Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 17(3): 749–778.
DOI: 10.3934/cam.2025030
Received: 31 December 2024
Revised: 09 July 2025
Accepted: 09 July 2025
Published: 26 August 2025

Research article

Local well-posedness for the initial boundary value problem of the scaling critical compressible Navier–Stokes equations in nearly half space

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Abstract: We consider the initial boundary value problem for the compressible Navier-Stokes equations of barotropic type. It is well-known that the equations maintain its structure nearly invariant under the scaling invariance and by the Fujita–Kato principle, the problem can be solved in the scaling invariant homogeneous Besov space locally in time for the Cauchy problem. We show the initial boundary value problem of the system with 0-Dirichlet boundary condition has a unique local solution in the nearly half Euclidean space. We employ the Lagrangian transform to transfer the equation in the Lagrangian coordinate, and flattening the boundary function to the half space and show the local well-posedness of the system. The key for the proof is employing end-point L^1 maximal regularity for the parabolic equations in half space shown in [1].

Keywords: the compressible Navier–Stokes equations; initial-boundary value problem; 0-Dirichlet boundary condition; maximal L^1-regularity; end-point estimate; scaling critical; the Besov space; the Lagrange transform; local well-posedness
Mathematics Subject Classification: primary 35K20, 35Q30,76N06, secondary 35K05, 35K61, 42B25

1. Introduction

We consider the initial and initial boundary value problem for the compressible Navier-Stokes equations in the perturbed half space:

Let $n \geq 2$, and let the velocity vector fluid $\bar{u}(t, y)$ and the density $\widetilde{\rho}(t, y)$ for $y \in \Omega(t)$ satisfy the compressible Navier–Stokes equations:

$$
\begin{cases}
\partial_t\widetilde{\rho} + \operatorname{div}(\widetilde{\rho}\bar{u}) = 0, & t > 0,\ y \in \Omega,\\
\widetilde{\rho}(\partial_t\bar{u} + \bar{u}\cdot\nabla\bar{u}) = \mathcal{L}\bar{u} - \nabla\bar{P}(\widetilde{\rho}), & t > 0,\ y \in \Omega,\\
\quad \bar{u}(t, y', 0) = 0, & t > 0,\ y \in \partial\Omega,\\
\quad \widetilde{\rho}(0, y) = 1 + \bar{\rho}_0(y), \quad \bar{u}(0, y) = \bar{u}_0(y), & \qquad y \in \Omega,
\end{cases}
\tag{1.1}
$$

where Ω is either the half space $\mathbb{R}^n_+ \equiv \{y \in \mathbb{R}^n; y = (y', y_n),\ y' \in \mathbb{R}^{n-1},\ y_n > 0\}$ or a perturbed half space, namely, for some boundary function $\eta_0 = \eta_0(y') : \mathbb{R}^{n-1} \to \mathbb{R}$, $\eta_0(y') \to 0$ as $|y'| \to 0$ and

$$
\Omega = \{y = (y', y_n); y_n > \eta_0(y'),\ y' \in \mathbb{R}^{n-1}\}.
$$

The Lamé operator $\mathcal{L}$ is given by

$$
\mathcal{L}\bar{u} \equiv \mu\Delta\bar{u} + (\mu + \lambda)\nabla\operatorname{div}\bar{u},
$$

where $\nabla = \nabla_y = (\partial_{y_1}, \partial_{y_2}, \cdots, \partial_{y_n})^{\mathsf{T}}$. The constants λ and μ denote the first and the second Lamé constants, respectively, and we assume that

$$
\mu > 0, \quad 2\mu + n\lambda > 0.
$$

Without losing generality, we consider the solutions around the constant state $(\widetilde{\rho}, \bar{u}) = (1, 0)$, so that the fluid density $\widetilde{\rho}$ can be written as

$$
\widetilde{\rho} = 1 + \bar{\rho}.
$$

We assume that the pressure satisfies

$$
\bar{P}(\tilde{\rho}) = \bar{P}(1 + \bar{\rho}) = \bar{P}(1) + \bar{P}'(1)\bar{\rho} + O(\bar{\rho}^2), \quad \bar{P}'(\cdot) > 0, \quad \bar{P}(1) = p_e,
$$

where $p_e = p_e(t, y)$ is the external pressure function. There are many literature on the existence and well-posedness theory for the initial and initial boundary value problem (1.1). The earlier result were mainly devoted to the classical solution or the solution in the Sobolev spaces. For instance, Nash [2], Itaya [3] considered a classical solution, Matsumura-Nishida [4, 5] provided a solution in the Sobolev spaces (see related results in [6–8]). *

It is well-known that the incompressible Navier–Stokes equations are invariant under some scaling transform for the velocity $\bar{u}(t, y)$ and the pressure $\bar{p}(t, y)$. Owing to this property, the Cauchy problem of the Navier–Stokes equations can be solved globally in time due to the method by Fujita–Kato [9]. In particular the well-posedness in the scaling critical spaces were developed by Kato [10] and Koch–Tataru [11], while the ill-posedness for the end-point Besov spaces were shown by Bourgain–Pavlović [12] and Wang [13]. See also for relevant regularity criteria by Serrin [14], Oyama [15] and Prodi [16]. Meanwhile the compressible Navier–Stokes equations are invariant under the following scaling transform, except for the pressure term $P(\widetilde{\rho}) = P(1 + \bar{\rho})$: For $\alpha > 0$,

$$
\begin{cases}
\widetilde{\rho}(t, y) \to \widetilde{\rho}_\alpha(t, y) \equiv \widetilde{\rho}(\alpha^2 t, \alpha y),\\
\bar{u}(t, y) \to \bar{u}_\alpha(t, y) \equiv \alpha\bar{u}(\alpha^2 t, \alpha y).
\end{cases}
$$

*Readers can find comprehensive overview in related monographs by Feileisl [17] and Lions [18].

The scaling invariant spaces in the Bochner class $L^\theta(\mathbb{R}_+;\dot{H}^s_p(\mathbb{R}^n;\mathbb{R}^n))$ is

$$\frac{2}{\theta}+\frac{n}{p}=s \quad \text{for } \bar{\rho}, \qquad \frac{2}{\theta}+\frac{n}{p}=1+s \quad \text{for } \bar{u}.$$

Danchin [19–22] have proved the global existence and uniqueness of the compressible Navier–Stokes equations in such scale critical space. More specifically, they have used a subspace of the following solution space: for $1 \le p < 2n$,

$$\bar{\rho}(t,y)\in L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)), \qquad \bar{u}(t,y)\in L^\infty(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap L^1(\mathbb{R}_+;\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)).$$

See also Charve–Danchin [23] and Haspot [24,25]. Recently, Guo–Yan–Zhang [26] have considered the well-posedness of (1.1) in the sense of Hadamard, and successfully show that the continuous dependence of the initial data in $1 < p < 2n$ (see for the compressible Navier-Stokes full system [27]). Note that for $p \ge 2n$, ill-posedness holds by the results [28–30]. On the other hand, the continuous dependence of the initial data is not well understood in a domain other than the whole space and the half space. In the whole space, when we employ the change of coordinates from the Eulerian coordinate to the Lagrangian coordinate, the transformed problem is shown to be well-posed in the homogeneous Besov space $\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)\times\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ for all $1 < p < 2n$, which is established by Danchin [22] and such a result was extended into the initial boundary value problem in the half-space by Danchin-Mucha [31] (see also [32]) and in a bounded domain by Danchin-Tolksdorff [33] (cf. for the incompressible case of a free boundary problem, Danchin-Hieber-Mucha-Talksdorff [34], Ogawa-Shimizu [35,36], and compressible case by Denisova-Solonnikov [37]).

In this paper, we consider the initial boundary value problem (1.1) and show the problem is local-in-time well-posed in the above scaling critical spaces for all $n-1 < p < 2n$ under the Lagrange coordinate in nearly half space Ω. Given the Eulerian coordinates $y \in \Omega$, we introduce the Lagrangian coordinates $\tilde{x}\in\Omega$ given by the system of the ordinary differential equations:

$$\frac{dy}{dt}=\bar{u}(t,y(t)),\quad t>0, \qquad y(0)=\tilde{x}. \tag{1.2}$$

If $\bar{u}(t,y)$ is Lipschitz continuous with respect to y, then (1.2) can be uniquely solved by

$$y(t)=\tilde{x}+\int_0^t \bar{u}(s,y(s,\tilde{x}))ds.$$

Setting

$$\begin{cases}\tilde{\rho}(t,\tilde{x})\equiv\bar{\rho}(t,y(t)),\\ \tilde{u}(t,\tilde{x})\equiv\bar{u}(t,y(t)),\end{cases} \tag{1.3}$$

and applying the Lagrangian coordinate to the problem (1.1), we may transform the system into a purely parabolic system, and the boundary condition remains as the 0-Dirichlet condition. Let the Jacobi matrix of the Lagrangian transform be denoted by $J(Du)$. Since $\tilde{u}(t,\tilde{x})=\bar{u}(t,y(t))$ in (1.3), we see that for $1\le i,j\le n$,

$$\frac{\partial y_i}{\partial\tilde{x}_j}=[J(Du)]_{i,j}=\delta_{ij}+\int_0^t\frac{\partial\tilde{u}_i}{\partial\tilde{x}_j}(s,\tilde{x})ds.$$

Hence, denoting the cofactor of $J(D\tilde{u})$ by $A = A_{ij} = \left[\operatorname{cof}(J(D\tilde{u}))\right]_{ij}$, the Jacobian by $|J(Du)(t)| \equiv \det J(Du(t,x))$, and the transposed matrix of $J(Du)$ by $J(Du)^{\mathsf{T}}$, respectively, we find that

$$\frac{d\tilde{x}}{dy} = (J(D\tilde{u})^{-1})^{\mathsf{T}} = |J(D\tilde{u})(t)|^{-1}A,$$
$$\nabla_y \bar{u} = ((J(D\tilde{u})^{-1})^{\mathsf{T}}\nabla_{\tilde{x}}\tilde{u} = |J(D\tilde{u})(t)|^{-1}A\nabla_{\tilde{x}}\tilde{u}.$$

By the cofactor expansion for the Jacobian,

$$\begin{aligned}\frac{d}{dt}|J(D\tilde{u})(t)| &= \sum_{i=1}^{n}\frac{\partial u_i}{\partial \tilde{x}_1}A_{i1} + \cdots + \sum_{i=1}^{n}\frac{\partial u_i}{\partial \tilde{x}_n}A_{in}\\ &= \sum_{i,j=1}^{n}\frac{\partial u_i}{\partial \tilde{x}_j}A_{ij} = \sum_{i,j=1}^{n}\frac{\partial u_i}{\partial \tilde{x}_j}\frac{\partial \tilde{x}_i}{\partial y_j}|J(D\tilde{u})(t)|\\ &= |J(Du)(t)|\operatorname{div}\bar{u} = |J(D\tilde{u})(t)|\operatorname{tr}(J(D\tilde{u})^{-1})^{\mathsf{T}}\nabla\tilde{u}.\end{aligned}$$

Observing $|J(D\tilde{u})(0)| = \det I = 1$, we note that the Jacobian satisfies

$$\begin{cases}\dfrac{d|J(D\tilde{u})(t)|}{dt} = |J(D\tilde{u})(t)|\operatorname{tr}(J(D\tilde{u})^{-1})^{\mathsf{T}}\nabla\tilde{u},\\ |J(D\tilde{u})(0)| = 1\end{cases}$$

and hence

$$|J(D\tilde{u})(t)| = \exp\Big(\int_0^t \operatorname{tr}(J(D\tilde{u}(s))^{-1})^{\mathsf{T}}\nabla\tilde{u}(s)\,ds\Big). \tag{1.4}$$

Note that we also have

$$\begin{aligned}|J(D\tilde{u})(t)| &= |J(D\tilde{u})(0)| + \int_0^t |J(D\tilde{u})(s)|\operatorname{tr}(J(D\tilde{u})(s))^{-1})^{\mathsf{T}}\nabla\tilde{u}(s)\,ds\\ &= 1 + \int_0^t \operatorname{tr}(\operatorname{cof} J(D\tilde{u}))\nabla\tilde{u}\,ds = 1 + \int_0^t A\nabla\cdot\tilde{u}\,ds.\end{aligned} \tag{1.5}$$

Meanwhile the transformed density ρ has to solve

$$\begin{cases}\partial_t\tilde{\rho} + (1+\tilde{\rho})\operatorname{tr}(J(D\tilde{u}(s))^{-1})^{\mathsf{T}}\nabla\tilde{u} = 0, & t>0, \quad \tilde{x}\in\Omega,\\ \tilde{\rho}(0,\tilde{x}) = \tilde{\rho}_0(\tilde{x}), & \tilde{x}\in\Omega,\end{cases}$$

and $\tilde{\rho}$ can be given by

$$\tilde{\rho}(t,\tilde{x}) = (1+\tilde{\rho}_0(x))\exp\Big(-\int_0^t \operatorname{tr}(J(D\tilde{u}(s))^{-1})^{\mathsf{T}}\nabla\tilde{u}(s)\,ds\Big) - 1. \tag{1.6}$$

Hence, by (1.4) and (1.6), we find that ρ is given by the velocity $\tilde{u}$ as

$$\tilde{\rho}(t,\tilde{x}) = (1+\tilde{\rho}_0(\tilde{x}))|J(D\tilde{u})(t)|^{-1} - 1. \tag{1.7}$$

For the Lagrangian framework, we refer to Denisova–Solonnikov [37].

According to the above observation, the original system (1.1) can be reduced to the following problem under the boundary condition on Ω:

$$\begin{cases} \partial_t \tilde{u} - \mu\Delta\tilde{u} - (\mu+\lambda)\nabla \mathrm{div}\,\tilde{u} = F(\tilde{\rho},\tilde{u}), & t>0, \quad \tilde{x}\in\Omega, \\ \tilde{\rho} = (1+\rho_0)|J(D\tilde{u})|^{-1} - 1, & t>0, \quad \tilde{x}\in\Omega, \\ \tilde{u}(t,\tilde{x}) = 0, & t>0, \quad \tilde{x}\in\partial\Omega, \\ \tilde{\rho}(0,\tilde{x}) = \bar{\rho}_0(\tilde{x}), \quad \tilde{u}(0,\tilde{x}) = \bar{u}_0(\tilde{x}), & \tilde{x}\in\Omega, \end{cases} \tag{1.8}$$

where

$$\begin{aligned} F(\tilde{\rho},\tilde{u}) \equiv & -(I - J(D\tilde{u})^{-1})\nabla\cdot\Big(\mu\big((J(D\tilde{u})^{-1})^{\mathsf{T}}\,\nabla\tilde{u} + (\nabla\tilde{u})^{\mathsf{T}}\,(J(D\tilde{u})^{-1})\big) \\ & \qquad\qquad + \big(\lambda\,\mathrm{tr}(J(D\tilde{u})^{-1})^{\mathsf{T}}\nabla\tilde{u} - P(1+\tilde{\rho})\big)I\Big) \\ & - \nabla\cdot\Big(\mu\big((I - J(D\tilde{u})^{-1})^{\mathsf{T}}\,\nabla\tilde{u} + (\nabla\tilde{u})^{\mathsf{T}}\,(I - J(D\tilde{u})^{-1})\big) \\ & \qquad\qquad + \big(\lambda\,\mathrm{tr}(J(D\tilde{u})^{-1} - I)^{\mathsf{T}}\nabla\tilde{u} - P(1+\tilde{\rho})\big)I\Big) - \tilde{\rho}\partial_t\tilde{u}. \end{aligned} \tag{1.9}$$

Before stating our results, we define the Besov space in the half-space (see for details, Bergh–Löfström [38], Peetre [16], Triebel [39, 40]).

Definition (The Besov spaces). Let $s\in\mathbb{R}$, $1\le p,\sigma\le\infty$. Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity for $x\in\mathbb{R}^n$, i.e., $\widehat{\phi}$ is the Fourier transform of a smooth radial function ϕ satisfying $\widehat{\phi}(\xi)\ge 0$, $\mathrm{supp}\,\widehat{\phi}\subset\{\xi\in\mathbb{R}^n \mid 2^{-1}<|\xi|<2\}$, and $\widehat{\phi_j}(\xi)=\widehat{\phi}(2^{-j}\xi)$, $\sum_{j\in\mathbb{Z}}\widehat{\phi_j}(\xi)=1$ for any $\xi\in\mathbb{R}^n\setminus\{0\}$ and $j\in\mathbb{Z}$, and $\widehat{\phi_{\widehat{0}}}(\xi)+\sum_{j\ge1}\widehat{\phi_j}(\xi)=1$ for any $\xi\in\mathbb{R}^n$, where $\widehat{\phi_{\widehat{0}}}(\xi)\equiv\widehat{\zeta}(|\xi|)$ with a low frequency cut-off $\widehat{\zeta}(r)=1$ for $0\le r<1$ and $\widehat{\zeta}(r)=0$ for $2<r$ (see [38]). For $s\in\mathbb{R}$, and $1\le p<\infty$, and $1\le\sigma\le\infty$, let $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ be the homogeneous Besov space given by the completion of smooth rapidly decreasing class $\mathcal{S}_0(\mathbb{R}^n)\equiv\{f\in\mathcal{S}(\mathbb{R}^n); \big|\hat{f}(\xi)\big|\to 0,\ |\xi|\to 0\}$ by the norm

$$\|f\|_{\dot{B}^s_{p,\sigma}} \equiv \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{s\sigma j}\|\phi_j * f\|_p^\sigma\Big)^{1/\sigma}, & 1\le\sigma<\infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\phi_j * f\|_p, & \sigma=\infty \end{cases}$$

and $B^s_{p,\sigma}(\mathbb{R}^n)$ be the inhomogeneous Besov space with the norm

$$\|f\|_{B^s_{p,\sigma}} \equiv \begin{cases} \|\phi_{\widehat{0}} * f\|_p + \Big(\sum_{j\ge1}^{\infty} 2^{s\sigma j}\|\phi_j * f\|_p^\sigma\Big)^{1/\sigma}, & 1\le\sigma<\infty, \\ \|\phi_{\widehat{0}} * f\|_p + \sup_{j=1}^{\infty} 2^{sj}\|\phi_j * f\|_p, & \sigma=\infty. \end{cases}$$

We make an obvious modification when $p=\infty$.

Concerning the Cauchy problem (1.1) in $\Omega=\mathbb{R}^n$ without boundary, the local-in-time well-posedness is obtained by Danchin [22] (cf. Haspot [24]).

Proposition 1.1 (Local well-posedness of the Cauchy problem [22])**.** *Let $n\ge 2$, $1<p<2n$, $\mu>0$, $2\mu+n\lambda>0$. Suppose that the initial domain is the whole Euclean space $\mathbb{R}^n$ without boundary and the initial data satisfy*

$$\bar{\rho}_0\in\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n), \quad \bar{u}_0\in\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$$

with $\inf_x \rho_0(x) > -c$ *for some* $0 < c < 1$.
Then there exists $T > 0$ *and a unique solution* $(\tilde{\rho}, \tilde{u})$ *to* (1.8) *for* $I = (0, T)$ *such that*

$$\tilde{\rho} \in C_b([0,T); \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)) \cap \dot{W}^{1,1}(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)),$$
$$\tilde{u} \in C_b([0,T); \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)) \cap \dot{W}^{1,1}(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)), \cap L^1(I; \dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))$$

with the estimates:

$$\inf_{t\in I, x\in\mathbb{R}^n} \tilde{\rho}(t,x) > -1$$

and

$$\begin{aligned} &\|\partial_t \tilde{\rho}\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))} + \|\nabla\tilde{\rho}\|_{L^\infty(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))} \\ &\quad + \|\partial_t \tilde{u}\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))} + \|D^2\tilde{u}\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))} \le C, \end{aligned}$$

where $C = C(n, p, T) > 0$. *Besides the system* (1.8) *is well-posed in the above shown class.*

Our main result here is to extend Danchin's result (Proposition 1.1) to the initial boundary problem (1.8)–(1.9) in the perturbed half space Ω and establish the local well-posedness in the critical Besov space. To state our result, we reformulate the problem in the half space by the following procedure.

We first introduce an extension function of the boundary function $\eta_0(\tilde{x}')$ into the whole domain Ω.

Definition. Let $1 \le q < \infty$. For $\eta_0 \in \dot{B}^{1+\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})$, set

$$E(x', x_n) \equiv (\operatorname{sech}(x_n|\nabla'|)\eta_0(x'), \quad (x', x_n) \in \mathbb{R}^n_+$$

so that

$$(\nabla' E(x', x_n), \partial_{x_n} E(x', x_n)) = (\operatorname{sech}(x_n|\nabla'|)\nabla'\eta_0(x'), \operatorname{sech}(x_n|\nabla'|)|\nabla'|\eta_0(x')), \quad x_n > 0, \tag{1.10}$$

where the operator $\operatorname{sech}(x_n|\nabla'|)$ is given by the Fourier multiplier

$$\operatorname{sech}(x_n|\nabla'|)f \equiv \mathcal{F}^{-1}_{\xi'}[\operatorname{sech}(x_n|\xi'|)\widehat{f}(\xi')],$$

$\nabla' = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_{n-1}})$, and $\mathcal{F}^{-1}_{\xi'}$ denotes the Fourier inverse transform from $\xi' \in \mathbb{R}^{n-1} \to x' \in \mathbb{R}^{n-1}$. We introduce the domain deformation (flattening) transform:

$$\mathcal{E} : \tilde{x} \in \Omega \mapsto x \in \mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n; x' \in \mathbb{R}^{n-1}, \ x_n > 0\} \tag{1.11}$$

given by

$$\begin{cases} \tilde{x}' = x', \\ \tilde{x}_n = x_n + E(x', x_n). \end{cases} \tag{1.12}$$

The Jacobi matrix $J(DE) = \partial\tilde{x}/\partial x$ of (1.12) with its determinant $1 + \partial_{x_n}E$. Since $\partial_{x_n}E(x', x_n) = \operatorname{sech}(x_n|\nabla'|)|\nabla'|\eta_0(x')$, under the smallness condition on $|\nabla'|\eta_0$, $\partial_{x_n}E > -1$ everywhere and the deformation $\mathcal{E}$ is bijective. If we set $\phi(x_n) = x_n + E(\cdot, x_n)$, then $\partial_{x_n}\phi = 1 + \partial_{x_n}E$ and is strictly positive under the smallness condition for $|\nabla'|\eta_0$, it means that $\phi(x_n)$ is invertible and monotone increasing with respect to x_n. Noting that $\phi(0) = E(x',0) = \eta_0(x')$, we find that $\mathcal{E}$ maps the domain $\{(\tilde{x}', \tilde{x}_n); \tilde{x}_n > \eta_0(\tilde{x}')\}$ into

$$\Big\{(x', x_n);\ x_n + E_n(x', x_n) > \eta_0(x')\Big\} = \{(x', x_n);\ x_n > 0\},$$

and the boundary $\partial\Omega = \{(\tilde{x}', \tilde{x}_n) \in \mathbb{R}^n;\ \tilde{x}_n = \eta_0(\tilde{x}')\}$ is transformed into the new boundary $\partial\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n;\ x_n = 0\}$. The component of the transposed inverse of the Jacobi matrix is given by

$$(J(DE)^{-1})^{\mathsf{T}} = \begin{pmatrix} 1 & 0 & \cdots & -\frac{\partial_1 E}{1+\partial_n E} \\ 0 & 1 & \cdots & -\frac{\partial_2 E}{1+\partial_n E} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - \frac{\partial_n E}{1+\partial_n E} \end{pmatrix}, \tag{1.13}$$

where we have denoted $\partial_j = \partial_{x_j}$ $(j = 1, 2, \cdots, n)$. The covariant derivatives for a function $K(x) = \tilde{K}(\tilde{x})$ $(1 \le j, k \le n)$, i.e., $\nabla K = (\partial_1 K, \partial_2 K, \cdots \partial_n K)^{\mathsf{T}}$ and a vector field $F(x', x_n) = \tilde{F}(\tilde{x}', \tilde{x}_n) : \mathbb{R}^n_+ \to \mathbb{R}^n$, are expressed from (1.13) by

$$(\nabla_E K)_j \equiv (\nabla K)_j + \big((J(DE)^{-1} - I)^{\mathsf{T}} \nabla K\big)_j = \partial_j K - \frac{\partial_j E}{1 + \partial_n E}\,\partial_n K, \tag{1.14}$$

$$\operatorname{div}_E F \equiv \operatorname{div} F + \operatorname{tr}\Big((J(DE)^{-1} - I)^{\mathsf{T}} \nabla F\Big) = \operatorname{div} F - \sum_{j=1}^{n} \frac{\partial_j E}{1 + \partial_n E} \partial_n F_j. \tag{1.15}$$

We also denote by $(\partial_E K)_j$ and $D_E K$ the corresponding covariant derivatives and the Jacobi matrix form for any function K, respectively. If E is sufficiently smooth, then it follows from (1.15) that

$$(1 + \partial_n E)\operatorname{div}_E F = (1 + \partial_n E)\operatorname{div} F - \nabla E \cdot (\partial_n F)$$

and

$$\begin{aligned} \operatorname{div} F &= \nabla E \cdot (\partial_n F) - (\partial_n E)\operatorname{div} F + (1 + \partial_n E)\operatorname{div}_E F \\ &= \partial_n(\nabla E \cdot F) - \operatorname{div}\big((\partial_n E) F\big) + (1 + \partial_n E)\operatorname{div}_E F, \end{aligned} \tag{1.16}$$

where the first and the second terms of the right hand side of (1.16) maintain their divergence form.

We introduce the redefined unknown functions:

$$\begin{cases} u(t, x) \equiv \tilde{u}(t, \tilde{x}), \\ p(t, x) \equiv \tilde{p}(t, \tilde{x}), \end{cases}$$

the Jacobi matrix is denoted by

$$J(D_E u)_{1 \le \ell, m \le n} = \left[\delta_{\ell,m} + \int_0^t \Big(\partial_\ell u_m(s, x) - \frac{\partial_\ell E(x)}{1 + \partial_n E(x)} \partial_n u_\ell(s, x)\Big) ds\right]_{1 \le \ell, m \le n}. \tag{1.17}$$

We also notice that the relation corresponding to (1.5) appears:

$$|J(D_E u)(t)| = |J(D_E u)(0)| + \int_0^t |J(D_E u)(s)| \operatorname{tr}\,(J(D_E u(s))^{-1})^{\mathsf{T}} \nabla u(s)\, ds$$

$$= 1 + \int_0^t \mathrm{tr}\,(\mathrm{cof}\, J(D_E u))\nabla u\, ds. \tag{1.18}$$

Applying the boundary flattening operation $\mathcal{E}$ given by (1.11)–(1.12) to the problem (1.8), the system is transformed into the following problem on the flat boundary domain $\mathbb{R}^n_+$:

$$\begin{cases} \partial_t u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u = f(u,E) + F(\rho,u,E), & t>0, \quad x\in\mathbb{R}^n_+, \\ \qquad \rho = (1+\rho_0)|J(D_E u)(t)|^{-1} - 1, & t>0, \quad x\in\mathbb{R}^n_+, \\ \qquad u(t,x) = 0, & t>0, \quad x\in\partial\mathbb{R}^n_+, \\ \qquad u(0,x',x_n) = \bar{u}_0(x', x_n - E(x',x_n)) \equiv u_0(x), & \qquad\quad x\in\mathbb{R}^n_+, \end{cases} \tag{1.19}$$

where the linear variable coefficient terms are expressed by $f(u,E)$ (cf. (1.16)), and it is given by

$$\begin{aligned} f(u,E) \equiv{}& \mu\,(\mathrm{div}_E - \mathrm{div}\,)(\nabla_E u) + \mu\,\mathrm{div}\,(\nabla_E - \nabla)u \\ &+ (\mu+\lambda)(\nabla_E - \nabla)\mathrm{div}_E\, u + (\mu+\lambda)\nabla(\mathrm{div}_E\, u - \mathrm{div}\, u) \\ ={}& \mu\,\mathrm{div}\,((J(DE)^{-1} - I)\nabla_E u) + \mu \mathrm{div}\Big((J(DE)^{-1} - I)^{\mathsf{T}}\nabla u\Big) \\ &+ (\mu+\lambda)\Big((J(DE)^{-1} - I)^{\mathsf{T}}\nabla\mathrm{div}_E\, u\Big) + (\mu+\lambda)\nabla\mathrm{div}\,((J(DE)^{-1} - I)^{\mathsf{T}})u)\Big) \\ \equiv{}& -\mu\Big(\mathrm{div}\Big(\frac{\nabla E}{1+\partial_n E}\partial_n u\Big) + \frac{\nabla E}{1+\partial_n E}\cdot\nabla(\partial_n u) - \frac{\nabla E}{1+\partial_n E}\cdot\partial_n\Big(\frac{\nabla E}{1+\partial_n E}\partial_n u\Big)\Big) \\ &- (\mu+\lambda)\Big(\nabla\Big(\frac{\partial_n u\cdot\nabla E}{1+\partial_n E}\Big) + \frac{\nabla E}{1+\partial_n E}\,\partial_n(\mathrm{div}\, u) - \frac{\nabla E}{1+\partial_n E}\,\partial_n\Big(\frac{\partial_n u\cdot\nabla E}{1+\partial_n E}\Big)\Big). \end{aligned} \tag{1.20}$$

The nonlinear terms (1.9) are transformed into the following form:

$$\begin{aligned} F(\rho,u,E) \equiv{}& -(I - J(D_E u)^{-1})\nabla\cdot\Big(\mu((J(D_E u)^{-1})^{\mathsf{T}}\,\nabla u + (\nabla u)^{\mathsf{T}}\,(J(D_E u)^{-1})) \\ &\qquad\qquad + (\lambda\,\mathrm{tr}(J(D_E u)^{-1})^{\mathsf{T}}\nabla u - P(1+\rho))I\Big) \\ &- \nabla\cdot\Big(\mu((I - J(D_E u)^{-1})^{\mathsf{T}}\,\nabla u + (\nabla u)^{\mathsf{T}}\,(I - J(D_E u)^{-1})) \\ &\qquad\qquad + (\lambda\,\mathrm{tr}(J(D_E u)^{-1} - I)^{\mathsf{T}}\nabla u - P(1+\rho))I\Big) - \rho\partial_t u. \end{aligned} \tag{1.21}$$

The notations div_E and ∇_E are defined by (1.14), (1.15) respectively and $J(D_E u)^{-1}$ denotes the inverse of the Jacobi matrix shown in (1.17), and $J(DE)^{-1}$ is given by (1.13).

We introduce the homogeneous Besov space on the half-Euclidean space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ as the set of all measurable functions f in $\mathbb{R}^n_+$ satisfying

$$\|f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)} \equiv \inf\left\{ \|\tilde{f}\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)} < \infty; \quad \begin{aligned} &\tilde{f} = \begin{Bmatrix} f(x',x_n) & (x_n>0) \\ \text{a proper extension} & (x_n\le 0) \end{Bmatrix}, \\ &\tilde{f} = c_n\sum_{j\in\mathbb{Z}}\phi_j * \tilde{f} \text{ in } \mathcal{S}'(\mathbb{R}^n) \end{aligned} \right\} < \infty, \tag{1.22}$$

where $c_n = (2\pi)^{-n/2}$. The inhomogeneous version $B^s_{p,\sigma}(\mathbb{R}^n_+)$ is analogously defined. Let $\mathcal{D}'(\Omega)$ denote the distributions over Ω and let $C_b(I;X)$ be a set of all bounded continuous functions from an interval I to a Banach space X. We also use $C_v(\mathbb{R}^n_+)$ (or $C_v(\mathbb{R}^{n-1})$) as a set of all continuous functions vanishing at $|x|\to\infty$.

Theorem 1.2 (Local well-posedness of the half space problem)**.** *Let* $n \geq 2$*,* $\mu > 0$*,* $2\mu + n\lambda > 0$*,* $n-1 < p < 2n$*,* $1 \leq q \leq p(n-1)/(n-p)$ $(n-1 < p < n)$*, and* $1 \leq q < p(n-1)/(p-n)$ $(n \leq p < 2n)$*. Suppose that the boundary function satisfies* $\eta_0 \in \dot{B}^{1+\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})$*, and for some small* $\varepsilon_0 > 0$*,*

$$\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})} \leq \varepsilon_0. \tag{1.23}$$

The initial data satisfy

$$\rho_0 \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+), \quad u_0 \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$$

and for some $0 < c < 1$*,* $\inf_{x\in\mathbb{R}^n_+} \rho_0(x) > -c$*.*
Then there exists $T > 0$ *and a unique solution* (ρ, u) *to* (1.19)–(1.21) *for* $I = (0, T)$ *such that*

$$\begin{aligned}
&\rho \in C_b([0,T); \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)), \quad \nabla\rho \in L^\infty(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)),\\
&u \in C_b([0,T); \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)), \quad D^2u \in L^1(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))
\end{aligned}$$

with the estimates $\inf_{t\in I, x\in\mathbb{R}^n_+} \rho(t,x) > -1$ *and*

$$\begin{aligned}
&\|\partial_t\rho\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|\nabla\rho\|_{L^\infty(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\qquad + \|\partial_t u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \leq C,
\end{aligned}$$

where $C = C(n, p, T) > 0$*. Besides the system* (1.19)–(1.21) *is well-posed in the above shown class.*

We notice that the range of p is limited in $p < 2n$ since even the Cauchy problem of the compressible Navier-Stokes equations is ill-posed in the homogenenous Besov spaces for $p \geq 2n$ (see Chen-Miao-Zhang [28] and Iwabuchi-Ogawa [30]).

The problem (1.1) is now solvable in the corresponding critical space if we introduce the space of pull backs of functions, owing to the fact that the ordinary differential equation (1.2) is uniquely solvable as well as the boundary flattening operation (1.11) is bijective under the smallness condition on the boundary function η_0 in (1.23). Let $\mathcal{E}$ be the map defined by (1.11)–(1.12) from $\Omega \to \mathbb{R}^n_+$.

Corollary 1.3 (Local well-posedness of the original problem)**.** *Under the same assumptiion on the data and* η_0 *in Theorem 1.2, let* $\bar{\rho}_0 = 1 + \tilde{\rho}_0$ *and* $\bar{u}_0$ *be such that* $\rho_0 = \tilde{\rho}_0 \circ \mathcal{E}^{-1} \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$ *and* $u_0 = \bar{u}_0 \circ \mathcal{E}^{-1} \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$*. If* (ρ, u) *is the local solution of* (1.8) *obtained in Theorem 1.2, then* $(\bar{\rho}, \bar{u})$ *solves* (1.1) *uniquely, where* $\bar{\rho} = 1 + \tilde{\rho}$ *and* $(\tilde{\rho}, \bar{u})$ *is the pull-back of* (ρ, u) *by the flattening operation* $\mathcal{E}$ *and the Lagrange transform.*

In what follows, we use the following notation. For $f \in \mathcal{S}_0(\mathbb{R}^n)$, let the Fourier and the inverse Fourier transform be defined by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) \equiv c_n \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \check{f}(x) = \mathcal{F}^{-1}[f](x) \equiv c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi,$$

respectively, where $c_n = (2\pi)^{-n/2}$. For f and g on $\mathbb{R}^n$, let the convolution of $f * g$ defined by

$$f * g = \int_{\mathbb{R}^n} f(x-y)g(y)dy,$$

where $\tilde{f}$ and $\tilde{g}$ are 0-extention of f and g, respectively. For $\Omega \subset \mathbb{R}^n$, $C_0^\infty(\Omega)$ denotes a set of infinity diffrentiable compactly supported functions on Ω. For a Banach space X and $I = (0, T)$ for some $T > 0$, $C_b(I; X)$ denotes a set of all continuously bounded X-valuded function of $t \in I$. $\{\phi_j\}_{j\in\mathbb{Z}}$ and $\{\psi_k\}_{k\in\mathbb{Z}}$ are the Littlewood-Palay dyadic decomposition of unity in space and time variables, respectively. Let A_E denote the cofactor of Jacobi matrix $J(D_E u)$ and $A_E\nabla \cdot u$ stands for the trace of the product of matricies A_E and Du, i.e., $\mathrm{tr}(A_E Du)$.

2. Preliminary

2.1. The homogeneous Besov spaces on the half-space

We recall the summary for the Besov spaces over a domain Ω near the half-Euclidean space $\mathbb{R}^n_+$. Let $\ell_0 = \{\{a_k\}_k; k \in \mathbb{N},\ a_k \in \mathbb{R},\ \lim_{k\to\infty} |a_k| = 0,\ \|\{a_k\}_k\|_{\ell_0} = \max_k |a_k|\} \subsetneq \ell_\infty$. It is well known that $(\ell_0)^* \simeq \ell_1$.

Definition. Let $\sigma = 0$ with $s \in \mathbb{R}$. Let

$$\dot{\mathcal{B}}^s_{\infty,0}(\mathbb{R}^n_+) \equiv \overline{C_0^\infty(\mathbb{R}^n_+)}^{\dot{B}^s_{\infty,0}(\mathbb{R}^n)}, \qquad \text{where } \|f\|_{\dot{B}^s_{\infty,0}} \equiv \left\|\{2^{sj}\|\phi_j * f\|_\infty\}_j\right\|_{\ell_0}.$$

Definition. Let $1 \le p < \infty$ and $1 \le \sigma < \infty$ with $s \in \mathbb{R}$.

$$\overset{\circ}{\dot{B}}{}^s_{p,\sigma}(\mathbb{R}^n_+) \equiv \overline{C_0^\infty(\mathbb{R}^n_+)}^{\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)},$$

where $\overline{X}^{\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)}$ stands for the closure of X by the Besov norm $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ (see Bahouri–Chemin–Danchin [41] and Bergh–Löfström [38]). It is shown that the above defined space coincides the space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ defined by the restriction in (1.22). Namely, the following proposition is shown by Triebel [39] and Danchin–Mucha [31] (see also [1]).

Proposition 2.1 ([31], [39])**.** *Let* $1 \le p < \infty$. (1) *For* $0 \le s,\ 1 \le \sigma < \infty$,

$$\dot{B}^{-s}_{p',\sigma'}(\mathbb{R}^n_+) \simeq \left(\overset{\circ}{\dot{B}}{}^s_{p,\sigma}(\mathbb{R}^n_+)\right)^*,$$

$$\dot{B}^{-s}_{1,1}(\mathbb{R}^n_+) \simeq \left(\dot{\mathcal{B}}^s_{\infty,0}(\mathbb{R}^n_+)\right)^*.$$

(2) *For* $-\infty < s \le 1/p$ *and for* $1 < \sigma < \infty$,

$$\overset{\circ}{\dot{B}}{}^s_{p,\sigma}(\mathbb{R}^n_+) \simeq \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+).$$

(3) *For* $-\infty < s < 1/p$ *and* $\sigma = 1$,

$$\overset{\circ}{\dot{B}}{}^s_{p,1}(\mathbb{R}^n_+) \simeq \dot{B}^s_{p,1}(\mathbb{R}^n_+).$$

We consider the restriction operator R_0 by multiplying a cut-off function $\chi_{\mathbb{R}^n_+}(x) = 1$ over $\mathbb{R}^n_+$ and otherwise 0, i.e., for $f \in \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$, we set $R_0 f = \chi_{\mathbb{R}^n_+}(x) f(x)$ in $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ if $s > 0$ and it is understood in a distribution sense. Let E_0 be the zero extension operator from $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ to $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$. Using Proposition 2.1, the following statement is a variant introduced by Triebel [39, p. 228].

Proposition 2.2 ([39]). *Let $1 \le p < \infty$, $1 \le \sigma < \infty$, and $-1 + 1/p < s < 1/p$. It holds that*

$$R_0 : \dot{B}^s_{p,\sigma}(\mathbb{R}^n) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+),$$
$$E_0 : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$$

and they are linear bounded operators. Besides, it holds that

$$R_0 E_0 = Id : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+),$$

where Id denotes the identity operator. Namely E_0 and R_0 are a retraction and a co-retraction, respectively.

The proof of Proposition 2.2 is along the same line of the proof in [39] (cf. [1]). Furthermore, Triebel [39, Theorem 2.9.1] states that

Proposition 2.3 (cf. [39, 40]). *Let $1 \le p < \infty$, $s \in \mathbb{R}$, and $f \in \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+)$. Then $\nabla f \in \dot{B}^s_{p,1}(\mathbb{R}^n_+)$ and $\nabla f \in \overset{\circ}{B}{}^s_{p,1}(\mathbb{R}^n_+)$ if $s < 1/p$. Conversely if $\nabla f \in \dot{B}^s_{p,1}(\mathbb{R}^n_+)$, then $f \in \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+)$ if $-1 + 1/p < s \le -1 + n/p$.*

In what follows, we restrict ourselves to the regularity range of the Besov spaces $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ in $-1 + 1/p < s < 1/p$ for $1 < p < \infty$ unless otherwise stated. According to Proposition 2.2, we may regard that any distribution in $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ under such restriction on s and p can be extended into a distribution by the 0-extention over whole space $\mathbb{R}^n$ and conversely.

2.2. *The Littlewood-Paley decomposition with separation of variables*

In order to split the variables $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}_+$, we introduce an x'-parallel decomposition and an x_n-parallel decomposition of the Littlewood–Paley type. In what follows, $\xi_n \in \mathbb{R}_+$denotes the Fourier parameter corresponding to $x_n \in \mathbb{R}^n_+$ and $\xi' \in \mathbb{R}^{n-1}$ for $x' \in \mathbb{R}^{n-1}$, respectively. We introduce $\{\overline{\Phi_m}\}_{m\in\mathbb{Z}}$ as a Littlewood–Paley dyadic frequency decomposition of unity in separated variables (ξ', ξ_n).

Definition (The Littlewood–Paley decomposition of separated variables). Let $\{\phi_m\}_{m\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity in $\mathbb{R}^{n-1}$. For $m \in \mathbb{Z}$, let

$$\begin{aligned} &\widehat{\zeta_m}(\xi_n) = \begin{cases} 1 \quad , & 0 \le |\xi_n| \le 2^m, \\ \text{smooth}, & 2^m < |\xi_n| < 2^{m+1}, \\ 0 \quad , & 2^{m+1} \le |\xi_n|, \end{cases} \\ &\widehat{\zeta_m}(\xi_n) - \widehat{\zeta_{m-1}}(\xi_n) \equiv \widehat{\phi_m}(\xi_n) \end{aligned} \tag{2.2}$$

and set

$$\widehat{\overline{\Phi_m}}(\xi) \equiv \widehat{\phi_m}(|\xi'|) \otimes \widehat{\zeta_{m-1}}(\xi_n) + \widehat{\zeta_m}(|\xi'|) \otimes \widehat{\phi_m}(\xi_n). \tag{2.3}$$

Then it is obvious from Figure 1 (restricted on the upper half region in $\mathbb{R}^n$) that

$$\sum_{m\in\mathbb{Z}} \widehat{\overline{\Phi_m}}(\xi) \equiv 1, \quad \xi = (\xi', \xi_n) \in \mathbb{R}^n \setminus \{0\}.$$

Definition (Varieties of the Littlewood–Paley dyadic decompositions). Let $(\tau, \xi', \xi_n) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ be Fourier adjoint variables corresponding to $(t, x', \eta) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$.

- $\{\phi_m(x)\}_{m\in\mathbb{Z}}$: the standard (annulus type) Littlewood–Paley dyadic decomposition by $x = (x', \eta) \in \mathbb{R}^n$.
- $\{\overline{\Phi_m}(x)\}_{m\in\mathbb{Z}}$: the Littlewood–Paley dyadic decomposition over $x = (x', \eta) \in \mathbb{R}^n$ given by (2.3).
- $\{\psi_k(t)\}_{k\in\mathbb{Z}}$: the Littlewood–Paley dyadic decompositions in $t \in \mathbb{R}$.
- $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ and $\{\phi_j(\eta)\}_{j\in\mathbb{Z}}$: the standard (annulus type) Littlewood–Paley dyadic decompositions in $x' \in \mathbb{R}^{n-1}$ and $\eta \in \mathbb{R}$, respectively.
- $\{\zeta_m(x')\}_{m\in\mathbb{Z}}$ and $\{\zeta_m(\eta)\}_{m\in\mathbb{Z}}$: the lower frequency smooth cut-off given by (2.2), respectively.
- Let $\widetilde{\phi_j} = \phi_{j-1} + \phi_j + \phi_{j+1}$ be the Littlewood–Paley dyadic decompositions with its j-neighborhood to ϕ_j.
- All the above defined decompositions are even functions.

We see that the norm of the Besov spaces on $\mathbb{R}^n$ defined by $\{\phi_m\}_{m\in\mathbb{Z}}$ is equivalent to the one from the Littlewood–Paley decomposition of direct sum type (cf. [1]), $\{\overline{\Phi_m}\}_{m\in\mathbb{Z}}$ over $\mathbb{R}^n$, and hence one can identify those norms as it appears the homogeneous Besov space over $\mathbb{R}^n_+$ as follows. In what follows, we freely use the retraction and coretraction operators as observed above, and for simplicity, we avoid reprised usage of them.

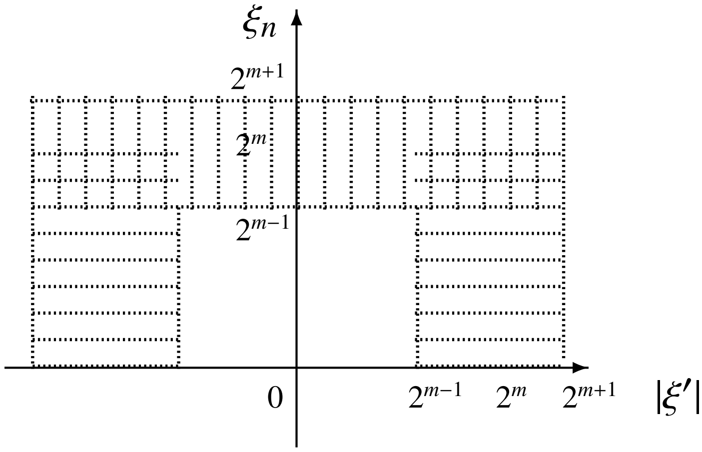

Figure 1. The support of Littlewood–Paley decomposition $\{\overline{\Phi_m}\}_{m\in\mathbb{Z}}$.

2.3. Bilinear estimates

We then give the useful bi-linear estimate which is used below. The following bilinear estimate is essentially obtained by Abidi-Paicu [42] in $\mathbb{R}^n$ and it can be extended into the half space $\mathbb{R}^n_+$ by Ogawa–Shimizu [35, 36].

Lemma 2.4 ([42], [36]). *Let $1 \le p, p_1, p_2, \sigma, \lambda_1, \lambda_2 \le \infty$, $1/p \le 1/p_1 + 1/p_2$, $p_1 \le \lambda_2$, $p_2 \le \lambda_1$, and*

$$\frac{1}{p} \le \frac{1}{p_1} + \frac{1}{\lambda_1} \le 1, \quad \frac{1}{p} \le \frac{1}{p_2} + \frac{1}{\lambda_2} \le 1,$$

and $s_1 + s_2 + n\inf(0, 1 - 1/p_1 - 1/p_2) > 0$ with $s_1 + n/\lambda_2 \le n/p_1$, $s_2 + n/\lambda_1 \le n/p_2$ (and hence $r = s_1 + s_2 - n(1/p_1 + 1/p_2 - 1/p) > -n + n/p$).

(1) *There exists $C > 0$ such that for any $f \in \dot{B}^{s_1}_{p_1,1}$ and $g \in \dot{B}^{s_2}_{p_2,1}$ the following estimate holds*

$$\|fg\|_{\dot{B}^r_{p,1}} \le C\|f\|_{\dot{B}^{s_1}_{p_1,1}}\|g\|_{\dot{B}^{s_2}_{p_2,1}}.$$

(2) *If* $1 \le p < \infty$, $1 \le q \le pn/(n-p)$ $(1 \le p < n)$, *and* $1 \le q < pn/(p-n)$ $(n \le p < \infty)$, *then for any* $f \in \dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)$ *and* $g \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$,

$$\|f\, g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} \le C\|f\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\|g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}. \tag{2.4}$$

(3) *In particular, for any* $f \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$ *and* $g \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$ *with* $1 \le p < 2n$, *the following estimate holds*

$$\|f\, g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} \le C\|f\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\|g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}.$$

See the possible range of exponents (p, q) to the estimate (2.4) in Figure 2 below. Hence regarding the region of possible choice of $1/q$, we see that

$$\begin{cases} \dfrac{1}{q} \ge \dfrac{1}{p} - \dfrac{1}{n}, & 1 < p < n, \\ \dfrac{1}{q} > -\dfrac{1}{p} + \dfrac{1}{n}, & n \le p < \infty. \end{cases}$$

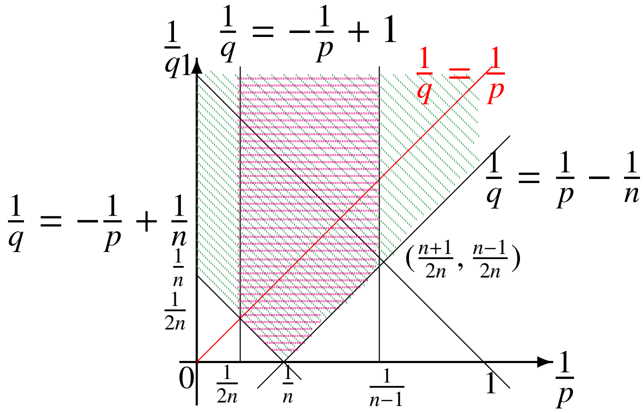

Figure 2. The possible range of exponents (p, q) in the bilinear estimate (2.4) (the red line for $p = q$).

3. Maximal L^1 regularity for the heat equation

We first recall endpoint maximal regularity for the initial boundary value problems to the heat equation. Let $I = (0, T)$ with $0 < T \le \infty$. Let $u = (u_1, u_2, \cdots, u_n)$ be a solution of the initial-boundary value problem of the heat equation with coefficients $\{a_{i,j}\}_{1 \le i,j \le n}$ and the inhomogeneous Dirichlet boundary condition in the half-space $\mathbb{R}^n_+ = \{x = (x', x_n); x' \in \mathbb{R}^{n-1}, x_n > 0\}$:

$$\begin{cases} \partial_t u - \sum_{1 \le i,j \le n} a_{ij}\partial_i\partial_j u = f, & t \in I, \quad x \in \mathbb{R}^n_+, \\ u(t, x', x_n)\big|_{x_n=0} = g(t, x'), & t \in I, \quad x' \in \mathbb{R}^{n-1}, \\ u(t, x)\big|_{t=0} = u_0(x), & x \in \mathbb{R}^n_+, \end{cases} \tag{3.1}$$

where $\partial_i \equiv \partial_{x_i}$ denotes the partial derivatives with respect to x_i $(1 \le i \le n)$, $u_0 = u_0(x)$, $f = f(t, x)$, and $g = g(t, x')$ are given initial, external force, and boundary data, respectively. The coefficient matrix

$\{a_{ij}(t,x)\}_{1\le i,j\le n}$ satisfies uniformly elliptic condition. Namely $\{a_{ij}\}$ is a real valued symmetric matrix such that for some constant $c>0$,

$$\sum_{1\le ij\le n} a_{ij}\xi_i\xi_j \ge c|\xi|^2 \tag{3.2}$$

for all $\xi\in\mathbb{R}^n$ with sufficient regularity in both t and x.

Maximal regularity for the initial boundary value problem of the parabolic type equation are well-estabilished in the results by Da Prato-Grisverd [43], Denk-Heiber-Prüss [44, 45], Dore [46], Dore-Venni [47], and Kunstmann-Weis [48]. Among others, Weis [49] obtained the necessary and sufficient condition for maximal regularity to the Cauchy problem of the abstruct Cauchy problem in the framework of UMD (unconditional martingale differences). Even non-UMD setting, Ogawa-Shimizu [50] showed maximal regulaity for the parabolic equation in non-reflexive Besov spaces. For 0-Dirichlet boundary data, Danchin–Mucha [31] obtained maximal L^1-regularity for the initial-boundary value problem (3.1) in the half-space with $a_{ij}(t,x)=\delta_{ij}$ and $g(t,x')\equiv 0$. Since the global estimate requires the base space for spatial variable x in the homogeneous Besov space, we introduce the homogeneous Lizorkin-Triebel-Bochner space over $\mathbb{R}_+\times\mathbb{R}^n_+$ (see for details Lizorkin [51], Peetre [52], Triebel [53], [39]).

Definition (The Lizorkin-Triebel-Bochner spaces). Let $s\in\mathbb{R}$, $1\le p,\sigma\le\infty$, and $X(\mathbb{R}^n_+)$ be a Banach space on $\mathbb{R}^n_+$ with the norm $\|\cdot\|_X$. Let $\{\psi_k\}_{k\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity for $t\in\mathbb{R}$. For $s\in\mathbb{R}$ and $1\le p<\infty$, $\dot F^s_{p,\sigma}(\mathbb{R};X)$ be the Lizorkin-Triebel-Bochner space with norm

$$\|\tilde f\|_{\dot F^s_{p,\sigma}(\mathbb{R};X)} \equiv \begin{cases} \Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{s\sigma k}\|\psi_k * \tilde f(t,\cdot)\|_X^\sigma\Big)^{1/\sigma}\Big\|_{L^p(\mathbb{R}_t)}, & 1\le\sigma<\infty,\\ \Big\|\sup_{k\in\mathbb{Z}} 2^{sk}\|\psi_k*\tilde f(t,\cdot)\|_X\Big\|_{L^p(\mathbb{R}_t)}, & \sigma=\infty. \end{cases}$$

Analogously above, we define the Lizorkin-Triebel-Bochner spaces $\dot F^s_{p,\sigma}(I;X)$ as the set of all measurable functions f on X satisfying

$$\|f\|_{\dot F^s_{p,\sigma}(I;X)} \equiv \inf\left\{\|\tilde f\|_{\dot F^s_{p,\sigma}(\mathbb{R};X)}<\infty;\ \tilde f=\begin{cases} f(t,x) & (t\in I)\\ \text{any extension} & (t\in\mathbb{R}\setminus I)\end{cases}\right\}.$$

We note that all the spaces of homogeneous type are understood as the Banach spaces by introducing the quotient spaces identifying all polynomial differences.

It is known that for $1\le q<\infty$, $\dot B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)$ satisfies $\mathcal{S}_0(\mathbb{R}^n)\hookrightarrow \dot B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)\hookrightarrow C_v(\mathbb{R}^n)$, where $\mathcal{S}_0$ denotes the rapidly decreasing smooth functions with vanishing at the origin of its Fourier transform and C_v denotes all continuous functions with vanishing at infinity, respectively (cf. [1, Proposition 2.3]).

Then end-point maximal regularity to the problem (3.1) is obtained by Danchin-Mucha [31] for $g\equiv 0$ case and Ogawa-Shimizu [1] for inhomogeneous boundary data.

Theorem 3.1 ([31], [1]). *Let $1<p<\infty$, $-1+1/p<s<1/p$, and assume that the coefficients $\{a_{ij}\}_{1\le i,j\le n}$ satisfy* (3.2). *Then the problem* (3.1) *admits a unique solution*

$$u\in C_b([0,\infty);\dot B^s_{p,1}(\mathbb{R}^n_+))\cap \dot W^{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+)),\quad \Delta u\in L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+)),$$

if and only if the external, initial, and boundary data in (3.1) *satisfy*

$$f\in L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+)),\quad u_0\in \dot B^s_{p,1}(\mathbb{R}^n_+),$$

$$g \in \dot{F}^{1-1/2p}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^{n-1})),$$

respectively. Besides, the solution u satisfies the following estimate for some constant $C_M > 0$ depending only on p, s, and n:

$$\begin{aligned}&\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\\ \le & C_M\big(\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|g\|_{\dot{F}^{1-1/2p}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|g\|_{L^1(\mathbb{R}_+;\dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^{n-1}))}\big).\end{aligned}$$

Remark. If $p = \infty$, the corresponding result holds for the homogeneous Besov space $\dot{\mathcal{B}}^s_{\infty,1}(\mathbb{R}^n) \equiv \overline{C^\infty_{00}(\mathbb{R}^n)}^{\dot{B}^s_{\infty,1}(\mathbb{R}^n)}$, where $C^\infty_{00}(\mathbb{R}^n)$ denotes all compactly supported C^∞-functions with zero at the origin of its Fourier transform and

$$\|f\|_{\dot{\mathcal{B}}^s_{\infty,1}(\mathbb{R}^n_+)} \equiv \inf\Big\{\|\widetilde{f}\|_{\dot{\mathcal{B}}^s_{\infty,1}(\mathbb{R}^n)} < \infty;\ \tilde{f} = \begin{cases} f(x', x_n) & (x_n > 0) \\ \text{any extension} & (x_n \le 0)\end{cases},\ \tilde{f} = \sum_{j\in\mathbb{Z}} \phi_j * \tilde{f} \text{ in } \mathcal{S}'\Big\},$$

instead of the Besov space $\dot{B}^s_{\infty,1}(\mathbb{R}^n_+)$ with imposing the compatibility condition if $(s, p) = (0, \infty)$. Note that $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n_+) \subset C_v(\mathbb{R}^n_+)$ for the endpoint case $(s, p) = (0, \infty)$.

We only show the estimate for $\mathbb{R}_+$ in time but a similar estimate for the finite time interval $I = (0, T)$ with $T < \infty$ is also available. In such a case, the restriction on the initial data u_0 can be relaxed into the class of inhomogeneous Besov spaces $B^s_{p,1}(\mathbb{R}^n_+) \supset \dot{B}^s_{p,1}(\mathbb{R}^n_+)$ and the constant appeared in the estimate can be estimated as $C_M \simeq O(\log T)$ $(T \to \infty)$ (see [50]).

4. Estimates for the boundary function

4.1. Estimate for the extension function of initial surface

First, we give an auxiliary estimate for the extension function given by the initial surface η_0.
First we show the estimate for the extension function E defined in (1.10).

Lemma 4.1. *Let $1 \le q < \infty$ and $\eta_0 \in \dot{B}^{1+(n-1)/q}_{q,1}(\mathbb{R}^{n-1})$. Then there exists a constant $C > 0$ such that*

$$\|\nabla E\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)} \le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}, \tag{4.1}$$

where ∇' denotes the gradient in $\mathbb{R}^{n-1}$ variables.

The above estimate is one of maximal regularity estimates for the half Laplacian heat semi-group in view of (1.10).
Proof of Lemma 4.1. Let us extend $\eta_0(x')$ into the whole space $\mathbb{R}^n$ by regarding $x_n \le 0$ as

$$\nabla\tilde{E}(x', x_n) = (\operatorname{sech}(x_n|\nabla'|)\nabla'\eta_0(x'),\ \operatorname{sech}(x_n|\nabla'|)|\nabla'|\eta_0(x')). \tag{4.2}$$

Then the above estimate can be proven by the restriction of the estimate for $\tilde{E}$. To see the estimate (4.1), we employ maximal trace regularity. Since $\nabla'\eta_0 \in \dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})$, $\nabla'\eta_0 = \sum_{m\in\mathbb{Z}} \phi_m * \nabla'\eta_0$ holds in

$\mathcal{S}'$, where ϕ_m denotes the Littlewood–Paley dyadic decomposition in $\mathbb{R}^{n-1}$ and it follows by the relation between the supports of the Fourier images of $\overline{\Phi_j}$ and ϕ_m that

$$\begin{aligned}\|\nabla\tilde{E}\|_{\dot{B}^{\frac{n}{q}}_{q,1}} &= \sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\Big\|\overline{\Phi_j} \underset{(x',x_n)}{*} (\mathrm{sech}\,(x_n|\nabla'|) \sum_{|m-j|\le 1} \phi_m \underset{(x')}{*} (\nabla', |\nabla'|)\eta_0)\Big\|_q \\ &= \sum_{j\in\mathbb{Z}} \sum_{|j-m|\le 1} 2^{\frac{n}{q}j}\Big\|\zeta_{j-1} \underset{(x_n)}{*} \mathrm{sech}\,(x_n|\nabla'|)(\phi_m \underset{(x')}{*} \phi_j \underset{(x')}{*} (\nabla', |\nabla'|)\eta_0)\Big\|_q \\ &\quad + \sum_{j\in\mathbb{Z}} \sum_{|j-m|\le 1} 2^{\frac{n}{q}j}\Big\|\phi_j \underset{(x_n)}{*} (\mathrm{sech}\,(x_n|\nabla'|)(\phi_m \underset{(x')}{*} \zeta_j \underset{(x')}{*} (\nabla', |\nabla'|)\eta_0))\Big\|_q \\ &\equiv I + II. \end{aligned} \tag{4.3}$$

Then for the ℓ-th component of the first term (written by I_ℓ) of the right hand side of (4.3) can be seen for all $\ell = 1, 2 \ldots, n-1$ that

$$\begin{aligned} I_\ell &\le 2\sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\|\zeta_{j-1}\|_{L^1_{x_n}(\mathbb{R}_+)}\Big(\int_{\mathbb{R}} \|\mathrm{sech}\,(x_n|\nabla'|)(\phi_j \underset{(x')}{*} \partial_\ell\eta_0)\|^q_{L^q(\mathbb{R}^{n-1})} dx_n\Big)^{1/q} \\ &\le C\sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\Big(\int_{\mathbb{R}} \|\mathrm{sech}\,(x_n|\nabla'|)\widetilde{\phi}_j\|^q_{L^1(\mathbb{R}^{n-1})}\|\phi_j \underset{(x')}{*} \partial_\ell\eta_0\|^q_{L^q(\mathbb{R}^{n-1})} dx_n\Big)^{1/q} \\ &\le C\sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}\Big(\int_{\mathbb{R}} e^{-2^j q|x_n|} dx_n\Big)^{1/q}\|\phi_j \underset{(x')}{*} \partial_\ell\eta_0\|_{L^q(\mathbb{R}^{n-1})} \\ &\le C\sum_{j\in\mathbb{Z}} 2^{\frac{n}{q}j}2^{-\frac{1}{q}j}\|\phi_j \underset{(x')}{*} \partial_\ell\eta_0\|_{L^q(\mathbb{R}^{n-1})} = C\|\partial_\ell\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}, \end{aligned}$$

where we set $\widetilde{\phi_j} = \phi_{j-1} + \phi_j + \phi_{j+1}$. The estimate for the second term II is along the similar way. Finally, we confirm that

$$\nabla\tilde{E}(x', x_n) = c_n^{-1}\sum_{j\in\mathbb{Z}} \phi_j \underset{(x',x_n)}{*} \nabla\tilde{E}(x', x_n) \quad \text{in } \mathcal{S}',$$

which is justified by [50, Proposition 2.1]. ❑

To show the estimates for the linear variable coefficient terms, we prepare the following basic lemma.

Lemma 4.2. *For* $1 \le q < \infty$, *let* $E(x, x_n)$ *is given by* (1.10) *and assume that for some small* $\varepsilon_0 > 0$*:*

$$\nabla'\eta_0 \in \dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1}). \tag{4.4}$$

Then there exists a constant $C > 0$ *such that*

$$\left.\begin{array}{r} \Big\|\dfrac{\nabla E}{1+\partial_n E}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)} \\ \Big\|\dfrac{\nabla' E}{\sqrt{1+|\nabla' E|^2}}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)} \\ \Big\|\dfrac{\sqrt{1+|\nabla' E|^2}-1}{\sqrt{1+|\nabla' E|^2}}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)} \end{array}\right\} \le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})},$$

where $\nabla' = (\partial_1, \partial_2, \cdots, \partial_{n-1})^{\mathsf{T}}$.

Proof of Lemma 4.2. We employ the extension function $\tilde{E}$ of E defined by (4.2). Each of the left hand side is expressed by a composite function of ∇E or $\nabla' E$ with some function $F(\cdot)$: $\mathbb{R}^m \to \mathbb{R}^m$, where $m = 1$, $n-1$, or n, where $F \in W^{n,\infty}(\mathbb{R}^m)$ with $F(0) = 0$, then the estimate for the composite function over $\mathbb{R}^n_+$ can be estimated by the extended function $F(\nabla\tilde{E})$ over $\mathbb{R}^n$, and it can be estimated by the homogeneous Besov spaces in whole space (see, e.g., [41]). For instance, the first case can be treated by setting

$$F(x) = \frac{x}{1 + x_n}$$

for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then $F^{(k)} \in L^\infty$ for all $0 \le k \le n+1$ and $F(0) = 0$. Hence we see by the definition of the Besov space on $\mathbb{R}^n_+$ that for $k = [n/q] + 1$,

$$\begin{aligned}\Big\|\frac{\nabla E}{1+\partial_n E}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)} &\le \Big\|\frac{\nabla \tilde{E}}{1+\partial_n \tilde{E}}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)} \\ &\le C\sum_{k=0}^{k}\|F^{(k)}\|_\infty\|\nabla\tilde{E}\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)} \le C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\end{aligned}$$

(cf. the proof of Lemma 4.1). The second and third estimates follow in a similar way. ❑

5. The estimates for the density

In this section, we show regularities of the density ρ by using the maximal regularity result for the velocity u.

Proposition 5.1. *Let $I = (0, T)$ for $T > 0$ and let $\rho_0 \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$, $\eta_0 \in \dot{B}^{1+\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})$. Assume that η_0 satisfies* (4.4) *and*

$$u \in C([0,T); \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap L^1(I; \dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))$$

and $|I| = T$ be small so that $\|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))} \le \varepsilon < 1$ for some $\varepsilon > 0$. Then the density $\rho = \rho(t,x)$ given by (1.7)*, i.e.,*

$$\rho(t,x) = (1 + \rho_0(x))\,|J(D_E u(t,x))|^{-1} - 1, \quad t \in I, x \in \mathbb{R}^n_+,$$

satisfies

$$\rho \in C([0,T); \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)).$$

Besides, the following estimates hold:

$$\|\rho\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C\Big(\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big), \tag{5.1}$$

$$\|\partial_t\rho\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C\Big(1 + \|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\Big)\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}, \tag{5.2}$$

where $C > 0$ is depending only on ε and n.

To prove Proposition 5.1, we prepare the following lemma.

Lemma 5.2. *Let η_0 satisfies* (4.4)*, and $u \in L^1(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))$, $\eta_0 \in \dot{B}^{1+\frac{n}{q}}_{q,1}(\mathbb{R}^{n-1})$, and $T > 0$ be small enought such that*

$$\int_0^T \|D^2 u(s)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} \, ds \ \le \varepsilon$$

for small $\varepsilon < 1$. Let $\operatorname{cof} J(D_E u)$ denote the cofactor matrix of $J(D_E u)$, and $(\operatorname{cof} J(D_E u)\nabla) \cdot u$ stands for $\sum_{j,k=1}^n \operatorname{cof} J(D_E u)_{jk} \partial_k u_j$. Then there exists constants $C = C(\varepsilon, n)$ such that the following estimates hold:

$$\Big\| \int_0^t (\operatorname{cof} J(D_E u)\nabla) \cdot u(s)\, ds \Big\|_{L^\infty(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C(1 + \|\nabla' \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}) \|D^2 u\|_{L^1(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}, \tag{5.3}$$

$$\Big\| \int_0^t (\operatorname{cof} J(D_E u)\nabla) \cdot u(s)\, ds \Big\|_{L^\infty(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C(1 + \|\nabla' \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}) \|D^2 u\|_{L^1(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}, \tag{5.4}$$

$$\Big\| (I - J(D_E u)^{-1})^{\mathsf{T}} \Big\|_{L^\infty(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C(1 + \|\nabla' \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}) \|D^2 u\|_{L^1(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}, \tag{5.5}$$

$$\Big\| \nabla (J(D_E u)^{-1})^{\mathsf{T}} \Big\|_{L^\infty(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C(1 + \|\nabla' \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}) \|D^2 u\|_{L^1(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))},$$

where $J(D_E u)^{-1}$ denotes the inverse of Jacobi matrix defined in (1.17).

Proof of Lemma 5.2. We only prove the case (5.3) and (5.5) since the other cases are proven by a similar way. We set the cofactor of the Jacobi matrix $A_E \equiv \operatorname{cof}\ J(D_E u)$, i.e.,

$$A_E \equiv \operatorname{cof} J(D_E u) = I + \sum_{k=1}^{n-1} C_k \prod_{\ell,m \le n}^{k} \Big(\int_0^t \Big(\frac{\partial u_m}{\partial x_\ell}(s) - \frac{\partial_\ell E(x)}{1 + \partial_n E(x)} \frac{\partial u_\ell}{\partial x_n}(s) \Big) ds \Big). \tag{5.6}$$

Since $\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$ and $\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)$ for $1 \le p, q \le \infty$ are the Banach algebra and (1.17), we have from (5.6) that

$$\begin{aligned}
&\Big\| \int_0^t (A_E \nabla) \cdot u(s)\, ds \Big\|_{L^\infty(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))} \\
&\le \Big\| \sum_{\ell=0}^{n-1} C_\ell \prod_{k,j \le n}^{\ell} \Big(\int_0^t \Big(\frac{\partial u_j}{\partial x_k}(s) - \frac{\partial_k E(x)}{1 + \partial_n E(x)} \partial_n u_j(s,x) \Big) ds \Big) \nabla \cdot u \Big\|_{L^1(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))} \\
&\le C \sum_{k=0}^{n-2} \Big(1 + \Big\| \frac{\partial_k E(x)}{1 + \partial_n E(x)} \Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}} \Big) \\
&\qquad \times \Big\| \int_0^t Du\, ds \Big\|_{L^\infty(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \Big\| \prod_{\ell.m \le n}^{\ell} \Big(\int_0^t \frac{\partial u_m}{\partial x_\ell}(s)\, ds \Big) \nabla \cdot u \Big\|_{L^1(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\le C \sum_{k=0}^{n-1} (1 + \|\nabla' \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}) \sup_{0<t<T} \Big\| \int_0^t Du\, ds \Big\|^k_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)} \|Du\|_{L^1(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))} \\
&\le C(1 + \|\nabla' \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}) \sum_{k=1}^{n-1} \|D^2 u\|^k_{L^1(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}.
\end{aligned}$$

By the assumption $\int_0^T \|D^2u(s)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)}\,ds \le \varepsilon < 1$, it holds that

$$
\begin{aligned}
&\Big\| \int_0^t (A_E\nabla)\cdot u(s)\,ds\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\le C(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\sum_{k=1}^{n-1}\varepsilon^{k-1}\|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\le \frac{C}{1-\varepsilon}(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))},
\end{aligned}
$$

which shows (5.3). The estiamte (5.4) can be shown in a similar way. By using the expression (1.18) and (5.6), it holds that

$$
\begin{aligned}
(I-J(D_Eu)^{-1})^{\mathsf{T}} &= \frac{1}{|J(D_Eu)|}\Big(|J(D_Eu)|I - A_E\Big)\\
&= \frac{1}{|J(D_Eu)|}\Big(\Big(\int_0^t \mathrm{tr}A_E\nabla u(s)\,ds\Big)I\\
&\qquad -\sum_{k=1}^{n-1}C_k\prod_{\ell,m\le n}^{k}\Big(\int_0^t\Big(\frac{\partial u_m}{\partial x_\ell}(s)-\frac{\partial_\ell E(x)}{1+\partial_n E(x)}\partial_n u_\ell(s,x)\Big)\,ds\Big)\Big).
\end{aligned}
\tag{5.7}
$$

The first term in the numerator of the right hand side of (5.7) can be estiamted by

$$
\begin{aligned}
&\Big\|\int_0^t A_E\nabla\cdot u(s)\,ds\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\le \sum_{k=1}^{n}\Big\|\prod_{\ell,m\le n}^{k}\Big(\int_0^t\Big(\frac{\partial u_m}{\partial x_\ell}(s)-\frac{\partial_\ell E(x)}{1+\partial_n E(x)}\partial_n u_\ell(s,x)\Big)\,ds\Big)\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&= \sum_{k=1}^{n}\Big(\Big(1+\Big\|\frac{\partial_k E(x)}{1+\partial_n E(x)}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}}\Big)\Big\|\int_0^t\nabla u(s)\,ds\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\Big)^k\\
&\le C(\varepsilon,n)(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}})\|D^2u\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}.
\end{aligned}
$$

The seconter term of the numerator in (5.7) is also estimated by a very similar way, while by using (1.18), the denominator can be seen as

$$
\begin{aligned}
\Big\|\frac{1}{|J(D_Eu)|}\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))} &\le \Big\|\sum_{k=0}^{\infty}(-1)^{k-1}\Big(\int_0^t \mathrm{tr}A_E\nabla u\,ds\Big)^k\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\le C(\varepsilon,n)(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})(1+\|D^2u\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}).
\end{aligned}
\tag{5.8}
$$

Combining (5.7)-(5.8), we obtain (5.5). ❑

Proof of Proposition 5.1.

$$
\rho(t,x) = \frac{1+\rho_0}{|J(D_Eu(t,x))|} - 1 = \frac{\rho_0}{|J(D_Eu(t,x))|} - \frac{\int_0^t (\mathrm{cof}\,J(D_Eu)\nabla)\cdot u(s)\,ds}{|J(D_Eu(t,x))|}.
$$

Combining

$$\Big\|\frac{\rho_0}{|J(D_E u)|}\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))} \le \Big\|\rho_0\sum_{k=0}^{\infty}(-1)^{k-1}\Big(\int_0^t (A_E\nabla)\cdot u\,ds\Big)^k\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}$$
$$\le \|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\sum_{k=0}^{\infty}\Big\|\int_0^t (A_E\nabla)\cdot u\,ds\Big\|^k_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}$$
$$\le C(\varepsilon,n)\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})(1+\|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))})$$

with (5.8), where we use the notation (5.6), we obtain (5.1). Since

$$\partial_t\rho(t,x) = -(1+\rho_0)\frac{(A_E\nabla)\cdot u}{(1+\int_0^t (A_E\nabla)\cdot u(s)\,ds)^2},$$

we have

$$\Big\|\frac{(A_E\nabla)\cdot u}{(1+\int_0^t (A\nabla)\cdot u(s)\,ds)^2}\Big\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}$$
$$\le \Big\|((A_E\nabla)\cdot u)\sum_{k=0}^{\infty}(-1)^k\Big\{2\int_0^t (A_E\nabla)\cdot u\,ds + (\int_0^t (A_E\nabla)\cdot u\,ds)^2\Big\}^k\Big\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}$$
$$\le \|(A_E\nabla)\cdot u\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\sum_{k=0}^{\infty}\Big\|\Big\{2\int_0^t (A_E\nabla)\cdot u\,ds + (\int_0^t (A_E\nabla)\cdot u\,ds)^2\Big\}^k\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}$$
$$\le C(\varepsilon,n)\,(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|D^2u\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))},$$

and hence

$$\big\|\partial_t\rho\big\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))} = \Big\|(1+\rho_0)\frac{(A_E\nabla)\cdot u}{(1+\int_0^t (A_E\nabla)\cdot u(s)\,ds)^2}\Big\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}$$
$$\le (1+\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))})\Big\|\frac{(A_E\nabla)\cdot u}{(1+\int_0^t (A_E\nabla)\cdot u(s)\,ds)^2}\Big\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}$$
$$\le C(\varepsilon,n)\,(1+\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))})(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|D^2u\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))},$$

which shows (5.2). Since

$$\nabla\rho(t,x) = \frac{\nabla\rho_0(x)}{|J(D_Eu(t,x))|} - (1+\rho_0(x))\frac{\int_0^t((\nabla A_E)\nabla)\cdot u(s)\,ds + \int_0^t (A_E\nabla)\cdot\nabla u(s)\,ds}{|J(D_Eu(t,x))|^2},$$

by using (5.3), we have

$$\Big\|\frac{\nabla\rho_0}{|J(Du)|}\Big\|_{L^\infty(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))} \le \|\nabla\rho_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\sum_{k=0}^{\infty}\Big\|\int_0^t (A_E\nabla)\cdot u\,ds\Big\|^k_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}$$
$$\le C(\varepsilon,n)\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}(1+\|D^2u\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}).$$

Combining the estimate

$$\begin{aligned}
&\Big\|\frac{\int_0^t((\nabla A_E)\nabla)\cdot u\,ds}{|J(Du(t,x))|^2}\Big\|_{L^\infty(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\le \Big\|C(n)\int_0^t\Big(\int_0^t \nabla D_E u\,ds\sum_{k=1}^{n-2}\big(\int_0^t Du\,ds\big)^k\Big)\nabla u\,ds\\
&\qquad\times\sum_{k=0}^{\infty}(-1)^k\Big(1+2\int_0^t(A_E\nabla)\cdot u\,ds+\Big(\int_0^t(A_E\nabla)\cdot u\,ds\Big)^2\Big)^k\Big\|_{L^\infty(I;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\le C(\varepsilon,n)(1+\|\nabla'\eta_0\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\Big\|\int_0^t D^2u\,ds\Big\|_{L^\infty(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\sum_{k=1}^{n-1}\Big\|\int_0^t Du\,ds\Big\|^k_{L^\infty(I;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\qquad\times(1+\|\nabla'\eta_0\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|\nabla u\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\Big(1+\|D^2u\|_{L^1(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\Big)\\
&\le C(\varepsilon,n)(1+\|\nabla'\eta_0\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|D^2u\|_{L^1(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))},
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\frac{\int_0^t(A_E\nabla)\cdot\nabla u(s)\,ds}{|J(Du(t,x))|^2}\Big\|_{L^\infty(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\le\Big\|\int_0^t \mathrm{tr}A_E D^2u(s)\,ds\\
&\qquad\times\sum_{k=0}^{\infty}(-1)^k\Big\{2\int_0^t(A_E\nabla)\cdot u\,ds+\big(\int_0^t(A_E\nabla)\cdot u\,ds\big)^2\Big\}^k\Big\|_{L^\infty(I;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\le C(\varepsilon,n)\Big\|\Big(I+\prod^{n-1}\big(\int_0^t Du\,ds\big)\Big)D^2u\Big\|_{L^1(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\Big(1+\|D^2u\|_{L^1(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\Big)\\
&\le C(\varepsilon,n)\Big(1+\Big\|\prod^{n-1}\big(\int_0^t Du\,ds\big)\Big\|_{L^\infty(I;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\Big)\|D^2u\|_{L^1(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\qquad\times\Big(1+\|D^2u\|_{L^1(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\Big)\\
&\le C(\varepsilon,n)(1+\|\nabla'\eta_0\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|D^2u\|_{L^1(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))},
\end{aligned}$$

we have

$$\begin{aligned}
&\Big\|\rho_0\frac{\int_0^t((\nabla A_E)\nabla)\cdot u(s)\,ds+\int_0^t(A_E\nabla)\cdot\nabla u(s)\,ds}{|J(Du(t,x))|^2}\Big\|_{L^\infty(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))}\\
&\le C(\varepsilon,n)(1+\|\nabla'\eta_0\|_{\dot B^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|\rho_0\|_{\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)}\|D^2u\|_{L^1(I;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))},
\end{aligned}$$

which completes the proof of Proposition 5.1. ❑

6. Estimates for perturbation terms

6.1. Estimate for the linear terms

We now consider the estimates for the linear variable coefficient terms defined in (1.20). All the estimate is based on the bilinear estimate in the homogeneous Besov space Lemma 2.4.

Proposition 6.1 (Estimates for linear variable coefficient terms)**.** *Let $n \geq 2$ and $1 \leq p < 2n$. For $u \in C(\bar{I}; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$, $\partial_t u$, $D^2 u$, $\nabla p \in L^1(I; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$, and E defined in* (1.10)*, let $f(\rho, u, E)$ be the terms defined in* (1.20). *Under the assumption $\|\nabla'\eta_0\|_{\dot{B}^{(n-1)/q}_{q,1}(\mathbb{R}^{n-1})}$ is small enough, the following estimates hold: For $1 \leq q < \frac{pn}{|p-n|}$,*

$$\|f(\rho,u,E)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \leq C\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\Big(\|D^2u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big). \tag{6.1}$$

Proof of Proposition 6.1. Recalling the definition of $f(\rho, u, E)$ and the covariant derivative (1.14), and applying Lemma 4.2, we find the estimate (6.1) follows by

$$\begin{aligned}
&\|f(\rho,u,E)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\leq \Big\|\operatorname{div}\Big(\frac{\nabla E}{1+\partial_n E}\partial_n u\Big)\Big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \Big\|\frac{\nabla E}{1+\partial_n E}\cdot\partial_n(\nabla_E u)\Big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\quad + \Big\|\frac{\nabla E}{1+\partial_n E}\partial_n p\Big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\leq C\Big\|\frac{\nabla E}{1+\partial_n E}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\Big(\big\|\partial_n u\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \big\|\nabla_E u\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \big\|\partial_n p\big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big) \\
&\leq C\Big\|\frac{\nabla E}{1+\partial_n E}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\Big(\big\|D^2 u\big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad + \Big\|\frac{\nabla E}{1+\partial_n E}\Big\|_{\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n_+)}\big\|D^2 u\big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \big\|\partial_n p\big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big) \\
&\leq C\big\|\nabla'\eta_0\big\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})}\Big(\big\|D^2 u\big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \big\|\nabla p\big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big),
\end{aligned}$$

where we apply Lemma 4.1 and notice that $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)$ is the Banach algebra and no restriction on the exponents p nor q. ❑

6.2. Estimates for the nonlinear terms

In this section, we treat the nonlinear terms defined in (1.8). To estimate the nonlinear term $F(\rho, u)$, we divide it into three parts:

$$F(\rho,u,E) \equiv F_u(u,E) + F_\rho(\rho,u,E) - \rho\partial_t u, \tag{6.2}$$

where

$$F_u(u,E) \equiv -\big(I - J(D_E u)^{-1}\big)^{\mathsf{T}}\nabla\cdot\Big(\mu\big((J(D_E u)^{-1})^{\mathsf{T}}\,\nabla u + (\nabla u)^{\mathsf{T}}\,J(D_E u)^{-1}\big)$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad + \lambda\, \mathrm{tr}(J(D_E u)^{-1})^{\mathsf{T}} \nabla u\, I\Big) \\
&\qquad + \nabla \cdot \Big(\mu((I - J(D_E u)^{-1})^{\mathsf{T}}\, \nabla u + (\nabla u)^{\mathsf{T}}\, (I - J(D_E u)^{-1})) \\
&\qquad\qquad\qquad + \lambda\, \mathrm{tr}(J(D_E u)^{-1} - I)^{\mathsf{T}} \nabla u\, I\Big), \\
F_\rho(\rho, u, E) &\equiv -\, \mathrm{tr}\, (J(D_E u)^{-1})^{\mathsf{T}} \nabla P(1 + \rho).
\end{aligned}
$$

Proposition 6.2 (Nonlinear estimate for $F_u(u, E)$ and $F_\rho(\rho, u, E)$)**.** *Let $n \geq 2$, $1 \leq p < 2n$, $I = (0, T)$ for $T > 0$. Let $u \in C([0, T); \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))$, $D^2 u \in L^1(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))$ ρ be defined by* (1.7) *with $\rho_0 \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$. We choose $T > 0$ such that $\|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \leq \varepsilon < 1$. Then the following estimates hold:*

$$
\begin{aligned}
&\|F_u(u, E)\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad \leq C(1 + \|\nabla' \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}, \\
&\|F_\rho(\rho, u, E)\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad \leq C(1 + \|\nabla' \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + C(1 + \|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)})\|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))},
\end{aligned}
$$

where $C > 0$ is depending only on ε and n.

Proof of Proposition 6.2. By Lemma 5.2, we have

$$
\begin{aligned}
&\|(I - J(D_E u)^{-1})^{\mathsf{T}} \nabla \cdot (J(D_E u)^{-1})^{\mathsf{T}}\, \nabla u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\quad \leq \|(I - J(D_E u)^{-1})^{\mathsf{T}}\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad\quad \times \Big(\|(\nabla(J(D_E u)^{-1})^{\mathsf{T}})\, \nabla u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|(J(D_E u)^{-1})^{\mathsf{T}}\, D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big) \\
&\quad \leq \|(I - J(D_E u)^{-1})^{\mathsf{T}}\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad\quad \times \Big(\|(\nabla(J(D_E u)^{-1})^{\mathsf{T}})\|_{L^\infty(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\nabla u\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad\qquad\qquad + \|(J(D_E u)^{-1})^{\mathsf{T}}\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big) \\
&\quad \leq C(1 + \|\nabla' \eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))},
\end{aligned}
$$

where $C > 0$ is depending only on ε and n. Other terms of $F_u(u, E)$ are estimated in a similar way.

For some $\theta \in (0, 1)$,

$$
P(1 + \rho) = P(1) + P'(1 + \theta\rho)\rho,
$$

we have $\nabla P(1 + \rho) = P'(1 + \theta\rho)\nabla\rho + P''(1 + \theta\rho)\theta\rho\nabla\rho$.

$$
\|(J(D_E u)^{-1})^{\mathsf{T}} \nabla P(1 + \rho)\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}
$$

$$\leq \Big\|\frac{1}{|J(D_E u)|}\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big(1+\Big\|\prod^{n-1}\Big(\int_0^t D_E u\,ds\Big)\Big\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)\|\nabla\rho\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}$$
$$\leq C\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + C(1+\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)})(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.$$

❑

Proposition 6.3 (Nonlinear estimate for $\rho\partial_t u$). *Let $n \geq 2$, $1 \leq p < 2n$, $I = (0,T)$ for $T > 0$. Let $u \in C([0,T);\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))$, $D^2u \in L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))$, ρ be defined by (1.7) with $\rho_0 \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$. We choose $T > 0$ such that $\|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \leq \varepsilon < 1$. We choose $T > 0$ such that $\|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \leq \varepsilon < 1$. Then the following estimate holds:*

$$\|\rho\partial_t u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \leq C\Big(\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + \|D^2u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)\|\partial_t u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}, \tag{6.5}$$

where $C > 0$ is depending only on ε and n.

Proof of Proposition 6.3. Substituting (5.1) to

$$\|\rho\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \leq \|\rho\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))},$$

we obtain (6.5). ❑

7. Proof of the local well-posedness

Proof of Theorem 1.2. For the given initial data $\rho_0 \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$ and $u_0 \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)$, we choose

$$M_0 = 4C_0\left(\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + \|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\right).$$

We define the complete metric space

$$\begin{aligned} X_T = \Big\{&\rho \in C_b([0,T);\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)), \nabla\rho \in L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)),\\ &u \in C_b([0,T);\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)), \quad D^2u \in L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)),\\ &\|(\rho,u)\|_{X_T} = \|(\rho,u)\|_{C_b} + \|(\rho,u)\|_T, \quad \|(\rho,u)\|_{C_b} \leq M_0, \quad \|(\rho,u)\|_T \leq M\Big\},\end{aligned}$$

where

$$\begin{aligned}\|(\rho,u)\|_{C_b} &= \sup_{t\in I}\|\rho(t)\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + \sup_{t\in I}\|u(t)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)},\\ \|(\rho,u)\|_T &= \|\partial_t\rho\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|\nabla\rho\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\end{aligned}$$

$$+ \|\partial_t u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.$$

Given $(\tilde{\rho}, \tilde{u}) \in X_T$, we consider the linear inhomogeneous initial-boundary value problem

$$\begin{cases} \partial_t u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u = f(\tilde{u}, E) + F_u(\tilde{u}, E) + F_\rho(\tilde{\rho}, \tilde{u}, E) + \tilde{u}\partial_t\tilde{\rho}, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \quad u(t, x', 0) = 0, & t > 0, \quad x' \in \mathbb{R}^{n-1}, \\ \rho = (1+\rho_0)|J(D_E\tilde{u})|^{-1} - 1, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \rho(0, x) = \rho_0(x), \quad u(0, x) = u_0(x), & \quad x \in \mathbb{R}^n_+, \end{cases} \tag{7.1}$$

where the nonlinear terms are defined by (6.2). We then define the map

$$\Phi : X_T \to X_T$$

by

$$(\tilde{\rho}, \tilde{u}) \mapsto (\rho, u) = \big(\Phi_\rho(\tilde{\rho}, \tilde{u}), \Phi_u(\tilde{\rho}, \tilde{u})\big)$$

and prove that Φ is contraction on X_T.

First we show that Φ is a self-map in X_T. Let (ρ, u) solve (7.1). Applying Theorem 1.2 to the equations (7.1), and using Proposition 6.1 to the purterbed term and Propositions 6.2, 6.3 to the nonlinear terms, we have

$$\begin{aligned} \|\Phi_u(\tilde{\rho}, \tilde{u})\|_{X_T} \le C\Big(&\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + (1 + \|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}) \\ &\times \sum_{k=1}^{\infty} \Big(\|\partial_t u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k\Big). \end{aligned}$$

Therefore, if we choose T so small such that

$$\begin{aligned} &C(1 + \|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)})(1 + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}}) \sum_{k=1}^{\infty} \Big(\|\partial_t u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k \\ \le\, &C(1 + \|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)})(1 + \|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}}) \sum_{k=1}^{\infty} M^k < M, \end{aligned}$$

then we obtain

$$\|\Phi(\tilde{\rho}, \tilde{u})\|_{X_T} \le M_0 + M,$$

which shows that Φ is a self-map on X_T. Next we consider the difference of two solutions (ρ_1, u_1), $(\rho_2, u_2) \in X$ whose initial data are the same (ρ_0, u_0) and set

$$\tilde{q}(t) = \tilde{\rho}_1(t) - \tilde{\rho}_2(t), \quad \tilde{w}(t) = \tilde{u}_1(t) - \tilde{u}_2(t).$$

We first proceed the estimate for the dencity. Since

$$\begin{aligned} \rho_1 &= (1+\rho_0)|J(D_E\tilde{u}_1)|^{-1} - 1, \\ \rho_2 &= (1+\rho_0)|J(D_E\tilde{u}_2)|^{-1} - 1, \end{aligned}$$

we see that

$$\begin{aligned}\rho_1 - \rho_2 &= (1+\rho_0)\Big(|J(D_E\tilde{u}_1)|^{-1} - |J(D_E\tilde{u}_1)|^{-1}\Big)\\ &= (1+\rho_0)\Big(e^{-\int_0^t \operatorname{tr}(J(D_E u_1(s))^{-1})^{\mathsf{T}}\nabla u(s)\,ds} - e^{-\int_0^t \operatorname{tr}(J(D_E u_2(s))^{-1})^{\mathsf{T}}\nabla u(s)\,ds}\Big)\end{aligned}$$

and it follows that

$$\begin{aligned}\|\rho_1-\rho_2\|_{L^\infty(I;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} &\le CM(1+\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)})(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|\tilde{u}_1-\tilde{u}_2\|_{L^1(I;\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\ &\le CM(1+\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)})(1+\|\nabla'\eta_0\|_{\dot{B}^{\frac{n-1}{q}}_{q,1}(\mathbb{R}^{n-1})})\|\tilde{w}\|_{L^1(I;\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.\end{aligned}$$

Then the difference $(\tilde{q},\tilde{w})$ satisfies the estimate

$$\begin{aligned}&\|\Phi_u(\tilde{\rho}_1,\tilde{u}_1)-\Phi_u(\tilde{\rho}_2,\tilde{u}_2)\|_{X_T}\\ &\le \sum_{k=0}^\infty \Big(\|\partial_t u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k\\ &\qquad\times\Big(\|\partial_t\tilde{q}\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2\tilde{w}\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)\\ &\le M\Big(\|\partial_t\tilde{q}\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2\tilde{w}\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)\\ &\le M\Big(\|\tilde{q}\|_{X_T}+\|\tilde{w}\|_{X_T}\Big).\end{aligned}$$

Therefore, if we choose $T>0$ so small

$$\begin{aligned}&C(1+\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)\cap\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)})\sum_{k=1}^\infty\Big(\|\partial_t u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(I;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k\\ &\le C(1+\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)\cap\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)})\sum_{k=1}^\infty M^k < \frac{1}{2},\end{aligned}$$

then it holds that

$$\|\Phi_u(\tilde{r}_1,\tilde{u_1})-\Phi_u(\tilde{r}_2,\tilde{u_2})\|_{X_T} \le \frac{1}{4}\|(\tilde{q},\tilde{w})\|_{X_T},$$

which shows the map

$$\Phi : X_T \to X_T$$

is contraction. By the fixed point theorem of Banach-Cacciopolli, there exists a unique fixed point (ρ,u) of the map Φ in X_T. The unique fixed point (ρ,u) satisfies (7.1) with the all right members changed into (ρ,u) and it is a local in time strong solution of (1.8). This completes the proof of Theorem 1.2. ❑

Dedicatory

This paper is dedicated to Professor Thomas C. Sideris on the occasion of his 70th birthday.

Acknowledgments

The first author is partially supported by JSPS grant-in-aid for Early Career Scientists #21K13821. The second author is partially supported by JSPS grant-in-aid for Scientific Research (S) #19H05597, Scientific Research (B) #23K25782, and Scientific Research (A) #25H00597. The third author is partially supported by JSPS grant-in-aid for Scientific Research (B) #21H00992, Scientific Research (B) #23K20804, and Scientific Research (B) #25K00916.

Author contributions

Authors are contributed equivalently on the process of sharing the idea of the main part of the paper. The type setting (Writing draft) was mainly done by Takayoshi Ogawa after the Surpervision (and Writing review) by Noboru Chikami and Senjo Shimizu.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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