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Research article

Hölder regularity for the trajectories of generalized charged particles in 1D

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Abstract: We proved Hölder regularity for the particle trajectories of an interacting particle system in one dimension. The particle velocities were given by the nonlocal and singular interactions with the other particles. Particle collisions occur in finite time. Prior to collisions the particle velocities become unbounded, and thus the trajectories fail to be of class C^1. Our Hölder-regularity result supplements earlier studies on the well-posedness of the particle system which implies only continuity of the trajectories. Moreover, it extends and unifies several of the previously obtained estimates on the trajectories. Our proof method relied on standard ODE techniques: we transformed the system into different variables to expose and exploit the hidden monotonicity properties.

Keywords: interacting particle systems; ODEs with singularities; regularity
Mathematics Subject Classification: 34E18, 74H30.

1. Introduction

We are interested in improving the regularity properties of the solution to a hybrid system of ODEs (see (1) below) which was recently proven to be well-posed [1, 2]. This system appears in plasticity theory as a model for the dynamics of crystallographic defects.

1.1. The formal governing equations

The system is an interacting particle system. It is formally given by

$$\begin{cases} \dfrac{dx_i}{dt} = \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{n} b_i b_j f(x_i - x_j) + b_i g(x_i), \qquad t \in (0,T),\ i = 1,\dots,n, \\ + \text{ annihilation upon collision,} \end{cases} \tag{1}$$

where $n \geq 1$ is the number of particles, $x_1 \leq \ldots \leq x_n$ are the time-dependent particle positions, and $b_1, \ldots, b_n \in \{-1, +1\}$ are the fixed signs of the particles. The given function $g : \mathbb{R} \to \mathbb{R}$ is an externally applied force, and the given odd function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ describes the interaction force between each pair of two particles. The interaction force is such that particles of the same sign repel and particles of opposite sign attract. Figure 1 illustrates typical choices of f and g; Assumption 1.1 below lists the properties we impose on them. Stereotypical examples of $g(x)$ are x and $\sin(x)$, and an example of $f(x)$ is x^{-k} for odd integers $k \geq 1$.

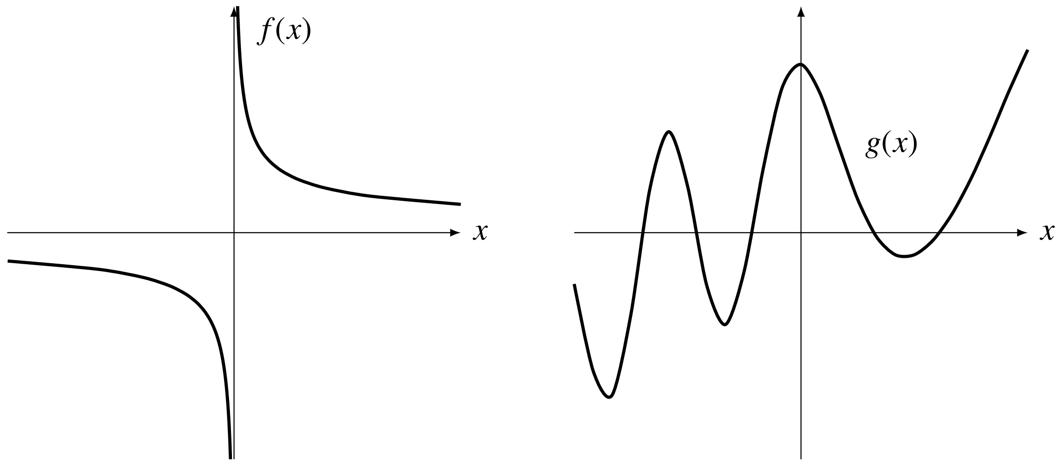

Figure 1. Examples of f and g.

The feature which makes (1) interesting is that f is singular at 0. As a result, particles of opposite sign collide in finite time and with unbounded velocity. Figure 2 sketches the particle dynamics including several collisions. The maximal time of existence of solutions to the system of ODEs is the first collision time τ_1. To extend it beyond collisions, the following collision rule is applied: whenever a pair of particles with opposite charge collide, they are removed (annihilated) from the system. The removal of pairs is done sequentially. As a consequence, not all colliding particles are necessarily removed; see Figure 2. The surviving particles continue to evolve by the system of ODEs (1). We refer to the combination of the system of ODEs and the annihilation rule in (1) as a hybrid system of ODEs.

1.2. Applications of (1)

We recall from [2] the applications of (1) that we have in mind. Besides being of interest to mathematical studies on interacting particle systems with multiple species such as [3–9], (1) appears as a simple microscopic model for describing the plasticity of metals. Indeed, plasticity is the emergent behavior of the dynamics of microscopic defects in the atomic lattice of metals. These defects are called dislocations. The textbooks [10, 11] describe dislocations in detail. In the simplest application of (1), x_i represents the position of a dislocation, b_i is its orientation, f describes the interaction force between dislocations, and g describes the effect of an external loading applied to the metal.

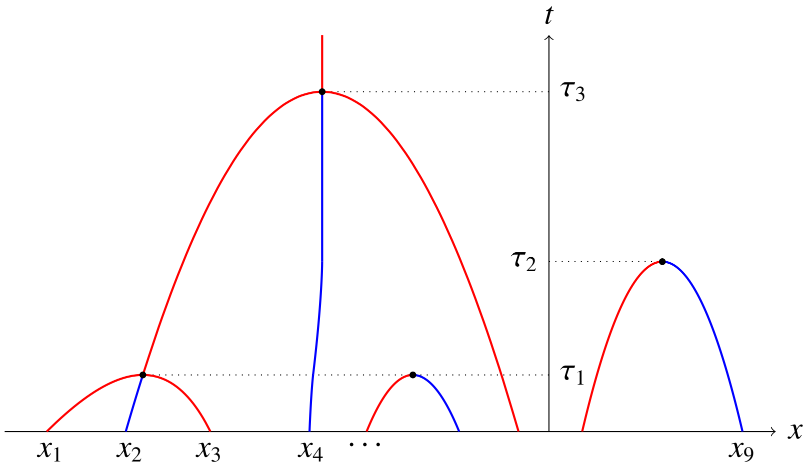

Figure 2. A sketch of the solution trajectories to (1) for $n = 9$. The color indicates the sign of the particles: red stands for positive and blue stands for negative. The black dots indicate the collision points.

Next we explain why we keep the interaction force f general. In addition to the application just mentioned, (1) is also used to describe the dynamics of dislocation *structures* such as dipoles (see [12, 13]) or dislocation walls (i.e., vertically periodic arrays of dislocations; see [14–16]). Both applications require a different f. Moreover, in the paper series by Patrizi, Valdinoci et al. (see, e.g., [17–21],) a different dislocation dynamics model is considered. This model describes the evolution of a phase field by a PDE. The analysis of this PDE relies on the construction of viscosity solutions. This construction is made by using the solution of (1). To prove the required estimates in their analysis, Hölder continuity on the solution of (1) would be a powerful tool to have. This application of (1) requires freedom in the choices of both f and g. In particular, it is needed that the singularity of f is allowed to be a power law for a range of values for the power.

All the applications mentioned above fit (1) under the general class of f and g we allow for in Assumption 1.1 below.

1.3. Known results on (1)

The works [1, 2] establish well-posedness of (1), derive estimates on the particle trajectories near collisions, and pass to the limit $n \to \infty$ in the rescaled time $t' = t/n$. In [1] this is done for the specific case of electrically charged particles (precisely, with $f(x) = \frac{1}{x}$ and $g \equiv 0$), and in [2] this is generalized to any f and g that satisfy a slightly weaker set of assumptions than Assumption 1.1 (we comment on this in Section 1.7). In this paper, we make the following standing assumptions on f and g:

Assumption 1.1. *$g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfies:*

(i) f is odd,

(ii) (singularity) there exists $a > 0$ such that

$$f = f_a + f_{\text{reg}}, \qquad f_a(x) := \frac{\operatorname{sgn}(x)}{|x|^a}$$

for some $f_{\text{reg}} \in W^{2,1}_{\text{loc}}(\mathbb{R} \setminus \{0\})$ with one-sided limits $f_{\text{reg}}(0\pm), f'_{\text{reg}}(0\pm) \in \mathbb{R}$, *(iii) (monotonicity) on* $(0, \infty)$*,* $f \geq 0$*,* $f' \leq 0$*, and* $f'' \geq 0$*.*

Next we briefly recall the results and arguments thereof given in [1, 2]. Standard ODE theory provides the existence and uniqueness of solutions up to the first collision time τ_1. It is only here that the Lipschitz continuity of g is required. The difficulty for obtaining global well-posedness of (1) is to get across τ_1 and later collision times. This was done by showing that at each collision time τ:

1. the limits $x_i(\tau-) := \lim_{t \uparrow \tau} x_i(t)$ exist, and
2. all particles that collide at τ at the same point $y \in \mathbb{R}$ must have *alternating* signs before τ.

The second statement implies that the annihilation rule leaves either no particles at y (when the number of colliding particles is even) or precisely one particle at y; see Figure 2. In both cases the ODE system can be restarted from τ after removing the annihilated particles. Iterating this procedure over all the collision times $\tau_1, \ldots, \tau_K$ (note that $K \leq \frac{n}{2}$), a global solution to (1) is constructed. By this construction, the particle trajectories on any interval of subsequent collision times $[\tau_{k-1}, \tau_k]$ (including $[\tau_0, \tau_1]$ and $[\tau_K, \tau_{K+1}]$ with $\tau_0 := 0$ and $\tau_{K+1} := T$) satisfy

$$x_i \in C([\tau_{k-1}, \tau_k]) \cap C^1([\tau_{k-1}, \tau_k)) \qquad \text{for all } 1 \leq k \leq K + 1. \tag{2}$$

We refrain from providing a precise definition for the solution to (1). We refer to [2, Definition 2.5] for such a definition. The definition is technical solely because of bookkeeping reasons; the definition keeps track of the annihilated particles, and extends their trajectories beyond their times of annihilation. Here, we simply remark that the solution is unique up to a relabelling of the particles (see, e.g., Figure 2, where at the three-particle collisions there is a choice of which of the two positive particles survives). Furthermore, since the solution is iteratively constructed over the collision times, it is sufficient to focus on the properties of the solution before and at τ_1.

The papers [1,2,19] established several estimates of the trajectories around τ_1. Before stating them, consider the following example: $n = 2$, $b_1 b_2 = -1$, $g \equiv 0$, and $f(x) = \operatorname{sgn}(x)|x|^{-a}$ (,i.e., $f_{\text{reg}} = 0$). Then, the solution to (1) can be computed explicitly, and is given by

$$x_2(t) + x_1(t) = x_2(0) + x_1(0),$$
$$x_2(t) - x_1(t) = c_a(\tau_1 - t)^{\frac{1}{1+a}},$$

for some explicit constants $\tau_1, c_a > 0$.

The estimates in [2, 19] demonstrate that many features of this power law behavior remain to hold for the trajectories for general n. In [19, Proposition 3.4] (with $f_{\text{reg}} = 0$) it is proven that

$$\min_{1 \leq i \leq n-1} x_{i+1}(t) - x_i(t) \tag{3}$$

is Hölder continuous in t with exponent $\frac{1}{1+a}$. In [2] it is shown that there exist constants $C, c > 0$ such that when particles $k, k+1, \ldots, \ell$ collide at the same point,

$$r_i(t) := x_{i+1}(t) - x_i(t) \leq C(\tau_1 - t)^{\frac{1}{1+a}} \qquad \text{for all } i = k, \ldots, \ell - 1, \tag{4}$$

$$x_k(t) - x_k(\tau_1) \leq -c(\tau_1 - t)^{\frac{1}{1+a}} \quad \text{and} \quad x_\ell(t) - x_\ell(\tau_1) \geq c(\tau_1 - t)^{\frac{1}{1+a}} \tag{5}$$

for all $t < \tau_1$ close enough to τ_1.

1.4. Main result

The properties cited above raise the question of whether x_i is Hölder continuous with exponent $\frac{1}{1+a}$. Note that by themselves these properties do not imply any Hölder regularity; to see this, consider for instance adding fast oscillations to the trajectories. This is not just a small technicality that was missed in the previous papers; the possibility of fast oscillations cannot be excluded from the arguments in the corresponding proofs.

Our main result states that the particle trajectories are indeed Hölder continuous with exponent $\frac{1}{1+a}$:

Theorem 1.2 (Main result)**.** *Let $n \geq 1$, $T > 0$, $b_1, \dots, b_n \in \{-1, +1\}$, f, g satisfy Assumption 1.1, and $x_1^\circ < \dots < x_n^\circ$ be initial conditions of the particles. For the solution of* (1) *starting at $x_1^\circ, \dots, x_n^\circ$, let $\tau_1, \dots, \tau_K < T$ be the collision times. Set $\tau_0 = 0$ and $\tau_{K+1} = T$. Then, for each $1 \leq k \leq K+1$, each particle trajectory on $[\tau_{k-1}, \tau_k]$ is Hölder continuous with exponent $\frac{1}{1+a}$.*

1.5. Classical approach to dealing with singularities

A classical approach to dealing with singular ODEs is to introduce new dependent and independent variables such that the vector field of the transformed system is locally continuous [22, 23]. Typically, in the new variables the system has an equilibrium such that solutions of interest are contained on an invariant manifold induced by the dynamics in a neighborhood of that equilibrium; see [24–26] for examples. Applying such a transformation to (1) on $[0, \tau_1]$, the equilibrium is given by the particle positions at τ_1, which will be asymptotically reached when the transformed time t' tends to ∞. More specifically, in the case $f(x) = \frac{1}{x}$ we can proceed by using a similar approach as the Kustaanheimo-Stiefel (KS) transformation [27, 28]. This transformation maps the independent variables to a higher dimensional space in which a simple change of independent variable leads to regularization of the equations. Finding the equilibria of these regularized equations is equivalent to a polynomial root-finding problem, and the problem of obtaining the existence of the invariant manifold reduces to an eigenvalue problem. Unfortunately, these problems are already difficult to solve analytically for $n \geq 5$. Yet, for numerical studies these regularized variables provide a promising alternative approach.

1.6. Idea of the proof

First, we show that without loss of generality we can treat each collision independently from the other particles. Essentially, this means that we may assume that all particles collide at $x = 0$ at τ_1. Second, for each collision, we show that it is sufficient to supplement (5) with a lower bound of the type

$$r_i(t) \geq c_1(\tau_1 - t)^{1/(1+a)} \qquad \text{for all } 1 \leq i \leq n-1. \tag{6}$$

The first step for showing (6) is to show that no two neighboring particles x_i and x_{i+1} can be too close together with respect to the distance to their other neighbors x_{i-1} and x_{i+2}, as otherwise x_i and x_{i+1} would collide first. This translates to a bound of the type

$$r_i \geq c_2 \min\{r_{i-1}, r_{i+1}\}.$$

This bound on itself is not enough to obtain (6), because it still allows for a configuration in which $r_i, r_{i+1} \ll r_{i-1}, r_{i+2}$. However, in such a situation, either x_i and x_{i+1} collide first, or x_{i+1} and x_{i+2} collide first, which would contradict that all particles collide at the same time. Generalizing this we obtain a bound of the type

$$\sum_{k=i}^{j-1} r_k = x_j - x_i \geq c_3 \min\{r_{i-1}, r_j\}$$

for all $i < j$. This is the key Lemma 4.3. From it and (5) we derive the desired bound (6) in Lemma 4.4 by an iteration argument. The proof of Lemma 4.3 relies on several quantitative bounds (see Lemmas 3.1, 3.2, and 3.3) which are inspired by the proofs in [1] and [2].

1.7. Discussion

Our main result, Theorem 1.2, generalizes and unifies in a clean, compact statement several previously obtained results such as the Hölder continuity of (3) and the estimates in (4).

Finally, we compare Assumption 1.1 on f, g to the assumptions made in [2, Assumption 2.2 and Theorem 2.7]. Assumption 1.1 is slightly stronger; Assumption 1.1(ii) is replaced in [2] by the weaker assumption that $cx^{-a} \leq f(x) \leq Cx^{-a}$ for some constants $c, C > 0$ and for all $x > 0$ small enough. We choose to impose Assumption 1.1(ii) because it fits naturally with the proof method used below and covers all applications for (1) given in [2].

1.8. Overview

The paper is organized as follows. In Section 2.1 we show that for proving Theorem 1.2 it is sufficient to zoom in on the trajectories at each collision separately. In Section 3 we exploit the monotonicity properties of f to strongly reduce the full dependence between the equations of the system of ODEs in (1) at the cost of differential inequalities rather than equalities. Based on those inequalities we prove Theorem 1.2 in Section 4.

2. Reduction to a single collision

In this section we show that it is sufficient to prove Theorem 1.2 for a single collision. Claim 2.1 states this in full detail.

Claim 2.1. *Without loss of generality we may assume in Theorem 1.2 that:*

(i) $k = 1$,

(ii) $b_i = (-1)^i$ *for all* $1 \leq i \leq n$,

(iii) $f_{\mathrm{reg}} = 0$,

(iv) $x_1^\circ, \ldots, x_n^\circ$ *are such that* $x_i(\tau) = 0$ *for all* $1 \leq i \leq n$ *where* $\tau \in (0, \tau_1]$ *can be chosen freely,*

(v) $x_1 < \ldots < x_n$ *on* $[0, \tau)$, *and*

(vi) there exist functions $F_i \in C([0, \tau])$ *such that*

$$\frac{dx_i}{dt} = \sum_{\substack{j=1 \\ j \neq i}}^{n} b_i b_j f(x_i - x_j) + F_i(t) \qquad \text{for all } t \in (0, \tau) \text{ and all } 1 \leq i \leq n. \tag{7}$$

Proof. Let the setting in Theorem 1.2 be given. Since the system (1) is essentially the same on each time interval of subsequent collision times, it is sufficient to prove the Hölder regularity on $[0, \tau_1]$. Since the particle trajectories are continuous on $[0, \tau_1]$, they are by definition of τ_1 separated on $[0, \tau_1)$. This separation has two consequences. First, since the particles are initially strictly ordered, they remain strictly ordered on $[0, \tau_1)$. Second, if two particles x_i and x_j do not collide at τ_1, then their trajectories are separated by a positive distance on $[0, \tau_1]$. By grouping together particles that collide at the same point at time τ_1, we can split (1) up into systems of ODEs of the form

$$\frac{dx_i}{dt} = \sum_{\substack{j \in I \\ j \neq i}} b_i b_j f_a(x_i - x_j) + F_i^I(t) \qquad i \in I, \tag{8}$$

where $I \subset \{1, \ldots, n\}$ contains the indices of the colliding particles (note that $I = \{k, \ldots, \ell\}$ for some $1 \leq k \leq \ell \leq n$ and that I may be a singleton),

$$F_i^I(t) := \sum_{\substack{j \in I \\ j \neq i}} b_i b_j f_{\mathrm{reg}}(x_i - x_j) + \sum_{j \notin I} b_i b_j f(x_i - x_j) + b_i g(x_i)$$

is continuous and bounded on $[0, \tau_1]$, and $b_i b_j = (-1)^{i+j}$ thanks to the “alternating signs” result mentioned in Section 1.3.

By considering $F_i^I(t)$ as a generic function in $C([0, \tau_1])$, the systems (8) indexed by I decouple. Hence, we may focus on a single, arbitrarily chosen system. Note that (8) is invariant under the swapping of all signs and invariant in shifting space. Hence, we may assume that all particles in (8) collide at $y = 0$. Finally, thanks to the known regularity in (2) it is sufficient to prove the Hölder regularity only on $[\tau_1 - \tau, \tau_1]$, where $\tau \in (0, \tau_1)$ can be chosen freely.

From this construction Claim 2.1 follows. Indeed, by shifting the time variable to $t' = t - \tau_1 + \tau$, we obtain (iv). By relabeling the particles, we can transform $I = \{k, \ldots, \ell\}$ to $\{1, \ldots, n'\}$ with $n' = \ell - k + 1 \in \{1, \ldots, n\}$; this demonstrates (vi). □

3. Estimates for the distance between particles

In this section we start from the setting in Claim 2.1. In particular, we consider system (7). We establish key estimates on the distance between particles on which our proof of Theorem 1.2 in Section 4 relies. We note that while we may assume $f_{\mathrm{reg}} = 0$, the estimates in this section also hold without this assumption provided that all three inequalities in Assumption 1.1(iii) on the monotonicity are strict inequalities.

Next we introduce notation. We reserve $C, c > 0$ to denote generic positive constants which do not depend on the relevant parameters and variables. We think of C as possibly large and c as possibly small. The values of C, c may change from line to line, but when they appear multiple times in the same display their values remain the same. When more than one generic constant appears in the same display, we use C', C'', c', etc., to distinguish them. When we need to keep track of certain constants, we use C_0, C_1, etc.

For the particle distances we introduce

$$r_{ji} := r_{j,i} := x_j - x_i \in \mathbb{R}, \qquad\qquad 1 \leq i, j \leq n,$$

$$r_i := r_{i+1,i} = x_{i+1} - x_i > 0, \qquad\qquad 1 \le i < n.$$

Note that $\operatorname{sgn}(r_{ji}) = \operatorname{sgn}(j - i)$.

To obtain estimates on r_{ji}, we start in Section 3.1 with examining the easier case of r_i. Afterwards, in Section 3.2 we treat the general case. We demonstrate how particle interactions can be grouped together such that they give a positive or negative contribution to $\dot{r}_{ji}$. Finally, in Section 3.3 we derive bounds on $\dot{r}_{ji}$.

3.1. Distance between neighboring particles

From the governing equations (7) we obtain

$$\begin{aligned}\dot{r}_i = \dot{x}_{i+1} - \dot{x}_i &= F_{i+1}(t) - F_i(t) + b_{i+1}b_i\big(f(x_{i+1} - x_i) - f(x_i - x_{i+1})\big) \\ &\quad + \sum_{\substack{k=1\\k\neq i,i+1}}^{n} [b_{i+1}b_k f(x_{i+1} - x_k) - b_i b_k f(x_i - x_k)].\end{aligned}$$

Using that f is odd and that $b_i = (-1)^i$, we can write this in terms of r_i and r_{ik} as

$$\dot{r}_i = -2f(r_i) + G_i(t) - \sum_{\substack{k=1\\k\neq i,i+1}}^{n} [b_i b_k g(r_{ik}; r_i)], \tag{9}$$

where

$$G_i(t) := F_{i+1}(t) - F_i(t)$$

describes the external forcing and

$$g(\cdot;\rho) : \mathbb{R} \setminus \{-\rho, 0\} \to \mathbb{R}, \qquad g(r;\rho) := f(\rho + r) + f(r),$$

is used to describe the effect of particles other than x_i and x_{i+1}.

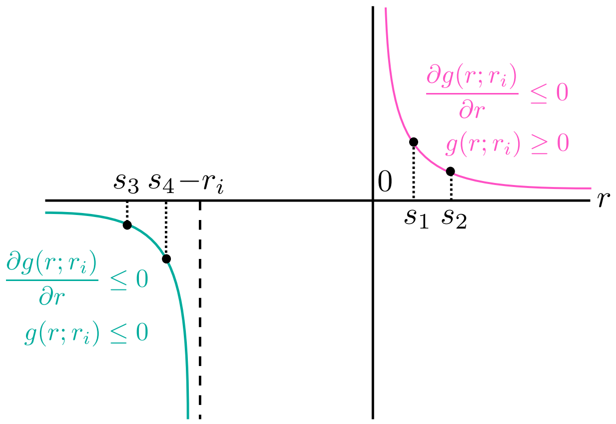

Figure 3. Properties of $g(r; r_i)$ in (9) for $r_i > 0$ fixed. The distance r_{ik} represented by r is contained in $(-\infty, -r_i)$ or $(0, \infty)$. The inequalities follow from the monotonicity properties of f.

Next we investigate how the presence of particles and particle pairs contribute to the sum in (9). In Figure 3 we present monotonicity properties of $g(\cdot; r_i)$. For convenience we assume that $b_i = 1$ (note that $b_{i+1} = -1$). The sign of g implies that a positive particle x_k to the left (i.e., $r_{ik} > 0$) tends to decrease r_i, i.e., its contribution to the sum in (9) is nonpositive. More generally, whether a particle x_k tends to increase or decrease r_i depends on b_k and whether x_k is to the left or to the right of the pair (x_i, x_{i+1}), but not on its distance to (x_i, x_{i+1}). In Table 1 below we list all four cases.

Next we consider a pair of neighboring particles (x_k, x_{k+1}) to either the left of x_i or to the right of x_{i+1}. Observe from Assumption 1.1(iii) (monotonicity) that for any $r_i > 0$:

1. for all $0 < s_1 < s_2$ we have $g(s_2; r_i) - g(s_1; r_i) \leq 0$, and
2. for all $s_3 < s_4 < -r_i$ we have $g(s_4; r_i) - g(s_3; r_i) \leq 0$.

The above properties are visualized in Figure 3. Hence, the pair (x_k, x_{k+1}) tends to either increase or decrease r_i independently of the positions of x_k and x_{k+1}; see Table 1 below. In (9) such a pair corresponds to two consecutive terms of the sum.

3.2. The distance between any pair of particles

Next we consider r_{ji} with $j > i$. Similar to the computation leading to (9), we obtain

$$\begin{aligned}\dot{r}_{ji} = \sum_{k=i}^{j-1} b_j b_k f(r_{jk}) - \sum_{k=i+1}^{j} b_i b_k f(r_{ik}) + \sum_{k\leq i-1} \Big[b_j b_k f(r_{jk}) - b_i b_k f(r_{ik}) \Big] \\ + \sum_{k\geq j+1} \Big[b_j b_k f(r_{jk}) - b_i b_k f(r_{ik}) \Big] + G_{ji}(t),\end{aligned} \tag{10}$$

where

$$G_{ji}(t) := F_j(t) - F_i(t).$$

Note that $\operatorname{sgn} f(r_{k\ell}) = \operatorname{sgn}(k - \ell)$ for all $k \neq \ell$. The first two sums in the right-hand side generalize the first term in the right-hand side of (9); not only do they account for the interaction between x_i and x_j, but they also account for each particle x_k in between x_i and x_j.

The latter two sums in (10) (note that the summands are the same) correspond to the single sum in (9). In fact, if the particles i and j have opposite sign, then the summand is similar to that in (9):

$$\Big[b_j b_k f(r_{jk}) - b_i b_k f(r_{ik}) \Big] = -b_i b_k g(r_{ik}; r_{ji}).$$

We recall from Section 3.1 that the sign of the summand is independent of the distance between the particles.

If b_i and b_j have the same sign, then the summand reads as

$$\Big[b_j b_k f(r_{jk}) - b_i b_k f(r_{ik}) \Big] = b_i b_k h(r_{ik}; r_{ji}),$$

where for any parameter $\rho > 0$ the function h is defined as

$$h(\cdot; \rho) : \mathbb{R} \setminus \{-\rho, 0\} \to \mathbb{R}, \qquad h(r; \rho) := f(\rho + r) - f(r);$$

see Figure 4. Note that the only difference between the expressions for h and g is the sign in front of $f(r)$. Hence, similar to $g(\cdot;\rho)$, on $(-\infty,-\rho)$ and $(0,\infty)$ the function $h(\cdot;\rho)$ has a sign and is monotone. The convexity of f yields the additional bound on h given by

$$h(r;\rho) = \int_r^{\rho+r} f'(s)\,ds \leq f'(\rho+r)\rho \leq 0 \qquad \text{for } r > 0 \tag{11}$$

and

$$h(r;\rho) \leq f'(r)\rho \leq 0 \qquad \text{for } r < -\rho.$$

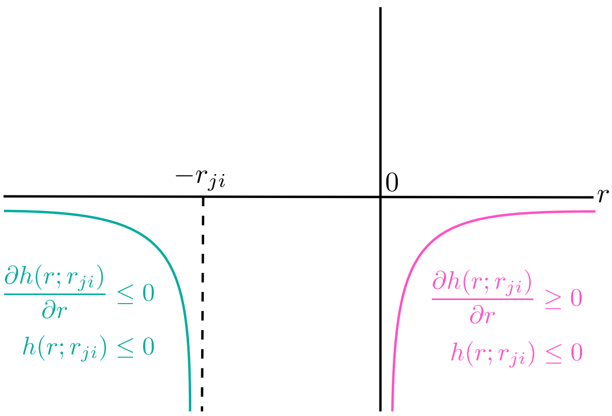

Figure 4. Properties $h(r;r_{ji})$ for r_{ji} fixed. The distance r_{ik} represented by r is contained in $(-\infty,-r_i)$ or $(0,\infty)$. In the case $b_i = b_j$ the interaction between r_{ik} and r_{ji} is determined by h.

Finally, Table 1 summarizes the above by giving an overview of the contribution to $\dot{r}_{ji}$ of the summand in the third and fourth sum in (10) for both single particles $k = \kappa$ and for two neighboring particles $k = \kappa, \kappa+1$. Table 1 will be the main tool in the results of the following section.

Table 1. Influence of one particle or a pair of particles on $\dot{r}_{ji}$. This influence depends on whether $b_i = -b_j$ or $b_i = b_j$. The sign of the contribution is unchanged if the sign of both b_i and b_κ are swapped.

particle(s)	configuration			$\dot{r}_{ji}$-contribution	
	b_i	b_κ	position κ	$b_i = -b_j$	$b_i = b_j$
x_κ	+	+	$\kappa < i$	−	−
x_κ	+	+	$\kappa > j$	+	−
x_κ	+	−	$\kappa < i$	+	+
x_κ	+	−	$\kappa > j$	−	+
$(x_\kappa, x_{\kappa+1})$	+	−	$\kappa+1 < i$	−	−
$(x_\kappa, x_{\kappa+1})$	+	−	$\kappa > j$	−	+
$(x_\kappa, x_{\kappa+1})$	+	+	$\kappa+1 < i$	+	+
$(x_\kappa, x_{\kappa+1})$	+	+	$\kappa > j$	+	−

3.3. Estimates on $\dot{r}_{ji}$

In this section we use Table 1 to establish the key estimates on $\dot{r}_{ji}$ from (10): in Lemma 3.1 we prove a lower bound for when $j = i + 1$ and in Lemmas 3.2 and 3.3 we provide upper bounds for general j.

Lemma 3.1. $\dot{r}_i \geq -2f(r_i) - C$ *for all* $i = 1, \ldots, n-1$.

Proof. Starting from (9) we apply $F_i(t) \geq -C$. We split the sum over k in $k \leq i-1$ and $k \geq i+2$; see also (10). Both sums can be treated analogously; we focus on the latter. If $b_i = 1$ and $n - i$ is odd, then this sum can be written as a sum over the particle pairs

$$(x_\kappa, x_{\kappa+1}) \;\text{ for }\; \kappa = i+2,\ i+4,\ i+6, \ldots, n-1.$$

Observe that $b_\kappa = b_i$. From Table 1 we see that each such pair yields a positive contribution to $\dot{r}_i$. Estimating these contributions from below by 0, Lemma 3.1 follows. If $b_i = -1$, then $b_\kappa = b_i$ and the contribution to $\dot{r}_i$ is the same as the previous case $b_i = 1$. Hence, by a similar argument we may remove all terms from the sum.

If $b_i = 1$ and $n - i$ is even, then by a similar argument we may remove all terms from the sum over $k \geq i+2$, except for $k = n$. Since $b_n = b_i$, we see from Table 1 that the summand at $k = n$ is positive, and thus Lemma 3.1 follows. Finally, for $b_i = -1$ we can proceed by a similar argument.

□

For the next lemmas we set

$$r_0 := \infty, \qquad r_n := \infty,$$

and use the conventions $f(r_0) = 0$ and $f(r_n) = 0$.

Lemma 3.2. *For all* $1 \leq i < j \leq n$, *if* $b_i = -b_j$, *then*

$$\dot{r}_{ji} \leq -2f(r_{ji}) + 2[f(r_j) + f(r_{i-1})] + C.$$

Proof. Starting from (10) we apply $F_{ji}(t) \leq C$. For the fourth sum in (10), we note that the summand at $k = j+1$ equals

$$b_j b_{j+1} f(r_j) + b_i b_{j+1} f(r_{j+1,i}) = f(r_j) + f(r_{j+1,i}) \leq 2f(r_j).$$

From Table 1 we see that each of the summands corresponding to

$$(x_\kappa, x_{\kappa+1}) \;\text{ for }\; \kappa = j+2, j+4, j+6, \ldots$$

yields a negative contribution to the sum as $b_\kappa = b_j = -b_i$; we estimate these contributions from above by 0. Finally, similar to the argument in the proof of Lemma 3.1, if the final term $k = n$ is not covered by the pairs above, then its contribution to the sum is negative. In that case we bound it from above by 0.

The third sum in (10) can be treated analogously; we obtain

$$\sum_{k \leq i-1} \Big[b_j b_k f(r_{jk}) + b_i b_k f(r_{ki}) \Big] \leq 2f(r_{i-1}).$$

For the first sum in (10) we note that the summand at $k = i$ equals $-f(r_{ji})$. The remaining number of summands is even. Grouping them as

$$(x_\kappa) \quad \text{for} \quad \kappa = j-2, j-4, j-6, \dots$$

the contribution of each term is negative. The second sum in (10) can be treated analogously; it is also bounded from above by $-f(r_{ji})$. The lemma follows by putting all estimates together. □

Lemma 3.3. *For all* $1 \le i < j \le n$, *if* $b_i = b_j$, *then*

$$\dot{r}_{ji} \le r_{ji} f'(r_{ji}) + f(r_j) + f(r_{i-1}) + C.$$

Proof. Again we start from (10) and apply $F_{ji}(t) \le C$. For the fourth sum in (10), note that the summand at $k = j+1$ equals

$$b_j b_{j+1} f(r_{j,j+1}) - b_i b_{j+1} f(r_{i,j+1}) = -f(r_{j+1,i+1}) + f(r_j) \le f(r_j).$$

From Table 1 we see that each of the summands corresponding to

$$(x_\kappa, x_{\kappa+1}) \quad \text{for} \quad \kappa = j+2, j+4, j+6, \dots$$

yields a negative contribution to the sum (note that $b_\kappa = b_i$); we estimate these contributions from above by 0. If $n - j$ is even then $k = n$ is not covered by the pairs above. In this case, $b_n = b_i$, hence the summand is negative and we bound it from above by 0.

Similarly, for the third sum of (10) we obtain

$$\sum_{k \le i-1} [b_j b_k f(r_{jk}) - b_i b_k f(r_{ik})] \le f(r_i).$$

Next we estimate the first sum of (10). Since $b_i = b_j$ the number of terms is even. Grouping pairs together, we write it as

$$\sum_{k=i}^{j-1} b_j b_k f(r_{jk}) = \sum_{\ell=0}^{\frac{j-i}{2}-1} f(r_{j,i+2\ell}) - f(r_{j,i+2\ell+1}) = \sum_{\ell=0}^{\frac{j-i}{2}-1} h(r_{j,i+2\ell+1}; r_{i+2\ell}). \tag{12}$$

Applying (11) to (12) and using the fact that f' is increasing we obtain

$$\sum_{k=i}^{j-1} b_j b_k f(r_{jk}) \le \sum_{\ell=0}^{\frac{j-i}{2}-1} f'(r_{j,i+2\ell}) r_{i+2\ell} \le f'(r_{ji}) \sum_{\ell=0}^{\frac{j-i}{2}-1} r_{i+2\ell}. \tag{13}$$

Similarly, we obtain for the second sum in (10) that

$$\sum_{k=i+1}^{j} b_i b_k f(r_{ki}) \le f'(r_{ji}) \sum_{\ell=0}^{\frac{j-i}{2}-1} r_{i+2\ell+1}. \tag{14}$$

Putting (13) and (14) together we obtain

$$\sum_{k=i}^{j-1} b_j b_k f(r_{jk}) + \sum_{k=i+1}^{j} b_i b_k f(r_{ki}) \le f'(r_{ji}) r_{ji}.$$

Putting the estimates together we obtain the lemma. □

4. Proof of the main theorem

The proof of Theorem 1.2 is given at the end of this section. First, we establish the key Lemma 4.3 on which this proof relies. We consider the setting in Claim 2.1. We combine Lemmas 3.1, 3.2, and 3.3 to obtain bounds on $r_{ji}(t)$; see Lemmas 4.1 and 4.2 below. Note from the upper bound in (4) that we may assume that r_i is as small as required by taking τ in Claim 2.1 small enough.

We start with two preparatory lemmas in which we exploit that $f(x) = \operatorname{sgn}(x)|x|^{-a}$:

Lemma 4.1. *For τ small enough and for $i = 1, \dots, n-1$*

$$r_i^a \dot{r}_i \geq -3 \quad \textit{on} \quad [0, \tau).$$

Proof. Using Lemma 3.1 we obtain

$$\dot{r}_i \geq \frac{-2}{r_i^a} - C.$$

Taking τ small enough such that $Cr_i^a \leq 1$ on $[0, \tau]$, Lemma 4.1 follows. □

For the next lemmas we recall that $r_0, r_n = \infty$ and that $r_0^{-a}, r_n^{-a} = 0$.

Lemma 4.2. *Let $b := \min(1, \frac{1}{2}a)$. For τ small enough and for all $1 \leq i < j \leq n$*

$$\dot{r}_{ji} \leq -\frac{b}{r_{ji}^a} + \frac{2}{r_j^a} + \frac{2}{r_{i-1}^a} \qquad \textit{on } [0, \tau).$$

Proof. We first consider $b_i = -b_j$. We use Lemma 3.2 to obtain

$$\dot{r}_{ji} \leq -\frac{2}{r_{ji}^a} + \frac{2}{r_j^a} + \frac{2}{r_{i-1}^a} + C.$$

Taking τ small enough such that $Cr_{ji}^a \leq 1$ on $[0, \tau]$, Lemma 4.2 follows.

For the other case $b_i = b_j$ the proof is similar: from Lemma 3.3 we get

$$\dot{r}_{ji} \leq -\frac{a}{r_{ji}^a} + \frac{1}{r_j^a} + \frac{1}{r_{i-1}^a} + C,$$

and then taking τ small enough such that $Cr_{ji}^a \leq \frac{a}{2}$ on $[0, \tau]$, Lemma 4.2 follows. □

Lemma 4.3. *Let b be given as in Lemma 4.2. For τ small enough and for all $t \in [0, \tau]$ and all $1 \leq i < j \leq n$ with $(j, i) \neq (n, 1)$ we have*

$$r_{ji}(t) \geq c_0 \min\{r_{i-1}(t), r_j(t)\}, \qquad c_0 := \min\left\{\left(\frac{b}{16}\right)^{\frac{1}{a}}, \left(\frac{b}{12}\right)^{\frac{1}{a+1}}\right\} > 0.$$

Proof. Fix $\tau > 0$ such that Lemmas 4.1 and 4.2 apply. Lemma 4.3 obviously holds at $t = \tau$. For $t \in [0, \tau)$ we reason by contradiction; suppose there exists $t_0 \in [0, \tau)$ such that

$$\alpha := r_{ji}(t_0) < c_0 \min\{r_{i-1}(t_0), r_j(t_0)\} =: c_0\beta.$$

Note that $\beta < \infty$ (indeed, while $r_0 = r_n = \infty$, the condition $(j, i) \neq (n, 1)$ implies that $r_{i-1}(t_0)$ or $r_j(t_0)$ is finite). Applying Lemma 4.1 and integrating over $[t_0, t]$, we obtain

$$\frac{1}{a+1}\left(r_k^{a+1} - (r_k^\circ)^{a+1}\right) \geq -3t$$

for $k \in \{i-1, j\}$. Rewriting this yields

$$r_k(t) \geq ([\beta^{a+1} - 3(a+1)(t-t_0)]_+)^{\frac{1}{a+1}}$$

for all $t \in [t_0, \tau]$. Let

$$t_1 := t_0 + \frac{\beta^{a+1}}{6(a+1)}$$

and note that

$$r_k \geq 2^{-\frac{1}{a+1}}\beta > 0 \quad \text{on} \quad [t_0, t_1]; \tag{15}$$

thus $t_1 < \tau$. Then, from Lemma 4.2 and (15) we obtain

$$\dot r_{ji} \leq -\frac{b}{r_{ji}^a} + 2^{2+\frac{a}{a+1}}\beta^{-a} \leq -\frac{b}{r_{ji}^a} + \frac{8}{\beta^a} \quad \text{on} \quad [t_0, t_1]. \tag{16}$$

We note from the following computation that the right-hand side in (16) is negative initially at $t = t_0$:

$$\frac{8}{\beta^a} < \frac{8c_0^a}{\alpha^a} \leq \frac{b}{2r_{ji}(t_0)^a}.$$

Moreover, the right-hand side in (16) decreases as r_{ji} decreases. Hence, we obtain

$$\dot r_{ji} \leq -\frac{b}{2r_{ji}^a} \quad \text{on} \quad [t_0, t_1].$$

Multiplying by r_{ji}^a and integrating over $[t_0, t]$, we obtain

$$\frac{1}{a+1}\left(r_{ji}(t)^{a+1} - \alpha^{a+1}\right) \leq -\frac{b}{2}(t - t_0).$$

Rewriting this, we get

$$r_{ji}(t) \leq \left[\alpha^{a+1} - \frac{b}{2}(a+1)(t-t_0)\right]_+^{\frac{1}{a+1}} \quad \text{on} \quad [t_0, t_1].$$

The first time t_2 at which the right-hand side hits 0 is

$$t_2 = t_0 + \frac{2}{b(a+1)}\alpha^{a+1} < t_0 + \frac{2}{b(a+1)}c_0^{a+1}\beta^{a+1} \leq t_0 + \frac{\beta^{a+1}}{6(a+1)} = t_1 < \tau.$$

Thus, $r_{ji}(t_2) = 0$ with $t_2 < \tau$, which contradicts $r_{ji} > 0$ on $[0, \tau)$. □

We obtain the desired lower bound on r_i (recall (6)) from (5) and Lemma 4.3 by an iteration argument:

Lemma 4.4. *For τ small enough there exists $c > 0$ such that $r_i(t) \geq c(\tau - t)^{\frac{1}{1+a}}$ for all $i = 1, \ldots, n-1$ and all $t \in [0, \tau]$.*

Proof. Take any $\tau > 0$ such that Lemma 4.3 applies. Let $1 \leq i \leq n-1$. Since the statement is obvious for $t = \tau$, it is sufficient to consider $t \in [0, \tau)$. By (5), it is enough to show that

$$r_i \geq c r_{n1} \tag{17}$$

for some $c > 0$. Note that for $n = 2$ we simply have $r_i = r_{n1}$, and thus we may assume $n \geq 3$.

We prove (17) by induction over finitely many steps. The induction statement is as follows: if $r_i \geq c r_{jk}$ for some $c > 0$ and some integers k, j satisfying $1 \leq k \leq i < j \leq n$ with $(j, k) \neq (n, 1)$, then there exists $c' > 0$ such that either

$$j \leq n-1 \qquad \text{and} \qquad r_i \geq c' r_{j+1,k}, \qquad \text{or} \tag{18a}$$

$$k \geq 2 \qquad \text{and} \qquad r_i \geq c' r_{j,k-1}. \tag{18b}$$

Observe that iterating the induction statement $(n-2)$ times yields (17). The conditions $(j, i) \neq (n, 1)$, $j \leq n-1$, and $k \geq 2$ are of little importance; they simply ensure that the induction does not go beyond the end particles x_1 and x_n.

Initially, i.e., for $k = i$ and $j = i + 1$, the condition in the induction statement is trivially satisfied with $c = 1$. Let the condition in the induction statement be satisfied for some c, j, k. By Lemma 4.3 we have

$$r_{jk} \geq c' \min\{r_{k-1}, r_j\}. \tag{19}$$

Suppose that the minimum is attained at r_j (note that this implies $j \leq n-1$). Then by the induction statement

$$r_{j+1,k} = r_j + r_{jk} \leq \Big(\frac{1}{c'} + 1\Big) r_{jk} \leq \frac{1}{c}\Big(\frac{1}{c'} + 1\Big) r_i,$$

and thus (18a) is satisfied. If the minimum in (19) is instead attained at r_{k-1}, then a similar argument shows that (18b) holds. □

Corollary 4.5. *For τ small enough and all $i = 1, \ldots, n-1$ we have that $r_i \in C^{1/(1+a)}([0, \tau])$.*

Proof. Take any $\tau > 0$ such that Lemma 4.4 applies. Let $1 \leq i \leq n-1$. Using Lemma 3.1 and Lemma 4.4 we obtain

$$\dot{r}_i \geq -\frac{2}{r_i^a} - C \geq \frac{-C'}{(\tau - t)^{\frac{a}{a+1}}}.$$

Then, using Lemma 3.2 we obtain

$$\dot{r}_i \leq -\frac{2}{r_i^a} + \frac{2}{r_{i-1}^a} + \frac{2}{r_{i+1}^a} + 2C \leq 0 + \frac{C'}{(\tau - t)^{\frac{a}{a+1}}}.$$

For any $0 \leq s < t \leq \tau$ we obtain

$$|r_i(t) - r_i(s)| \leq \int_s^t |\dot{r}_i(h)|\, dh \leq \int_s^t \frac{C}{(\tau - h)^{\frac{a}{a+1}}}\, dh = C'\Big((\tau - s)^{\frac{1}{a+1}} - (\tau - t)^{\frac{1}{a+1}}\Big)$$

$$= C' \frac{t-s}{(\tau-s)^{\frac{a}{a+1}} + (\tau-t)^{\frac{a}{a+1}}} \leq C' \frac{t-s}{(t-s)^{\frac{a}{a+1}}} = C'(t-s)^{\frac{1}{a+1}}.$$

□

Finally, with Corollary 4.5 we prove Theorem 1.2:

Proof of Theorem 1.2. Consider the setting in Claim 2.1 and take $\tau > 0$ such that Corollary 4.5 holds. It is left to prove that $x_i \in C^{1/(1+a)}([0,\tau])$ for all $i = 1, \dots, n$.

Let $M = \sum_{i=1}^n x_i$. Consider the variable transformation from $(x_1, \dots, x_n)$ to $(M, r_1, r_2, \dots, r_{n-1})$. Note that this transformation is linear and bijective. Hence, it is sufficient to show that $M, r_1, r_2, \dots, r_{n-1} \in C^{1/(1+a)}([0,\tau])$. For the variables r_i this is given by Corollary 4.5. For M we compute from (7) (note that f is odd) $\dot{M} = \sum_{i=1}^n F_i$, which is uniformly bounded on $(0,\tau)$. Hence, M is Lipschitz continuous on $[0,\tau]$. □

Author contributions

All authors equally contributed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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