; H^3\right).
\tag{3.58}
$$

For φ_t, from

$$
\varphi_t = -v\cdot\nabla\varphi - \frac{\delta - 1}{2}\tilde{\varphi}\mathrm{div}v,
$$

we only need to consider the term $\tilde{\varphi}\mathrm{div}v$. Due to

$$
\tilde{\varphi}\mathrm{div}v \in L^2\left([0, T^*]; H^3\right), \quad (\tilde{\varphi}\mathrm{div}v)_t \in L^2\left([0, T^*]; H^1\right),
$$

and the Sobolev embedding theorem, we have

$$
\tilde{\varphi}\mathrm{div}v \in C\left([0, T^*]; H^2\right),
$$

which implies that

$$
\varphi_t \in C\left([0, T^*]; H^2\right).
$$

Similar arguments can be used to deal with the regularities of ϕ and we can get that

$$
\phi \in C\left([0, T^*]; H^3\right), \quad \phi_t \in C\left([0, T^*]; H^2\right).
\tag{3.59}
$$

Using similar methods as in the proof of Lemma 3.1, we can get that $\Phi \in C\left([0, T^*]; H^3\right)$.

For velocity u, from the regularity shown in (3.53) and Sobolev's embedding theorem, we obtain that

$$u \in C\left([0, T^*]; H^2\right) \cap C\left([0, T^*]; \text{weak} - H^3\right). \tag{3.60}$$

Then from Lemma 2.4, for any $s' \in [2, 3)$, we have

$$\|u\|_{s'} \leq C|u|_2^{1-\frac{s'}{3}} \|u\|_3^{\frac{s'}{3}}.$$

Together with the upper bound shown in (3.45) and the time continuity (3.60), we have

$$u \in C([0, T^*]; H^{s'}). \tag{3.61}$$

Finally, we consider u_t. From equations $(3.46)_3$ we have

$$u_t = -v \cdot \nabla u - \frac{2A\gamma}{\gamma - 1} \tilde{\phi} \nabla \phi - \alpha \varphi^2 L(u) + \frac{\delta}{\delta - 1} Q(v) \cdot \nabla \varphi^2 + \nabla \Phi.$$

From (3.53), we have

$$\varphi^2 L(u) \in L^2\left([0, T^*]; H^2\right), \quad \left(\varphi^2 L(u)\right)_t \in L^2\left([0, T^*]; L^2\right),$$

which means that

$$\varphi^2 L(u) \in C\left([0, T^*]; H^1\right). \tag{3.62}$$

Combining (3.47), (3.58)–(3.59), (3.61) and (3.62), we deduce that

$$u_t \in C\left([0, T^*]; H^1\right). \tag{3.63}$$

Hence we complete the proof. $\qquad \square$

3.5. Proof of Theorem 3.1

Based on the classical iteration scheme and the existence results for the linearized problem obtained in Section 3.4, we are ready to prove Theorem 3.1. As in Section 3.3, we define constants c_0 and c_i ($i = 1, 2, 3$), and assume that

$$1 + \|\varphi_0\|_3 + \|W_0\|_3 \leq c_0.$$

Let $\left(\varphi^0, W^0 = \left(\phi^0, u^0\right)\right)$, with the regularities

$$\varphi^0 \in C\left([0, T^*]; H^3\right), \quad \phi^0 \in C\left([0, T^*]; H^3\right), \quad \varphi^0 \nabla^4 u^0 \in L^2\left([0, T^*]; L^2\right),$$

$$u^0 \in C([0, T^*]; H^{s'}) \cap L^\infty\left([0, T^*]; H^3\right) \text{ for any } s' \in [2, 3),$$

be the solution to the problem

$$\begin{cases} X_t + u_0 \cdot \nabla X = 0 & \text{in } (0, +\infty) \times \mathbb{T}^3, \\ Y_t + u_0 \cdot \nabla Y = 0 & \text{in } (0, +\infty) \times \mathbb{T}^3, \\ Z_t - X^2 \Delta Z = 0 & \text{in } (0, +\infty) \times \mathbb{T}^3, \\ (X, Y, Z)|_{t=0} = (\varphi_0, \phi_0, u_0) & \text{in } \mathbb{T}^3. \end{cases} \tag{3.64}$$

We take a time $T^{**} \in (0, T^*]$ small enough such that

$$\sup_{0 \le t \le T^{**}} \left(\|\varphi^0(t)\|_1^2 + \|\phi^0(t)\|_1^2 + \|u^0(t)\|_1^2 \right) + \int_0^{T^{**}} \left| \varphi^0 \nabla^2 u^0 \right| dt \le c_1^2,$$

$$\sup_{0 \le t \le T^{**}} \left(\left| \varphi^0(t) \right|_{D^2}^2 + \left| \phi^0(t) \right|_{D^2}^2 + \left| u^0(t) \right|_{D^2}^2 \right) + \int_0^{T^{**}} \left| \varphi^0 \nabla^3 u^0 \right| dt \le c_2^2, \qquad (3.65)$$

$$\sup_{0 \le t \le T^{**}} \left(\left| \varphi^0(t) \right|_{D^3}^2 + \left| \phi^0(t) \right|_{D^3}^2 + \left| u^0(t) \right|_{D^3}^2 \right) + \int_0^{T^{**}} \left| \varphi^0 \nabla^4 u^0 \right| dt \le c_3^2.$$

Proof. We prove the existence, uniqueness, and time-continuity in three steps.

Step 1. Existence. Let $(\tilde{\varphi}, \tilde{\phi}, v) = \left(\varphi^0, \phi^0, u^0 \right)$. We define $\left(\varphi^1, W^1, \Phi^1 \right)$ as a strong solution to problem (3.46). Then we construct approximate solutions

$$\left(\varphi^{k+1}, W^{k+1}, \Phi^{k+1} \right) = \left(\varphi^{k+1}, \phi^{k+1}, u^{k+1}, \Phi^{k+1} \right)$$

inductively. By assuming that $\left(\varphi^k, W^k, \Phi^k \right)$ was defined for $k \ge 1$, let $\left(\varphi^{k+1}, W^{k+1}, \Phi^{k+1} \right)$ be the unique solution to problem (3.46) with $(\tilde{\varphi}, \tilde{\phi}, v)$ replaced by $\left(\varphi^k, W^k \right)$ as follows:

$$\begin{cases} \varphi_t^{k+1} + u^k \cdot \nabla \varphi^{k+1} + \frac{\delta-1}{2} \varphi^k \mathrm{div} u^k = 0, \\[2mm] A_0 W_t^{k+1} + \sum_{j=1}^3 A_j(W^k) \partial_j W^{k+1} + (\varphi^{k+1})^2 \mathbb{L}(W^{k+1}) = \mathbb{Q}(W^k) \cdot H(\varphi^{k+1}) + a_1 \begin{pmatrix} 0 \\ \nabla \Phi^{k+1} \end{pmatrix}, \\[2mm] \Delta \Phi^{k+1} = (\varphi^{k+1})^{\frac{2}{\delta-1}} - b, \\[2mm] (\varphi^{k+1}, W^{k+1}, \Phi^{k+1})|_{t=0} = (\varphi_0(x), W_0(x), \Phi_0), \quad x \in \mathbb{T}^3. \end{cases} \qquad (3.66)$$

It follows from Lemma 3.6 that the sequence $\left(\varphi^k, W^k, \Phi^k \right)$ satisfies the uniform a priori estimates (3.45) for $0 \le t \le T^{**}$. Then, from (3.66), we can obtain that

$$\begin{cases} \bar{\varphi}_t^{k+1} + u^k \cdot \nabla \bar{\varphi}^{k+1} + \bar{u}^k \cdot \nabla \varphi^k + \frac{\delta-1}{2} (\bar{\varphi}^k \, \mathrm{div} \, u^{k-1} + \varphi^k \, \mathrm{div} \, \bar{u}^k) = 0, \\[2mm] A_0 \bar{W}_t^{k+1} + \sum_{j=1}^3 A_j(W^k) \partial_j \bar{W}^{k+1} + (\varphi^{k+1})^2 \mathbb{L}(\bar{W}^{k+1}) \\[2mm] = \sum_{j=1}^3 A_j(\bar{W}^k) \partial_j W^k - \bar{\varphi}^{k+1} (\varphi^{k+1} + \varphi^k) \mathbb{L}(W^k) + \mathbb{Q}(W^k) \cdot (\mathbb{H}(\varphi^{k+1}) - \mathbb{H}(\varphi^k)) \\[2mm] \quad + \mathbb{Q}(\bar{W}^k) \cdot \mathbb{H}(\varphi^k) + a_1 \begin{pmatrix} 0 \\ \nabla \bar{\Phi}^{k+1} \end{pmatrix}, \\[2mm] \Delta \bar{\Phi}^{k+1} = (\varphi^{k+1})^{\frac{2}{\delta-1}} - (\varphi^k)^{\frac{2}{\delta-1}}. \end{cases} \qquad (3.67)$$

First, we consider $|\bar{\varphi}^{k+1}|_2$. Multiplying $(3.67)_1$ by $2\bar{\varphi}^{k+1}$ and integrating over $\mathbb{T}^3$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} |\bar{\varphi}^{k+1}|_2^2 = -2 \int (u^k \cdot \nabla \bar{\varphi}^{k+1} + \bar{u}^k \cdot \nabla \varphi^k + \frac{\delta_1 - 1}{2} \left(\bar{\varphi}^k \, \mathrm{div} \, u^{k-1} + \varphi^k \, \mathrm{div} \, \bar{u}^k \right) \bar{\varphi}^{k+1}$$

$$\le C |\nabla u^k|_\infty |\bar{\varphi}^{k+1}|_2^2 + C |\bar{\varphi}^{k+1}|_2 (|\bar{u}^k|_2 |\nabla \varphi^k|_\infty + |\bar{\varphi}^k|_2 |\nabla u^{k-1}|_\infty + |\varphi^k \, \mathrm{div} \, \bar{u}^k|_2),$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\varphi}^{k+1}(t)|_2^2 \le C_\nu|\bar{\varphi}^{k+1}(t)|_2^2 + \nu\left(|\bar{u}^k(t)|_2^2 + |\bar{\varphi}^k(t)|_2^2 + |\varphi^k \operatorname{div}\bar{u}^k(t)|_2^2\right) \tag{3.68}$$

with $C_\nu = C\left(1 + \nu^{-1}\right)$ and $0 < \nu \le \frac{1}{10}$ is a constant.

Furthermore, from $(3.67)_3$ and Lagrange's mean value theorem, we can easily deduce that

$$|\Delta\bar{\Phi}^{k+1}| = \frac{2}{\delta-1}(\theta^{k+1})^{\frac{3-\delta}{\delta-1}}|\bar{\varphi}^{k+1}| \le C|\bar{\varphi}^{k+1}|, \tag{3.69}$$

due to $\delta \in (1,2] \cup \{3\}$, where θ^{k+1} is between φ^{k+1} and φ^k.

It follows from (3.69) and the Poincaré inequality that

$$\left\|\bar{\Phi}^{k+1}\right\|_2 \le C\left|\bar{\varphi}^{k+1}\right|_2. \tag{3.70}$$

Next, we consider $|\bar{W}^{k+1}|_2$. Multiplying $(3.66)_3$ by $2\bar{W}^{k+1}$ and integrating over $\mathbb{T}^3$, we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int (\bar{W}^{k+1})^\top A_0 \bar{W}^{k+1} + 2a_1\alpha|\varphi^{k+1}\nabla\bar{u}^{k+1}|_2^2 + 2a_1(\alpha+\beta)|\varphi^{k+1}\operatorname{div}\bar{u}^{k+1}|_2^2 \\
&\le \int (\bar{W}^{k+1})^\top \operatorname{div}A(W^k)\bar{W}^{k+1} + \int \sum_{j=1}^3 (\bar{W}^{k+1})^\top A_j(\bar{W}^k)\partial_j W^k \\
&\quad - 2a_1\frac{\delta}{\delta-1}\int \nabla(\varphi^{k+1})^2 \cdot Q(\bar{u}^{k+1}) \cdot \bar{u}^{k+1} - 2a_1\int (\bar{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k) \cdot L(u^k)) \cdot \bar{u}^{k+1} \\
&\quad - 2a_1\frac{\delta-1}{\delta}\int \nabla(\bar{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k)) \cdot Q(u^k) \cdot \bar{u}^{k+1} \\
&\quad + 2a_1\frac{\delta-1}{\delta}\int \nabla(\varphi^k)^2 \cdot (Q(u^k)-Q(u^{k-1})) \cdot \bar{u}^{k+1} + a_1\int \nabla\bar{\Phi} \cdot \bar{u}^{k+1} \\
&= \sum_{i=1}^7 J_i.
\end{aligned} \tag{3.71}$$

It follows from Lemma 2.1, (3.21), Hölder's inequality, Young's inequality, and $(3.70)_2$ that

$$\begin{aligned}
J_1 &= \int \left(\bar{W}^{k+1}\right)^\top \operatorname{div}A(W^k)\bar{W}^{k+1} \le C|\nabla W^k|_\infty|\bar{W}^{k+1}|_2^2 \le C|\bar{W}^{k+1}|_2^2, \\
J_2 &= \int \sum_{j=1}^3 A_j\left(\bar{W}^k\right)\partial_j W^k \cdot \bar{W}^{k+1} \\
&\le C|\nabla W^k|_\infty|\bar{W}^k|_2|\bar{W}^{k+1}|_2 \le C\nu^{-1}|\bar{W}^{k+1}|_2^2 + \nu|\bar{W}^k|_2^2, \\
J_3 &= -2a_1\frac{\delta}{\delta-1}\int \nabla(\varphi^{k+1})^2 \cdot Q(\bar{u}^{k+1}) \cdot \bar{u}^{k+1} \\
&\le C|\nabla\varphi^{k+1}|_\infty|\varphi^{k+1}\nabla\bar{u}^{k+1}|_2|\bar{u}^{k+1}|_2 \\
&\le C|\bar{W}^{k+1}|_2^2 + \frac{a_1\alpha}{10}|\varphi^{k+1}\nabla\bar{u}^{k+1}|_2^2, \\
J_4 &= -2a_1\int (\bar{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k) \cdot L(u^k)) \cdot \bar{u}^{k+1}
\end{aligned} \tag{3.72}$$

$$\leq C|\bar{\varphi}^{k+1}|_2|\bar{u}^{k+1}|_2|\varphi^k\nabla^2 u^k|_\infty + C|\bar{\varphi}^{k+1}|_2|\varphi^{k+1}\bar{u}^{k+1}|_6|\nabla^2 u^k|_3$$

$$\leq C|\bar{\varphi}^{k+1}|_2|\bar{u}^{k+1}|_2|\varphi^k\nabla^2 u^k|_\infty + C|\bar{\varphi}^{k+1}|_2^2|\nabla^2 u^k|_3^2 + \frac{\alpha}{10}|\varphi^{k+1}\nabla\bar{u}^{k+1}|_2^2$$

$$\leq C|\bar{\varphi}^{k+1}|_2^2 + C(1+|\varphi^{k+1}\nabla^4 u^k|_2)|\bar{u}^{k+1}|_2^2 + \frac{\alpha}{10}|\varphi^{k+1}\nabla\bar{u}^{k+1}|_2^2,$$

$$J_5 = -2a_1\frac{\delta-1}{\delta}\int \nabla(\bar{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k))\cdot Q(u^k)\cdot\bar{u}^{k+1}$$

$$\leq C|\nabla^2 u^k|_6|\bar{\varphi}^{k+1}|_2|\varphi^{k+1}\bar{u}^{k+1}|_3 + C|\bar{\varphi}^{k+1}|_2|\varphi^k\nabla^2 u^k|_\infty|\bar{u}^{k+1}|_2$$

$$\quad + C|\bar{\varphi}^{k+1}|_2|\nabla u^k|_\infty|\varphi^{k+1}\nabla\bar{u}^{k+1}|_2 + C|\bar{\varphi}^{k+1}|_2|\varphi^{k+1}\nabla\bar{u}^{k+1}|_2|\nabla u^k|_\infty$$

$$\quad + C|\bar{\varphi}^{k+1}|_2^2|\nabla u^k|_\infty|\nabla\bar{u}^{k+1}|_\infty$$

$$\leq C|\bar{\varphi}^{k+1}|_2^2 + C(1+|\varphi^k\nabla^4 u^k|_2^2)|\bar{W}^{k+1}|_2^2 + \frac{a_1\alpha}{10}|\varphi^{k+1}\nabla\bar{u}^{k+1}|_2^2,$$

$$J_6 = 2a_1\frac{\delta-1}{\delta}\int \nabla(\varphi^k)^2\cdot(Q(u^k)-Q(u^{k-1}))\cdot\bar{u}^{k+1}$$

$$\leq C|\nabla\varphi^k|_\infty|\varphi^k\nabla\bar{u}^k|_2|\bar{u}^{k+1}|_2 \leq C\nu^{-1}|\bar{W}^{k+1}|_2^2 + \nu|\varphi^k\nabla\bar{u}^k|_2^2,$$

$$J_7 = a_1\int \nabla\bar{\Phi}\cdot\bar{u}^{k+1} \leq C|\nabla\Phi|_2|\bar{u}^{k+1}|_2 \leq C|\nabla\Phi|_2^2 + C|\bar{u}^{k+1}|_2^2$$

$$\leq C\left|\bar{\varphi}^{k+1}\right|_2^2 + C|\bar{u}^{k+1}|_2^2.$$

Then, from (3.71) and (3.72), it yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (\bar{W}^{k+1})^\top A_0\bar{W}^{k+1} + a_1\alpha|\varphi^{k+1}\nabla\bar{u}^{k+1}|_2^2$$

$$\leq C\left(\nu^{-1}+|\varphi^k\nabla^4 u^k|_2^2\right)|\bar{W}^{k+1}|_2^2 + C|\bar{\varphi}^{k+1}|_2^2 + \nu\left(|\varphi^k\nabla\bar{u}^k|_2^2 + |\bar{\varphi}^k|_2^2 + |\bar{W}^k|_2^2\right). \tag{3.73}$$

We denote

$$S^{k+1}(t) = \sup_{s\in[0,t]}|\bar{W}^{k+1}(s)|_2^2 + \sup_{s\in[0,t]}|\bar{\varphi}^{k+1}(s)|_2^2.$$

From (3.68), (3.70), and (3.73), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \left((\bar{W}^{k+1})^\top A_0\bar{W}^{k+1} + |\bar{\varphi}^{k+1}(s)|_2^2\right) + |\varphi^{k+1}\nabla\bar{u}^{k+1}|_2^2$$

$$\leq E_\nu^k(|\bar{W}^{k+1}|_2^2 + |\bar{\varphi}^{k+1}|_2^2) + \nu\left(|\varphi^k\nabla\bar{u}^k|_2^2 + |\bar{\varphi}^k|_2^2 + |\bar{W}^k|_2^2\right), \tag{3.74}$$

for some E_ν^k such that $\int_0^t E_\nu^k\mathrm{d}s \leq C + C(1+\frac{1}{\nu})t$. According to Gronwall's inequality, one has

$$S^{k+1} + \int_0^t |\varphi^{k+1}\nabla\bar{u}^{k+1}|_2^2\mathrm{d}s$$

$$\leq C\nu\int_0^t (|\varphi^k\nabla\bar{u}^k|_2^2 + |\bar{\varphi}^k|_2^2 + |\bar{W}^k|_2^2)\mathrm{d}s\cdot\exp(C+C(1+\frac{1}{\nu})t) \tag{3.75}$$

$$\leq \left(C\nu\int_0^t |\varphi^k\nabla\bar{u}^k|_2^2\mathrm{d}s + Ct\nu\sup_{s\in[0,t]}(|\bar{W}^k|_2^2 + |\bar{\varphi}^k|_2^2)\right)\exp(C+C(1+\frac{1}{\nu})t).$$

We can choose $\nu_0 > 0$ and $T_* \in (0,\min(1,T^{**}))$ small enough such that

$$C\nu_0\exp C \leq \frac{1}{8}, \quad \exp(C(1+\frac{1}{\nu})T_*) \leq 2,$$

which yields that

$$\sum_{k=1}^{+\infty}\left(S^{k+1}(T_*) + \int_0^{T_*} |\varphi^{k+1}\nabla\bar{u}^{k+1}|_2^2\right) \le C < +\infty. \tag{3.76}$$

It follows from (3.45), (3.70), and (3.76) that (φ^k, W^k, Φ^k) converges to a limit (φ, W, Φ) in the following strong sense:

$$(\varphi^k, W^k, \Phi^k) \to (\varphi, W, \Phi) \quad \text{in} \quad L^\infty(0, T_*; H^2(\mathbb{T}^3)). \tag{3.77}$$

Due to the local estimates (3.45) and the lower-continuity of norm for weak or weak* convergence, we know that (φ, W, Φ) satisfies the estimates (3.45). According to the strong convergence in (3.76), we can show that (φ, W, Φ) is a weak solution of (3.1) in the sense of distribution with the regularities:

$$\begin{aligned}
&\varphi \in L^\infty\left([0, T^*]; H^3\right), \varphi_t \in L^\infty\left([0, T^*]; H^2\right), \phi \in L^\infty\left([0, T^*]; H^3\right), \phi_t \in L^\infty\left([0, T^*]; H^2\right), \\
&\Phi \in L^\infty\left([0, T^*]; H^4\right), u \in L^\infty\left([0, T^*]; H^3\right), \varphi\nabla^4 u \in L^2\left([0, T^*]; L^2\right), \\
&u_t \in L^\infty\left([0, T^*]; H^1\right) \cap L^2\left([0, T^*]; D^2\right).
\end{aligned} \tag{3.78}$$

Thus the existence of strong solutions is proved.

Step 2. Uniqueness and time-continuity. It can be obtained via the same arguments used in the proof of Lemma 3.6. $\qquad\square$

3.6. Proof of Theorem 1.1

Proof. Now we are ready to prove Theorem 1.1 and the proof is divided into two steps.

Step 1. Existence of regular solutions. First, for the initial assumption (1.8), it follows from Theorem 3.1 that there exists a positive time T_* such that problem (3.1) has a unique strong solution (φ, ϕ, u) in $[0, T_*] \times \mathbb{T}^3$ satisfying the regularities in (3.3), which means that

$$\left(\rho^{\frac{\delta-1}{2}}, \rho^{\frac{\gamma-1}{2}}\right) = (\varphi, \phi) \in C^1\left((0, T_*) \times \mathbb{T}^3\right), \quad \text{and} \quad (u, \nabla u) \in C\left((0, T_*) \times \mathbb{T}^3\right).$$

Noticing that $\rho = \varphi^{\frac{2}{\delta-1}}$ and $\frac{2}{\delta-1} \ge 1$, it is easy to show that

$$\rho \in C^1\left((0, T_*) \times \mathbb{T}^3\right). \tag{3.79}$$

Second, the system $(3.1)_2$ for $W = (\phi, u)$ could be written as

$$\begin{cases} \phi_t + u \cdot \nabla\phi + \frac{\gamma-1}{2}\phi \operatorname{div} u = 0, \\ u_t + u \cdot \nabla u + \frac{A\gamma}{\gamma-1}\nabla\phi^2 + \varphi^2 Lu = \nabla\varphi^2 \cdot Q(u) + \nabla\Phi. \end{cases} \tag{3.80}$$

Multiplying $(3.80)_1$ by $\frac{\partial\rho}{\partial\phi}(t, x) = \frac{2}{\gamma-1}\phi^{\frac{3-\gamma}{\gamma-1}}(t, x) \in C\left((0, T_*) \times \mathbb{T}^3\right)$ on both sides, we get the continuity equation in $(1.1)_1$:

$$\rho_t + u \cdot \nabla\rho + \rho \operatorname{div} u = 0.$$

Multiplying $(3.80)_2$ by $\phi^{\frac{2}{\gamma-1}} = \rho(t, x) \in C^1\left((0, T_*) \times \mathbb{T}^3\right)$ on both sides, we get the momentum equation in $(1.1)_2$:

$$\rho u_t + \rho u \cdot \nabla u + \nabla P = \operatorname{div}\left(\mu(\rho)\left(\nabla u + (\nabla u)^\top\right) + \lambda(\rho)\operatorname{div} uI_3\right) + \rho\nabla\Phi.$$

Finally, recalling that ρ can be represented by the formula

$$\rho(t,x) = \rho_0(U(0,t,x)) \exp\left(\int_0^t \operatorname{div} u(s, U(s,t,x))\mathrm{d}s\right),$$

where $U \in C^1\left([0,T_*] \times [0,T_*] \times \mathbb{T}^3\right)$ is the solution to the initial value problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}U(t,s,x) = u(s, U(s,t,x)), & 0 \le s \le T_*, \\ U(t,t,x) = x, & 0 \le t \le T_*, \quad x \in \mathbb{T}^3, \end{cases}$$

it is obvious that

$$\rho(t,x) \ge 0, \quad \forall\, (t,x) \in (0,T_*) \times \mathbb{T}^3.$$

That is to say, (ρ, u, Φ) satisfies problem (1.1) in the sense of distributions, and has the regularities shown in Definition 1.1, which means that the Cauchy problem (1.1)–(1.2) has a unique regular solution (ρ, u, Φ).

Step 2. The smoothness of regular solutions. Now we will show that the regular solution that we obtained in the above step is indeed a classical one in positive time $(0, T_*]$.

Due to the definition of regular solution and the classical Sobolev embedding theorem, we immediately know that

$$(\rho, \nabla\rho, \rho_t, u, \nabla u, \nabla\Phi) \in C\left([0,T_*] \times \mathbb{T}^3\right).$$

Now we only need to prove that

$$(u_t, \operatorname{div}\mathbb{S}) \in C\left((0,T_*] \times \mathbb{T}^3\right).$$

Next, we first give the continuity of u_t. We differentiate $(3.80)_2$ with respect to t :

$$u_{tt} + \varphi^2 L u_t = -\left(\varphi^2\right)_t L u - (u \cdot \nabla u)_t - \frac{A\gamma}{\gamma-1}\nabla\left(\phi^2\right)_t + \left(\nabla\varphi^2 \cdot Q(u)\right)_t + \nabla\Phi_t, \tag{3.81}$$

which, along with (3.3), easily implies that

$$u_{tt} \in L^2\left([0,T_*]; L^2\right). \tag{3.82}$$

Applying ∂_x^ζ ($|\zeta| = 2$) to (3.81), multiplying the resulting equations by $\partial_x^\zeta u_t$ and integrating over $\mathbb{T}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\partial_x^\zeta u_t\right|_2^2 + \alpha\left|\varphi\nabla\partial_x^\zeta u_t\right|_2^2 + (\alpha+\beta)\left|\varphi\operatorname{div}\partial_x^\zeta u_t\right|_2^2 \\
&= \int\left(-\nabla\varphi^2 \cdot Q\left(\partial_x^\zeta u_t\right) - \left(\partial_x^\zeta\left(\varphi^2 L u_t\right) - \varphi^2 L\partial_x^\zeta u_t\right)\right) \cdot \partial_x^\zeta u_t \\
&\quad + \int\left(-\partial_x^\zeta\left(\left(\varphi^2\right)_t L u\right) - \partial_x^\zeta(u \cdot \nabla u)_t - \frac{A\gamma}{\gamma-1}\partial_x^\zeta\nabla\left(\phi^2\right)_t\right) \cdot \partial_x^\zeta u_t \\
&\quad + \int\partial_x^\zeta\left(\nabla\varphi^2 \cdot Q(u)\right)_t \cdot \partial_x^\zeta u_t + \int\nabla\partial_x^\zeta\Phi_t \cdot \partial_x^\zeta u_t \\
&\triangleq \sum_{i=1}^7 P_i.
\end{aligned} \tag{3.83}$$

It follows from Hölder's inequality, Lemma 2.1, and Young's inequality that

$$
\begin{aligned}
P_1 &= \int \left(-\nabla\varphi^2 \cdot Q\left(\partial_x^\zeta u_t\right) \right) \cdot \partial_x^\zeta u_t \\
&\le C \left|\varphi\nabla^3 u_t\right|_2 \left|\nabla^2 u_t\right|_2 |\nabla\varphi|_\infty \le \frac{\alpha}{20} \left|\varphi\nabla^3 u_t\right|_2^2 + C \left|u_t\right|_{D^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
P_2 &= \int -\left(\partial_x^\zeta\left(\varphi^2 L u_t\right) - \varphi^2 L \partial_x^\zeta u_t\right) \cdot \partial_x^\zeta u_t \\
&\le C \left(\left|\varphi\nabla^3 u_t\right|_2 |\nabla\varphi|_\infty + |\nabla\varphi|_\infty^2 |u_t|_{D^2} + \left|\nabla^2\varphi\right|_3 \left|\varphi\nabla^2 u_t\right|_6 \right) |u_t|_{D^2} \\
&\le \frac{\alpha}{20} \left|\varphi\nabla^3 u_t\right|_2^2 + C |u_t|_{D^2}^2,
\end{aligned}
\tag{3.84}
$$

$$
\begin{aligned}
P_3 &= \int -\partial_x^\zeta\left(\left(\varphi^2\right)_t L u\right) \cdot \partial_x^\zeta u_t \\
&\le C\Big(\left|\nabla^2\varphi\right|_3 |Lu|_6 |\varphi_t|_\infty |u_t|_{D^2} + \left|\varphi\nabla^2 u_t\right|_6 |\varphi_t|_{D^2} |Lu|_3 \\
&\quad + |\nabla\varphi|_\infty |\nabla\varphi_t|_6 |Lu|_3 |u_t|_{D^2} + \left|\varphi\nabla^3 u\right|_6 |\nabla\varphi_t|_3 |u_t|_{D^2} \\
&\quad + |\nabla\varphi|_\infty |\varphi_t|_\infty \left|\nabla^3 u\right|_2 |u_t|_{D^2} + |\varphi_t|_\infty |u_t|_{D^2} \left|\varphi\nabla^4 u\right|_2 \Big) \\
&\le \frac{\alpha}{20} \left|\varphi\nabla^3 u_t\right|_2^2 + C |u_t|_{D^2}^2 + C \left|\varphi\nabla^4 u\right|_2^2 + C,
\end{aligned}
$$

$$
\begin{aligned}
P_4 &= \int -\partial_x^\zeta(u \cdot \nabla u)_t \cdot \partial_x^\zeta u_t \\
&\le C \left(\|u_t\|_1 + |u_t|_{D^2} \right) \|u\|_3 - \int (u \cdot \nabla)\partial_x^\zeta u_t \cdot \partial_x^\zeta u_t \\
&\le C + C |u_t|_{D^2}^2 + C|\nabla u|_\infty \left|\partial_x^\zeta u_t\right|_2^2 \le C + C |u_t|_{D^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
P_5 &= \int -\frac{A\gamma}{\gamma-1} \partial_x^\zeta \nabla\left(\phi^2\right)_t \cdot \partial_x^\zeta u_t \\
&\le C\Big(\left|\nabla^2\phi_t\right|_2 \left|\phi\nabla^3 u_t\right|_2 + |\phi_t|_\infty \left|\nabla^3\phi\right|_2 \left|\nabla^2 u_t\right|_2 \\
&\quad + C \left|\nabla^2\phi\right|_6 |\nabla\phi_t|_3 \left|\nabla^2 u_t\right|_2 + \left|\nabla^2\phi_t\right|_2 |\nabla\phi|_\infty |\nabla u_t|_2 \Big) \\
&\le \frac{\alpha}{20} |\phi\nabla^3 u_t|_2^2 + C(1 + |\partial_x^\zeta u_t|^2),
\end{aligned}
$$

$$
\begin{aligned}
P_6 &= \int \partial_x^\zeta\left(\nabla\varphi^2 \cdot Q(u)\right)_t \cdot \partial_x^\zeta u_t \\
&\le \frac{\alpha}{20} |\varphi\nabla^3 u_t|_2^2 + C|\partial_x^\zeta u_t|_2^2 + C|\varphi\nabla^4 u|_2^2 + C,
\end{aligned}
$$

$$
P_7 = \int \nabla\partial_x^\zeta \Phi_t \cdot \partial_x^\zeta u_t \le C|\partial_x^\zeta u_t|_2|\nabla^3\Phi_t|_2 \le C|\partial_x^\zeta u_t|_2^2 + C.
$$

It follows from (3.83)–(3.84) that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left|\nabla^2 u_t\right|_2^2 + \frac{\alpha}{2} \left|\varphi\nabla^3 u_t\right|_2^2 \le C \left|\nabla^2 u_t\right|_2^2 + C \left|\varphi\nabla^4 u\right|_2^2 + C.
\tag{3.85}
$$

Then multiplying both sides of (3.85) with t and integrating over $[\tau, t]$ for any $\tau \in (0, t)$, one gets

$$
t |u_t|_{D^2}^2 + \int_\tau^t s \left|\varphi\nabla^3 u_t\right|_2^2 \, \mathrm{d}s \le C\tau |u_t(\tau)|_{D^2}^2 + C(1 + t).
\tag{3.86}
$$

According to the definition of the regular solution, we know that

$$\nabla^2 u_t \in L^2\left([0, T_*]; L^2\right),$$

which, along with Lemma 2.8, implies that there exists a sequence s_k such that

$$s_k \to 0 \quad \text{and} \quad s_k \left|\nabla^2 u_t\left(s_k, \cdot\right)\right|_2^2 \to 0, \quad \text{as } k \to +\infty.$$

Then, letting $\tau = s_k \to 0$ in (3.86), we have

$$t \left|u_t\right|_{D^2}^2 + \int_0^t s \left|\varphi \nabla^3 u_t\right|_2^2 \, \mathrm{d}s \leq C(1 + t) \leq C,$$

which implies that

$$t^{\frac{1}{2}} u_t \in L^\infty\left([0, T_*]; H^2\right). \tag{3.87}$$

Based on the classical Sobolev embedding theorem

$$L^\infty\left([0, T]; H^1\right) \cap W^{1,2}\left([0, T]; H^{-1}\right) \hookrightarrow C\left([0, T]; L^q\right), \tag{3.88}$$

for any $q \in (3, 6)$, from (3.82) and (3.87), we have

$$t u_t \in C\left([0, T_*]; W^{1,4}\right),$$

which implies that $u_t \in C\left((0, T_*] \times \mathbb{T}^3\right)$. Finally, we consider the continuity of div $\mathbb{S}$. Denote $\mathbb{N} = \varphi^2 L u - \nabla \varphi^2 \cdot Q(u)$. Based on (3.3) and (3.87), we have

$$t \mathbb{N} \in L^\infty\left(0, T_*; H^2\right).$$

From $\mathbb{N}_t \in L^2\left(0, T_*; L^2\right)$ and (3.88), we obtain $t \mathbb{N} \in C\left([0, T_*]; W^{1,4}\right)$, which implies that

$$\mathbb{N} \in C\left((0, T_*] \times \mathbb{T}^3\right).$$

Since $\rho \in C\left([0, T_*] \times \mathbb{T}^3\right)$ and div $\mathbb{S} = \rho \mathbb{N}$, then we obtain the desired conclusion. □

Author contributions

The authors have made the same contributions. All authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Conflict of interest

The authors declare there is no conflict of interest.

References

1. P. A. Markowich, C. A. Ringhofer, C. Schmeiser, *Semiconductor Equations*, Springer, 1990. https://doi.org/10.1007/978-3-7091-6961-2

2. S. Kawashima, *Systems of a Hyperbolic-Parabolic composite type, with applications to the equations of Magnetohydrodynamics*, Ph.D. thesis, Kyoto University, 1983.

3. A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.*, **20** (1980), 67–104. https://doi.org/10.1215/kjm/1250522322

4. Y. Cho, H. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, *J. Math. Pures Appl.*, **83** (2004), 243–275. https://doi.org/10.1016/j.matpur.2003.11.004

5. Y. Cho, H. Kim, On classical solutions of the compressible Navier–Stokes equations with nonnegative initial densities, *Manuscripta Math.*, **120** (2006), 91–129. https://doi.org/10.1007/s00229-006-0637-y

6. X. Huang, J. Li, Z. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations, *Comm. Pure Appl. Math.*, **65** (2012), 549–585. https://doi.org/10.1002/cpa.21382

7. Q. Jiu, M. Li, Y. Ye, Global classical solution of the Cauchy problem to 1D compressible Navier–Stokes equations with large initial data, *J. Differential Equations*, **257** (2014), 311–350. https://doi.org/10.1016/j.jde.2014.03.014

8. H. Gong, J. Li, X.-G. Liu, X. Zhang, Local well-posedness of isentropic compressible Navier–Stokes equations with vacuum, *Commun. Math. Sci.*, **18** (2020), 1891–1909. https://doi.org/10.4310/CMS.2020.v18.n7.a4

9. X. Huang, On local strong and classical solutions to the three-dimensional barotropic compressible Navier–Stokes equations with vacuum, *Sci. China Math.*, **64** (2021), 1771–1788. https://doi.org/10.1007/s11425-019-9755-3

10. D. Bresch, B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, *Comm. Math. Phys.*, **238** (2003), 211–223. https://doi.org/10.1007/s00220-003-0859-8

11. D. Bresch, A. Vasseur, C. Yu, Global existence of entropy-weak solutions to the compressible Navier–Stokes equations with non-linear density-dependent viscosities, *J. Eur. Math. Soc.*, **24** (2022), 1791–1837. https://doi.org/10.4171/jems/1143

12. Q. Jiu, Z. Xin, The cauchy problem for 1D compressible flows with density-dependent viscosity coefficients, *Kinet. Relat. Models*, **1** (2008), 313–330. https://doi.org/10.3934/krm.2008.1.313

13. J. Li, Z. Xin, Global existence of weak solutions to the barotropic compressible Navier–Stokes flows with degenerate viscosities, preprint, arXiv: 1504.06826.

14. A. Mellet, A. Vasseur, On the barotropic compressible Navier–Stokes equations, *Comm. Partial Differential Equations*, **32** (2007), 431–452. https://doi.org/10.1080/03605300600857079

15. A. Vasseur, C. Yu, Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations, *Invent. Math.*, **206** (2016), 935–974. https://doi.org/10.1007/s00222-016-0666-4

16. Y. Li, R. Pan, S. Zhu, On classical solutions to 2D shallow water equations with degenerate viscosities, *J. Math. Fluid Mech.*, **19** (2017), 151–190. https://doi.org/10.1007/s00021-016-0276-3

17. Y. Li, R. Pan, S. Zhu, On classical solutions for viscous polytropic fluids with degenerate viscosities and vacuum, *Arch. Ration. Mech. Anal.*, **234** (2019), 1281–1334. https://doi.org/10.1007/s00205-019-01412-6

18. Y. Geng, Y. Li, S. Zhu, Vanishing viscosity limit of the Navier–Stokes equations to the Euler equations for compressible fluid flow with vacuum, *Arch. Ration. Mech. Anal.*, **234** (2019), 727–775. https://doi.org/10.1007/s00205-019-01401-9

19. Z. Xin, S. Zhu, Global well-posedness of regular solutions to the three-dimensional isentropic compressible Navier–Stokes equations with degenerate viscosities and vacuum, *Adv. Math.*, **393** (2021), 108072. https://doi.org/10.1016/j.aim.2021.108072

20. Z. Xin, S. Zhu, Well-posedness of three-dimensional isentropic compressible Navier–Stokes equations with degenerate viscosities and far field vacuum, *J. Math. Pures Appl.*, **152** (2021), 94–144. https://doi.org/10.1016/j.matpur.2021.05.004

21. Y. Cao, H. Li, S. Zhu, Global regular solutions for one-dimensional degenerate compressible Navier–Stokes equations with large data and far field vacuum, *SIAM J. Math. Anal.*, **54** (2022), 4658–4694. https://doi.org/10.1137/21M1464609

22. P. Germain, P. Lefloch, Finite energy method for compressible fluids: the Navier–Stokes–Korteweg model, *Comm. Pure Appl. Math.*, **69** (2016), 3–61. https://doi.org/10.1002/cpa.21622

23. Z. Guo, H.-L. Li, Z. Xin, Lagrange structure and dynamics for solutions to the spherically symmetric compressible Navier–Stokes equations, *Comm. Math. Phys.*, **309** (2012), 371–412. https://doi.org/10.1007/s00220-011-1334-6

24. P.-L. Lions, *Mathematical Topics in Fluid Mechanics: Compressible Models*, volume 2, Oxford University Press, New York, 1998.

25. T. Yang, H. Zhao, A vacuum problem for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity, *J. Differential Equations*, **184** (2002), 163–184. https://doi.org/10.1006/jdeq.2001.4140

26. D. Donatelli, Local and global existence for the coupled Navier–Stokes–Poisson problem, *Quart. Appl. Math.*, **61** (2003), 345–361. https://doi.org/10.1090/qam/1976375

27. H.-L. Li, A. Matsumura, G. Zhang, Optimal decay rate of the compressible Navier–Stokes—Poisson system in $\mathbb{R}^3$, *Arch. Ration. Mech. Anal.*, **196** (2010), 681–713. https://doi.org/10.1007/s00205-009-0255-4

28. X. Zheng, Global well-posedness for the compressible Navier–Stokes–Poisson system in the L^p framework, *Nonlinear Anal.*, **75** (2012), 4156–4175. https://doi.org/10.1016/j.na.2012.03.006

29. Z. Tan, Y. Wang, Y. Wang, Stability of steady states of the Navier—Stokes—Poisson equations with non-flat doping profile, *SIAM J. Math. Anal.*, **47** (2015), 179–209. https://doi.org/10.1137/130950069

30. S. Liu, X. Xu, J. Zhang, Global well-posedness of strong solutions with large oscillations and vacuum to the compressible Navier–Stokes–Poisson equations subject to large and non-flat doping profile, *J. Differential Equations*, **269** (2020), 8468–8508. https://doi.org/10.1016/j.jde.2020.06.006

31. H. Liu, T. Luo, H. Zhong, Global solutions to compressible Navier–Stokes–Poisson and Euler–Poisson equations of plasma on exterior domains, *J. Differential Equations*, **269** (2020), 9936–10001. https://doi.org/10.1016/j.jde.2020.07.005

32. Y. Chen, B. Huang, X. Shi, Global well-posedness of classical solutions to the compressible Navier–Stokes–Poisson equations with slip boundary conditions in 3D bounded domains, *J. Math. Fluid Mech.*, **26** (2024), 41. https://doi.org/10.1007/s00021-024-00875-2

33. B. Ducomet, Š. Nečasová, A. Vasseur, On global motions of a compressible barotropic and selfgravitating gas with density-dependent viscosities, *Z. Angew. Math. Phys.*, **61** (2010), 191–211. https://doi.org/10.1007/s00033-009-0035-x

34. B. Ducomet, Š. Nečasová, A. Vasseur, On spherically symmetric motions of a viscous compressible barotropic and selfgravitating gas, *J. Math. Fluid Mech.*, **13** (2011), 191–211. https://doi.org/10.1007/s00021-009-0010-5

35. A. Zlotnik, On global in time properties of the symmetric compressible barotropic Navier—Stokes–Poisson flows in a vacuum, *Int. Math. Ser.*, **7** (2008), 329–376. https://doi.org/10.1007/978-0-387-75219-8_7

36. Y. Ye, C. Dou, Global weak solutions to 3D compressible Navier–Stokes–Poisson equations with density-dependent viscosity, *J. Math. Anal. Appl.*, **455** (2017), 180–211. https://doi.org/10.1016/j.jmaa.2017.05.044

37. H. Yu, Global existence and decay of strong solution to the 3D compressible Navier–Stokes—Poisson equations with large data, *J. Differential Equations*, **444** (2025), 113672. https://doi.org/10.1016/j.jde.2025.113672

38. O. Ladyzenskaja, N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.

39. A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer, 1986. https://doi.org/10.1007/978-1-4612-1116-7

40. J. Simon, Compact sets in $L^p(0, T; B)$, *Ann. Mat. Pura. Appl.*, **146** (1987), 65–96. https://doi.org/10.1007/BF01762360

41. J. Boldrini, M. Rojas-Medar, E. Fernández-Cara, Semi-Galerkin approximation and regular solutions to the equations of the nonhomogeneous asymmetric fluids, *J. Math. Pure Appl.*, **82** (2003), 1499–1525. https://doi.org/10.1016/j.matpur.2003.09.005

42. C. Evans, *Partial Differential Equations*, American Mathematical Society, 2010. https://doi.org/10.1090/gsm/019