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Research article

The sharp lifespan for a system of multiple speed wave equations: Radial case

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Abstract: Ohta examined a system of multiple speed wave equations with small initial data and demonstrated a finite time blowup. We show, in the radial case, that the same system exists almost globally with the same lifespan as a lower bound. To do this, we use integrated local energy estimates, r^p weighted local energy estimates, the Morawetz estimate that results from using the scaling vector field as a multiplier, and mixed-speed ghost weights.

Keywords: wave equations; local energy estimates; multiple speed; almost global existence
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1. Introduction

This article focuses on establishing the sharp lifespan, in the radial case, for a multiple-speed system of wave equations with small initial data introduced in [1]. Based on [2], we know that there exists a constant c so that quasilinear wave equations of the form

$$\begin{cases} \Box u = Q(u, \partial u, \partial^2 u), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ u(0,\,\cdot\,) = \varepsilon f, & \partial_t u(0,\,\cdot\,) = \varepsilon g, \end{cases} \tag{1.1}$$

where $f, g \in C_c^\infty(\mathbb{R}^3)$ and Q vanishes to second order at the origin, have solutions on $[0, T_*]$ with $T_* \geq c/\varepsilon^2$ if ε is sufficiently small. With the additional assumption that $(\partial_u^2 Q)(0,0,0) = 0$, which in essence rules out u^2 terms and leaves $u\partial u$ nonlinearities at the lowest order, almost global existence $T_* \geq \exp(c/\varepsilon)$ was proved. The latter result was partly extended to systems of equations in [3].

In the case of multiple-speed systems of wave equations

$$\Box_{c_I} u^I = Q^I(\partial u, \partial^2 u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3,$$

almost global existence was established in [4]. Here

$$\Box_c = \partial_t^2 - c^2 \Delta$$

denotes the d'Alembertian at speed c. Like in the single-speed case [5, 6], if the nonlinearity satisfies a null condition, then global existence can be recovered. In the semilinear case, if

$$Q^I(\partial u) = B^{I,\alpha\beta}_{JK}\partial_\alpha u^J \partial_\beta u^K + C(\partial u),$$

where C vanishes to third order and repeated variables are implicitly summed using the Einstein convention, the null condition is only necessary when $c_I = c_J = c_K$. See, e.g., [7–11]. The reason that it suffices to only have an assumption on the same-speed interactions is that solutions to the wave equation enjoy additional decay off of the light cone, and when there are differing speeds, one of the factors will be away from its light cone, thus contributing to more rapid decay.

For multiple speed analogs of (1.1) with $(\partial_u^2 Q)(0,0,0) = 0$, one may wonder if global existence can be recovered provided no quadratic nonlinear term in the equation for u^I has factors that are both at the same speed c_I. More precisely, if we truncate to quadratic level for semilinear equations

$$\Box_{c_I} u^I = A^{I,\alpha}_{JK} u^J \partial_\alpha u^K + B^{I,\alpha\beta}_{JK}\partial_\alpha u^J \partial_\beta u^K,$$

it may be reasonable to expect global existence provided that

$$A^{I,\alpha}_{JK} = 0 = B^{I,\alpha\beta}_{JK}, \quad \text{whenever } c_I = c_J = c_K. \tag{1.2}$$

In a somewhat surprising result, [1] demonstrated that (1.2) is not a sufficient condition for global existence for small initial data. Indeed, for $c > 1$, the following system was considered:

$$\begin{cases} \Box v = w\partial_t v, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \Box_c w = (\partial_t v)^2, & \\ v(0,\,\cdot\,) = \varepsilon v_{(0)}, & \partial_t v(0,\,\cdot\,) = \varepsilon v_{(1)}, \\ w(0,\,\cdot\,) = \varepsilon w_{(0)}, & \partial_t w(0,\,\cdot\,) = \varepsilon w_{(1)}. \end{cases} \tag{1.3}$$

It was established that there are smooth, compactly supported initial data $v_{(j)}, w_{(j)}$ and a constant c_0 so that the lifespan T_* satisfies $T_* \le \exp(c_0/\varepsilon^2)$.

The current study seeks to show the reverse inequality. That is, for any data $v_{(0)}, v_{(1)}, w_{(0)}, w_{(1)}$ that are smooth and compactly supported, we seek to show that there is a constant $\tilde{c}$ so that $T_\varepsilon \ge \exp(\tilde{c}/\varepsilon^2)$. Without loss of generality, we take the supports of the data to be contained within the unit ball $\{|x| \le 1\}$. We shall also use time translation symmetry and take the initial data on the time slice $t = 4$. In the current article we consider only the radial case: $v_{(j)}(x) = v_{(j)}(|x|)$, $w_{(j)}(x) = w_{(j)}(|x|)$.

Using the assumption of radial symmetry, we can reduce the question at hand to a problem in $(1+1)$-dimensions. Indeed, by conjugating, we have

$$\Box_c w(t,r) = r^{-1}(\partial_t^2 - c^2\partial_r^2)(rw).$$

If we set $V(t,r) = rv(t,r)$ and $W(t,r) = rw(t,r)$, we can instead seek sufficiently regular solutions to the $(1+1)$-dimensional initial-value boundary-value problem

$$\begin{cases} \Box V = r^{-1} W\partial_t V, & (t,r) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ \Box_c W = r^{-1}(\partial_t V)^2, & \\ W(t,0) = V(t,0) = 0, & \text{for all } t, \\ V(4,\,\cdot\,) = \varepsilon V_{(0)}, & \partial_t V(4,\,\cdot\,) = \varepsilon V_{(1)}, \\ W(4,\,\cdot\,) = \varepsilon W_{(0)}, & \partial_t W(4,\,\cdot\,) = \varepsilon W_{(1)}. \end{cases} \tag{1.4}$$

Here (when applied to V, W) we understand $\Box_c = \partial_t^2 - c^2\partial_r^2$ to be the $(1+1)$-dimensional d'Alembertian.

If we extend $V_{(j)}$, $W_{(j)}$ in an odd fashion, $V_{(j)}(-r) = -V_{(j)}(r)$, $W_{(j)}(-r) = -W_{(j)}(r)$, then it is straightforward to check that V, W extend oddly, and we can instead seek to solve

$$\begin{cases} \Box V = x^{-1} W \partial_t V, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ \Box_c W = x^{-1}(\partial_t V)^2, \\ V(4,\,\cdot\,) = \varepsilon V_{(0)}, \quad \partial_t V(4,\,\cdot\,) = \varepsilon V_{(1)}, \\ W(4,\,\cdot\,) = \varepsilon W_{(0)}, \quad \partial_t W(4,\,\cdot\,) = \varepsilon W_{(1)}. \end{cases} \tag{1.5}$$

The main theorem is a statement of almost global existence for (1.5).

Theorem 1.1. *Suppose that* $V_{(j)}, W_{(j)} \in C_c^\infty(\mathbb{R})$ *and* $V_{(j)}(x) = -V_{(j)}(-x), W_{(j)}(x) = -W_{(j)}(-x)$ *for* $j = 0, 1$. *Then there exist constants* $\tilde{c}, \varepsilon_0 > 0$ *such that when* $0 < \varepsilon < \varepsilon_0$, (1.5) *has a unique solution* $(V, W) \in C^\infty([4, T_\varepsilon] \times \mathbb{R})$, *where*

$$T_\varepsilon = \exp(\tilde{c}/\varepsilon^2). \tag{1.6}$$

As indicated above, Theorem 1.1 leads to the following corollary, showing that, in the radial case, the upper bound on the lifespan in [1] is sharp.

Corollary 1.2. *Suppose that* $v_{(j)}, w_{(j)} \in C_c^\infty(\mathbb{R}^3)$, *and that* $v_{(j)}(x) = v_{(j)}(|x|), w_{(j)}(x) = w_{(j)}(|x|)$ *for* $j = 0, 1$. *Then there exist constants* $\tilde{c}, \varepsilon_0 > 0$ *such that when* $0 < \varepsilon < \varepsilon_0$, (1.3) *has a unique solution* $(v, w) \in C^\infty([0, T_\varepsilon] \times \mathbb{R}^3)$ *with* T_ε *as in* (1.6).

As the current study is a proof of concept, we, for simplicity of exposition, restrict ourselves to smooth and compactly supported initial data. The current proof relies on the smallness of the compactly supported data in an H^{11} norm. As the techniques will not approach sharp regularity, we have not endeavored to minimize this. The assumption of compact support can very likely be dropped as long as the data decay sufficiently fast at infinity. While anticipated to only be technical in nature, this change would add some complication to the decompositions used in the next section as the two speeds interact outside of the speed 1 light cone.

Our proof relies on the method of invariant vector fields with adaptations to a restricted set of vector fields, which is necessitated as the Lorentz boosts have an associated speed and only commute with the d'Alembertian of the same speed. This was pioneered in, e.g., [4, 10–16]. The method is further simplified in the radial case, as the only relevant vector fields are derivatives and the scaling vector field. And in the study at hand, we will rely solely on the scaling vector field to avoid issues with commuting derivatives with the singular weight introduced in the one-dimensional reduction.

Like [12] and [15], we will call upon a class of integrated local energy estimates in order to prove long-time existence. These estimates go back to the seminal work [17]. In the current setting, a number of variants are used. These include the r^p-weighted estimates of [18], ghost-weighted estimates from [19], and integrated estimate variants of [20], where the scaling vector field is used as a multiplier.

To obtain the requisite decay, rather than using the classical Klainerman–Sobolev estimate [21], which introduces Lorentz boosts, or the weighted Sobolev estimate [21], which provides decay in $|x|$ but fails to capture the added decay off of the light cone, as was done in [12] and [15], we will instead use a class of space-time Klainerman–Sobolev estimates of [22]. The space-time nature of these decay bounds meshes well with the integrated local energy estimates.

In the estimates that we prove, we allow for ghost weights that are associated with a different speed than the equation. This works in the non-radial case (see the forthcoming work [23]) provided that the speed within the ghost weight exceeds that of the speed of the equation. In the radial case, however, no restriction on the speeds is needed. This is the primary place where we rely upon the radiality assumption. We anticipate examining the general case in a future study.

After posting this result, we learned of the paper [24]. Corollary 1.2 is a direct consequence of the main result of [24]. We thank Mengyun Liu and Chengbo Wang for alerting us to this. Our physical space techniques are quite distinct from the approach of [24], which is rooted in the fundamental solution. We believe the current methods present a path toward dropping the assumption of radial symmetry and to the study of these equations in the presence of background geometry. As such, we believe that the proof contained herein is of independent interest.

2. Notation and decay estimates

For $x \in \mathbb{R}$, we denote $r = |x|$. We will use the standard null coordinates

$$u = t - r, \quad \underline{u} = t + r, \quad \partial_u = \frac{1}{2}(\partial_t - \partial_r), \quad \partial_{\underline{u}} = \frac{1}{2}(\partial_t + \partial_r).$$

The above correspond to speed 1, but at wave speed c, we will instead have

$$u_c = ct - r, \quad \underline{u}_c = ct + r, \quad \partial_{u_c} = \frac{1}{2c}(\partial_t - c\partial_r), \quad \partial_{\underline{u}_c} = \frac{1}{2c}(\partial_t + c\partial_r).$$

We denote the scaling vector field by

$$S = t\partial_t + r\partial_r = u\partial_u + \underline{u}\partial_{\underline{u}} = u_c\partial_{u_c} + \underline{u}_c\partial_{\underline{u}_c}. \tag{2.1}$$

It will be important for later purposes to note

$$[\partial_{u_c}, S] = \partial_{u_c}, \quad [\partial_{\underline{u}_c}, S] = \partial_{\underline{u}_c}, \quad [\Box, S] = 2\Box.$$

For $k \in \mathbb{N}$, we use the notation

$$|S^{\le k} w| = \sum_0^k |S^j w|.$$

The space-time Klainerman–Sobolev-type estimates of [22] will be the principal source of decay. See also [25]. The only difference herein is allowing for $c \neq 1$ and simplifications that result from the $(1 + 1)$-dimensional regime. As such, we will be brief in the presentation.

If at speed c, the compactly supported data (at $t = 4$) are taken to be supported in the unit ball, then the components of the solution will be supported in $C^c = \{(t, x) \,:\, t \in [4, T_\varepsilon],\ r \le ct - (4c - 1)\}$. We now dyadically decompose C^c in both t and in either r or u_c depending on the proximity to the light cone. We first decompose in t and set

$$C^c_\tau = \{(t, r) \in \mathbb{R}_+ \times \mathbb{R}_+ \,:\, t \in [4, T_\varepsilon] \cap [\tau, 2\tau],\ r \le ct - (4c - 1)\}.$$

We further break into

$$C^{c,R=1}_\tau = C^c_\tau \cap \{r \le 2\}, \quad C^{c,R}_\tau = C^c_\tau \cap \{R \le r \le 2R\} \text{ when } 1 < R,$$

and

$$C_\tau^{c,U_c} = C_\tau^c \cap \{U_c \le ct - r \le 2U_c\} \text{ when } 1 < U_c.$$

We finally set

$$C_\tau^{c,c\tau/2} = C_\tau^c \cap \{ct - r \ge c\tau/2\} \cap \{r \ge c\tau/2\}.$$

Throughout $\tau, R, U_c, U_1 := U$ will be understood to range over dyadic values. This gives

$$C_\tau^c = \Big(\bigcup_{1 \le R \le c\tau/4} C_\tau^{c,R} \Big) \cup \Big(\bigcup_{2 \le U_c \le c\tau/4} C_\tau^{c,U_c} \Big) \cup C_\tau^{c,c\tau/2}. \tag{2.2}$$

We will let $\tilde{C}_\tau^{c,R}$ and $\tilde{C}_\tau^{c,U_c}$ denote slight enlargements (in both scales) to allow for tails of cutoff functions. On the components of the decomposition (and their enlargements), we have

$$\langle r \rangle \approx R, \quad t \approx \tau, \quad \langle u_c \rangle \approx \tau \quad \text{ on } C_\tau^{c,R},\ 1 \le R \le c\tau/4,$$

$$r \approx \tau, \quad t \approx \tau, \quad \langle u_c \rangle \approx U_c \quad \text{ on } C_\tau^{c,U_c},\ 2 \le U_c \le c\tau/4.$$

The remaining region $C_\tau^{c,c\tau/2}$ may be thought of as either $R = c\tau/2$ or $U_c = c\tau/2$.

The following are space-time analogs of the Klainerman–Sobolev estimates:

Lemma 2.1 ([22]). *Suppose $W \in C^2([4, T_\varepsilon] \times \mathbb{R})$ is an odd function. If $c > 0$, $\tau \ge 4$, $1 \le R \le c\tau/2$, $2 \le U_c \le c\tau/4$, then*

$$\|r^{-\frac{1}{2}} W\|_{L_t^\infty L_r^\infty(C_\tau^{c,R})} \lesssim \frac{1}{\tau^{\frac{1}{2}} R} \|S^{\le 1} W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,R})} + \frac{1}{\tau^{\frac{1}{2}}} \|\partial_r S^{\le 1} W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,R})}, \tag{2.3}$$

$$\|W\|_{L_t^\infty L_r^\infty(C_\tau^{c,U_c})} \lesssim \frac{1}{\tau^{\frac{1}{2}} U_c^{\frac{1}{2}}} \|S^{\le 1} W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,U_c})} + \frac{U_c^{\frac{1}{2}}}{\tau^{\frac{1}{2}}} \|\partial_r S^{\le 1} W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,U_c})}. \tag{2.4}$$

Here and throughout, L_r^2 is simply the 1-dimensional norm:

$$\|f\|_{L_r^2}^2 = \int_0^\infty |f(r)|^2 \, dr.$$

Proof. On C_τ^{c,U_c} regions and on $C_\tau^{c,R}$ regions for $R > 1$, the result follows from [22] in one spatial dimension. On the $C_\tau^{c,R=1}$ regions, however, a different change of variables is required to avoid picking up vector fields other than S. Some care is also taken to assist with the singular behavior at $r = 0$ that was introduced in the reduction to one dimension.

Let $\beta : [0, \infty) \to [0, 1]$ be a smooth cutoff function such that $\beta(y) = 1$ for $y \in [1, 2]$ and $\beta(y) = 0$ for $y \in [0, 1 - \delta] \cup [2 + \delta, \infty)$, where $0 < \delta \ll 1$. We examine $\beta(t/\tau) W(t, r)$. We first change variables to $t = e^s$ and $r = \rho e^s$. Applying the Fundamental Theorem of Calculus in s and ρ (relying upon the fact that $S^{\le 1} W$ is odd and hence vanishes at $r = 0$), we have

$$|\beta(e^s/\tau) W(e^s, \rho e^s)| \lesssim \int_0^\rho \int_{-\infty}^{\log t} |\partial_\zeta \partial_s (\beta(e^s/\tau) W(e^s, \zeta e^s))| \, ds \, d\zeta.$$

Since $\partial_s(W(e^s, \zeta e^s)) = (S W)(e^s, \zeta e^s)$, upon converting back to (t, r)-coordinates, we have

$$|\beta(t/\tau)W(t,r)| \lesssim \tau^{-2} \int\int_{[\tau(1-\delta),t]\times[0,r]} |S^{\le 1}W(t,z)|\,dz\,dt \\ + \tau^{-1} \int\int_{[\tau(1-\delta),t]\times[0,r]} |(\partial_r S^{\le 1}W)(t,z)|\,dz\,dt. \quad (2.5)$$

Applying the Schwarz inequality to the right side and multiplying through by $r^{-\frac{1}{2}}$ then yields

$$\|r^{-\frac{1}{2}}W\|_{L_t^\infty L_r^\infty(C_\tau^{c,R=1})} \lesssim \frac{1}{\tau^{\frac{3}{2}}}\|S^{\le 1}W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,R=1})} + \frac{1}{\tau^{\frac{1}{2}}}\|\partial_r S^{\le 1}W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,R=1})},$$

which is stronger than (2.3) when $R = 1$. □

By writing $\partial_r = \partial_{\underline{u}_c} - \partial_{u_c}$ and using (2.1), we can obtain the following corollary. See, e.g., [25].

Corollary 2.2. *Suppose $W \in C^2([4, T_\varepsilon] \times \mathbb{R})$ is an odd function. If $c > 0$, $\tau \ge 4$, $1 \le R \le c\tau/2$, $2 \le U_c \le c\tau/4$, then*

$$\|r^{-\frac{1}{2}}W\|_{L_t^\infty L_r^\infty(C_\tau^{c,R})} \lesssim \frac{1}{\tau^{\frac{1}{2}}R}\|S^{\le 2}W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,R})} + \frac{1}{\tau^{\frac{1}{2}}}\|\partial_{\underline{u}_c} S^{\le 1}W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,R})}, \quad (2.6)$$

$$\|W\|_{L_t^\infty L_r^\infty(C_\tau^{c,U_c})} \lesssim \frac{1}{\tau^{\frac{1}{2}}U_c^{\frac{1}{2}}}\|S^{\le 2}W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,U_c})} + \frac{\tau^{\frac{1}{2}}}{U_c^{\frac{1}{2}}}\|\partial_{\underline{u}_c} S^{\le 1}W\|_{L_t^2 L_r^2(\tilde{C}_\tau^{c,U_c})}. \quad (2.7)$$

Using (2.1), we also obtain the following bounds on $\partial V := (\partial_t, \partial_r)V$, which appeared previously in [25]. They are space-time variants of estimates originally from [4] and [26].

Lemma 2.3. *Suppose $V \in C^3([4, T_\varepsilon] \times \mathbb{R})$ is an odd function. If $\tau \ge 4$, $1 \le R \le \tau/2$, and $2 \le U \le \tau/4$, then*

$$\|V\|_{L_t^\infty L_r^\infty(C_\tau^{1,R})} \lesssim \frac{1}{\tau^{\frac{1}{2}}R^{\frac{1}{2}}}\|S^{\le 2}V\|_{L_t^2 L_r^2(\tilde{C}_\tau^{1,R})} + \frac{R^{\frac{1}{4}}}{\tau^{\frac{1}{2}}}\|r^{\frac{1}{4}}\partial_{\underline{u}} S^{\le 1}V\|_{L_t^2 L_r^2(\tilde{C}_\tau^{1,R})}, \quad (2.8)$$

$$\|V\|_{L_t^\infty L_r^\infty(C_\tau^{1,U})} \lesssim \frac{1}{\tau^{\frac{1}{2}}U^{\frac{1}{2}}}\|S^{\le 2}V\|_{L_t^2 L_r^2(\tilde{C}_\tau^{1,U})} + \frac{\tau^{\frac{1}{2}}}{U^{\frac{1}{2}}}\|\partial_{\underline{u}} S^{\le 1}V\|_{L_t^2 L_r^2(\tilde{C}_\tau^{1,U})}, \quad (2.9)$$

$$\|\partial V\|_{L_t^\infty L_r^\infty(C_\tau^{1,R})} \lesssim \frac{1}{\tau^{\frac{1}{2}}R^{\frac{1}{2}}}\|\partial S^{\le 2}V\|_{L_t^2 L_r^2(\tilde{C}_\tau^{1,R})} + \frac{R^{\frac{1}{4}}}{\tau^{\frac{1}{2}}}\|r^{\frac{1}{4}}\Box S^{\le 1}V\|_{L_t^2 L_r^2(\tilde{C}_\tau^{1,R})}, \quad (2.10)$$

$$\|\partial V\|_{L_t^\infty L_r^\infty(C_\tau^{1,U})} \lesssim \frac{1}{\tau^{\frac{1}{2}}U^{\frac{1}{2}}}\|\partial S^{\le 2}V\|_{L_t^2 L_r^2(\tilde{C}_\tau^{1,U})} + \frac{\tau^{\frac{1}{2}}}{U^{\frac{1}{2}}}\|\Box S^{\le 1}V\|_{L_t^2 L_r^2(\tilde{C}_\tau^{1,U})}. \quad (2.11)$$

Proof. Outside of the $R = 1$ case, the above follow from [22] and [25]. When $R = 1$, we argue as in (2.5). At this point, a different application of the Schwarz inequality yields

$$\beta(t/\tau)V(t,r) \lesssim \frac{1}{\tau^{\frac{3}{2}}}\|S^{\le 1}V\|_{L_t^2 L_r^2(\tilde{C}_\tau^{1,R=1})} + \frac{r^{\frac{1}{4}}}{\tau^{\frac{1}{2}}}\|\tilde{r}^{\frac{1}{4}}\partial_{\tilde{r}} S^{\le 1}V\|_{L_t^2 L_{\tilde{r}}^2(\tilde{C}_\tau^{1,R=1})}.$$

Using (2.1) then allows us to recover (2.8). A subsequent application of (2.1), as in [25], yields (2.10). □

It will be of utmost importance to track the availability to sum over the dyadic ranges R, τ, U, U_c. To this end, we shall use notation such as

$$\|W\|^2_{\ell^2_\tau \ell^2_{U\le\tau/4} L^2_t L^2_r(C^{1,U}_\tau)} = \sum_\tau \sum_{U\le\tau/4} \|W\|^2_{L^2_t L^2_r(C^{1,U}_\tau)}, \quad \|W\|^2_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)} = \sup_{U\ge 1} \sum_\tau \|W\|^2_{L^2_t L^2_r(C^{1,U}_\tau)}.$$

Other variants, such as over $C^{1,R}_\tau$, C^{c,U_c}_τ, etc., are similarly defined.

3. Local energy estimates

In this section we shall gather the energy estimates that will be used in the sequel. These are variants of the integrated local energy estimates, which originated in [17]. See also [12, 15, 27], and [28] for subsequent generalizations and applications to nonlinear equations. These ideas are combined with an integrated variant of the classical estimate obtained by using the scaling vector field as a multiplier, as was done in [20]. We also utilize a ghost weight, as introduced in [19]. Similar to [23], we use ghost weights where the speed does not necessarily coincide with the given equation. For the speed c component, our principal estimate is an r^p-weighted (with $p = 1$) integrated local energy estimate from [18]. It is combined with a ghost weight, as was done in [3]. We pair this with a Hardy inequality that takes advantage of the multiple speeds.

For weighting functions, we shall use, for a parameter $\theta \ge 1$,

$$\sigma_\theta(y) = \frac{y}{|y| + \theta}, \quad \sigma'_\theta(y) = \frac{\theta}{(|y| + \theta)^2}, \tag{3.1}$$

which is bounded and C^1. Moreover, we note that

$$\sigma'_\theta(y) \gtrsim \theta^{-1} \quad \text{on } \{\langle y\rangle \approx \theta\}. \tag{3.2}$$

In order to help control the singularity at $r = 0$ that results from the reduction to one dimension, we also note

$$\frac{d}{dr}(\sigma_R(r))^{\frac12} \gtrsim r^{-\frac12} R^{-\frac12} \quad \text{on } \{\langle r\rangle \approx R\}. \tag{3.3}$$

Using $(\sigma_R(r))^\delta$ with $0 < \delta < 1$ as a weight in order to gain added control at $r = 0$ has appeared previously in, e.g., [29].

We shall use the following proposition when $p = 0$ and $p = 1$. In the former case, this corresponds to the classical integrated local energy estimates and variants that are available using the ghost weight. For $p = 1$, these are instead variants of [20] where $S = t\partial_t + r\partial_r$ is used as a multiplier. The fact that no upper bound on p is necessary is a consequence of the radiality assumption. The angular terms necessitate $p \le 2$ in more general cases. We start with a lemma, which indicates that the "good" derivative portion of the calculation works independently.

Lemma 3.1. *Suppose that $p \ge 0$, $V \in C^2([4, T] \times \mathbb{R})$, and that there exists $\tilde R > 0$ so that $V(t, x) \equiv 0$ whenever $|x| > \tilde R$. Then*

$$\|r^{-\frac14}\langle r\rangle^{-\frac14}\langle \underline{u}\rangle^{\frac p2}\partial_{\underline{u}} V\|^2_{\ell^\infty_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)} + \|\langle u\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac p2}\partial_{\underline{u}} V\|^2_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)} + \|\langle u_c\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac p2}\partial_{\underline{u}} V\|^2_{\ell^\infty_{U_c} \ell^2_\tau L^2_t L^2_r(C^{c,U_c}_\tau)}$$
$$\lesssim \|\langle r\rangle^{\frac p2}\partial_{\underline{u}} V(4, \cdot)\|^2_{L^2_r} + \int_4^T \int_0^\infty \langle \underline{u}\rangle^p |\Box V| |\partial_{\underline{u}} V| \, dr\, dt. \tag{3.4}$$

Proof. We consider

$$\int_4^T \int_0^\infty e^{(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_U(t-r)} e^{-\sigma_{U_c}(ct-r)}(1+t+r)^p \Box V(t,r)\,(\partial_t+\partial_r)V(t,r)\,dr\,dt$$
$$= \frac{1}{2}\int_4^T \int_0^\infty e^{(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_U(t-r)} e^{-\sigma_{U_c}(ct-r)}(1+t+r)^p(\partial_t-\partial_r)\Big((\partial_t+\partial_r)V\Big)^2\,dr\,dt. \quad (3.5)$$

Integrating by parts gives that this is equivalent to

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty e^{(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_U(T-r)} e^{-\sigma_{U_c}(cT-r)}(1+T+r)^p\Big((\partial_t+\partial_r)V(T,r)\Big)^2\,dr\\
&\quad - \frac{1}{2}\int_0^\infty e^{(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_U(4-r)} e^{-\sigma_{U_c}(4c-r)}(5+r)^p\Big((\partial_t+\partial_r)V(4,r)\Big)^2\,dr\\
&\quad + \frac{1}{2}\int_4^T e^{-\sigma_U(t)} e^{-\sigma_{U_c}(ct)}(1+t)^p(\partial_r V(t,0))^2\,dt\\
&\quad + \frac{1}{2}\int_4^T \int_0^\infty \frac{d}{dr}(\sigma_R(r))^{\frac{1}{2}} e^{(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_U(t-r)} e^{-\sigma_{U_c}(ct-r)}(1+t+r)^p\Big((\partial_t+\partial_r)V\Big)^2\,dr\,dt\\
&\quad + \int_4^T \int_0^\infty e^{(\sigma_R(r))^{\frac{1}{2}}} \sigma_U'(t-r) e^{-\sigma_U(t-r)} e^{-\sigma_{U_c}(ct-r)}(1+t+r)^p\Big((\partial_t+\partial_r)V\Big)^2\,dr\,dt\\
&\quad + \frac{c+1}{2}\int_4^T \int_0^\infty e^{(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_U(t-r)} \sigma_{U_c}'(ct-r) e^{-\sigma_{U_c}(ct-r)}(1+t+r)^p\Big((\partial_t+\partial_r)V\Big)^2\,dr\,dt. \quad (3.6)
\end{aligned}$$

We drop the non-negative first and third terms. We subsequently can restrict the range of the fourth, fifth, and sixth terms so that (3.2) can be applied. We also use that σ_θ is bounded uniformly in θ. For example,

$$\int_4^T \int_0^\infty e^{(\sigma_R(r))^{\frac{1}{2}}} \sigma_U'(t-r) e^{-\sigma_U(t-r)} e^{-\sigma_{U_c}(ct-r)}(1+t+r)^p\Big((\partial_t+\partial_r)V\Big)^2\,dr\,dt$$
$$\gtrsim U^{-1}\int\int_{\{\langle t-r\rangle\approx U\}}(1+t+r)^p\Big((\partial_t+\partial_r)V\Big)^2\,dr\,dt.$$

Taking appropriate supremums in R, U using the boundedness of σ_θ then yields (3.4). □

While the previous lemma worked independently, to get similar control on the remaining derivatives, we must examine both the "good" derivative $\partial_t + \partial_r$ and the "bad" derivative $\partial_t - \partial_r$ in unison. In the radial case, this results from the $r = 0$ boundary behavior. In more general situations, there is interaction amongst the angular behavior as well.

Proposition 3.2. *Suppose that $p \geq 0$, $V \in C^2([4,T]\times\mathbb{R})$, and that there exists $\tilde{R} > 0$ so that $V(t,x) \equiv 0$ whenever $|x| > \tilde{R}$. Then*

$$\begin{aligned}
&\|r^{-\frac{1}{4}}\langle r\rangle^{-\frac{1}{4}}\langle \underline{u}\rangle^{\frac{p}{2}}\partial_{\underline{u}}V\|^2_{\ell^\infty_R\ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)} + \|r^{-\frac{1}{4}}\langle r\rangle^{-\frac{1}{4}}\langle u\rangle^{\frac{p}{2}}\partial_u V\|^2_{\ell^\infty_R\ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)} + \|\langle u_c\rangle^{-\frac{1}{2}}\langle \underline{u}\rangle^{\frac{p}{2}}\partial_{\underline{u}}V\|^2_{\ell^\infty_{U_c}\ell^2_\tau L^2_t L^2_r(C^{c,U_c}_\tau)}\\
&\quad + \|\langle u_c\rangle^{-\frac{1}{2}}\langle u\rangle^{\frac{p}{2}}\partial_u V\|^2_{\ell^\infty_{U_c}\ell^2_\tau L^2_t L^2_r(C^{c,U_c}_\tau)} + \|\langle u\rangle^{-\frac{1}{2}}\langle \underline{u}\rangle^{\frac{p}{2}}\partial_{\underline{u}}V\|^2_{\ell^\infty_U\ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}\\
&\lesssim \|\langle r\rangle^{\frac{p}{2}}\partial V(4,\cdot)\|^2_{L^2_r} + \int_4^T \int_0^\infty \langle \underline{u}\rangle^p|\tilde{\Box}V||\partial_{\underline{u}}V|\,dr\,dt + \int_4^T \int_0^\infty \langle u\rangle^p|\tilde{\Box}V||\partial_u V|\,dr\,dt. \quad (3.7)
\end{aligned}$$

Proof. We argue much like in the preceding proof but instead examine

$$\int_4^T \int_0^\infty e^{-(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_{U_c}(ct-r)}(1+|t-r|)^p \Box V(t,r)\,(\partial_t - \partial_r)V(t,r)\,dr\,dt$$
$$= \frac{1}{2}\int_4^T \int_0^\infty e^{-(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_{U_c}(ct-r)}(1+|t-r|)^p(\partial_t+\partial_r)\Big((\partial_t-\partial_r)V\Big)^2\,dr\,dt. \quad (3.8)$$

Integrating by parts shows that this is equal to

$$\frac{1}{2}\int_0^\infty e^{-(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_{U_c}(cT-r)}(1+|T-r|)^p\Big((\partial_t-\partial_r)V(T,r)\Big)^2\,dr$$
$$-\frac{1}{2}\int_0^\infty e^{-(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_{U_c}(4c-r)}(1+|4-r|)^p\Big((\partial_t-\partial_r)V(4,r)\Big)^2\,dr - \frac{1}{2}\int_4^T e^{-\sigma_{U_c}(ct)}(1+t)^p(\partial_r V(t,0))^2\,dt$$
$$+\frac{1}{2}\int_4^T\int_0^\infty \frac{d}{dr}(\sigma_R(r))^{\frac{1}{2}} e^{-(\sigma_R(r))^{\frac{1}{2}}} e^{-\sigma_{U_c}(ct-r)}(1+|t-r|)^p\Big((\partial_t-\partial_r)V\Big)^2\,dr\,dt$$
$$+\frac{c-1}{2}\int_4^T\int_0^\infty e^{-(\sigma_R(r))^{\frac{1}{2}}}\sigma'_{U_c}(ct-r)e^{-\sigma_{U_c}(ct-r)}(1+|t-r|)^p\Big((\partial_t-\partial_r)V\Big)^2\,dr\,dt. \quad (3.9)$$

If it were not for the $r = 0$ boundary term, we could argue as above to obtain our estimate. Here we instead must sum (3.6) and (3.9). The former provides the desired control in the $r = 0$ boundary term. Then using (3.1) and (3.2) as in the preceding lemma concludes the proof. □

We next proceed with the speed c estimates. This first estimate combines the r^p weighting of [18] with the ghost weighting of [19], as was done in [3]. Here, however, we use a ghost weight at speed 1 despite working with a solution to a speed c wave equation.

Proposition 3.3. *Suppose that $c > 0$, $W \in C^2([4,T]\times\mathbb{R})$, and that there exists $\tilde{R} > 0$ so that $W(t,x) \equiv 0$ whenever $|x| > \tilde{R}$. Then*

$$\|\partial_{\underline{u}_c} W\|^2_{\ell^2_\tau \ell^2_R L^2_t L^2_r(C^{1,R}_\tau)} + \|\langle u\rangle^{-\frac{1}{2}} r^{\frac{1}{2}}\partial_{\underline{u}_c} W\|^2_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}$$
$$\lesssim \|r^{\frac{1}{2}}\partial_{\underline{u}_c} W(4,\,\cdot\,)\|^2_{L^2_r} + \int_4^T\int_0^\infty r|\Box_c W||\partial_{\underline{u}_c} W|\,dr\,dt. \quad (3.10)$$

Proof. We now start with

$$\int_4^T\int_0^\infty r e^{-\sigma_U(t-r)}\Box_c W(t,r)(\partial_t + c\partial_r)W(t,r)\,dr\,dt = \frac{1}{2}\int_4^T\int_0^\infty r e^{-\sigma_U(t-r)}(\partial_t - c\partial_r)\Big((\partial_t+c\partial_r)W\Big)^2\,dr\,dt,$$

which, upon integrating by parts, is seen to be the same as

$$\frac{1}{2}\int_0^\infty r e^{-\sigma_U(T-r)}\Big((\partial_t+c\partial_r)W(T,r)\Big)^2\,dr - \frac{1}{2}\int_0^\infty r e^{-\sigma_U(4-r)}\Big((\partial_t+c\partial_r)W(4,r)\Big)^2\,dr$$
$$+\frac{c}{2}\int_4^T\int_0^\infty e^{-\sigma_U(t-r)}\Big((\partial_t+c\partial_r)W\Big)^2\,dr\,dt + \frac{1+c}{2}\int_4^T\int_0^\infty r\sigma'_U(t-r)e^{-\sigma_U(t-r)}\Big((\partial_t+c\partial_r)W\Big)^2\,dr\,dt.$$

After noting the first term is non-negative, we may drop it. The proof is then completed by restricting the range of the last term so that (3.2) may be applied and using the boundedness of σ_θ. □

The preceding lemma will be paired with two Hardy inequalities as W appears in our nonlinearity without a derivative. The first is a space-time variant of the standard Hardy inequality. See, e.g., [3].

Lemma 3.4. *Suppose that $c > 0$, $W \in C^1([4,T] \times \mathbb{R})$ is an odd function, and that there exists $\tilde{R} > 0$ so that $W(t,x) \equiv 0$ whenever $|x| > \tilde{R}$. Then*

$$\| |x|^{-1} W \|_{L^2_t L^2_x([4,T]\times\mathbb{R})} \lesssim \| |x|^{-\frac{1}{2}} W(4,\,\cdot\,) \|_{L^2_x} + \|\partial_{\underline{u}_c} W\|_{L^2_t L^2_x([4,T]\times\mathbb{R})}. \tag{3.11}$$

Proof. We write

$$\int_4^T \int r^{-2} W^2 \, dx\, dt = -\frac{2}{c} \int_4^T \int_0^\infty (\partial_t + c\partial_r) r^{-1} \cdot W^2 \, dr\, dt.$$

As W is C^1 and odd, the Mean Value Theorem gives that $W = O(r)$ near $r = 0$. Thus, these integrals are well-defined and $r^{-1}W^2(t,r) \to 0$ as $r \to 0$. Integrating by parts reveals

$$\int_4^T \int r^{-2} W^2 \, dx\, dt + \frac{2}{c} \int_0^\infty r^{-1} (W(T,r))^2 \, dr$$
$$= \frac{2}{c} \int_0^\infty r^{-1} (W(4,r))^2 \, dr + \frac{4}{c} \int_4^T \int_0^\infty r^{-1} W (\partial_t + c\partial_r) W \, dr\, dt. \tag{3.12}$$

The Schwarz inequality shows that

$$\frac{4}{c} \int_4^T \int_0^\infty r^{-1} W (\partial_t + c\partial_r) W \, dr\, dt \lesssim \|r^{-1} W\|_{L^2_t L^2_x([4,T]\times\mathbb{R})} \|\partial_{\underline{u}_c} W\|_{L^2_t L^2_x([4,T]\times\mathbb{R})}$$
$$\le \frac{1}{2} \|r^{-1} W\|^2_{L^2_t L^2_x([4,T]\times\mathbb{R})} + C \|\partial_{\underline{u}_c} W\|^2_{L^2_t L^2_x([4,T]\times\mathbb{R})}.$$

Plugging this into (3.12) and absorbing the $\|r^{-1}W\|_{L^2_t L^2_x}$ term back into the left side yields the desired estimate. □

The second Hardy inequality that we rely upon takes advantage of the multiple speed structure.

Lemma 3.5. *Suppose $c > 1$, $W \in C^1([4,T] \times \mathbb{R})$ is an odd function, and that there exists $\tilde{R} > 0$ so that $W(t,x) \equiv 0$ when $|x| > \tilde{R}$. Then*

$$\| \langle u \rangle^{-\frac{1}{2}} |x|^{-\frac{1}{2}} W \|_{\ell^\infty_U \ell^2_\tau L^2_t L^2_x (C^{1,U}_\tau)} \lesssim \| |x|^{-\frac{1}{2}} W(4,\,\cdot\,) \|_{L^2_x} + \|\partial_{\underline{u}_c} W\|_{L^2_t L^2_x([4,T]\times\mathbb{R})}. \tag{3.13}$$

Proof. The argument here is similar to the preceding, but we take advantage of the difference in speeds between the weight $\langle u \rangle^{-1}$ and the speed c of the equation. To this end, using (3.2), we observe

$$\iint_{\{\langle t-r \rangle \approx U\}} \langle u \rangle^{-1} r^{-1} W^2 \, dx\, dt \le \frac{2}{1-c} \int_4^T \int_0^\infty r^{-1} (\partial_t + c\partial_r)(e^{\sigma_U(t-r)}) \cdot W^2 \, dr\, dt$$

and use integration by parts to see that the right side is equivalent to

$$\frac{2}{1-c} \int_0^\infty r^{-1} e^{\sigma_U(T-r)} (W(T,r))^2 \, dr + \frac{2}{c-1} \int_0^\infty r^{-1} e^{\sigma_U(4-r)} (W(4,r))^2 \, dr$$
$$- \frac{2c}{c-1} \int_4^T \int_0^\infty r^{-2} e^{\sigma_U(t-r)} W^2 \, dr\, dt + \frac{4}{c-1} \int_4^T \int_0^\infty r^{-1} e^{\sigma_U(t-r)} W (\partial_t + c\partial_r) W \, dr\, dt. \tag{3.14}$$

Using the Cauchy-Schwarz inequality to bound

$$\begin{aligned}\frac{4}{c-1}\int_4^T\int_0^\infty r^{-1}e^{\sigma_U(t-r)}W(\partial_t+c\partial_r)W\,dr\,dt\\ \le \frac{c}{2(c-1)}\int_4^T\int_0^\infty r^{-2}e^{\sigma_U(t-r)}W^2\,dr\,dt + C\int_4^T\int_0^\infty e^{\sigma_U(t-r)}(\partial_{\underline{u}_c}W)^2\,dr\,dt,\end{aligned}\tag{3.15}$$

plugging this into (3.14), and omitting nonpositive terms from the right produces the desired result. □

4. Proof of Theorem 1.1

To solve (1.5), we set up an iteration. Let $W_0 \equiv V_0 \equiv 0$, and let V_j, W_j solve

$$\begin{cases}\Box V_j = x^{-1}W_{j-1}\partial_t V_{j-1},\\ \Box_c W_j = x^{-1}(\partial_t V_{j-1})^2,\\ V_j(4,\,\cdot\,) = \varepsilon V_{(0)}, \quad \partial_t V_j(4,\,\cdot\,) = \varepsilon V_{(1)},\\ W_j(4,\,\cdot\,) = \varepsilon W_{(0)}, \quad \partial_t W_j(4,\,\cdot\,) = \varepsilon W_{(1)}.\end{cases}\tag{4.1}$$

We shall show that the sequence $((V_j, W_j))_{j=0}^\infty$ is Cauchy on $[4, T_\varepsilon]\times\mathbb{R}$ in an appropriate sense, and by standard results the limit is the desired solution.

4.1. Boundedness

We first show a uniform boundedness of the sequence $((V_j, W_j))$. Let

$$\begin{aligned}M_j &= \|r^{-\frac14}\langle r\rangle^{-\frac14}\langle u\rangle^{\frac12}S^{\le 7}\partial_u V_j\|_{\ell^\infty_R\ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)} + \|\langle u_c\rangle^{-\frac12}\langle u\rangle^{\frac12}S^{\le 7}\partial_u V_j\|_{\ell^\infty_{U_c}\ell^2_\tau L^2_t L^2_r(C^{c,U_c}_\tau)}\\
&\quad + \|r^{-\frac14}\langle r\rangle^{-\frac14}\langle \underline{u}\rangle^{\frac12}S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^\infty_R\ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)} + \|\langle u_c\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12}S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^\infty_{U_c}\ell^2_\tau L^2_t L^2_r(C^{c,U_c}_\tau)}\\
&\quad + \|\langle u\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12}S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^\infty_U\ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)} + \|r^{-\frac14}\langle r\rangle^{-\frac14}S^{\le 10}\partial V_j\|_{\ell^\infty_R\ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)}\\
&\quad + \|\langle u_c\rangle^{-\frac12}S^{\le 10}\partial V_j\|_{\ell^\infty_{U_c}\ell^2_\tau L^2_t L^2_r(C^{c,U_c}_\tau)} + \|\langle u\rangle^{-\frac12}S^{\le 10}\partial_{\underline{u}} V_j\|_{\ell^\infty_U\ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}\\
&\quad + \|r^{\frac12}\langle u\rangle^{-\frac12}S^{\le 10}\partial_{\underline{u}_c} W_j\|_{\ell^\infty_U\ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)} + \|r^{-\frac12}\langle u\rangle^{-\frac12}S^{\le 10} W_j\|_{\ell^\infty_U\ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}\\
&\quad + \|S^{\le 10}\partial_{\underline{u}_c} W_j\|_{\ell^2_R\ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)} + \|r^{-1}S^{\le 10} W_j\|_{\ell^2_R\ell^2_\tau L^2_t L^2_r(C^{c,R}_\tau)}.\end{aligned}\tag{4.2}$$

We will label these terms $(I)_j, (II)_j, \dots, (XII)_j$.

Since $\Box V_1 \equiv \Box_c W_1 \equiv 0$, the smallness of the initial data along with (3.7), (3.10), (3.11), and (3.13) imply that there exists C_0 so that

$$M_1 \le C_0\varepsilon.$$

Moreover, V_1 is supported within C^1. Under the assumption that

$$M_k \le 2C_0\varepsilon, \quad \text{for } 0 \le k \le j-1,\tag{4.3}$$

and that supp $V_{j-1} \subset C^1$, we shall show that there exists a fixed constant C such that

$$M_j^2 \leq (C_0\varepsilon)^2 + C(2C_0\varepsilon)^2(\log(2+T_\varepsilon))^{\frac{1}{2}} M_j \tag{4.4}$$

for $(t,r) \in [4, T_\varepsilon] \times \mathbb{R}_+$. It will also follow immediately that supp $V_j \subset C^1$. Using (1.6) and absorbing the M_j back into the left side, this implies

$$M_j^2 \leq 2(C_0\varepsilon)^2 + \tilde{C}\tilde{c}\varepsilon^2.$$

Provided the $\tilde{c}$ in (1.6) is chosen to be sufficiently small, this shows that

$$M_j \leq 2C_0\varepsilon$$

and completes the inductive proof of boundedness.

4.1.1. Decay bounds

To aid in the proof of (4.4), we shall first show some auxiliary decay bounds provided (4.3) holds. In particular, we start with proofs that, for $k \leq j-1$,

$$\|\langle \underline{u}_c\rangle^{\frac{1}{2}} r^{-\frac{1}{2}} S^{\leq 7} W_k\|_{\ell^2_\tau \ell^2_{R\leq c\tau/2} L^\infty_t L^\infty_r (C^{c,R}_\tau)} \lesssim \varepsilon, \tag{4.5}$$

$$\|\langle u_c\rangle^{\frac{1}{2}} \langle r\rangle^{-\frac{1}{2}} S^{\leq 7} W_k\|_{\ell^2_\tau \ell^2_{U_c\leq c\tau/4} L^\infty_t L^\infty_r (C^{c,U_c}_\tau)} \lesssim \varepsilon, \tag{4.6}$$

$$\|\langle \underline{u}\rangle S^{\leq 5} \partial V_k\|_{\ell^\infty_R \ell^2_{\tau\geq 2R} L^\infty_t L^\infty_r (C^{1,R}_\tau)} \lesssim \varepsilon, \tag{4.7}$$

$$\|\langle u\rangle S^{\leq 5} \partial V_k\|_{\ell^\infty_U \ell^\infty_{\tau\geq 4U} L^\infty_t L^\infty_r (C^{1,U}_\tau)} \lesssim \varepsilon. \tag{4.8}$$

We shall provide the proofs of these in order. The first two are easy corollaries of Corollary 2.2.

Proof of (4.5). This is an immediate consequence of (2.6) and (4.3). We apply (2.6) to $S^{\leq 7} W_k$ and note that if W_k is odd, then so is $S W_k$. Then, indeed,

$$\begin{aligned}
&\|\langle \underline{u}_c\rangle^{\frac{1}{2}} r^{-\frac{1}{2}} S^{\leq 7} W_k\|_{\ell^2_\tau \ell^2_{R\leq c\tau/2} L^\infty_t L^\infty_r (C^{c,R}_\tau)} \\
&\qquad \lesssim \|\langle r\rangle^{-1} S^{\leq 9} W_k\|_{\ell^2_\tau \ell^2_{R\leq c\tau/2} L^2_t L^2_r (\tilde{C}^{c,R}_\tau)} + \|S^{\leq 8} \partial_{\underline{u}_c} W_k\|_{\ell^2_\tau \ell^2_{R\leq c\tau/2} L^2_t L^2_r (\tilde{C}^{c,R}_\tau)} \\
&\qquad \lesssim (XII)_k + (XI)_k,
\end{aligned}$$

and the bound follows from (4.3). □

Proof of (4.6). Applying (2.7), we have

$$\begin{aligned}
&\|\langle u_c\rangle^{\frac{1}{2}} \langle r\rangle^{-\frac{1}{2}} S^{\leq 7} W_k\|_{\ell^2_\tau \ell^2_{U_c\leq c\tau/4} L^\infty_t L^\infty_r (C^{c,U_c}_\tau)} \\
&\qquad \lesssim \|\langle r\rangle^{-1} S^{\leq 9} W_k\|_{\ell^2_\tau \ell^2_{U_c\leq c\tau/4} L^2_t L^2_r (\tilde{C}^{c,U_c}_\tau)} + \|S^{\leq 8} \partial_{\underline{u}_c} W_k\|_{\ell^2_\tau \ell^2_{U_c\leq c\tau/4} L^2_t L^2_r (\tilde{C}^{c,U_c}_\tau)} \\
&\qquad \lesssim (XII)_k + (XI)_k,
\end{aligned}$$

and (4.6) is then a direct consequence of (4.3). □

We now proceed to the pointwise bounds for V_k with $k \leq j-1$ using Lemma 2.3. We first consider the bound away from the light cone.

Proof of (4.7). Applying (2.10) and commuting S with $\Box$ and ∂, we have

$$\|\langle \underline{u}\rangle S^{\le 5}\partial V_k\|_{\ell^\infty_R \ell^2_{\tau\ge 2R} L^\infty_t L^\infty_r(C^{1,R}_\tau)} \lesssim \|\langle r\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial V_k\|_{\ell^\infty_R \ell^2_{\tau\ge 2R} L^2_t L^2_r(\tilde C^{1,R}_\tau)} + \|\langle r\rangle^{\frac14} r^{\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 6}\Box V_k\|_{\ell^\infty_R \ell^2_{\tau\ge 2R} L^2_t L^2_r(\tilde C^{1,R}_\tau)}.$$

Splitting ∂ into $(\partial_u, \partial_{\underline{u}})$, we bound the first term by

$$\|\langle r\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_k\|_{\ell^\infty_R \ell^2_{\tau\ge 2R} L^2_t L^2_r(\tilde C^{1,R}_\tau)} + \|\langle r\rangle^{-\frac12}\langle u\rangle^{\frac12} S^{\le 7}\partial_u V_k\|_{\ell^\infty_R \ell^2_{\tau\ge 2R} L^2_t L^2_r(\tilde C^{1,R}_\tau)} \lesssim (III)_k + (I)_k,$$

where we exploit that $u \approx \underline{u}$ on $\tilde C^{1,R}_\tau$ regions with $R \le \tau/2$. This term is $O(\varepsilon)$ by (4.3).

Moving to the second term, we crudely apply the product rule to $S^{\le 6}\Box V_k$, giving

$$|S^{\le 6}\Box V_k| \le r^{-1}|S^{\le 6} W_{k-1}||S^{\le 6}\partial_t V_{k-1}|, \tag{4.9}$$

where we note that $|S r^{-1}| = r^{-1}$. On $\tilde C^{1,R}_\tau$ with $R \le \tau/2$, we have

$$\langle r\rangle \lesssim \langle \underline{u}_c\rangle \approx \langle \underline{u}\rangle \approx \langle u\rangle \approx \tau.$$

This allows us to bound:

$$\begin{aligned}\|\langle r\rangle^{\frac14} r^{\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 6}\Box V_k\|_{\ell^\infty_R \ell^2_{\tau\ge 2R} L^2_t L^2_r(\tilde C^{1,R}_\tau)} &\lesssim \|r^{-\frac12}\langle \underline{u}_c\rangle^{\frac12} S^{\le 6} W_{k-1}\|_{\ell^\infty_\tau \ell^\infty_{R\le \tau/2} L^\infty_t L^\infty_r(\tilde C^{1,R}_\tau)} \\ &\times \Big(\|r^{-\frac14}\langle r\rangle^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 6}\partial_{\underline{u}} V_{k-1}\|_{\ell^\infty_R \ell^2_{\tau\ge 2R} L^2_t L^2_r(\tilde C^{1,R}_\tau)} + \|r^{-\frac14}\langle r\rangle^{-\frac14}\langle u\rangle^{\frac12} S^{\le 6}\partial_u V_{k-1}\|_{\ell^\infty_R \ell^2_{\tau\ge 2R} L^2_t L^2_r(\tilde C^{1,R}_\tau)}\Big).\end{aligned}$$

We note that since $c > 1$, each $\tilde C^{1,R}_\tau$ with $R \le \tau/2$ is contained in a finite number of regions $C^{c,\tilde R}_{\tilde\tau}$ with $\tilde R \le c\tilde\tau/2$. This allows us to apply (4.5) to the first factor in the right side, which shows that this is $\lesssim \varepsilon[(III)_{k-1} + (I)_{k-1}]$. The desired bound is then a consequence of (4.3). □

We finally consider the pointwise bound for ∂V_k away from the light cone.

Proof of (4.8). Fixing U and τ with $U \le \tau/4$, we start with an application of (2.11), which gives that

$$\|\langle u\rangle S^{\le 5}\partial V_k\|_{L^\infty_t L^\infty_r(C^{1,U}_\tau)} \lesssim \|\langle u\rangle^{\frac12}\langle \underline{u}\rangle^{-\frac12} S^{\le 7}\partial V_k\|_{L^2_t L^2_r(\tilde C^{1,U}_\tau)} + \|\langle u\rangle^{\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 6}\Box V_k\|_{L^2_t L^2_r(\tilde C^{1,U}_\tau)}. \tag{4.10}$$

For the first term, we have

$$\begin{aligned}\|\langle u\rangle^{\frac12}\langle \underline{u}\rangle^{-\frac12} S^{\le 7}\partial V_k\|_{L^2_t L^2_r(\tilde C^{1,U}_\tau)} &\lesssim \|\langle r\rangle^{-\frac12}\langle u\rangle^{\frac12} S^{\le 7}\partial_u V_k\|_{L^2_t L^2_r(\tilde C^{1,U}_\tau)} + \|\langle r\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_k\|_{L^2_t L^2_r(\tilde C^{1,U}_\tau)} \\ &\le (I)_k + (III)_k,\end{aligned}$$

and the bound is an immediate consequence of (4.3).

For the second term in (4.10), we use (4.9). In order to use (4.5) and (4.6), we will consider separately when $\tilde C^{1,U}_\tau$ intersects $\bigcup_{R\le c\tilde\tau/2} C^{c,R}_{\tilde\tau}$ and $\bigcup_{U_c\le c\tilde\tau/4} C^{c,U_c}_{\tilde\tau}$. Away from the speed c light cone, we have

$$\begin{aligned}\|\langle u\rangle^{\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 6}\Box V_k\|_{L^2_t L^2_r(\tilde C^{1,U}_\tau \cap (\bigcup_{\tilde\tau}\bigcup_{R\le c\tilde\tau/2} C^{c,R}_{\tilde\tau}))} &\lesssim \|\langle \underline{u}_c\rangle^{\frac12} r^{-\frac12} S^{\le 6} W_{k-1}\|_{\ell^\infty_{\tilde\tau} \ell^\infty_{R\le c\tilde\tau/2} L^\infty_t L^\infty_r(C^{c,R}_{\tilde\tau})} \\ &\times \Big(\|\langle r\rangle^{-\frac12}\langle u\rangle^{\frac12} S^{\le 6}\partial_u V_{k-1}\|_{L^2_t L^2_r(\tilde C^{1,U}_\tau)} + \|\langle r\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 6}\partial_{\underline{u}} V_{k-1}\|_{L^2_t L^2_r(\tilde C^{1,U}_\tau)}\Big),\end{aligned}$$

as $\tilde{C}^{1,U}_{\tau}$ only intersects $\bigcup_{R\le c\tilde{\tau}/2} C^{c,R}_{\tilde{\tau}}$ for a finite number of $\tilde{\tau}$. Since $\tilde{C}^{1,U}_{\tau}$ (for fixed U and τ) is contained in a finite number of dyadic regions $C^{1,R}_{\tilde{\tau}}$, it follows, using (4.5), that this is

$$\lesssim \varepsilon[(I)_{k-1} + (III)_{k-1}].$$

After applying (4.3), supremums can be taken over U and τ to obtain the desired result.

Near the speed c light cone, we instead see

$$\begin{aligned}\|\langle u\rangle^{\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 6}\Box V_k\|_{L^2_tL^2_r(\tilde{C}^{1,U}_{\tau}\cap(\bigcup_{\tilde{\tau}}\bigcup_{U_c\le c\tilde{\tau}/4} C^{c,U_c}_{\tilde{\tau}}))} &\lesssim \|\langle u_c\rangle^{\frac12} r^{-\frac12} S^{\le 6} W_{k-1}\|_{\ell^2_{\tilde{\tau}}\ell^2_{U_c\le c\tilde{\tau}/4}L^\infty_tL^\infty_r(C^{c,U_c}_{\tilde{\tau}})}\\ &\times\Big(\|\langle u_c\rangle^{-\frac12}\langle u\rangle^{\frac12} S^{\le 6}\partial_u V_{k-1}\|_{\ell^\infty_{\tilde{\tau}}\ell^\infty_{U_c}L^2_tL^2_r(C^{c,U_c}_{\tilde{\tau}})} + \|\langle u_c\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 6}\partial_{\underline{u}} V_{k-1}\|_{\ell^\infty_{\tilde{\tau}}\ell^\infty_{U_c}L^2_tL^2_r(\tilde{C}^{c,U_c}_{\tilde{\tau}})}\Big).\end{aligned}$$

By (4.6), this is bounded by

$$\varepsilon[(II)_{k-1} + (IV)_{k-1}].$$

Thus, an application of (4.3) and taking supremums over U, τ yields the result. □

4.1.2. Bound on terms $(I)_j, \dots, (V)_j$

By applying (3.7) (with $p = 1$) to $S^{\le 7}V_j$, we see that

$$\begin{aligned}(I)^2_j + \cdots + (V)^2_j \le (C_0\varepsilon)^2 + C\int_4^{T_\varepsilon}\int_0^\infty \langle \underline{u}\rangle |S^{\le 7}\Box V_j||\partial_{\underline{u}} S^{\le 7}V_j|\,dr\,dt\\ + C\int_4^{T_\varepsilon}\int_0^\infty \langle u\rangle |S^{\le 7}\Box V_j||\partial_u S^{\le 7}V_j|\,dr\,dt.\end{aligned}\tag{4.11}$$

We need to show that the latter two terms are bounded by

$$C\varepsilon^2(\log(2+T_\varepsilon))^{\frac12} M_j.\tag{4.12}$$

Due to the finite speed of propagation, we note $\partial_t V_{j-1}$, and hence $\Box V_j$, vanishes for $r \ge t-3$.

We shall first decompose these integrals using (2.2) at speed c and note that $C^1 \subset C^c$. For the first of the integrals in (4.11), we have

$$\begin{aligned}\|\langle \underline{u}\rangle S^{\le 7}\Box V_j\cdot S^{\le 7}\partial_{\underline{u}}V_j\|_{L^1_tL^1_r(C^1)} &\lesssim \|\langle \underline{u}\rangle S^{\le 7}\Box V_j\cdot S^{\le 7}\partial_{\underline{u}}V_j\|_{\ell^1_\tau L^1_tL^1_r(C^{c,R=1}_\tau)}\\ &+ \|\langle \underline{u}\rangle S^{\le 7}\Box V_j\cdot S^{\le 7}\partial_{\underline{u}}V_j\|_{\ell^1_\tau\ell^1_{1<R\le c\tau/2}L^1_tL^1_r(C^{c,R}_\tau)} + \|\langle \underline{u}\rangle S^{\le 7}\Box V_j\cdot S^{\le 7}\partial_{\underline{u}}V_j\|_{\ell^1_\tau\ell^1_{U_c\le c\tau/4}L^1_tL^1_r(C^{c,U_c}_\tau)}.\end{aligned}\tag{4.13}$$

A naive application of the product rule gives that

$$|S^{\le 7}\Box V_j| \le r^{-1}|S^{\le 7}W_{j-1}|\Big(|S^{\le 7}\partial_{\underline{u}}V_{j-1}| + |S^{\le 7}\partial_u V_{j-1}|\Big),$$

which we will apply in each instance.

For the first term on the right side of (4.13), since $\langle \underline{u}\rangle \approx \langle u\rangle$ on $C^{c,R=1}_\tau$, we have

$$\begin{aligned}\|\langle \underline{u}\rangle S^{\le 7}\Box V_j\cdot S^{\le 7}\partial_{\underline{u}}V_j\|_{\ell^1_\tau L^1_tL^1_r(C^{c,R=1}_\tau)} &\lesssim \|\langle \underline{u}_c\rangle^{\frac12} r^{-\frac12} S^{\le 7}W_{j-1}\|_{\ell^2_\tau L^\infty_t L^\infty_r(C^{c,R=1}_\tau)}\\ &\times\Big(\|\langle r\rangle^{-\frac14} r^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}}V_{j-1}\|_{\ell^\infty_\tau L^2_tL^2_r(C^{c,R=1}_\tau)} + \|\langle r\rangle^{-\frac14} r^{-\frac14}\langle u\rangle^{\frac12} S^{\le 7}\partial_u V_{j-1}\|_{\ell^\infty_\tau L^2_tL^2_r(C^{c,R=1}_\tau)}\Big)\end{aligned}$$

$$\times \|\langle r\rangle^{-\frac14} r^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^2_\tau L^2_t L^2_r(C^{c,R=1}_\tau)},$$

which, using (4.5), is bounded by $C\varepsilon[(III)_{j-1} + (I)_{j-1}](III)_j$. Since we have (4.3), this term is controlled by (4.12) as desired.

For the next term in (4.13), we use the Schwarz inequality and the facts that $\langle \underline{u}\rangle \approx \tau$ and $\langle r\rangle \lesssim \tau$ on $C^{c,R}_\tau$ with $R \le c\tau/2$ to see that

$$\begin{aligned}
&\|\langle \underline{u}\rangle S^{\le 7}\Box V_j \cdot S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^1_\tau \ell^1_{1<R\le c\tau/2} L^1_t L^1_r(C^{c,R}_\tau)} \\
&\quad\lesssim \|\langle \underline{u}_c\rangle^{\frac12} r^{-\frac12} S^{\le 7} W_{j-1}\|_{\ell^2_\tau \ell^2_{1<R\le c\tau/2} L^\infty_t L^\infty_r(C^{c,R}_\tau)} \|\langle r\rangle^{-\frac14} r^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_R L^2_t L^2_r(C^{c,R}_\tau)} \\
&\qquad\times \|\langle r\rangle^{-\frac14} r^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^2_\tau \ell^2_{1<R\le c\tau/2} L^2_t L^2_r(C^{c,R}_\tau)} \\
&\quad+ \|\langle \underline{u}_c\rangle^{\frac12} r^{-\frac12} S^{\le 7} W_{j-1}\|_{\ell^2_\tau \ell^2_{1<R\le c\tau/2} L^\infty_t L^\infty_r(C^{c,R}_\tau)} \|\langle r\rangle^{-\frac12}\langle u\rangle^{\frac12} S^{\le 7}\partial_u V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_R L^2_t L^2_r(C^{c,R}_\tau)} \\
&\qquad\times \|\langle u\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^2_\tau \ell^2_{1<R\le c\tau/2} L^2_t L^2_r(C^{c,R}_\tau)}.
\end{aligned} \tag{4.14}$$

For the last factor in each term, we sum back up and re-decompose in terms of speed 1 regions to see

$$\begin{aligned}
&\|r^{-\frac14}\langle r\rangle^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^2_\tau \ell^2_{1<R\le c\tau/2} L^2_t L^2_r(C^{c,R}_\tau)} + \|r^{-\frac14}\langle r\rangle^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^2_\tau \ell^2_{U_c\le c\tau/4} L^2_t L^2_r(C^{c,U_c}_\tau)} \\
&\qquad\lesssim \|r^{-\frac14}\langle r\rangle^{-\frac1r}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{L^2_t L^2_r(C^1)} \\
&\qquad\lesssim (\log(2+T_\varepsilon))^{\frac12}\|r^{-\frac14}\langle r\rangle^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^\infty_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)}
\end{aligned} \tag{4.15}$$

and similarly

$$\begin{aligned}
&\|\langle u\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^2_\tau \ell^2_{1<R\le c\tau/2} L^2_t L^2_r(C^{c,R}_\tau)} + \|\langle u\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^2_\tau \ell^2_{U_c\le c\tau/4} L^2_t L^2_r(C^{c,U_c}_\tau)} \\
&\qquad\lesssim (\log(2+T_\varepsilon))^{\frac12}\|\langle u\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}.
\end{aligned} \tag{4.16}$$

Applying (4.5), it then follows that the left side of (4.14) is bounded by

$$C\varepsilon(III)_{j-1}(\log(2+T_\varepsilon))^{\frac12}(III)_j + C\varepsilon(I)_{j-1}(\log(2+T_\varepsilon))^{\frac12}(V)_j,$$

which, owing to (4.3), is controlled by (4.12).

The last term in (4.13) is handled similarly, but we now must rely upon (4.6). Here we use the fact that $r \approx \tau \approx \langle \underline{u}\rangle$ on C^{c,U_c}_τ with $1 \le U_c \le c\tau/4$. Indeed,

$$\begin{aligned}
&\|\langle \underline{u}\rangle S^{\le 7}\Box V_j \cdot S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^1_\tau \ell^1_{U_c\le c\tau/4} L^1_t L^1_r(C^{c,U_c}_\tau)} \\
&\quad\lesssim \|\langle u_c\rangle^{\frac12} r^{-\frac12} S^{\le 7} W_{j-1}\|_{\ell^2_\tau \ell^2_{U_c\le c\tau/4} L^\infty_t L^\infty_r(C^{c,U_c}_\tau)} \|\langle u_c\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_{U_c} L^2_t L^2_r(C^{c,U_c}_\tau)} \\
&\qquad\times \|\langle r\rangle^{-\frac14} r^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^2_\tau \ell^2_{U_c\le c\tau/4} L^2_t L^2_r(C^{c,U_c}_\tau)} \\
&\quad+ \|\langle u_c\rangle^{\frac12} r^{-\frac12} S^{\le 7} W_{j-1}\|_{\ell^2_\tau \ell^2_{U_c\le c\tau/4} L^\infty_t L^\infty_r(C^{c,U_c}_\tau)} \|\langle u_c\rangle^{-\frac12}\langle u\rangle^{\frac12} S^{\le 7}\partial_u V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_{U_c} L^2_t L^2_r(C^{c,U_c}_\tau)} \\
&\qquad\times \|\langle u\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}} V_j\|_{\ell^2_\tau \ell^2_{U_c\le c\tau/4} L^2_t L^2_r(C^{c,U_c}_\tau)}.
\end{aligned} \tag{4.17}$$

Using (4.6), (4.15), and (4.16), it follows that this is

$$\lesssim \varepsilon(IV)_{j-1}(\log(2+T_\varepsilon))^{\frac12}(III)_j + \varepsilon(IV)_{j-1}(\log(2+T_\varepsilon))^{\frac12}(V)_j.$$

The inductive hypothesis (4.3) then gives that this is bounded by (4.12) as desired.

The same strategy bounds the second integral in (4.11). In fact, since $\langle u\rangle \lesssim \langle \underline{u}\rangle$ in all regions, the argument can be simplified. Indeed, we have

$$\begin{aligned}\|\langle u\rangle S^{\leq 7}\Box V_j\cdot S^{\leq 7}\partial_u V_j\|_{\ell^1_\tau \ell^1_{R\leq c\tau/2}L^1_t L^1_r(C^{c,R}_\tau)} &\lesssim \|\langle \tau\rangle^{\frac{1}{2}} r^{-\frac{1}{2}} S^{\leq 7} W_{j-1}\|_{\ell^2_\tau \ell^2_{R\leq c\tau/2}L^\infty_t L^\infty_r(C^{c,R}_\tau)}\\ &\times\Big(\|\langle r\rangle^{-\frac{1}{4}} r^{-\frac{1}{4}}\langle \underline{u}\rangle^{\frac{1}{2}} S^{\leq 7}\partial_{\underline{u}} V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_R L^2_t L^2_r(C^{c,R}_\tau)} + \|\langle r\rangle^{-\frac{1}{4}} r^{-\frac{1}{4}}\langle u\rangle^{\frac{1}{2}} S^{\leq 7}\partial_u V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_R L^2_t L^2_r(C^{c,R}_\tau)}\Big)\\ &\times \|\langle r\rangle^{-\frac{1}{4}} r^{-\frac{1}{4}}\langle u\rangle^{\frac{1}{2}} S^{\leq 7}\partial_u V_j\|_{\ell^2_\tau \ell^2_{R\leq c\tau/2}L^2_t L^2_r(C^{c,R}_\tau)}. \end{aligned}\tag{4.18}$$

Using a direct analog of (4.15) and (4.5), this is

$$\lesssim \varepsilon((III)_{j-1} + (I)_{j-1})(\log(2+T_\varepsilon))^{\frac{1}{2}}(I)_j,$$

which is in turn controlled by (4.12).

And

$$\begin{aligned}\|\langle u\rangle S^{\leq 7}\Box V_j\cdot S^{\leq 7}\partial_u V_j\|_{\ell^1_\tau \ell^1_{U_c\leq c\tau/4}L^1_t L^1_r(C^{c,U_c}_\tau)} &\lesssim \|\langle u_c\rangle^{\frac{1}{2}} r^{-\frac{1}{2}} S^{\leq 7} W_{j-1}\|_{\ell^2_\tau \ell^2_{U_c\leq c\tau/4}L^\infty_t L^\infty_r(C^{c,U_c}_\tau)}\\ &\times\Big(\|\langle u_c\rangle^{-\frac{1}{2}}\langle \underline{u}\rangle^{\frac{1}{2}} S^{\leq 7}\partial_{\underline{u}} V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_{U_c} L^2_t L^2_r(C^{c,U_c}_\tau)} + \|\langle u_c\rangle^{-\frac{1}{2}}\langle u\rangle^{\frac{1}{2}} S^{\leq 7}\partial_u V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_{U_c} L^2_t L^2_r(C^{c,U_c}_\tau)}\Big)\\ &\times \|\langle r\rangle^{-\frac{1}{4}} r^{-\frac{1}{4}}\langle u\rangle^{\frac{1}{2}} S^{\leq 7}\partial_u V_j\|_{\ell^2_\tau \ell^2_{U_c\leq c\tau/4}L^2_t L^2_r(C^{c,U_c}_\tau)}. \end{aligned}\tag{4.19}$$

Using the analog of (4.16) and (4.6), this is

$$\lesssim \varepsilon((IV)_{j-1} + (II)_{j-1})(\log(2+T_\varepsilon))^{\frac{1}{2}}(I)_j,$$

which is in turn controlled by (4.12), after applying (4.3). This completes the proof of the boundedness of terms $(I)_j,\dots,(V)_j$.

4.1.3. Bound on terms $(VI)_j,\dots,(VIII)_j$

We now proceed to considering the high-order energy bounds for V_j. We begin by applying (3.7) (with $p=0$) to $S^{\leq 10}V_j$ to obtain

$$(VI)_j^2 + (VII)_j^2 + (VIII)_j^2 \leq (C_0\varepsilon)^2 + C\int_4^{T_\varepsilon}\int |\Box S^{\leq 10}V_j||\partial S^{\leq 10}V_j|\,dr\,dt.$$

Again, we need to show that the last term is bounded by (4.12). This time, however, we need to apply the product rule more carefully. We note that in each term, there will be one factor with no more than half of the 10 total vector fields. As such, we have

$$|S^{\leq 10}\Box V_j| \leq r^{-1}|S^{\leq 5}W_{j-1}||S^{\leq 10}\partial V_{j-1}| + r^{-1}|S^{\leq 10}W_{j-1}||S^{\leq 5}\partial V_{j-1}|.$$

When W is lower order, we shall initially decompose in speed c regions. In $C^{c,R}_\tau$ regions, since $\langle r\rangle \lesssim \tau$, we bound

$$\|r^{-1}S^{\leq 5}W_{j-1}S^{\leq 10}\partial V_{j-1}S^{\leq 10}\partial V_j\|_{\ell^1_\tau \ell^1_{R\leq c\tau/2}L^1_t L^1_r(C^{c,R}_\tau)} \lesssim \|\langle \underline{u}_c\rangle^{\frac{1}{2}} r^{-\frac{1}{2}} S^{\leq 5} W_{j-1}\|_{\ell^2_\tau \ell^2_{R\leq c\tau/2}L^\infty_t L^\infty_r(C^{c,R}_\tau)}$$

$$\times \|r^{-\frac{1}{4}}\langle r\rangle^{-\frac{1}{4}} S^{\le 10}\partial V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_R L^2_t L^2_r(C^{c,R}_\tau)} \|r^{-\frac{1}{4}}\langle r\rangle^{-\frac{1}{4}} S^{\le 10}\partial V_j\|_{\ell^2_\tau \ell^2_{R\le c\tau/2} L^2_t L^2_r(C^{c,R}_\tau)}. \quad (4.20)$$

Arguing as in (4.15), the last factor satisfies

$$\|r^{-\frac{1}{4}}\langle r\rangle^{-\frac{1}{4}} S^{\le 10}\partial V_j\|_{\ell^2_\tau \ell^2_{R\le c\tau/2} L^2_t L^2_r(C^{c,R}_\tau)} + \|r^{-\frac{1}{4}}\langle r\rangle^{-\frac{1}{4}} S^{\le 10}\partial V_j\|_{\ell^2_\tau \ell^2_{U_c\le c\tau/4} L^2_t L^2_r(C^{c,U_c}_\tau)}$$
$$\lesssim (\log(2+T_\varepsilon))^{\frac{1}{2}} \|r^{-\frac{1}{4}}\langle r\rangle^{-\frac{1}{4}} S^{\le 10}\partial V_j\|_{\ell^\infty_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)}. \quad (4.21)$$

Using this and (4.5), we see that the left side of (4.20) is

$$\lesssim \varepsilon (VI)_{j-1} (\log(2+T_\varepsilon))^{\frac{1}{2}} (VI)_j,$$

and thus, due to (4.3), is controlled by (4.12). In the C^{c,U_c}_τ regions, we instead have

$$\|r^{-1} S^{\le 5} W_{j-1} S^{\le 10}\partial V_{j-1} S^{\le 10}\partial V_j\|_{\ell^1_\tau \ell^1_{U_c\le c\tau/4} L^1_t L^1_r(C^{c,U_c}_\tau)} \lesssim \|\langle u_c\rangle^{\frac{1}{2}}\langle r\rangle^{-\frac{1}{2}} S^{\le 5} W_{j-1}\|_{\ell^2_\tau \ell^2_{U_c\le c\tau/4} L^\infty_t L^\infty_r(C^{c,U_c}_\tau)}$$
$$\times \|\langle u_c\rangle^{-\frac{1}{2}} S^{\le 10}\partial V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_{U_c} L^2_t L^2_r(C^{c,U_c}_\tau)} \|r^{-\frac{1}{4}}\langle r\rangle^{-\frac{1}{4}} S^{\le 10}\partial V_j\|_{\ell^2_\tau \ell^2_{U_c\le c\tau/4} L^2_t L^2_r(C^{c,U_c}_\tau)}. \quad (4.22)$$

By (4.6) and (4.21), this is

$$\lesssim \varepsilon (VII)_{j-1} (\log(2+T_\varepsilon))^{\frac{1}{2}} (VI)_j,$$

which is in turn bounded by (4.12) upon applying (4.3).

When ∂V is lower order, we instead use the speed 1 decomposition. Away from the light cone, since $\langle r\rangle \le \tau$ on $C^{1,R}_\tau$ with $R \le \tau/2$, we obtain

$$\|r^{-1} S^{\le 5}\partial V_{j-1} S^{\le 10} W_{j-1} S^{\le 10}\partial V_j\|_{\ell^1_R \ell^1_{\tau\ge 2R} L^1_t L^1_r(C^{1,R}_\tau)}$$
$$\lesssim \|\langle \underline{u}\rangle S^{\le 5}\partial V_{j-1}\|_{\ell^\infty_R \ell^2_{\tau\ge 2R} L^\infty_t L^\infty_r(C^{1,R}_\tau)} \|r^{-1} S^{\le 10} W_{j-1}\|_{\ell^2_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)}$$
$$\times \|\langle r\rangle^{-\frac{1}{2}} S^{\le 10}\partial V_j\|_{\ell^\infty_R \ell^\infty_\tau L^2_t L^2_r(C^{1,R}_\tau)}.$$

Here we have used the additional power $\langle \underline{u}\rangle^{-\frac{1}{2}} \le \langle r\rangle^{-\frac{1}{2}}$ to control the remaining dyadic sum over R. We may apply (4.7) to then see that this is $\lesssim \varepsilon (XI)_{j-1} (VI)_j$, which by (4.3) is better than the required bound (4.12).

When ∂V is lower order and we are near the speed 1 light cone, we instead have

$$\|r^{-1} S^{\le 5}\partial V_{j-1} S^{\le 10} W_{j-1} S^{\le 10}\partial V_j\|_{\ell^1_U \ell^1_{\tau\ge 4U} L^1_t L^1_r(C^{1,U}_\tau)}$$
$$\lesssim \|\langle u\rangle S^{\le 5}\partial V_{j-1}\|_{\ell^\infty_\tau \ell^\infty_{U\ge\tau/4} L^\infty_t L^\infty_r(C^{1,U}_\tau)} \|\langle u\rangle^{-\frac{1}{2}} r^{-\frac{1}{2}} S^{\le 10} W_{j-1}\|_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}$$
$$\times \|\langle r\rangle^{-\frac{1}{2}} S^{\le 10}\partial V_j\|_{\ell^2_\tau \ell^2_{U\le\tau/4} L^2_t L^2_r(C^{1,U}_\tau)}.$$

Here we have again gained ℓ^2_U summability from the extra factor of $\langle u\rangle^{-\frac{1}{2}}$. We note that the last factor satisfies

$$\|\langle r\rangle^{-\frac{1}{2}} S^{\le 10}\partial V_j\|_{\ell^2_\tau \ell^2_{U\le\tau/4} L^2_t L^2_r(C^{1,U}_\tau)} \lesssim \|\langle r\rangle^{-\frac{1}{2}} S^{\le 10}\partial V_j\|_{\ell^2_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)}$$
$$\lesssim (\log(2+T_\varepsilon))^{\frac{1}{2}} \|\langle r\rangle^{-\frac{1}{2}} S^{\le 10}\partial V_j\|_{\ell^\infty_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)}. \quad (4.23)$$

It follows from (4.8) that

$$\|r^{-1} S^{\le 5}\partial V_{j-1} S^{\le 10} W_{j-1} S^{\le 10}\partial V_j\|_{\ell^1_U \ell^1_{\tau\ge 4U} L^1_t L^1_r(C^{1,U}_\tau)} \lesssim \varepsilon (X)_{j-1} (\log(2+T_\varepsilon))^{\frac{1}{2}} (VI)_j.$$

This gives the bound by (4.12) upon using (4.3).

4.1.4. Bound on terms $(IX)_j, \dots, (XII)_j$

Applying (3.10), (3.11), and (3.13) to $S^{\le 10}W_j$ gives

$$(IX)_j^2 + \cdots + (XII)_j^2 \le (C_0\varepsilon)^2 + C\int_4^{T_\varepsilon}\int r|\Box_c S^{\le 10}W_j||\partial_{\underline{u}_c} S^{\le 10}W_j|\,dr\,dt.$$

We again seek to show that the last term is bounded by (4.12). The decomposition of the integral will be at speed 1 throughout. The product rule yields

$$|S^{\le 10}\Box_c W_j| \le r^{-1}|S^{\le 5}\partial V_{j-1}||S^{\le 10}\partial V_{j-1}|.$$

Away from the speed 1 light cone, we have

$$\begin{aligned}\|rS^{\le 10}\Box_c W_j \cdot S^{\le 10}\partial_{\underline{u}_c} W_j\|_{\ell^1_\tau \ell^1_{R\le\tau/2}L^1_t L^1_r(C^{1,R}_\tau)} &\lesssim \|\langle \underline{u}\rangle S^{\le 5}\partial V_{j-1}\|_{\ell^\infty_R \ell^\infty_{\tau\ge 2R}L^\infty_t L^\infty_r(C^{1,R}_\tau)}\\ &\times \|\langle r\rangle^{-\frac12} S^{\le 10}\partial V_{j-1}\|_{\ell^\infty_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)}\|S^{\le 10}\partial_{\underline{u}_c} W_j\|_{\ell^2_\tau \ell^2_R L^2_t L^2_r(C^{1,R}_\tau)}.\end{aligned}$$

Here, we have used that $\langle r\rangle \lesssim \langle \underline{u}\rangle$. Moreover, the factor of $\langle \underline{u}\rangle^{-\frac12}$ is used to control the remaining ℓ^2_R summation. As (4.7) gives that this is $\lesssim \varepsilon(VI)_{j-1}(XI)_j$, (4.3) shows that these terms are controlled by (4.12) (without the logarithmic factor, in fact).

Near the speed 1 light cone, we use the Schwarz inequality to bound

$$\begin{aligned}\|rS^{\le 10}\Box_c W_j \cdot S^{\le 10}\partial_{\underline{u}_c} W_j\|_{\ell^1_\tau \ell^1_{U\le\tau/4}L^1_t L^1_r(C^{1,U}_\tau)} &\lesssim \|\langle u\rangle S^{\le 5}\partial V_{j-1}\|_{\ell^\infty_U \ell^\infty_{\tau\ge 4U}L^\infty_t L^\infty_r(C^{1,U}_\tau)}\\ &\times \|\langle r\rangle^{-\frac12} S^{\le 10}\partial V_{j-1}\|_{\ell^2_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}\|r^{\frac12}\langle u\rangle^{-\frac12} S^{\le 10}\partial_{\underline{u}_c} W_j\|_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}.\end{aligned}$$

The additional $\langle u\rangle^{-\frac12}$ allowed us to absorb a square summation in U. Upon applying (4.23) and (4.8), these final terms are

$$\lesssim \varepsilon(\log(2+T_\varepsilon))^{\frac12}(VI)_{j-1}(X)_j.$$

Thus, (4.3) gives that they are controlled by (4.12). This completes the proof of (4.4).

4.2. Convergence

We now show that the sequence $((V_j, W_j))$ converges by showing that it is Cauchy in an appropriate norm. With this aim, we set

$$\begin{aligned}A_j = {}&\|r^{-\frac14}\langle r\rangle^{-\frac14}\langle u\rangle^{\frac12} S^{\le 7}\partial_u(V_j - V_{j-1})\|_{\ell^\infty_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)} + \|\langle u_c\rangle^{-\frac12}\langle u\rangle^{\frac12} S^{\le 7}\partial_u(V_j - V_{j-1})\|_{\ell^\infty_{U_c} \ell^2_\tau L^2_t L^2_r(C^{c,U_c}_\tau)}\\ &+ \|r^{-\frac14}\langle r\rangle^{-\frac14}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}}(V_j - V_{j-1})\|_{\ell^\infty_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)} + \|\langle u_c\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}}(V_j - V_{j-1})\|_{\ell^\infty_{U_c} \ell^2_\tau L^2_t L^2_r(C^{c,U_c}_\tau)}\\ &+ \|\langle u\rangle^{-\frac12}\langle \underline{u}\rangle^{\frac12} S^{\le 7}\partial_{\underline{u}}(V_j - V_{j-1})\|_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)} + \|r^{-\frac14}\langle r\rangle^{-\frac14} S^{\le 10}\partial(V_j - V_{j-1})\|_{\ell^\infty_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)}\\ &+ \|\langle u_c\rangle^{-\frac12} S^{\le 10}\partial(V_j - V_{j-1})\|_{\ell^\infty_{U_c} \ell^2_\tau L^2_t L^2_r(C^{c,U_c}_\tau)} + \|\langle u\rangle^{-\frac12} S^{\le 10}\partial_{\underline{u}}(V_j - V_{j-1})\|_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}\\ &+ \|r^{\frac12}\langle u\rangle^{-\frac12} S^{\le 10}\partial_{\underline{u}_c}(W_j - W_{j-1})\|_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)} + \|r^{-\frac12}\langle u\rangle^{-\frac12} S^{\le 10}(W_j - W_{j-1})\|_{\ell^\infty_U \ell^2_\tau L^2_t L^2_r(C^{1,U}_\tau)}\\ &+ \|S^{\le 10}\partial_{\underline{u}_c}(W_j - W_{j-1})\|_{\ell^2_R \ell^2_\tau L^2_t L^2_r(C^{1,R}_\tau)} + \|r^{-1} S^{\le 10}(W_j - W_{j-1})\|_{\ell^2_R \ell^2_\tau L^2_t L^2_r(C^{c,R}_\tau)},\end{aligned} \quad (4.24)$$

and we will show that, for each j,

$$A_j \leq \frac{1}{2} A_{j-1}. \tag{4.25}$$

We note that

$$\Box(V_j - V_{j-1}) = r^{-1}\Big((W_{j-1} - W_{j-2})\partial_t V_{j-1} + W_{j-2}\partial_t(V_{j-1} - V_{j-2})\Big)$$

and

$$\Box_c(W_j - W_{j-1}) = r^{-1}\partial_t(V_{j-1} - V_{j-2})(\partial_t V_{j-1} + \partial_t V_{j-2}).$$

For this system, we call upon the same arguments as in the proof of (4.4). Doing so yields

$$A_j \lesssim A_{j-1}(M_{j-1} + M_{j-2}),$$

and applying (4.4) with ε sufficiently small completes the proof.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author Contributions

Marvin Koonce: Draft writing Section 4, writing-review & editing; Jason Metcalfe: Draft writing Section 1–3, writing-review & editing, research supervision.

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Conflict of interest

The authors declare there is no conflict of interest.

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